Introduction

In the last 50 years, dynamical systems have become one of the main objects of study in mathematics, with many applications outside mathematics. It is a huge subject that can be considered from many points of view. There are discrete and continuous dynamical systems; there are local and global dynamical systems; there are one-dimensional dynamical systems and infinitely-dimensional dynamical systems; there are measurable dynamical systems, topological dynamical systems, smooth dynamical systems—and there are holomorphic dynamical systems.

This book is devoted to a (relatively small) portion of the (quite vast) area of holomorphic dynamical systems: one-dimensional dynamical systems on Riemann surfaces; more specifically, on hyperbolic Riemann surfaces. The investigation of one-dimensional holomorphic dynamical systems started in the second half of the nineteenth century, more or less in the same years when Poincaré began to understand the importance of dynamical systems and started to investigate them in earnest. About 150 years later, the field of one-dimensional holomorphic dynamical systems is still a very active area of research, both on nonhyperbolic Riemann surfaces (mainly the Riemann sphere $\widehat{\mathbb{C}}$ and the complex plane $\mathbb{C}$) and on hyperbolic Riemann surfaces (the unit disk $\mathbb{D}$ and all Riemann surfaces whose universal cover is the disk), with several new papers appearing every year. There are many books describing the basics of holomorphic dynamics on the Riemann sphere (see, e. g., [287]); on the other hand, the only book devoted to holomorphic dynamics on hyperbolic Riemann surfaces as far as I know is [3], that has been out of print since at least 20 years ago (more about this later). By the way, I do not know of any introductory book on the dynamics of holomorphic functions on the plane, a curious hole in the literature.

Let me now describe a bit more precisely what this book is about. A *discrete holomorphic dynamical system* is given by a holomorphic self-map f of a complex manifold M. (A *continuous holomorphic dynamical system* is instead usually given by a holomorphic vector field on a complex manifold; in this book, however, we shall take a slightly different point of view, as I shall explain below.) As often happens with dynamical systems, the object of study is classical, in this case holomorphic maps; it is the kind of questions that one asks on these objects that characterize the field. Namely, we associate to f the sequence $\{f^\nu\}$ of iterates of f, where f^ν is the composition of f with itself $\nu \in \mathbb{N}$ times; we also associate to each point $z \in X$ its *orbit* $\{f^\nu(z)\}$. In dynamical systems we are then interested in the *asymptotic behavior* of the sequence of iterates, that is, what happens as ν goes to infinity: is the sequence convergent? If it is not convergent, can we anyway describe the set of accumulation points? What about single orbits, do they all have the same behavior or different points that can behave differently? What about stability, that is, what happens if we perturb the starting point of the orbit or even the original functions? Do chaotic behaviors appear? And so on. This book shall try and give some answers for dynamical systems defined on hyperbolic Riemann surfaces. In this case, the Montel theorem prevents the appearance of chaos;

https://doi.org/10.1515/9783110601978-201

so, as we shall see, the flavor of the theory and the kind of results we shall obtain is quite different from the case of holomorphic dynamical systems on nonhyperbolic Riemann surfaces—even though recently it has been discovered that the theory developed for hyperbolic Riemann surfaces can be useful for understanding the behavior of dynamical systems in the complex plane (see, e. g., [38]).

As anticipated above, the investigation of this subject began with the works of Schröder [368, 369] in 1870 and Kœnigs [244] in 1883. They were mainly interested in the local situation for holomorphic functions of one variable. Let z_0 be a point of the complex plane $\mathbb{C}$ and f a holomorphic function defined in a neighborhood of z_0 such that $f(z_0) = z_0$. Then the behavior of the sequence of iterates of f near z_0 depends on the value of the derivative of f at z_0. More specifically, if $|f'(z_0)| < 1$ every point z sufficiently close to z_0 is attracted by z_0 (i. e., $f^\nu(z) \to z_0$ as $\nu \to +\infty$) while if $|f'(z_0)| > 1$ the points are repelled away from z_0—or, if you prefer, they are attracted by z_0 under the action of f^{-1}, which is defined in a neighborhood of z_0. Finally, if $|f'(z_0)| = 1$ (and there is a bounded neighbourhood of z_0 sent into itself by f; otherwise more complicated things can happen), the behavior of $\{f^\nu\}$ is cyclic, with a finite period if $f'(z_0)$ is a root of unity. As we shall see in Chapter 4, this local behavior has global repercussions; in a very precise sense, if $0 < |f'(z_0)| < 1$ then the linear map given by the multiplication by $f'(z_0)$ is a good model for the dynamics of f. In 1904, Böttcher [71] was able to give a model also when $f'(z_0) = 0$. Furthermore, for global maps in hyperbolic Riemann surfaces necessarily $|f'(z_0)| \leq 1$ and when $|f'(z_0)| = 1$ then f is an automorphism with simple dynamics; so in our context, the more interesting case is when f has no fixed points.

The first really deep work on global holomorphic dynamical systems has been done by Julia [215] in 1918. He investigated the dynamics of rational functions defined on the Riemann sphere $\widehat{\mathbb{C}}$ and discovered that the global behavior of the sequence of iterates is both complicated and fascinating. Near fixed points it is possible to adapt and clarify the local description, but new phenomena arise, linked for instance to the distribution of periodic points (i. e., fixed points of f^ν with $\nu > 1$). A main problem was the description of the *Julia set* of f, that is, of the set of points $z_0 \in \widehat{\mathbb{C}}$ such that the sequence of iterates $\{f^\nu\}$ is not equicontinuous in any neighborhood of z_0. The idea is that if $\{f^\nu\}$ is equicontinuous in a neighborhood of z_0 then there is a subsequence $\{f^{\nu_k}\}$ converging uniformly near z_0 and then the behavior of the sequence of iterates is somehow under control. In other words, the Julia set is in some sense the singular set for the asymptotic behavior of $\{f^\nu\}$; it is the set where chaotic behavior appears.

Slightly later, in a series of papers Fatou [146–148] extended and deepened Julia's work, also investigating holomorphic dynamical systems on the complex plane generated by transcendental entire functions [149]. Again, a main role is played by the Julia set, defined replacing the notion of equicontinuity by the notion of normality: the *Fatou set*, the complement of the Julia set, is the largest open subset where the sequence of iterates is normal in the sense of Montel.

After Fatou, the study of dynamical systems generated by rational and entire functions momentarily lost its impetus. Besides the works of Cremer [132, 133], Siegel [378], Töpfer [391], and Baker [29–32], mainly devoted to the study of periodic points, both locally and globally by using Nevanlinna's distribution value theory, and Brolin [86], devoted to a deep investigation of the iteration of polynomials of low degree, and a few others, nothing really new appeared.

The situation changed completely in the 1970s and 1980s when the work of Brjuno, Hermann, Sullivan, Douady, Hubbard, and many others shed a completely new light on the topic, showing its deep relationship with the theory of quasi-conformal mappings and opening the gates for a flood of exciting new and deep results that is still going on nowadays, thanks to so many mathematicians (including some Fields medalists) that it is impossible to list their names here.

However, this is not the subject of this book. As hinted above, a main source of complexity in the study of holomorphic dynamical systems on $\widehat{\mathbb{C}}$ and $\mathbb{C}$ is that the sequence of iterates is not normal everywhere, and thus chaos appears. On the other hand, the Montel theorem implies that on hyperbolic Riemann surfaces the whole sequence of iterates is normal everywhere. This completely changes the situation. In fact, by normality, the sequence of iterates is relatively compact in a suitable function space and the compactness has strong consequences on the dynamics of f. For instance, as mentioned before, if f is a holomorphic self-map of a hyperbolic Riemann surface with a fixed-point z_0, then $|f'(z_0)| \le 1$; moreover, $|f'(z_0)| = 1$ if and only if f is an automorphism and $f'(z_0) = 1$ if and only if f is the identity. This can be obtained by noticing that the sequence of iterates should have a converging subsequence and, therefore, the coefficients of the Taylor expansion of f^ν at z_0 cannot tend to infinity as $\nu \to +\infty$; since $(f^\nu)'(z_0) = f'(z_0)^\nu$, we get $|f'(z_0)| \le 1$ and from this it is not too difficult to prove the rest of the assertion (see Theorem 3.1.10). It should be remarked that the strength of this approach was completely understood only after its application (due to H. Cartan [104, 105] and to Carathéodory [98] in the 1930s) to the theory of holomorphic maps of several complex variables, probably because in one variable it was initially somehow concealed by the Schwarz–Pick lemma.

Thus we have the hope to be able to understand the holomorphic dynamics on hyperbolic Riemann surfaces by using the Montel theorem and the Schwarz–Pick lemma. As already remarked by Julia [215], if f is a holomorphic function of $\mathbb{D}$ into itself with a fixed-point $z_0 \in \mathbb{D}$, then the behavior of $\{f^\nu\}$ can be easily derived by the Schwarz–Pick lemma: if $|f'(z_0)| < 1$, then z_0 is globally attractive (and not just locally attractive as already proved by Kœnigs) and if $|f'(0)| = 1$ then f is a non-Euclidean rotation about z_0.

The new ideas needed to study what happens when f has no fixed points were provided by Wolff [414–416] and Denjoy [135] in 1926. Let $\tau \in \partial\mathbb{D}$; then as $z \in \mathbb{D}$ tends to τ, the Poincaré disks of center z and fixed Euclidean radius tend to a *horocycle* at τ, that is to an Euclidean disk internally tangent to $\partial\mathbb{D}$ at τ. Then Wolff proved a sort of Schwarz lemma for holomorphic functions without fixed points, using the horocycles:

if f sends $\mathbb{D}$ into itself without fixed points, then there exists a unique point $\tau \in \partial\mathbb{D}$ (called the *Wolff point* of f) such that f sends every horocycle at τ into itself. Knowing this, it is then not to difficult to prove, using the Montel theorem, that the sequence of iterates $\{f^\nu\}$ converges, uniformly on compact sets, to the constant map sending all $\mathbb{D}$ in τ; this is the *Wolff–Denjoy theorem*.

For a multiply connected domain $D \subset \mathbb{C}$ different from $\mathbb{C}^*$ and, more generally, for multiply connected hyperbolic Riemann surfaces the dynamics has been described by Heins [184, 191] first in 1941 and then with more details in 1988. If f has a fixed point, the local picture forces the global one, exactly as in $\mathbb{D}$. If instead f has no fixed points, then the sequence of iterates tends to the boundary. In particular, if the boundary of D is sufficiently regular, then either the sequence of iterates converges, uniformly on compact sets, to a constant map $\tau \in \overline{D}$ or f is an automorphism.

With a few partial exceptions, the study of continuous holomorphic dynamical systems on Riemann surfaces started much later. A sequence of iterates can be interpreted as a homomorphism from the semigroup $\mathbb{N}$ endowed with the sum to the semigroup of holomorphic self-maps of the Riemann surface endowed with the composition. From this point of view, a continuous holomorphic dynamical system is a *one-parameter semigroup*, that is a continuous homomorphism from the semigroup $\mathbb{R}^+$ endowed with the sum to the semigroup of holomorphic self-maps of the Riemann surface endowed with the composition. Again, we are interested in the asymptotic behavior. On the unit disk, Berkson and Porta [61] in 1978 showed that one-parameter semigroups can be recovered as the flow of a semicomplete holomorphic vector field, the *infinitesimal generator* of the semigroup. Shortly later, Heins [190] in 1981 has been able to classify one-parameter semigroups on all Riemann surfaces, showing that the only interesting ones are on $\mathbb{D}$; furthermore, using the results of Berkson and Porta, he was able to give neat geometric representations of one-parameter semigroups on $\mathbb{D}$.

This was more or less the state of the art in 1989 when [3] was published. The first part of that book was devoted to holomorphic dynamical systems in one complex variable; the second part of the book dealt with the theory of holomorphic dynamical systems in several complex variables, a subject that (with a few notable exceptions) was just starting to be developed at that time. Thirty years have passed; also considering that [3] went soon out of print, a few years ago I thought that it was time for a new updated edition. My expectation was that the second part of the book would have needed a thorough rewriting, because the landscape of the field in several variables has changed a lot in the intervening years; but I also thought that the updating of the first part should have been a much easier affair, because in one variable the theory seemed to be already more or less complete at the end of 1980s.

Well, I was wrong. The book you have in your hands is the updated version of only the first part of [3] and it is about three times longer than the original, going from about 100 pages to more than 300 pages. What happened is that in the last 30 years, even though the basic of the subject of course remained the same, many new exciting developments have appeared and many new applications have been discovered;

moreover, a new light has been shed on results that were already known in 1989 but that I then left out because it was not yet clear (at least to me) how important they were. A partial list of the new results included here is: the multi-point Schwarz–Pick lemma discovered by Beardon and Minda [48] in 2004; the Burns–Krantz theorem [91] published in 1994 on the boundary rigidity of holomorphic self-maps of $\mathbb{D}$, with the generalization given by Bracci, Kraus, and Roth [82] in 2020; the study of random iteration on hyperbolic Riemann surfaces, started essentially in the 1990s by Gill and others but whose main results were obtained by Beardon, Carne, Minda, and Ng [52] in 2004, by Keen and Lakic [224, 223, 225] in 2006 and by Short, Christodoulou and myself [116, 10] in 2021; the whole theory of models, starting from the fundamental work of Pommerenke [338], Baker–Pommerenke [34] and Cowen [126] at the beginning of the 1980s and then revised and completed by Arosio and Bracci [22] in 2016; the study of backward dynamics done by Bracci and Poggi-Corradini [76, 327] in 2003; and so on. Furthermore, the study of continuous holomorphic dynamical systems literally exploded, producing so many new results that to present even just the most important ones would need yet another book—that, luckily, has already appeared [80]. (And yes, in a few years you will also get the updated edition of the second part of [3]. I hope.)

Some comments and remarks about the structure of this work are in order. I have written this book keeping in mind two different goals (and audiences). First of all, this is intended as a reference book on holomorphic dynamical systems on hyperbolic Riemann surfaces and related topics. During my own investigations, I found many beautiful theorems never presented in book form; furthermore, the whole theory seemed to me requiring a comprehensive exposition collecting several results scattered around in the literature. This allowed a unified exposition of the main results and a clearer discussion of the threads connecting them.

So, the first audience of this book is mainly composed by researchers in holomorphic dynamical systems; they will find an up-to-date description of the field, open problems to solve and many references to several topics not discussed here. But, as already anticipated, I also had another goal in mind. This book would also like to be an introduction to this area for, say, first-year Ph. D. students (or for good master students, too), giving them both a sample of typical features and techniques, presented from scratch starting from the Schwarz lemma, as well as motivations provided by the historical development of the theory.

Also for this reason, I tried to keep prerequisites to a minimum. Besides a good knowledge of the basics of function theory of one complex variable, only a good topological background (up to covering spaces and the fundamental group) is needed. Sometimes we shall use notions or results from ordinary differential equations, differential geometry or measure theory, but whenever an external result is needed I have tried to always give a precise statement and a reference to a place where a proof can be found. Moreover, the Appendix contains statements and proofs of a few classical results not always covered by standard introductory courses in complex analysis.

Let us now briefly describe the actual content of this book; more details can be found in the introductions to each chapter.

Chapter 1 is a thorough introduction to geometric function theory on hyperbolic Riemann surfaces. We shall discuss the Schwarz–Pick lemma, including the multipoint version; the Poincaré metric and distance; the structure of the automorphism group of $\mathbb{D}$ and, more generally, of hyperbolic Riemann surfaces; the Montel, Vitali, and Picard theorems in full generality; the classification of Riemann surfaces; the boundary behavior of the universal covering map of multiply connected hyperbolic domains; the Ahlfors–Schwarz–Pick lemma; and much more.

In Chapter 2, we introduce the horocycles in $\mathbb{D}$ and their main properties, the Julia and Wolff lemmas that are boundary versions of the Schwarz lemma. We use them to study the angular derivative of holomorphic self-maps of $\mathbb{D}$ into itself, proving the Julia–Wolff–Carathéodory theorem, and then to investigate the structure of the automorphism group of a hyperbolic Riemann surface. We shall also prove the Lindelöf theorem on the existence of nontangential limits and the Burns–Krantz theorem on the boundary rigidity of holomorphic self-maps of $\mathbb{D}$.

Chapter 3 is devoted to discrete dynamics on hyperbolic Riemann surfaces. We start by describing the theory for holomorphic functions with a fixed point; we present two proofs of the Wolff–Denjoy theorem; we develop the iteration theory on hyperbolic Riemann surfaces and its version in finitely connected hyperbolic domains; we study the stability of the Wolff point; we introduce the notion of model for a holomorphic self-map, proving its existence and uniqueness; and we study random iteration on hyperbolic Riemann surfaces.

In Chapter 4, we concentrate our attention on the dynamics in the unit disk, where we can get deeper results. We study in detail how the orbits approach the Wolff point; we prove a complete classification of the possible models that can arise for holomorphic self-maps of $\mathbb{D}$, including ways to detect the model just looking at the map; we study the backward dynamics, understanding what happens to the orbits in the past and not just in the future; and we get a few results about the existence of common fixed points for commuting maps.

In Chapter 5, we investigate the one-parameter semigroups of holomorphic functions on a Riemann surface. In particular, we present the results of Berkson–Porta and Heins cited above about the existence and properties of the infinitesimal generator, the classification of one-parameter semigroups on Riemann surfaces other than $\mathbb{D}$ and the geometrical realization of one-parameter semigroups on $\mathbb{D}$.

Finally, as anticipated, to help the reader the Appendix contains the statement and proofs of a number of less standard results in real and complex analysis of one variable that we happen to use in the book.

Each section of each chapter ends with notes, containing history, comments, remarks, indications of related topics, and references to the bibliography. I tried to systematically trace who did what when and indeed the list of references includes more than 400 entries from 1826 to 2022. However, I am painfully aware that this list is not complete and I apologize in advance to anybody I forgot to mention or that did not receive a correct attribution.

Let us end this (long) introduction with the pleasant duty of acknowledgments. First and foremost, I would like to deeply thank my wife, Adele, and my sons, Leonardo, Jacopo, and Niccolò that supported me (in both the Italian meanings of the word: sustain and endure) during all these years, not complaining too much when their respectively husband and father disappeared in a mathematical hole with a faraway look clearly showing that he was not listening to the much more important things that they were saying to him. But when I emerge from the hole, I look at them and I am proud of who they are and have become. And I apologize, at least most of the time, and they forgive me, most of the times.

I thankfully and fondly remember the late Edoardo Vesentini, my Ph. D. advisor, that so many years ago trusted me when I told him that I wanted to study holomorphic dynamical systems (a field that at the time I called iteration theory). If I had a good start in my mathematical career, it is because of him.

The complete list of friends and colleagues that helped me and accompanied me in this 30-year long journey would occupy too much space to be printed here—and I consider myself a very lucky person for having such an extensive list. A special role in my mathematical and personal life has been played by Filippo Bracci, Chiara de Fabritiis, Graziano Gentili, Giorgio Patrizio, Jasmin Raissy, Tamara Servi, and Francesca Tovena but I extend a heartfelt thanks to all of you on the list. I could not have done it without you, really.

A special thanks goes to my editors at de Gruyter, in particular to Apostolos Damialis, Steve Elliot, and Nadja Schedensack that put up with all my delays for so many years. At last, this book is completed; let us see how many years I will need for the next one.

Last but not least, I would like to thank my mother, Silvana, 88 years old but still going strong and still trying to understand what I actually do for a living. Teaching, is clear. Administration, is understandable. Mathematical research... but she always trusted me no matter what. Thanks mom, this book is for you.