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Abstract. We correct a gap in two lemmas in [2], providing a new proof of the main results of that paper for hyperbolic and strongly elliptic self-maps of a bounded strongly convex domain with C^2 boundary.

We have found a gap in the proofs of Lemmas 2.2 and 2.5 of our paper [2]. In this note we fill these gaps, giving a proof of the main results using different arguments.

More precisely we prove the following version of [2, Theorem 0.1]:

Theorem 1. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be either hyperbolic or strongly elliptic, with Wolff point $\tau \in \overline{D}$. Let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step. Then:*

(i) *the sequence $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$;*
(ii) *if $\sigma \neq \tau$ then σ is repelling;*
(iii) *$\sigma \neq \tau$ if and only if $\{z_k\}$ goes to σ inside a K-region, that is, there exists $M > 0$ so that $z_k \in K_p(\sigma, M)$ eventually, where p is any point in D.*

Remark 2. *If f is strongly elliptic then clearly $\sigma \neq \tau$. We conjecture that $\sigma \neq \tau$ in the hyperbolic case too.*

Remark 3. *The following proof does not work in the parabolic case, considered in the original version of* [2, Theorem 0.1]. *Thus the behavior of backward orbits for a parabolic self-map is still not understood, even (as far as we know) in the unit ball of $\mathbb{C}^n$ (see* [4]*).*

Proof. The proof is divided into two cases according to whether f is hyperbolic or strongly elliptic. We will freely use the notations introduced in [2].

Hyperbolic case.

We begin by proving part (i) following the approach already indicated in [2, Remark 2.1].

Lemma 4. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step $a > 0$. Then $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$.*

Proof. First of all, recall that [3, Lemma 2.4 and Remark 3] yields a constant $C_1 > 0$ such that

$$\|z_k - z_{k+1}\|^2 + |\langle z_k - z_{k+1}, z_k \rangle| \leq \frac{C_1^2}{1-\hat{a}^2} d(z_k, \partial D), \tag{1}$$

and so

$$\|z_k - z_{k+1}\| \leq \frac{C_1}{\sqrt{1-\hat{a}^2}} \sqrt{d(z_k, \partial D)} \leq \frac{C_1}{1-\hat{a}} \sqrt{d(z_k, \partial D)}\ , \tag{2}$$

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where $\hat{a} = \tanh a \in (0,1)$. On the other hand, given $p \in D$ the triangular inequality and the upper estimate [1, Theorem 2.3.51] on the boundary behaviour of the Kobayashi distance yield a constant $C_2 > 0$ such that

$$\tfrac{1}{2}\log h_{\tau,p}(z_k) \le k_D(p, z_k) \le C_2 - \tfrac{1}{2}\log d(z_k, \partial D) ,$$

that is

$$d(z_k, \partial D) \le \frac{e^{2C_2}}{h_{\tau,p}(z_k)} , \tag{3}$$

and thus

$$\|z_k - z_{k+1}\| \le \frac{C}{1-\hat{a}}\sqrt{\frac{1}{h_{\tau,p}(z_k)}} , \tag{4}$$

for a suitable $C > 0$. Therefore using [2, (2.1)] we obtain that for every $k, m \ge 0$ we have

$$\begin{aligned} \|z_k - z_{k+m}\| &\le \sum_{j=k}^{k+m-1} \|z_j - z_{j+1}\| \le \frac{C}{1-\hat{a}} \frac{1}{\sqrt{h_{\tau,p}(z_k)}} \sum_{j=0}^{m-1} \beta_\tau^{j/2} \\ &\le \frac{C}{1-\hat{a}} \frac{1}{1-\beta_\tau^{1/2}} \frac{1}{\sqrt{h_{\tau,p}(z_k)}} . \end{aligned} \tag{5}$$

Since $h_{p.\tau}(z_k) \to +\infty$ as $k \to +\infty$ by [2, Lemma 2.6] it follows that $\{z_k\}$ is a Cauchy sequence in $\mathbb{C}^n$, converging to a point σ, necessarily belonging to ∂D by [2, Lemma 2.1]. The proof is then completed by quoting [2, Lemma 2.3]. □

The following lemma, whose proof is identical to the proof of [2, Lemma 2.4], allows us to control the dilation coefficient at the limit of a backward orbit, giving in particular part (ii) of Theorem 1 in the hyperbolic case.

Lemma 5. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in$* $\mathrm{Hol}(D, D)$ *be hyperbolic or parabolic with Wolff point $\tau \in \partial D$ and dilation coefficient* $0 < \beta_\tau \le 1$. *Let $\sigma \in \partial D \setminus \{\tau\}$ be a boundary fixed point with finite dilation coefficient β_σ. Then*

$$\beta_\sigma \ge \frac{1}{\beta_\tau} \ge 1 .$$

In particular, if f is hyperbolic then σ is repelling.

Proof. Argue as in the proof of [2, Lemma 2.4]. □

To deal with K-regions, we need the following remark.

Remark 6. *In strongly convex domains K-regions are comparable to* Stein admissible approach regions $A(\sigma, M)$ of vertex $\sigma \in \partial D$ and aperture $M > 1$*:*

$$A(\sigma, M) = \{z \in D \mid \|z-\sigma\|^2 < Md(z, \partial D), |\langle z-\sigma, n_\sigma\rangle| < Md(z, \partial D)\} , \tag{6}$$

where n_σ is the outer unit normal vector to ∂D at σ. Here "comparable" means that for every $\sigma \in \partial D$ there exists a neighbourhood $U \subset \mathbb{C}^n$ of σ such that for any $M > 1$ and $p \in D$ there are M_1, $M_2 > 1$ such that

$$A(\sigma, M_1) \cap U \subseteq K_p(\sigma, M) \cap U \subseteq A(\sigma, M_2) \cap U ;$$

see, e.g., [1, Propositions 2.7.4, 2.7.6 and p. 380].

We can now prove the first half of Theorem 1.(iii) for the hyperbolic case.

Lemma 7. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau < 1$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a > 0$ converging to $\sigma \in \partial D \setminus \{\tau\}$. Then for every $p \in D$ there exists $M > 0$ such that $z_k \in K_p(\sigma, M)$ eventually.*

Proof. Fix $p \in D$. By Remark 6 it suffices to prove that there exists $M > 1$ such that $\{z_k\}$ converges to σ inside an admissible approach region $A(\sigma, M)$.

Set $t_k := h_{\tau,p}(z_k)$. Thanks to [2, (2.1)] we have

$$\frac{1}{t_{k+m}} \leq \beta_\tau^m \frac{1}{t_k} \tag{7}$$

for all k, $m \geq 0$. Moreover, thanks to [1, Corollary 2.3.55], since $\sigma \neq \tau$, there exists $\varepsilon > 0$ and $K > 0$ such that for any $w \in D \cap B(\tau, \varepsilon)$ and $k \in \mathbb{N}$ such that $z_k \in D \cap B(\sigma, \varepsilon)$ we have

$$k_D(z_k, w) \geq -\tfrac{1}{2} \log d(z_k, \partial D) - \tfrac{1}{2} \log d(w, \partial D) + K ,$$

where $B(x, \varepsilon)$ is the Euclidean ball of center x and radius ε.

On the other hand, [1, Theorem 2.3.51] yields $c_1 \in \mathbb{R}$ such that

$$k_D(w, p) \leq c_1 - \tfrac{1}{2} \log d(w, \partial D)$$

for any $w \in D$. So for $w \in D \cap B(\tau, \varepsilon)$ and k sufficiently large we have

$$k_D(z_k, w) - k_D(w, p) \geq -\tfrac{1}{2} \log d(z_k, \partial D) - \tfrac{1}{2} \log d(w, \partial D) + \tfrac{1}{2} \log d(w, \partial D) - c_1 + K ,$$

which implies

$$t_k = h_{\tau,p}(z_k) = \lim_{w \to \tau} [k_D(z_k, w) - k_D(w, p)] \geq -\tfrac{1}{2} \log d(z_k, \partial D) + K - c_1 ,$$

that is

$$\frac{1}{t_k} \leq \widetilde{C}_1 d(z_k, \partial D), \tag{8}$$

for some $\widetilde{C}_1 > 0$.

Therefore, thanks to (5), for all $m \geq 0$ and k large enough we have

$$\|z_k - z_{k+m}\| \leq \frac{C\widetilde{C}_1}{1 - \hat{a}} \frac{1}{1 - \beta_\tau^{1/2}} \sqrt{d(z_k, \partial D)} \tag{9}$$

for some $C > 0$, where $\hat{a} = \tanh a$, and letting m tend to infinity we obtain that for k sufficiently large there is $M_1 > 1$ such that

$$\|z_k - \sigma\| < M_1 \sqrt{d(z_k, \partial D)}. \tag{10}$$

On the other hand, up to translating the domain, without loss of generality we can assume that D contains the origin. In particular, D being bounded and strongly convex, we can replace n_σ by σ in the definition of $A(\sigma, M)$. Therefore, to conclude the proof it suffices to prove that there exists $M_2 > 1$ such that

$$|\langle z_k - \sigma, \sigma \rangle| \leq M_2 d(z_k, \partial D)$$

for k large enough. Now

$$|\langle z_j - z_{j+1}, z_j - \sigma \rangle| \leq \|z_j - z_{j+1}\| \|z_j - \sigma\|,$$

and so, thanks to (1), (8) and (10), for k large enough and $m \geq 0$ we have

$$
\begin{aligned}
|\langle z_k - z_{k+m}, \sigma\rangle| &\leq \sum_{j=k}^{k+m-1} |\langle z_j - z_{j+1}, \sigma\rangle| \\
&\leq \sum_{j=k}^{k+m-1} \Big(|\langle z_j - z_{j+1}, z_j - \sigma\rangle| + |\langle z_j - z_{j+1}, z_j\rangle| \Big) \\
&\leq \sum_{j=k}^{k+m-1} \Big(\|z_j - z_{j+1}\| \|z_j - \sigma\| + \frac{C_1^2}{1-\hat{a}^2} d(z_j, \partial D) \Big) \qquad (11) \\
&\leq \sum_{j=k}^{k+m-1} \Big(\frac{M_1 C_1}{1-\hat{a}} d(z_j, \partial D) + \frac{C_1^2}{1-\hat{a}^2} d(z_j, \partial D) \Big) \\
&\leq C' \sum_{j=k}^{k+m-1} d(z_j, \partial D),
\end{aligned}
$$

for some $C' > 0$. Arguing as in (5), using (3), (7) and (8) we obtain

$$|\langle z_k - z_{k+m}, \sigma\rangle| \leq M_2 d(z_k, \partial D)$$

for $m \geq 0$, k large enough and for some $M_2 > 1$. Letting m tend to infinity we finally have

$$|\langle z_k - \sigma, \sigma\rangle| \leq M_2 d(z_k, \partial D).$$

as claimed. □

The following lemma completes the proof of Theorem 1.(iii):

Lemma 8. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be hyperbolic with Wolff point $\tau \in \partial D$ and dilation coefficient $0 < \beta_\tau < 1$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step converging to $\sigma \in \partial D \setminus \{\tau\}$ inside a K-region. Then $\sigma \neq \tau$.*

Proof. Assume, by contradiction, that $\sigma = \tau$. Fix $p \in D$, and let $M > 1$ be such that $z_k \in K_p(\tau, M)$. Given $\varepsilon > 0$, [1, Lemma 2.7.1] yields $r > 0$ such that if $k_D(z_k, p) \geq r$ then $z_k \in E_p(\tau, \varepsilon)$, that is $h_{\tau,p}(z_k) < \varepsilon$. Since $k_D(z_k, p) \to +\infty$, it follows that $h_{\tau,p}(z_k) \to 0$ as $k \to +\infty$. But [2, Lemma 2.6] implies that $h_{\tau,p}(z_k) \to +\infty$, contradiction. □

Strongly elliptic case. We start by proving by contradiction that any backward orbit has to accumulate to the boundary of the domain D.

Lemma 9. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D, D)$ be strongly elliptic with Wolff point $p \in D$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$. Then $z_k \to \partial D$ as $k \to +\infty$.*

Proof. Define $\ell_k > 0$ by setting $\frac{1}{2}\log\ell_k = k_D(z_k, p)$. Since f is strongly elliptic, we have

$$k_D(z_k, p) < k_D(z_{k+1}, p)\ ,$$

and thus the sequence $\{\ell_k\}$ is strictly increasing. Assume, by contradiction, that it has a finite limit ℓ_∞. This means that every limit point z_∞ of the sequence $\{z_k\}$ satisfies $k_D(z_\infty, p) = \frac{1}{2}\log\ell_\infty$. But $f(z_\infty)$ is a limit point of the sequence $\{f(z_k)\} = \{z_{k-1}\}$ and thus we again have $k_D\big(f(z_\infty), p\big) = \frac{1}{2}\log\ell_\infty$, which is impossible by [2, Lemma 1.1] because f is strongly elliptic. Therefore $\ell_\infty = +\infty$, which means that $z_k \to \partial D$. □

This allows us to prove the following key result.

Lemma 10. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic with Wolff point $p \in D$. Let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step. Then there exists a constant $0 < c < 1$ such that*

$$k_D(z_k, p) - k_D(z_{k+1}, p) \le \tfrac{1}{2}\log c < 0$$

for all $k \in \mathbb{N}$.

Proof. Assume, by contradiction, that for every $0 < c < 1$ there is $k(c) \in \mathbb{N}$ such that

$$k_D(z_{k(c)}, p) - k_D(z_{k(c)+1}, p) > \tfrac{1}{2}\log c\ ,$$

that is

$$k_D(z_{k(c)+1}, p) - k_D\big(f(z_{k(c)+1}), p\big) < -\tfrac{1}{2}\log c\ .$$

Consider the sequences $\{z_{k(1-\frac{1}{j})+1}\}$ and $\{z_{k(1-\frac{1}{j})} = f(z_{k(1-\frac{1}{j})+1})\}$. Thanks to Lemma 9, we know that both these sequences accumulate on ∂D; therefore, by extracting subsequences, we can find a subsequence $\{z_{k_j}\}$ such that $z_{k_j} \to \sigma_1 \in \partial D$, $f(z_{k_j}) \to \sigma_2 \in \partial D$ as $j \to +\infty$ and

$$\lim_{j\to+\infty}\big[k_D(z_{k_j}, p) - k_D\big(f(z_{k_j}), p\big)\big] \le 0\ .$$

If $\sigma_1 \neq \sigma_2$, then [1, Corollary 2.3.55], together with the fact that $\{z_k\}$ has bounded Kobayashi step, lead to a contradiction since for k large enough there is $K \in \mathbb{R}$ such that

$$a \ge k_D\big(z_{k_j}, f(z_{k_j})\big) \ge -\tfrac{1}{2}\log d(z_{k_j}, \partial D) - \tfrac{1}{2}\log d\big(f(z_{k_j}), \partial D\big) + K$$

whereas the right-hand side tends to infinity. Therefore, $\sigma_1 = \sigma_2$ and we have

$$\liminf_{z\to\sigma_1}\big[k_D(z, p) - k_D\big(f(z), p\big)\big] \le 0\ .$$

Then we can apply [1, Proposition 2.4.15, Theorem 2.4.16 and Proposition 2.7.15], obtaining that σ_1 is a boundary fixed point and that for any $R > 0$ we have $f\big(E_p(\sigma_1, R)\big) \subseteq E_p(\sigma_1, R)$. We can then choose $R < 1$ so that $p \notin \overline{E_p(\sigma_1, R)}$, and let $w \in \overline{E_p(\sigma_1, R)}$ be a point closest to p with respect to the Kobayashi distance. Since $f(w) \in \overline{E_p(\sigma_1, R)}$ this means that $k_D\big(f(w), p\big) \ge k_D(w, p)$, which is impossible because $w \neq p$ and f is strongly elliptic. □

We can now prove, using the argument already suggested in [2, Remark 2.2], that the whole backward orbit converges to a boundary fixed point $\sigma \in \partial D$, which is obviously different from the Wolff point $p \in D$.

Lemma 11. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic with Wolff point $p \in D$, and let $\{z_k\} \subset D$ be a backward orbit with bounded Kobayashi step $a = \frac{1}{2}\log\alpha$. Then $\{z_k\}$ converges to a boundary fixed point $\sigma \in \partial D$ with $\beta_\sigma \le \alpha$.*

Proof. Without loss of generality, we can assume that $z_0 \neq p$. We consider $s_k > 0$ defined by setting $-\frac{1}{2}\log s_k = k_D(z_k, p)$. Taking the constant $0 < c < 1$ given by the Lemma 10, we therefore have

$$-\tfrac{1}{2}\log s_k + \tfrac{1}{2}\log s_{k+1} \le \tfrac{1}{2}\log c\ ,$$

that is

$$s_{k+1} \le c s_k\ . \tag{12}$$

Therefore $s_{k+m} \le c^m s_k$ for every $k, m \in \mathbb{N}$, and using again (1) and [1, Theorem 2.3.51] as in the proof of Lemma 4, for all $j \in \mathbb{N}$ we obtain

$$\|z_j - z_{j+1}\| \le \frac{C}{1-\hat{a}}\sqrt{s_j}$$

for a suitable $C>0$, where $\hat{a}=\tanh a$. Arguing exactly as in (5) we then obtain that

$$\|z_k - z_{k+m}\| \le \frac{C}{1-\hat{a}}\frac{1}{1-c^{1/2}}\sqrt{s_k}, \tag{13}$$

for any $m \ge 0$ and k large enough. So $\{z_k\}$ is a Cauchy sequence in $\mathbb{C}^n$ converging to a point $\sigma \in \partial D$ by Lemma 10, and the assertion follows from [2, Lemma 2.3]. □

The following general result proves Theorem 1.(ii) in the strongly elliptic case.

Lemma 12. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic with Wolff point $p \in D$. If $\sigma \in \partial D$ is a boundary fixed point then $\beta_\sigma > 1$.*

Proof. Since p is a fixed point of f, we already know that

$$\tfrac{1}{2}\log\beta_\sigma = \liminf_{z\to\sigma}\bigl[k_D(z,p) - k_D\bigl(f(z),p\bigr)\bigr] \ge 0\;.$$

Assume, by contradiction, that $\beta_\sigma = 1$. Then [1, Proposition 2.4.15, Theorem 2.4.16 and Proposition 2.7.15] yields $f\bigl(E_p(\sigma,R)\bigr) \subseteq E_p(\sigma,R)$ for any $R>0$ because σ is a boundary fixed point. Choose $R<1$ so that $p \notin \overline{E_p(\sigma,R)}$, and let $w \in \overline{E_p(\sigma,R)}$ be a point closest to p with respect to the Kobayashi distance. Since $f(w) \in \overline{E_p(\sigma,R)}$ this means that $k_D\bigl(f(w),p\bigr) \ge k_D(w,p)$, which is impossible because $w \ne p$ and f is strongly elliptic. □

We conclude by proving Theorem 1.(iii) in the strongly elliptic case.

Lemma 13. *Let $D \Subset \mathbb{C}^n$ be a bounded strongly convex domain with C^2 boundary. Let $f \in \mathrm{Hol}(D,D)$ be strongly elliptic, with Wolff point $p \in D$. Let $\{z_k\} \subset D$ be a backward orbit for f with bounded Kobayashi step converging to $\sigma \in \partial D$. Then for every $q \in D$ there exists $M>0$ such that $z_k \in K_q(\sigma,M)$ eventually.*

Proof. It suffices again to prove that there exists $M>1$ such that $\{z_k\}$ converges to σ inside an admissible approach region $A(\sigma,M)$.

Without loss of generality, we can assume that $z_0 \ne p$. We consider again $s_k>0$ defined by setting $-\frac{1}{2}\log s_k = k_D(z_k,p)$. Thanks to (12), there is a constant $0<c<1$ such that

$$s_{k+m} \le c^m s_k \tag{14}$$

for all $k,m \ge 0$.

Now, [1, Theorem 2.3.51, Theorem 2.3.52] yield constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ such that

$$\widetilde{C}_1 d(z_j,\partial D) \le s_j \le \widetilde{C}_2 d(z_j,\partial D) \tag{15}$$

for all $j \in \mathbb{N}$, and so plugging this in (13) we have

$$\|z_k - z_{k+m}\| \le \frac{C}{1-\hat{a}}\frac{1}{1-c}\sqrt{s_k} \le \frac{C}{1-\hat{a}}\frac{1}{1-c}\sqrt{\widetilde{C}_2}\sqrt{d(z_k,\partial D)}$$

for any $m \ge 0$ and k large enough. Letting m tend to infinity we then obtain

$$\|z_k - \sigma\| \le M_1\sqrt{d(z_k,\partial D)}, \tag{16}$$

for some $M_1 > 1$.

On the other hand, up to translating the domain, without loss of generality we can assume that D contains the origin. In particular, since D is bounded and strongly convex we can replace n_σ by σ in the definition of $A(\sigma,M)$. Therefore, it suffices to prove that there exists $M_2>1$ such that

$$|\langle z_k - \sigma, \sigma\rangle| \le M_2 d(z_k,\partial D)$$

for k large enough. But this follows by arguing as in the proof of Lemma 7 using s_k instead of t_k, thanks to (14) and (15). □

This concludes the proof of Theorem 1 in both cases. □

References

[1] M. Abate, **Iteration theory of holomorphic maps on taut manifolds** Mediterranean Press, Rende (1989)
[2] M. Abate, J. Raissy, *Backward iteration in strongly convex domains*, Adv. in Math., **228**, Issue 5, pp. 2837–2854 (2011)
[3] M. Abate, A. Saracco, *Carleson measures and uniformly discrete sequences in strongly pseudoconvex domains.* J. Lond. Math. Soc. (2) 83, 587–605 (2011)
[4] O. Ostapyuk, *Backward iteration in the unit ball.* Illinois J. Math. 55, 1569–1602 (2011)

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