Chapter 2.1
Taut manifolds

We are now ready to enter the main part of this book, devoted to several complex variables. Up to now, we mainly worked on hyperbolic Riemann surfaces; therefore the first problem we must solve is to find out which kind of manifold is the right multidimensional analogue of hyperbolic Riemann surfaces. We recall that in this book every manifold is, unless otherwise stated, connected, Hausdorff and second countable, and that a submanifold of a manifold X is always imbedded, i.e., it is a subset of X which is a manifold with the structure inherited by X.

We saw that the main property of hyperbolic Riemann surfaces was Montel's theorem: a family of holomorphic functions into a hyperbolic Riemann surface is always normal. As often happens in mathematics, a characterization gives rise to a definition: a *taut manifold* is, essentially, a complex manifold X such that $\mathrm{Hol}(Y, X)$ is normal for every complex manifold Y. In the first section we shall describe several examples of taut manifolds (and other examples will be introduced in chapter 2.3), showing that, at least, we are working with a sufficiently large amount of complex manifolds. In the second section we shall show that function theory on taut manifolds is akin to function theory on hyperbolic Riemann surfaces, thus justifying our choice; indeed, we shall generalize practically all consequences of Montel's theorem we presented in sections 1.1.4 and 1.3.1. The knowledgeable reader will recognize in it a fancy version of classical function theory on bounded domains developed mainly by H. Cartan and Carathéodory in the thirties.

The last section is devoted to introducing iteration theory on taut manifolds. The situation, this time, is quite different from the one-dimensional case. The main reason is, roughly speaking, that the image of a limit of a sequence of iterates should be a closed connected submanifold; therefore, in the one-dimensional case the image can be only a point or the whole manifold, whereas in several variables there are many intermediate possibilities. For the moment, we shall limit ourselves to a preliminary discussion of the situation; the results we get will trace the way to the full development we shall undertake in chapter 2.4.

A final warning: in this chapter we shall talk for the first time of pseudoconvex domains. This book is by no means an introduction to the subject; you must already know (at least by hearsay) the basic theory. If you don't, you may probably read this chapter, and the next one, without losing anything important, but you will be inesorably stuck in chapter 2.3. So please, if you need to, look at Krantz [1982], or at an analogous book, before it is too late.

2.1.1 *Definitions and examples*

In the first part of this book we singled out the hyperbolic Riemann surfaces, so to have at our disposal Montel's theorem and its corollaries. As already mentioned, in this section we shall introduce taut manifolds, the multidimensional versions of hyperbolic Riemann surfaces (but see also chapter 2.3), and other related concepts. We shall moreover describe several examples, leaving function theory to the next section.

We recall the definition of normality. Let X, Y be two complex manifolds; a family $\mathcal{F}$ of holomorphic maps from X to Y is *normal* if every sequence in $\mathcal{F}$ admits either a convergent subsequence or a compactly divergent subsequence.

Normality is a compactness condition. Given a complex manifold X, denote by X^* its one-point compactification (see Kelley [1955], p. 150). Since X is Hausdorff, locally compact, connected and second countable, X^* is Hausdorff, locally compact, second countable and hence metrizable. In particular, for every complex manifold Y the space $C^0(Y, X^*)$ is second countable, and a subset of $C^0(Y, X^*)$ is compact iff it is sequentially compact. It follows then that a family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is normal iff $\mathcal{F} \cup \{\infty\} \subset C^0(Y, X^*)$ is relatively compact, where we are denoting by ∞ both the point at infinity of X^* and any map of constant value ∞.

A complex manifold X is *taut* if $\mathrm{Hol}(\Delta, X)$ is a normal family. Since $\mathrm{Hol}(\Delta, X)$ is closed in $C^0(\Delta, X)$, this is equivalent to say that $\mathrm{Hol}(\Delta, X) \cup \{\infty\} \subset C^0(\Delta, X^*)$ is compact. The usefulness of this definition is that Δ is such a standard space that tautness implies the compactness of $\mathrm{Hol}(Y, X) \cup \{\infty\}$ for every complex manifold Y, that is exactly the property we would like to work with. The proof relies on the Ascoli-Arzelà theorem, as usual, and thus is divided in two parts: first, we prove the equicontinuity with respect to some suitable distance; second, we prove the closure of $\mathrm{Hol}(Y, X) \cup \{\infty\}$ in $C^0(Y, X^*)$.

Proposition 2.1.1: *Let X be a complex manifold, and d a distance on X compatible with its topology. Assume* $\mathrm{Hol}(\Delta, X)$ *is equicontinuous with respect to d; then* $\mathrm{Hol}(Y, X)$ *is equicontinuous with respect to d for every complex manifold Y.*

Proof: Assume, by contradiction, that there is a complex manifold Y such that $\mathrm{Hol}(Y, X)$ is not equicontinuous. Hence there exist $\varepsilon > 0$, a point $z_0 \in Y$ and sequences $\{z_\nu\} \subset Y$ and $\{f_\nu\} \subset \mathrm{Hol}(Y, X)$ such that $z_\nu \to z_0$ and $d\big(f_\nu(z_\nu), f_\nu(z_0)\big) \geq \varepsilon$ for all $\nu \in \mathbf{N}$. Choosing a suitable local coordinate system we can then assume Y to be the euclidean unit ball B of some $\mathbf{C}^n$, and take $z_0 = 0$.

Define $g_\nu \in \mathrm{Hol}(\Delta, X)$ by $g_\nu(\zeta) = f_\nu\big(\zeta z_\nu / \|z_\nu\|\big)$. Then $\|z_\nu\| \to 0$ as $\nu \to +\infty$, and

$$d\big(g_\nu(\|z_\nu\|), g_\nu(0)\big) = d\big(f_\nu(z_\nu), f_\nu(0)\big) \geq \varepsilon$$

for all $\nu \in \mathbf{N}$, and hence $\mathrm{Hol}(\Delta, X)$ is not equicontinuous, contradiction, **q.e.d.**

And so

Theorem 2.1.2: *Let X be a taut complex manifold. Then* $\mathrm{Hol}(Y, X)$ *is a normal family for every complex manifold Y.*

Proof: Fix a distance d on X^* compatible with its topology. The tautness of X implies that $\mathrm{Hol}(\Delta, X) \cup \{\infty\}$ is compact in $C^0(\Delta, X^*)$ and thus, by the Ascoli-Arzelà theorem, it is equicontinuous with respect to d. Therefore, by Proposition 2.1.1, $\mathrm{Hol}(Y, X)$ is equicontinuous with respect to d for every complex manifold Y.

Now suppose there is a complex manifold Y such that $\mathrm{Hol}(Y,X)$ is not normal. So $\mathrm{Hol}(Y,X)\cup\{\infty\}$ is not a compact subset of $C^0(Y,X^*)$; since it is equicontinuous with respect to d, it follows that $\mathrm{Hol}(Y,X)\cup\{\infty\}$ cannot be closed in $C^0(Y,X^*)$, by the Ascoli-Arzelà theorem. In particular, there is a sequence $\{f_\nu\}\subset \mathrm{Hol}(Y,X)$ converging toward a map $f\in C^0(Y,X^*)$ which is neither in $C^0(Y,X)$ nor the constant map ∞, because $\mathrm{Hol}(Y,X)$ is closed in $C^0(Y,X)$. So there are a point $z_0\in Y$ such that $f(z_0)=\infty$, and f is not identically ∞ in any neighbourhood of z_0. Hence, choosing a suitable system of local coordinates, without loss of generality we can assume Y to be the unit ball B of some $\mathbf{C}^n$, and $z_0=0$.

Since $f\not\equiv\infty$, there is $z_1\in B$ such that $f(z_1)\neq\infty$. Define $g_\nu\in\mathrm{Hol}(\Delta,X)$ and $g\in C^0(\Delta,X^*)$ setting $g_\nu(\zeta)=f_\nu(\zeta z_1/\|z_1\|)$ and $g(\zeta)=f(\zeta z_1/\|z_1\|)$. Then g does not belong to $C^0(\Delta,X)\cup\{\infty\}$ and $g_\nu\to g$ as $\nu\to+\infty$, that is $\mathrm{Hol}(\Delta,X)\cup\{\infty\}$ is not closed in $C^0(\Delta,X^*)$, contradiction, **q.e.d.**

Corollary 2.1.3: *A Riemann surface is taut iff it is hyperbolic.*

Proof: Theorems 1.1.43, 2.1.2 and Proposition 1.1.52, **q.e.d.**

Before giving other examples of taut manifolds, we introduce two notions strictly linked to tautness.

Let D be a domain of a complex manifold X. We shall say that D is *tautly imbedded* in X if $\mathrm{Hol}(\Delta,D)$ is relatively compact in $\mathrm{Hol}(\Delta,X)$; that D has *simple boundary* if every holomorphic map $\varphi\colon\Delta\to X$ such that $\varphi(\Delta)\subset\overline{D}$ and $\varphi(\Delta)\cap\partial D\neq\emptyset$ is constant. Note that every domain in a Riemann surface has simple boundary, by the open map theorem.

Again, choosing Δ is just a matter of convenience:

Proposition 2.1.4: *Let D be a domain of a complex manifold X. Then:*
(i) *D is tautly imbedded in X iff* $\mathrm{Hol}(Y,D)$ *is relatively compact in* $\mathrm{Hol}(Y,X)$ *for every complex manifold Y;*
(ii) *D has simple boundary iff every holomorphic map $f\colon Y\to X$ such that $f(Y)\subset\overline{D}$ and $f(Y)\cap\partial D\neq\emptyset$ is constant, where Y is any complex manifold.*

Proof: (i) One direction is obvious. Conversely, assume D tautly imbedded in X; in particular, since $\mathrm{Hol}(\Delta,D)$ contains the constant maps, D is relatively compact in X.

Fix a distance d on X inducing the manifold topology. Since the closure of $\mathrm{Hol}(Y,D)$ in $\mathrm{Hol}(Y,X)$ is contained in $C^0(Y,\overline{D})$ and $\overline{D}$ is compact, by the Ascoli-Arzelà theorem $\mathrm{Hol}(Y,D)$ is relatively compact in $\mathrm{Hol}(Y,X)$ iff it is equicontinuous with respect to d. The assertion then follows from Proposition 2.1.1.

(ii) Again, one direction is clear. Conversely, assume there is a complex manifold Y and a non-constant holomorphic map $f\colon Y\to X$ such that $f(Y)\subset\overline{D}$ and $f(Y)\cap\partial D\neq\emptyset$. Choosing a suitable system of local coordinates, we can assume Y to be the unit ball B of some $\mathbf{C}^n$, and $f(0)\in\partial D$. Then apply the usual trick: since f is not constant, there is $z_1\in B$ such that $f(0)\neq f(z_1)$. Then $\varphi\colon\Delta\to X$ defined by $\varphi(\zeta)=f\big(\zeta z_1/\|z_1\|\big)$ is a non-constant holomorphic map such that $\varphi(\Delta)\subset\overline{D}$ and $\varphi(\Delta)\cap\partial D\neq\emptyset$, and thus D has not simple boundary, **q.e.d.**

A domain $D \subset X$ which is taut as complex manifold need not to be tautly imbedded (it can even be not relatively compact: think of H^+ in $\mathbf{C}$). The best we can do is:

Proposition 2.1.5: *Let X be a taut manifold, and $D \subset\subset X$ a relatively compact domain. Then D is tautly imbedded in X.*

Proof: Since no sequence in $\mathrm{Hol}(\Delta, D)$ can be compactly divergent in X, $\mathrm{Hol}(\Delta, D)$ is relatively compact in $\mathrm{Hol}(\Delta, X)$, **q.e.d.**

On the other hand, a tautly imbedded domain need not to be taut:

Proposition 2.1.6: *A tautly imbedded domain $D \subset X$ is taut iff for every holomorphic map $\varphi \in \overline{\mathrm{Hol}(\Delta, D)} \subset \mathrm{Hol}(\Delta, X)$ we have either $\varphi(\Delta) \subset D$ or $\varphi(\Delta) \subset \partial D$. In particular, a tautly imbedded domain with simple boundary is taut.*

Proof: Assume D taut, and take a sequence $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, D)$ which is converging toward $\varphi \in \mathrm{Hol}(\Delta, X)$. Since D is taut, either $\varphi \in \mathrm{Hol}(\Delta, D)$ or, up to a subsequence, $\{\varphi_\nu\}$ is compactly divergent, and thus $\varphi(\Delta) \subset \partial D$.

Conversely, take a sequence $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, D)$. Since D is tautly imbedded in X, there is a subsequence $\{\varphi_{\nu_j}\}$ converging toward a map $\varphi \in \mathrm{Hol}(\Delta, X)$. Hence, by assumption, either $\varphi(\Delta) \subset D$ — and so $\varphi_{\nu_j} \to \varphi$ in $\mathrm{Hol}(\Delta, X)$ — or $\varphi(\Delta) \subset \partial D$ — and thus $\{\varphi_{\nu_j}\}$ is compactly divergent, **q.e.d.**

At this point, the knowledgeable reader may begin to smell pseudoconvexity approaching. The link is provided by the classical *Kontinuätssatz:*

Theorem 2.1.7: *Let D be a bounded domain in $\mathbf{C}^n$. Then D is pseudoconvex iff for every family $\mathcal{F} \subset \mathrm{Hol}(\Delta, D) \cap C^0(\overline{\Delta}, \overline{D})$ such that $\bigcup\limits_{\varphi \in \mathcal{F}} \varphi(\partial\Delta) \subset\subset D$ we have $\bigcup\limits_{\varphi \in \mathcal{F}} \varphi(\overline{\Delta}) \subset\subset D$.*

A proof can be found, e.g., in Krantz [1982].

Comparing Proposition 2.1.6 with Theorem 2.1.7 one can suspect that every bounded taut domain is pseudoconvex. To confirm this suspicion, we first characterize tautly imbedded domains in $\mathbf{C}^n$, thus officially opening our list of several variables examples:

Proposition 2.1.8: *A domain $D \subset \mathbf{C}^n$ is tautly imbedded in $\mathbf{C}^n$ iff it is bounded.*

Proof: We have already observed that a tautly imbedded domain is always relatively compact. Conversely, let $D \subset\subset \mathbf{C}^n$ be bounded, and take a sequence $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, D)$. Denote by $p_j\colon \mathbf{C}^n \to \mathbf{C}$ the projection onto the j-th coordinate; then there exists a bounded domain $\Omega \subset\subset \mathbf{C}$ such that $p_j(D) \subset\subset \Omega$ for every $j = 1, \ldots, n$. Now, applying Montel's theorem to $\{p_j \circ \varphi_\nu\} \in \mathrm{Hol}(\Delta, \Omega)$ for $j = 1, \ldots, n$, we can extract a subsequence $\{\varphi_{\nu_k}\}$ such that $\{p_j \circ \varphi_{\nu_k}\}$ is converging in $\mathrm{Hol}(\Delta, \Omega)$ for every $j = 1, \ldots, n$. Thus $\{\varphi_{\nu_k}\}$ is converging in $\mathrm{Hol}(\Delta, \mathbf{C}^n)$, **q.e.d.**

Then

Proposition 2.1.9: *Every bounded taut domain $D \subset \mathbf{C}^n$ is pseudoconvex.*

Proof: Let $\mathcal{F} \subset \mathrm{Hol}(\Delta, D) \cap C^0(\overline{\Delta}, \overline{D})$ be a family of continuous maps holomorphic in Δ such that $\bigcup_{\varphi\in\mathcal{F}} \varphi(\partial\Delta)$ is relatively compact in D. In particular, $\bigcup_{\varphi\in\mathcal{F}} \varphi(\partial\Delta)$ is bounded; hence, by the maximum principle, $\bigcup_{\varphi\in\mathcal{F}} \varphi(\overline{\Delta})$ is bounded too, and, by Proposition 2.1.8, $\mathcal{F}$ is relatively compact in $\mathrm{Hol}(\Delta, \mathbf{C}^n)$. In particular, since $\bigcup_{\varphi\in\mathcal{F}} \varphi(\partial\Delta) \subset\subset D$, no sequence in $\mathcal{F}$ can be compactly divergent. By the tautness of D, $\mathcal{F}$ is then relatively compact in $\mathrm{Hol}(\Delta, D)$ — and thus in $C^0(\Delta, D)$. It follows that $\bigcup_{\varphi\in\mathcal{F}} \varphi(\overline{\Delta}) \subset\subset D$, and D is pseudoconvex, **q.e.d.**

As a consequence, taut domains are by no means generic; in particular, not every tautly imbedded domain is taut.

Now we proceed toward the construction of examples of taut manifolds. We shall be mainly interested in taut domains; so we begin looking for conditions allowing the application of Proposition 2.1.6.

Let D be a domain in a complex manifold X. A *peak function* for D at a point $x \in \partial D$ is a holomorphic function f defined in a neighbourhood of $\overline{D}$ such that $f(x) = 1$ and $|f(z)| < 1$ for all $z \in \overline{D} \setminus \{x\}$. If $|f(z)| < 1$ only for $z \in D$, f will be a *weak peak function.* A *local (weak) peak function* for D at x is a (weak) peak function for $D \cap U$ at x, where U is a neighbourhood of x in X. Then

Proposition 2.1.10: *Let D be a domain in a complex manifold X. Then*
(i) *if D is tautly imbedded in X and there is a local weak peak function for D at each point of ∂D, then D is taut;*
(ii) *if there is a local peak function for D at each point of ∂D, then D has simple boundary.*

Proof: (i) By Proposition 2.1.6, it suffices to show that for every $\varphi \in \mathrm{Hol}(\Delta, X)$ such that $\varphi(\Delta) \subset \overline{D}$ we have either $\varphi(\Delta) \subset D$ or $\varphi(\Delta) \subset \partial D$.

Let $E = \varphi^{-1}(\partial D)$. If E is empty, we are done. Otherwise, let $\zeta_0 \in E$ such that $x_0 = \varphi(\zeta_0) \in \partial D$. Let U be a neighbourhood of x_0 in X such that there exists a weak peak function λ for $D \cap U$ at x_0, and V a neighbourhood of ζ_0 such that $\varphi(V) \subset U$. Then $\lambda\circ\varphi$ is holomorphic in V and $|\lambda\circ\varphi|$ attains its maximum at ζ_0. Therefore $\lambda\circ\varphi \equiv \lambda\bigl(\varphi(\zeta_0)\bigr)$ on V, and hence $\varphi(V) \subset \partial D$. In other words, $V \subset E$; therefore, E is open, is obviously closed and, since Δ is connected, $E = \Delta$, and we are done.

(ii) The same argument works, using local peak functions instead of local weak peak functions, **q.e.d.**

And so we finally get true several variables examples of taut domains:

Corollary 2.1.11: *If $D \subset\subset \mathbf{C}^n$ is a convex domain, and $x \in \partial D$, then there exists a weak peak function for D at x. In particular, every bounded convex domain of $\mathbf{C}^n$ is taut.*

Proof: In fact, there is a complex linear functional $\phi\colon \mathbf{C}^n \to \mathbf{C}$ such that $\mathrm{Re}\,\phi(x) > \mathrm{Re}\,\phi(z)$ for all $z \in D$. Then $\lambda = \exp\bigl(\phi - \phi(x)\bigr)$ is a weak peak function for D at x, and the latter assertion follows from Proposition 2.1.10, **q.e.d.**

In this corollary we made use of global peak functions, whereas Proposition 2.1.10 requires only local peak functions. The quick reader will immediately leap to the conclusion: every strongly pseudoconvex domain is taut, for it is locally convex. To prove this assertion, and to fix notations from now on, we digress a bit listing some definitions.

A domain $\Omega \subset \mathbf{R}^N$ has C^r *boundary* (or is a C^r *domain*), where $r = 1\ 2, \ldots, \infty, \omega$ (and C^ω means real analytic), if there is a C^r function $\rho\colon \mathbf{R}^N \to \mathbf{R}$ such that:

(i) $\Omega = \{x \in \mathbf{R}^N \mid \rho(x) < 0\}$,
(ii) $\partial\Omega = \{x \in \mathbf{R}^N \mid \rho(x) = 0\}$ and
(iii) $\operatorname{grad}\rho$ is not vanishing on $\partial\Omega$.

ρ is a *defining function* for Ω; it is easy to check that if ρ_1 is another defining function for Ω then there is a never vanishing C^r function $\psi\colon \mathbf{R}^N \to \mathbf{R}^+$ such that

$$\rho_1 = \psi\rho. \tag{2.1.1}$$

If $\Omega \subset \mathbf{R}^N$ is a C^r domain with defining function ρ, $\partial\Omega$ is a C^r manifold embedded in $\mathbf{R}^N$. In particular, for every $x \in \partial\Omega$ the tangent space of $\partial\Omega$ at x can be identified with the kernel of $d\rho_x$ (which by (2.1.1) is independent of the chosen defining function), that is

$$T_x(\partial\Omega) = \left\{ v \in \mathbf{R}^N \;\middle|\; \sum_{j=1}^N \frac{\partial\rho}{\partial x_j}(x)\, v_j = 0 \right\}.$$

The *outer unit normal vector* $\mathbf{n}_x$ at x is the unit vector parallel to $-\operatorname{grad}\rho(x)$; thus $T_x(\partial\Omega)$ is just the hyperplane orthogonal to $\mathbf{n}_x$.

If $\rho\colon \mathbf{R}^N \to \mathbf{R}$ is a C^2 function, the *Hessian* $H_{\rho,x}$ of ρ at $x \in \mathbf{R}^N$ is the symmetric bilinear form

$$\forall v,\, w \in \mathbf{R}^N \qquad H_{\rho,x}(v,w) = \sum_{h,k=1}^N \frac{\partial^2\rho}{\partial x_h \partial x_k}(x)\, v_h w_k.$$

It is easy to check that a C^2 domain $\Omega \subset \mathbf{R}^N$ is convex iff for every $x \in \partial\Omega$ the symmetric bilinear form $H_{\rho,x}$ is positive semidefinite on $T_x(\partial\Omega)$, where ρ is any defining function for Ω (see, e.g., Krantz [1982], p. 102). We shall say that a C^2 domain $\Omega \subset \mathbf{R}^n$ is *strongly* (or *strictly*) *convex* at $x \in \partial\Omega$ if for some C^2 defining function ρ for Ω the Hessian $H_{\rho,x}$ is positive definite on $T_x(\partial\Omega)$; that Ω is *strongly* (or *strictly*) *convex* if it is so at each point of $\partial\Omega$. Note that, by (2.1.1), the definition is independent of the chosen defining function; it can even be proved that every strongly convex domain Ω has a C^2 defining function ρ such that $H_{\rho,x}$ is positive definite on the whole of $\mathbf{R}^N$ for every $x \in \partial\Omega$ (see Krantz [1982], p. 101). It is easy to check that a strongly convex domain Ω is strongly convex in the elementary sense, that is $tx + (1-t)y \in \Omega$ for every $t \in (0,1)$ and $x, y \in \overline{\Omega}$, $x \neq y$; anyway, in this book a strongly convex domain will always be a C^2 domain satisfying the previous definition. Finally, a convex domain which is not strongly convex will be sometimes called *weakly convex*.

Now we move on to the complex case. Let $D \subset\subset \mathbf{C}^n$ be a bounded domain with C^2 boundary and defining function $\rho\colon \mathbf{C}^n \to \mathbf{R}$. The *complex tangent space* $T_x^{\mathbf{C}}(\partial D)$ of ∂D at $x \in \partial D$ is the kernel of $\partial\rho_x$, that is

$$T_x^{\mathbf{C}}(\partial D) = \left\{ v \in \mathbf{C}^n \;\middle|\; \sum_{j=1}^n \frac{\partial\rho}{\partial z_j}(x)\, v_j = 0 \right\}.$$

As usual, $T_x^{\mathbf{C}}(\partial D)$ does not depend on the particular defining function. Note furthermore that the outer unit normal vector $\mathbf{n}_x$ is parallel to the complex gradient vector

$$\frac{\partial\rho}{\partial\bar z}(x) = \left(\frac{\partial\rho}{\partial\bar z_1}(x), \ldots, \frac{\partial\rho}{\partial\bar z_n}(x)\right).$$

The following lemma is now quite clear:

Lemma 2.1.12: *Let $D \subset\subset \mathbf{C}^n$ be a bounded strongly convex domain in $\mathbf{C}^n$, and take $x \in \partial D$. Then there exists a peak function for D at x; in particular, D has simple boundary.*

Proof: Let $\rho\colon \mathbf{C}^n \to \mathbf{R}$ be a defining function for D, and define $\phi\colon \mathbf{C}^n \to \mathbf{C}$ by

$$\phi(z) = \sum_{j=1}^n \frac{\partial\rho}{\partial\bar z_j}(x)\,(z_j - x_j).$$

Now

$$\operatorname{Re}\phi(z) = \frac{d}{dt}\Big[\rho\big(tz + (1-t)x\big)\Big]_{t=0}.$$

Since D is strongly convex, if $z \in \overline{D} \setminus \{x\}$ we have

$$\forall t \in (0,1) \qquad\qquad \rho\big(tz + (1-t)x\big) < 0;$$

in particular, $\operatorname{Re}\phi(z) > 0$ for all $z \in \overline{D} \setminus \{x\}$. Therefore $\lambda(z) = \exp\big(-\phi(z)\big)$ is a peak function for D at x, **q.e.d.**

The *Levi form* $L_{\rho,x}$ of ρ at $x \in \mathbf{C}^n$ is the hermitian form

$$\forall v,\, w \in \mathbf{C}^n \qquad\qquad L_{\rho,x}(v,w) = \sum_{h,k=1}^n \frac{\partial^2\rho}{\partial z_h \partial\bar z_k}(x)\, v_h \overline{w_k}.$$

Then a bounded C^2 domain $D \subset\subset \mathbf{C}^n$ is *strongly* (or *strictly*) *pseudoconvex* at a point $x \in \partial D$ if for some (and hence all) C^2 defining function ρ for D the Levi form $L_{\rho,x}$ is positive definite on $T_x^{\mathbf{C}}(\partial D)$; D is *strongly* (or *strictly*) *pseudoconvex* if it is so at each point of ∂D. We stress that, in this book, a strongly pseudoconvex domain is always bounded and with C^2 boundary. Note that if D is strongly pseudoconvex then there is again a defining function ρ for D such that the Levi form $L_{\rho,x}$ is positive definite on $\mathbf{C}^n$ for every $x \in \partial D$ (Krantz [1982], p. 109).

The link between strongly pseudoconvex domains and strongly convex domains is provided by the following (well known) result:

Proposition 2.1.13: *A bounded C^2 domain $D \subset\subset \mathbf{C}^n$ is strongly pseudoconvex iff for every $x \in \partial D$ there is a neighbourhood $U \subset \mathbf{C}^n$ and a biholomorphism $\Phi: U \to \Phi(U)$ such that $\Phi(U \cap D)$ is strongly convex.*

Proof: Assume first D strongly pseudoconvex, and let $\rho: \mathbf{C}^n \to \mathbf{R}$ be a C^2 defining function such that $L_{\rho,x}$ is positive definite on $\mathbf{C}^n$ for every $x \in \partial D$. Fix $x_0 \in \partial D$; up to an affine transformation of $\mathbf{C}^n$, we can assume $x_0 = 0$ and $\mathbf{n}_{x_0} = \partial\rho/\partial\bar{z}(x_0) = (-1, 0, \ldots, 0)$. Then the second-order Taylor expansion of ρ about x_0 is

$$\rho(z) = 2\ \mathrm{Re}\left\{ -z_1 + \frac{1}{2} \sum_{h,k=1}^{n} \frac{\partial^2 \rho}{\partial z_h \partial z_k}(0)\, z_h z_k \right\} + L_{\rho,0}(z, z) + o(\|z\|^2).$$

Define $\Phi: \mathbf{C}^n \to \mathbf{C}^n$ by

$$\begin{cases} \Phi_1(z) = z_1 - \dfrac{1}{2} \displaystyle\sum_{h,k=1}^{n} \frac{\partial^2 \rho}{\partial z_h \partial z_k}(0)\, z_h z_k, \\ \Phi_j(z) = z_j \quad \text{for} \quad j = 2, \ldots, n. \end{cases} \tag{2.1.2}$$

Since $d\Phi_0 = \mathrm{id}$, Φ is a biholomorphism between a neighbourhood V of the origin and its image. Furthermore

$$\forall w \in \Phi(V) \qquad \rho\big(\Phi^{-1}(w)\big) = -2\ \mathrm{Re}\, w_1 + L_{\rho,0}(w, w) + o(\|w\|^2).$$

Therefore the Hessian of $\rho \circ \Phi^{-1}$ at the origin in the real coordinates is the Levi form of ρ at 0, which is positive definite. So $\Phi(V \cap D)$ is strongly convex at $\Phi(x_0)$ and thus, by the continuity of the second-order derivatives, we can find a neighbourhood $U \subset\subset V$ of x_0 such that $\Phi(U \cap D)$ is strongly convex.

To prove the converse, it suffices to show that a strongly convex domain is strongly pseudoconvex, and that strong pseudoconvexity is preserved by biholomorphic maps.

Let $D \subset\subset \mathbf{C}^n$ be a C^2 domain with defining function ρ. Then taking $x \in \partial D$ and $v \in \mathbf{C}^n$ we have

$$H_{\rho,x}(v, v) = 2\, L_{\rho,x}(v, v) + 2\ \mathrm{Re} \sum_{h,k=1}^{n} \frac{\partial^2 \rho}{\partial z_h \partial z_k}(x)\, v_h v_k.$$

Assume now D strongly convex at $x \in \partial D$. Then if $v \in T_x^{\mathbf{C}}(\partial D) \subset T_x(\partial D)$, with $v \neq 0$, we have $iv \in T_x(\partial D)$ too, and thus

$$L_{\rho,x}(v, v) = \frac{1}{4} \Big[H_{\rho,x}(v, v) + H_{\rho,x}(iv, iv) \Big] > 0,$$

that is D is strongly pseudoconvex at x.

Finally, if $\Phi: U \to V$ is a biholomorphism of a neighbourhood U of $x \in \partial D$ we have

$$\forall v, w \in \mathbf{C}^n \qquad L_{\rho,x}(v, w) = L_{\rho \circ \Phi^{-1}, \Phi(x)}\big(d\Phi_x(v), d\Phi_x(w)\big),$$

and so strong pseudoconvexity is preserved by biholomorphic maps, **q.e.d.**

In particular, every strongly convex domain is strongly pseudoconvex, and conversely every strongly pseudoconvex domain is locally convex.

We do not want to spend words about the importance and omnipresence of strongly pseudoconvex domains in complex analysis; any book on function theory of several complex variables will do the job incommensurably better. Anyway, it should be clear why the following corollary provides us with quite an ample list of taut domains:

Corollary 2.1.14: *Every strongly pseudoconvex domain in* $\mathbf{C}^n$ *has simple boundary and, in particular, is taut.*

Proof: Let D be a strongly pseudoconvex domain in $\mathbf{C}^n$, and $x \in \partial D$. Then there exists a neighbourhood U of x and a holomorphic injective map $\phi: U \to \mathbf{C}^n$ such that $\phi(U \cap D)$ is strongly convex. Hence using Lemma 2.1.12 we can construct a local peak function for D at x, and the assertion follows from Proposition 2.1.10, **q.e.d.**

So we have plenty of taut domains. Examples of generic taut manifolds can be constructed using the following

Lemma 2.1.15: (i) *A closed submanifold* Y *of a taut manifold* X *is taut.*
(ii) *The product of two taut manifolds is taut.*

Proof: (i) It suffices to notice that $\mathrm{Hol}(\Delta, Y)$ is closed in $\mathrm{Hol}(\Delta, X)$.

(ii) Let X_1 and X_2 be two taut manifolds, and denote by $p_j: X_1 \times X_2 \to X_j$ the projection ($j = 1, 2$). Then the assertion follows remarking that

$$\mathrm{Hol}(\Delta, X_1 \times X_2) = \mathrm{Hol}(\Delta, X_1) \times \mathrm{Hol}(\Delta, X_2)$$

and that a sequence $\{\varphi_\nu\} \subset \mathrm{Hol}(\Delta, X_1 \times X_2)$ is compactly divergent iff at least one of the sequences $\{p_j \circ f_\nu\} \subset \mathrm{Hol}(\Delta, X_j)$, for $j = 1, 2$, is compactly divergent, **q.e.d.**

In chapter 2.3 we shall describe other techniques for the construction of taut manifolds. For instance, it will turn out that any manifold covered by a strongly pseudoconvex domain is taut, as well as a great deal of homogeneous manifolds.

2.1.2 *Function theory on taut manifolds*

In this section we shall generalize to taut manifolds and tautly imbedded domains several theorems we proved in sections 1.1.4 and 1.3.1 concerning hyperbolic Riemann surfaces.

To unify the exposition, we introduce the following (quite artificial) definition: a complex manifold X is *relatively taut* if there exists a compact connected metric space $(\overline{X}, d)$ such that

(i) X is an open dense topological subspace of $\overline{X}$, and
(ii) $\mathrm{Hol}(\Delta, X)$ is relatively compact in $C^0(\Delta, \overline{X})$.

We shall say that X is *relatively taut in* $\overline{X}$; furthermore, for every complex manifold Y we shall denote by $\mathrm{Hol}(Y, \overline{X})$ the closure of $\mathrm{Hol}(Y, X)$ in $C^0(Y, \overline{X})$.

Both taut manifolds (where $\overline{X}$ is the one-point compactification) and tautly imbedded domains (where $\overline{X}$ is the topological closure in the larger manifold) are relatively taut, and a great deal of function theory on taut manifolds depends only on this fact.

A compact manifold is relatively taut iff it is taut; however, in general being a taut manifold is stronger than being relatively taut. For instance, if X is taut then $\mathrm{Hol}(Y, X)$ is locally compact for every complex manifold Y (copy the proof of Corollary 1.1.44), whereas this is not true if X is only relatively taut. For instance, take $X = B^2 \setminus \{0\}$, where B^2 is the unit euclidean ball of $\mathbf{C}^2$. X is relatively taut in $\overline{B^2}$ equipped with the usual euclidean distance, by Proposition 2.1.8. Define $f_{\delta,\varepsilon}$, $f \in \mathrm{Hol}(\Delta, X)$ by $f(\zeta) = (0, (\zeta + 1)/2)$, and

$$f_{\delta,\varepsilon}(\zeta) = \left(\delta, \frac{(1-\delta^2)^{1/2}}{2-\varepsilon}\,(\zeta + 1 - \varepsilon)\right),$$

for $0 < \delta < 1$ and $0 < \varepsilon < 1$. Then every neighbourhood of f in $\mathrm{Hol}(\Delta, X)$ for the compact-open topology contains a sequence $\{f_{\delta_\nu,\varepsilon}\}$, with $\delta_\nu \to 0$ and $\varepsilon > 0$ small, converging to

$$f_\varepsilon(\zeta) = \left(0, \frac{1}{2-\varepsilon}\,(\zeta + 1 - \varepsilon)\right).$$

Since f_ε belongs to $\mathrm{Hol}(\Delta, \overline{B^2})$ but not to $\mathrm{Hol}(\Delta, X)$, f cannot have compact neighbourhoods in $\mathrm{Hol}(\Delta, X)$, and $\mathrm{Hol}(\Delta, X)$ is not locally compact.

It is worth remarking another difference between taut and tautly imbedded manifolds on one side and generic relatively taut manifolds on the other side. We shall say that a complex manifold X relatively taut in $\overline{X}$ *fulfills the identity principle* in $\overline{X}$ if for any complex manifold Y and maps f, $g \in \mathrm{Hol}(Y, \overline{X})$ so that there is an open subset $A \subset Y$ where f and g coincide, we have $f \equiv g$ on Y. Clearly, taut manifolds (for $\mathrm{Hol}(Y, X^*) = \mathrm{Hol}(X, Y) \cup \{\infty\}$) and tautly imbedded domains (thanks to the usual identity principle applied to the larger manifold) fulfill the identity principle; on the other hand, this is not true for any relatively taut manifold.

To give an example, we first remark that if X is relatively taut in $\overline{X}$, then it is relatively taut in its one-point compactification X^*; this is easily seen using the continuous surjective map $p\colon \overline{X} \to X^*$ obtained by setting $p|_X = \mathrm{id}_X$ and $p(\overline{X} \setminus X) = \{\infty\}$. Now take

$$X = B^2 \setminus \{(0, w) \mid |w| \le 1/2\}. \tag{2.1.3}$$

Since X is tautly imbedded in $\mathbf{C}^2$, it is relatively taut in X^*. Then the maps f and $g \in \mathrm{Hol}(\Delta, X^*)$ given by

$$f(\zeta) = \begin{cases} p(0, \zeta) & \text{if } |\zeta| > 1/2, \\ \infty & \text{if } |\zeta| \le 1/2, \end{cases}$$

and $g \equiv \infty$ show that X does not fulfill the identity principle in X^*.

Coming back to the general situation, the formalization of what we did in the first section gives:

Lemma 2.1.16: *Let X be a complex manifold contained, as an open dense topological subspace, in a compact connected metric space $(\overline{X}, d)$. Then:*
(i) *X is relatively taut in $\overline{X}$ iff $\mathrm{Hol}(\Delta, X)$ is equicontinuous with respect to d;*
(ii) *if X is relatively taut in $\overline{X}$, then $\mathrm{Hol}(Y, X)$ is relatively compact in $C^0(Y, \overline{X})$ for every complex manifold Y, and thus $\mathrm{Hol}(Y, X)$ is equicontinuous with respect to d.*

Proof: (i) Apply the Ascoli-Arzelà Theorem 1.1.42.
(ii) Proposition 2.1.1 and the Ascoli-Arzelà theorem, **q.e.d.**

We can now start our study of function theory on taut manifolds, beginning with an easy fact (cf. Corollary 1.1.41):

Corollary 2.1.17: *Let X be a relatively taut manifold. Then the topology of pointwise convergence on $\mathrm{Hol}(Y, X)$ coincides with the compact-open topology for every complex manifold Y.*

Proof: Quote Kelley [1955], p. 232, using Lemma 2.1.16.(ii), **q.e.d.**

The second result is the generalization of *Vitali's theorem* (cf. Theorem 1.1.45):

Proposition 2.1.18: *Let X be a manifold relatively taut in $\overline{X}$ fulfilling the identity principle, and Y another complex manifold. Let $\{f_\nu\}$ be a sequence of maps in $\mathrm{Hol}(Y, X)$; assume there is a non-empty open set $A \subset Y$ such that $\{f_\nu(z)\}$ converges in $\overline{X}$ for every $z \in A$. Then $\{f_\nu\}$ converges uniformly on compact sets to an element $f \in \mathrm{Hol}(Y, \overline{X})$. In particular, if X is taut and $f(A) \cap X \neq \emptyset$ then $f \in \mathrm{Hol}(Y, X)$.*

Proof: Since X is relatively taut, $\{f_\nu\}$ must have at least one limit point in $\mathrm{Hol}(Y, \overline{X})$; by the identity principle, it can have at most one, and the assertion follows, **q.e.d.**

Using the manifold X defined in (2.1.3) and its one-point compactification, it is easy to see that Proposition 2.1.18 does not hold for relatively taut manifolds not fulfilling the identity principle.

Our next aim is the generalization of Theorem 1.3.4 to relatively taut complex manifolds. To state the result in its greatest generality, we need a topological lemma:

Lemma 2.1.19: *Let X, Y be two locally compact locally connected Hausdorff topological spaces, and $\{\varphi_\nu\}$ a sequence of continuous open maps of X into Y, converging for the compact-open topology to a continuous map $\varphi: X \to Y$. Suppose that $z_0 \in X$ is an isolated point of $\varphi^{-1}\big(\varphi(z_0)\big)$. Then for any neighbourhood U of z_0 there is a $\nu_0 \in \mathbf{N}$ such that $\varphi(z_0) \in \varphi_\nu(U)$ for $\nu \geq \nu_0$.*

Proof: Suppose the assertion is false. Then, up to a subsequence, we can choose a neighbourhood U of z_0 such that $\overline{U} \subset X$ is compact, $\overline{U} \cap \varphi^{-1}\big(\varphi(z_0)\big) = \{z_0\}$ and $\varphi(z_0) \notin \varphi_\nu(\overline{U})$ for all $\nu \in \mathbf{N}$. In particular, $\varphi(\partial U)$ is compact and $\varphi(z_0) \notin \varphi(\partial U)$. Hence we can find a neighbourhood V of $\varphi(\partial U)$ and a connected neighbourhood P of $\varphi(z_0)$ in Y such that $V \cap P = \emptyset$.

Now, since $\varphi_\nu \to \varphi$ for the compact-open topology, there is a $\nu_0 \in \mathbf{N}$ such that $\varphi_\nu(\partial U) \subset V$ for all $\nu \geq \nu_0$. Since φ_ν is open, we have $\partial\varphi_\nu(U) \subset \varphi_\nu(\partial U)$. Therefore $\varphi_\nu(U)$ is a relatively compact open set in Y with $\partial\varphi_\nu(U) \subset V$ for all $\nu \geq \nu_0$.

We claim that $\partial\varphi_\nu(U) \cap P \neq \emptyset$ if ν is large enough; this will yield the sought contradiction. Since $\varphi(z_0) \in P$, we have $\varphi_\nu(z_0) \in P$ for ν large enough. On the other hand, $\varphi(z_0) \notin \varphi_\nu(U)$, by assumption. If $\partial\varphi_\nu(U) \cap P = \emptyset$, we would have

$$P = \big[\varphi_\nu(U) \cap P\big] \cup \big[\big(Y \setminus \varphi_\nu(U)\big) \cap P\big],$$

and we would have written P as the union of two non void (for $\varphi_\nu(z_0)$ belongs to the first, and $\varphi(z_0)$ to the second) disjoint open subsets, whereas P is connected. Hence $\partial\varphi_\nu(U) \cap P \neq \emptyset$ for ν large enough, and the lemma is proved, **q.e.d.**

This lemma is the topological background of Corollary 1.1.36. Indeed, in the one-variable situation, every φ_ν is automatically open, and $\varphi^{-1}\big(\varphi(z_0)\big)$ is automatically discrete, and so Corollary 1.1.36 becomes a quantitative version of Lemma 2.1.19.

Using Lemma 2.1.19 we can prove the following fact (cf. Lemma 1.3.1):

Lemma 2.1.20: *Let X be relatively taut in $\overline{X}$, and $f \in \mathrm{Hol}(X, X)$. Then id_X can be a limit point of the sequence of iterates of f only if $f \in \mathrm{Aut}(X)$.*

Proof: Choose a subsequence $\{f^{k_\nu}\}$ converging to id_X. We can assume that $\{f^{k_\nu - 1}\}$ converges to a map $g \in \mathrm{Hol}(X, \overline{X})$ such that

$$\forall z \in X \qquad g\big(f(z)\big) = \lim_{\nu\to\infty} f^{k_\nu - 1}\big(f(z)\big) = z.$$

Therefore, if we set $Y = g^{-1}(X)$, it follows that Y is a not empty open submanifold of X; furthermore

$$\forall z \in Y \qquad f\big(g(z)\big) = \lim_{\nu\to\infty} f\big(f^{k_\nu - 1}(z)\big) = z.$$

Thus $f\colon X \to Y$ is a biholomorphism, and g is its inverse.

In particular, f is an open map. Set $\varphi_\nu = f^{k_\nu}$, and choose $z_0 \in X$. Then $\{\varphi_\nu\}$ is a sequence of open maps of X into itself converging to id_X; applying Lemma 2.1.19 we obtain $z_0 \in \varphi_\nu(X) \subset f(X)$ for ν large enough. In other words, f is onto, $Y = X$, and we are done, **q.e.d.**

Again, a non-periodic automorphism f such that the identity is a limit point of $\{f^k\}$ will be called *pseudoperiodic*. Another definition we shall need is the following: if A is a linear operator on a (finite dimensional) vector space, the *spectrum* $\mathrm{sp}(A)$ of A is the set of eigenvalues of A.

We are now ready for the announced generalization of Theorem 1.3.4, the *Cartan-Carathéodory theorem*:

Theorem 2.1.21: *Let X be a relatively taut manifold fulfilling the identity principle, and take $f \in \mathrm{Hol}(X, X)$ with a fixed point $z_0 \in X$. Then:*
(i) *the spectrum of df_{z_0} is contained in $\overline{\Delta}$;*
(ii) $|\det df_{z_0}| \le 1$*;*
(iii) $df_{z_0} = \mathrm{id}$ *iff f is the identity;*
(iv) *$T_{z_0}X$ admits a df_{z_0}-invariant splitting $T_{z_0}X = L_N \oplus L_U$ such that the spectrum of $df_{z_0}|_{L_N}$ is contained in Δ, the spectrum of $df_{z_0}|_{L_U}$ is contained in $\partial\Delta$ and $df_{z_0}|_{L_U}$ is diagonalizable;*
(v) $|\det df_{z_0}| = 1$ *iff f is an automorphism.*

Proof: Before starting the proof, we remark that, by equicontinuity, for every neighbourhood $U \subset\subset X$ of z_0 we can find another neighbourhood $V \subset\subset U$ of z_0 such that $g(V) \subset U$ for all $g \in \mathrm{Hol}(X, \overline{X})$ such that $g(z_0) = z_0$. In particular, this holds for any limit point of the sequence $\{f^k\}$ of iterates of f.

Since X is relatively taut in $\overline{X}$, there is a subsequence $\{f^{k_\nu}\}$ of iterates of f converging to a map $h \in \mathrm{Hol}(X, \overline{X})$. Clearly, $h(z_0) = z_0$. Furthermore, by the previous observation h is holomorphic near z_0; therefore dh_{z_0} is well-defined and $(df_{z_0})^{k_\nu} = d(f^{k_\nu})_{z_0} \to dh_{z_0}$ as $\nu \to +\infty$. In particular, if $\lambda \in \mathbf{C}$ is an eigenvalue of df_{z_0}, then the sequence $\{\lambda^{k_\nu}\}$ converges to an eigenvalue of dh_{z_0}. Therefore $|\lambda| \le 1$, and (i) and (ii) are proved.

We shall prove (iii) in a slightly more general case, that is for maps $h \in \mathrm{Hol}(X, \overline{X})$ such that $h(z_0) = z_0$. We have already remarked that such an h is holomorphic in a neighbourhood of z_0; in particular, dh_{z_0} is well defined.

So take $h \in \mathrm{Hol}(X, \overline{X})$ such that $h(z_0) = z_0$ and $dh_{z_0} = \mathrm{id}$. By equicontinuity, we can find two coordinate neighbourhoods $U \subset\subset V \subset\subset X$ of z_0 such that $h^k(U) \subset V$ for all $k \in \mathbf{N}$. By the identity principle, it suffices to show that $h|_U \equiv \mathrm{id}_U$.

Let $\phi\colon V \to D$ be a biholomorphism between V and a bounded domain D of some $\mathbf{C}^n$ containing the origin such that $\phi(z_0) = 0$. Then, if we set $D_1 = \phi(U)$ and $\psi = \phi \circ h \circ \phi^{-1}$, we have $\psi(0) = 0$, $d\psi_0 = \mathrm{id}$ and $\psi^k(D_1) \subset\subset D$ for all $k \in \mathbf{N}$.

Assume, by contradiction, $\psi|_{D_1} \not\equiv \mathrm{id}_{D_1}$. Then we can write

$$\psi(z) = z + P_j(z) + o(\|z\|^j),$$

where P_j is a non-zero homogeneous polynomial of degree j. One sees easily, by induction on k, that

$$\forall k \in \mathbf{N} \qquad \psi^k(z) = z + k\,P_j(z) + o(\|z\|^j); \tag{2.1.4}$$

therefore no subsequence of $\{\psi_k\}$ can converge (for $\{kP_j\}$ does not have converging subsequences), and this is impossible, for D is tautly imbedded in $\mathbf{C}^n$. In conclusion, $\psi|_{D_1} = \mathrm{id}_{D_1}$, $h|_U = \mathrm{id}_U$ and, by the identity principle, (iii) is proved.

To prove (iv), let $\{\lambda_1, \dots, \lambda_p\} \subset \overline{\Delta}$ be the eigenvalues of df_{z_0}, and let

$$T_{z_0}X = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_p}$$

be the Jordan decomposition of $T_{z_0}X$ with respect to df_{z_0}. Set $L_N = \bigoplus_{|\lambda_\nu|<1} J_{\lambda_\nu}$ and $L_U = \bigoplus_{|\lambda_\nu|=1} J_{\lambda_\nu}$; it suffices to show that $df_{z_0}|_{L_U}$ is diagonalizable. If not, in the Jordan canonical form of $df_{z_0}|_{L_U}$ there should be a block of the form

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix},$$

where $|\lambda| = 1$. Then $(df_{z_0}|_{L_U})^k$ should have a corresponding block of the form

$$\begin{pmatrix} \lambda^k & k\lambda^{k-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & k\lambda^{k-1} \\ 0 & & & \lambda^k \end{pmatrix}. \tag{2.1.5}$$

Since $(df_{z_0}|_{L_U})^{k_\nu} \to dh_{z_0}|_{L_U}$, the entries of $(df_{z_0}|_{L_U})^{k_\nu}$ should be bounded by a constant independent of ν, and hence (2.1.5) yields a contradiction.

It remains to prove (v). If f is an automorphism, (ii) yields $|\det df_{z_0}| = 1$ at once. Conversely, assume $|\det df_{z_0}| = 1$; in particular, $T_{z_0}X = L_U$, and so df_{z_0} in Jordan canonical form is diagonal. Then we can find a subsequence, still denoted by $\{f^{k_\nu}\}$, converging to a map $h \in \mathrm{Hol}(X, \overline{X})$ such that $h(z_0) = z_0$ and $dh_{z_0} = \mathrm{id}$. But we saw that this implies $h = \mathrm{id}_X$, and so we can apply Lemma 2.1.20 to infer that f is an automorphism, **q.e.d.**

The subspace L_U of $T_{z_0}X$ defined in (iv) is called the *unitary space* of f at the fixed point z_0, because $df_{z_0}|_{L_U}$ acts isometrically with respect to any hermitian product on L_U. On the other hand, the subspace L_N is called the *nilpotent space* of f at z_0 because $(df_{z_0}|_{L_N})^k \to 0$ as $k \to +\infty$, for $\mathrm{sp}(df_{z_0}|_{L_N}) \subset \Delta$.

An immediate consequence of the Cartan-Carathéodory theorem is *Cartan's uniqueness theorem:*

Corollary 2.1.22: *Let X be a relatively taut manifold fulfilling the identity principle, and $z_0 \in X$. Then if $f, g \in \mathrm{Aut}(X)$ are such that $f(z_0) = g(z_0) = z_0$ and $df_{z_0} = dg_{z_0}$, it follows that $f \equiv g$ on X.*

Proof: Apply Theorem 2.1.21.(iii) to $g^{-1} \circ f$, **q.e.d.**

In other words, automorphisms with a fixed point are completely determined by their differential at that point.

There is another nice corollary of Theorem 2.1.21, that we shall need in section 2.2.1. A domain $D \subset \mathbf{C}^n$ is *circular* if $e^{i\theta}z \in D$ for every $\theta \in \mathbf{R}$ and $z \in D$. Then

Corollary 2.1.23: *Let $D \subset\subset \mathbf{C}^n$ be a bounded circular domain of $\mathbf{C}^n$ containing 0. Let $f \in \mathrm{Aut}(D)$ be such that $f(0) = 0$. Then f is linear.*

Proof: Fix $\theta \in \mathbf{R}$ and define $h_\theta \colon D \to D$ by $h_\theta(z) = f^{-1}\big(e^{-i\theta} f(e^{i\theta}z)\big)$. Clearly $h_\theta(0) = 0$ and

$$dh_\theta(0) = df^{-1}(0) \cdot e^{-i\theta} \cdot df(0) \cdot e^{i\theta} = \mathrm{id}\,.$$

By Theorem 2.1.21.(iii) $h_\theta = \mathrm{id}_D$, that is

$$f(e^{i\theta}z) = e^{i\theta}f(z)$$

for all $\theta \in \mathbf{R}$ and $z \in D$. Therefore the linear term in the expansion of f in homogeneous polynomials is the only one different from zero, **q.e.d.**

We are left with the generalization of Proposition 1.1.46 and Corollary 1.1.47. For the sake of simplicity, we leave the general setting of relatively taut manifolds proving

Proposition 2.1.24: *Let X be a taut manifold. Then* $\mathrm{Aut}(X)$ *is closed in* $\mathrm{Hol}(X,X)$*, and* $\mathrm{Aut}_{z_0}(X)$ *is compact for all $z_0 \in X$.*

Proof: Let $\{f_\nu\}$ be a sequence of automorphisms of X converging to $f \in \mathrm{Hol}(X,X)$. Set $g_\nu = f_\nu^{-1}$, and choose $z_0 \in X$. Since $g_\nu\bigl(f_\nu(z_0)\bigr) = z_0$ for all $\nu \in \mathbf{N}$, and moreover $f_\nu(z_0) \to f(z_0) \in X$, no subsequence of $\{g_\nu\}$ can be compactly divergent. Therefore, up to a subsequence we can assume $g_\nu \to g \in \mathrm{Hol}(X,X)$, and it is clear that g is the inverse of f, **q.e.d.**

Actually, this is also true for tautly imbedded domains. The proof relies on another version of Corollary 1.1.36:

Proposition 2.1.25: *Let X be a complex manifold, and $\{f_\nu\}$ a sequence of holomorphic functions on X converging to a holomorphic function $f\colon X \to \mathbf{C}$. Assume $f_\nu(z) \neq 0$ for all $z \in X$ and $\nu \in \mathbf{N}$. Then either $f \equiv 0$ or $f(z) \neq 0$ for all $z \in X$.*

Proof: Assume, by contradiction, both $f \not\equiv 0$ and the existence of $z_0 \in X$ with $f(z_0) = 0$; we now apply the usual trick. Choosing a suitable system of local coordinates centered about z_0, we can assume X to be the euclidean unit ball B of some $\mathbf{C}^n$, and $z_0 = 0$. Since $f \not\equiv 0$, there is $z_1 \in B$ such that $f(z_1) \neq 0$; define g_ν, $g \in \mathrm{Hol}(\Delta, \mathbf{C})$ by

$$g_\nu(\zeta) = f_\nu(\zeta z_1/\|z_1\|) \qquad \text{and} \qquad g(\zeta) = f(\zeta z_1/\|z_1\|).$$

Then every g_ν has no zeroes on Δ and $g_\nu \to g$; hence, by Corollary 1.1.36, either $g \equiv 0$ or $g(\zeta) \neq 0$ for every $\zeta \in \Delta$. But $g(0) = 0$ and $g(\|z_1\|) \neq 0$, contradiction, **q.e.d.**

Then:

Theorem 2.1.26: *Let $D \subset\subset X$ be a tautly imbedded domain in a complex manifold X, and $\{f_\nu\}$ a sequence of automorphisms of D converging to a holomorphic map $f\colon D \to X$. Then either $f \in \mathrm{Aut}(D)$ or $f(D) \subset \partial D$ and f is everywhere degenerate on D.*

Proof: Let $E = \{z \in D \mid df_z \text{ is invertible}\}$. E is clearly open; we claim it is also closed in D. Take $z_0 \in D \setminus E$ (if $D = E$ there is nothing to prove), and choose a coordinate neoghbourhood U of $f(z_0)$, and a relatively compact coordinate neighbourhood $V \subset\subset D$ of z_0 such that $f(V) \subset\subset U$. In particular, we have $f_\nu(V) \subset\subset U$ for ν large enough. Since

U and V are coordinate neighbourhoods, we can trivialize TU and TV so to define unambiguously $j_\nu = \det(df_\nu)$ and $j = \det(df)$ from V into $\mathbf{C}$ for ν large enough. Every j_ν (if defined) has no zeroes in V; moreover, $j_\nu \to j$ and $j(z_0) = 0$. Therefore Proposition 2.1.25 implies $j|_V \equiv 0$, and so $V \subset D \setminus E$.

In conclusion, since D is connected, either $E = \emptyset$ or $E = D$; it remains to show that in the first case $f(D) \subset \partial D$, and that in the second case $f \in \mathrm{Aut}(D)$.

Assume $E = \emptyset$ and, by contradiction, that $f(D) \cap D \neq \emptyset$. Choose $z_0 \in D$ so that $f(z_0) = w_0 \in D$. Let $g_\nu = f_\nu^{-1}$; up to a subsequence we can assume that $\{g_\nu\}$ converges to a holomorphic map $g\colon D \to X$. Since $f_\nu(z_0) \to f(z_0) = w_0$, we have

$$g(w_0) = \lim_{\nu\to\infty} g_\nu\big(f(z_0)\big) = \lim_{\nu\to\infty} g_\nu\big(f_\nu(z_0)\big) = z_0 \in D.$$

Hence there is a small neighbourhood $U \subset D$ of w_0 such that $g(U)$ is relatively compact in D. But then, for ν large enough $g_\nu(U)$ is relatively compact in D and for every $z \in U$

$$f\big(g(z)\big) = \lim_{\nu\to\infty} f_\nu\big(g_\nu(z)\big) = z.$$

Therefore $(df)_{g(z)} \cdot (dg)_z = \mathrm{id}$ for $z \in U$; in particular, $g(U) \subset E$, contradiction, and the first part of the assertion is proved.

Now assume $E = D$. Then f is open and locally injective; by Lemma 2.1.19, $f(D) \subset \bigcup_\nu f_\nu(D) = D$. To show that $f \in \mathrm{Aut}(D)$ it is then enough to produce an inverse. Again, up to a subsequence we can assume that $g_\nu = f_\nu^{-1}$ converges to a holomorphic function $g\colon D \to X$. If $z \in D$ we have

$$g\big(f(z)\big) = \lim_{\nu\to\infty} g_\nu\big(f_\nu(z)\big) = z.$$

Hence dg is invertible on $f(D)$; then the argument already used implies that dg is everywhere invertible and, by Lemma 2.1.19, $g(D) \subset D$. But then it is clear that g is the inverse of f, and the theorem is proved, **q.e.d.**

Corollary 2.1.27: *Let $D \subset\subset X$ be a tautly imbedded domain in a complex manifold X. Then $\mathrm{Aut}(D)$ is closed in $\mathrm{Hol}(D, D)$, and $\mathrm{Aut}_{z_0}(D)$ is compact for all $z_0 \in D$.*

2.1.3 *Limit points of iterates*

Up to now, the function theory on taut manifolds was not that different from what we saw for hyperbolic Riemann surfaces. In this section we shall see that this is not anymore the case when we begin to work seriously with iterates.

The core of the difference lies in the following observation. Let X be a complex manifold, and $f \in \mathrm{Hol}(X, X)$ such that its sequence of iterates converges to a holomorphic function $\rho\colon X \to X$. The sequence $\{f^{2k}\}$ tends to ρ too; it follows that $\rho^2 = \rho$. If X is a Riemann surface, this implies that ρ is either the identity or a constant function — for the image of ρ coincides with $\mathrm{Fix}(\rho)$ and thus it is either X or one point. This is not true if the dimension of X is greater than 1: in general, maps $\rho \in \mathrm{Hol}(X,X)$ such that $\rho^2=\rho$ can be quite complicated. For instance, let B^2 be the unit euclidean ball in $\mathbf{C}^2$. The power series

$$1-\sqrt{1-t}=\sum_{k=1}^{\infty} c_k t^k$$

is converging for $|t|<1$ and has $c_k>0$ for all $k\geq 1$. Take $g_k \in \mathrm{Hol}(B^2,\mathbf{C})$ such that $|g_k(z,w)|\leq c_k$ for all $(z,w)\in B^2$, and define $\phi\in \mathrm{Hol}(B^2,\Delta)$ by

$$\phi(z,w)=z+\sum_{k=1}^{\infty} g_k(z,w)\, w^{2k}.$$

Then $\rho(z,w)=\big(\phi(z,w),0\big)$ always satisfies $\rho^2=\rho$, and it can be very far from being constant…

This section is mainly devoted to elucidate the relationship between limit points of sequences of iterates and maps ρ such that $\rho^2=\rho$. Our results will be applied in chapter 2.4 to investigate thoroughly iteration theory on taut manifolds.

We begin with a definition. Let X be a complex manifold. A *holomorphic retraction* of X — or, with terminology borrowed by semigroup theory, an *idempotent* of $\mathrm{Hol}(X,X)$ — is a holomorphic map $\rho\colon X\to X$ such that $\rho^2=\rho$. The image of a holomorphic retraction is said to be a *holomorphic retract* of X.

We saw that holomorphic retractions can be quite complicated. However, the holomorphic retracts are never too wild:

Lemma 2.1.28: *Let X be a complex manifold, and $\rho\colon X\to X$ a holomorphic retraction of X. Then the image of ρ is a closed submanifold of X.*

Proof: Let $M=\rho(X)$ be the image of ρ, and take $z_0\in M$. Choose an open neighbourhood U of z_0 in X contained in a local chart for X at z_0. Then $V=\rho^{-1}(U\cap M)\cap U$ is an open neighbourhood of z_0 contained in a local chart such that $\rho(V)\subset V$. Therefore without loss of generality we can assume that X is a bounded domain D in $\mathbf{C}^n$.

Set $P=d\rho_{z_0}\colon \mathbf{C}^n\to\mathbf{C}^n$, and define $\varphi\colon D\to\mathbf{C}^n$ by

$$\varphi=\mathrm{id}_D+(2P-\mathrm{id}_D)\circ(\rho-P).$$

Since $d\varphi_{z_0}=\mathrm{id}$, φ defines a local chart in a neighbourhood of z_0. Now $P^2=P$ and $\rho^2=\rho$; hence

$$\begin{aligned}\varphi\circ\rho&=\rho+(2P-\mathrm{id}_D)\circ\rho^2-(2P-\mathrm{id}_D)\circ P\circ\rho\\ &=P\circ\rho=P+P\circ(2P-\mathrm{id}_D)\circ(\rho-P)=P\circ\varphi.\end{aligned}$$

Therefore in this local chart ρ becomes linear, and M is a submanifold near z_0. By the arbitrariness of z_0, the assertion follows, **q.e.d.**

And now we can prove the fundamental

Theorem 2.1.29: *Let X be a taut manifold, and $f \in \mathrm{Hol}(X, X)$. Assume that the sequence $\{f^k\}$ of iterates of f is not compactly divergent. Then there exist a submanifold M of X and a holomorphic retraction $\rho: X \to M$ such that every limit point $h \in \mathrm{Hol}(X, X)$ of $\{f^k\}$ is of the form*

$$h = \gamma \circ \rho, \tag{2.1.6}$$

where γ is an automorphism of M. Moreover, even ρ is a limit point of the sequence $\{f^k\}$.

Proof: Let $\{f^{k_\nu}\}$ be a subsequence of $\{f^k\}$ converging to $h \in \mathrm{Hol}(X, X)$. We can assume that also $p_\nu = k_{\nu+1} - k_\nu$ and $q_\nu = p_\nu - k_\nu = k_{\nu+1} - 2k_\nu$ tend to $+\infty$ as $\nu \to +\infty$, and that $\{f^{p_\nu}\}$ and $\{f^{q_\nu}\}$ are either converging or compactly divergent. Now for all $z \in X$

$$\lim_{\nu \to \infty} f^{p_\nu}\big(f^{k_\nu}(z)\big) = \lim_{\nu \to \infty} f^{k_{\nu+1}}(z) = h(z);$$

therefore $\{f^{p_\nu}\}$ cannot be compactly divergent, and thus converges to a map $\rho \in \mathrm{Hol}(X, X)$ such that

$$h \circ \rho = \rho \circ h = h. \tag{2.1.7}$$

Next, for all $z \in X$

$$\lim_{\nu \to \infty} f^{q_\nu}\big(f^{k_\nu}(z)\big) = \lim_{\nu \to \infty} f^{p_\nu}(z) = \rho(z).$$

Hence neither $\{f^{q_\nu}\}$ can be compactly divergent, and converges to a map $g \in \mathrm{Hol}(X, X)$ such that

$$g \circ h = h \circ g = \rho. \tag{2.1.8}$$

In particular

$$\rho^2 = \rho \circ \rho = g \circ h \circ \rho = g \circ h = \rho,$$

and ρ is a holomorphic retraction of X onto a submanifold M, by Lemma 2.1.28. Now (2.1.7) implies $h(X) \subset M$. Since $g \circ \rho = \rho \circ g$, we have $g(M) \subset M$ and (2.1.8) yields

$$g \circ h|_M = h \circ g|_M = \mathrm{id}_M;$$

hence $\gamma = h|_M \in \mathrm{Aut}(M)$ and (2.1.7) becomes (2.1.6).

Now, let $\{f^{k'_\nu}\}$ be another subsequence of $\{f^k\}$ converging to a map $h' \in \mathrm{Hol}(X, X)$. Arguing as before, we can assume $s_\nu = k'_\nu - k_\nu$ and $t_\nu = k_{\nu+1} - k'_\nu$ tending to $+\infty$ as $\nu \to +\infty$, and that $\{f^{s_\nu}\}$ and $\{f^{t_\nu}\}$ converge to holomorphic maps $\alpha \in \mathrm{Hol}(X, X)$, respectively $\beta \in \mathrm{Hol}(X, X)$ such that

$$\alpha \circ h = h \circ \alpha = h' \qquad \text{and} \qquad \beta \circ h' = h' \circ \beta = h. \tag{2.1.9}$$

Then $h(X) = h'(X)$, and so M does not depend on the particular converging subsequence.

It remains to show that ρ itself does not depend on the chosen subsequence. Write $h = \gamma_1 \circ \rho_1$, $h' = \gamma_2 \circ \rho_2$, $\alpha = \gamma_3 \circ \rho_3$ and $\beta = \gamma_4 \circ \rho_4$, where ρ_1, ρ_2, ρ_3 and ρ_4 are holomorphic retractions of X onto M, and γ_1, γ_2, γ_3 and γ_4 are automorphisms of M. Then $h \circ h' = h' \circ h$ and $\alpha \circ \beta = \beta \circ \alpha$ together with (2.1.9) become

$$\begin{gathered}\gamma_1 \circ \gamma_2 \circ \rho_2 = \gamma_2 \circ \gamma_1 \circ \rho_1 \\ \gamma_3 \circ \gamma_1 \circ \rho_1 = \gamma_1 \circ \gamma_3 \circ \rho_3 = \gamma_2 \circ \rho_2 \\ \gamma_4 \circ \gamma_2 \circ \rho_2 = \gamma_2 \circ \gamma_4 \circ \rho_4 = \gamma_1 \circ \rho_1 \\ \gamma_3 \circ \gamma_4 \circ \rho_4 = \gamma_4 \circ \gamma_3 \circ \rho_3\end{gathered} \tag{2.1.10}$$

Writing ρ_2 in function of ρ_1 using the first and the second equation in (2.1.10) we find $\gamma_3 = \gamma^{-1} \circ \gamma_2$. Writing ρ_1 in function of ρ_2 using the first and the third equation, we get $\gamma_4 = \gamma_2^{-1} \circ \gamma_1$. Hence $\gamma_3 = \gamma_4^{-1}$ and the fourth equation yields $\rho_3 = \rho_4$. But then, using the second and third equation we obtain

$$\rho_2 = \gamma_3^{-1} \circ \gamma_1^{-1} \circ \gamma_2 \circ \rho_2 = \rho_3 = \rho_4 = \gamma_4^{-1} \circ \gamma_2^{-1} \circ \gamma_1 \circ \rho_1 = \rho_1,$$

and we are done, **q.e.d.**

The manifold M whose existence is asserted in the previous theorem is called the *limit manifold* of the map f, and its dimension the *limit multiplicity* of f; analogously, the holomorphic retraction is called the *limit retraction* of f. These concepts are defined as soon as the sequence $\{f^k\}$ is not compactly divergent.

If f has a fixed point, its limit multiplicity can be easily computed:

Corollary 2.1.30: *Let X be a taut manifold, and take $f \in \mathrm{Hol}(X, X)$ such that $f(z_0) = z_0$ for some $z_0 \in X$. Then the unitary space of f at z_0 is the tangent space at z_0 of the limit manifold of f. In particular, the limit multiplicity of f is the number of eigenvalues of df_{z_0} contained in $\partial\Delta$, counted according to their multiplicity.*

Proof: Let $\rho\colon X \to M$ be the limit retraction; clearly, $z_0 \in M$. Fix a subsequence $\{f^{k_\nu}\}$ converging to ρ; in particular,

$$(df_{z_0})^{k_\nu} \longrightarrow d\rho_{z_0}. \tag{2.1.11}$$

Let $T_{z_0}X = L_N \oplus L_U$ be the df_{z_0}-invariant splitting given by Theorem 2.1.21.(iv); then (2.1.11) implies $d\rho_{z_0}|_{L_N} = 0$ and $d\rho_{z_0}|_{L_U} = \mathrm{id}$, for df_{z_0} acts diagonally on L_U and $\mathrm{sp}(df_{z_0}|_{L_U}) \subset \partial\Delta$. Hence $L_U = T_{z_0}M$, and the assertion follows, **q.e.d.**

Another observation we shall need later on is that f acts on its limit manifold:

Corollary 2.1.31: *Let X be a taut manifold; take $f \in \mathrm{Hol}(X, X)$ such that $\{f^k\}$ is not compactly divergent, and let $\rho\colon X \to M$ be its limit retraction. Then $f(M) \subset M$ and $\varphi = f|_M$ is an automorphism of M.*

Proof: Since $f \circ \rho = \rho \circ f$, it is clear that $f(M) \subset M$. Let $\{f^{k_\nu}\}$ be a subsequence of iterates converging to ρ; then $f^{k_\nu+1} \to \varphi \circ \rho$ as $\nu \to +\infty$, and the assertion follows from Theorem 2.1.29, **q.e.d.**

Theorem 2.1.29 outlines the direction we should follow in the study of the sequence of iterates. We should understand when the sequence $\{f^k\}$ is compactly divergent, and what happens in that case, and we should find out when, and whether, its only limit point is the limit retraction. We shall undertake this study in chapter 2.4; for the moment we shall limit ourselves to two consequences of Theorem 2.1.29.

The first one actually is an application of the proof of Theorem 2.1.29. It is the generalization of Proposition 1.3.14:

Corollary 2.1.32: *Let X be a relatively taut manifold without compact submanifolds of dimension greater than zero. Let $f \in \mathrm{Hol}(X, X)$ be such that $f(X) \subset\subset X$. Then f has a unique fixed point $z_0 \in X$, and the sequence of iterates of f converges to z_0.*

Proof: Since $f(X)$ is relatively compact in X, the sequence $\{f^k\}$ of iterates of f is relatively compact in $\mathrm{Hol}(X, X)$. Then, arguing as in the proof of Theorem 2.1.29, we can find a subsequence of iterates converging to a holomorphic retraction $\rho: X \to X$. Now, $\rho(X)$ should be a closed submanifold of X contained in $\overline{f(X)}$, i.e., a compact connected submanifold of X; by the assumption, then, $\rho(X)$ is a point $z_0 \in X$. Since $f \circ \rho = \rho \circ f$, it follows that z_0 is a fixed point of f. Finally, the same argument used in the proof of Theorem 2.1.29 — namely, (2.1.7) — shows that z_0 is the unique limit point of $\{f^k\}$, that is $f^k \to z_0$, **q.e.d.**

Any manifold X where $\mathrm{Hol}(X, \mathbf{C})$ separates points is without compact submanifolds of dimension greater than zero. So Corollary 2.1.32 applies, for instance, to bounded domains in $\mathbf{C}^n$.

The second application we present is the generalization of Corollary 1.3.21. We first need the

Proposition 2.1.33: *Let $D \subset\subset \mathbf{C}^n$ be a taut domain, and $f \in \mathrm{Hol}(D, D)$ such that $\{f^k\}$ is not compactly divergent. Assume that the q-th cohomology group $H^q(D, \mathbf{C})$ is non-trivial and finite dimensional, and that the induced map $f^*: H^q(D, \mathbf{C}) \to H^q(D, \mathbf{C})$ is not nilpotent. Then the limit multiplicity of f is at least q.*

Proof: By the universal coefficient theorem, $H_q(D, \mathbf{C}) = H_q(D, \mathbf{Z}) \otimes \mathbf{C}$; therefore we can choose a basis $\{\sigma_1, \ldots, \sigma_d\}$ of $H_q(D, \mathbf{C})$ contained in $H_q(D, \mathbf{Z})$, and the dual basis $\{\omega_1, \ldots, \omega_d\}$ of $H^q(D, \mathbf{C})$ is contained in $H^q(D, \mathbf{Z})$. Furthermore, since D is pseudoconvex (Proposition 2.1.9), we can represent $\omega_1, \ldots, \omega_d$ by holomorphic forms on D (see, e.g., Krantz [1982], p. 237).

Since f^* is not nilpotent, up to reordering $\{\omega_1, \ldots, \omega_d\}$ we may choose a subsequence $\{f^{k_\nu}\}$ converging to the limit retraction $\rho \in \mathrm{Hol}(D, D)$ such that

$$(f^*)^{k_\nu}\omega_1 = \sum_{j=1}^{d} c_\nu^j \omega_j \neq 0,$$

where every c_ν^j is integer. Moreover, we can also assume that for some r we have $c_\nu^r \neq 0$ for all $\nu \in \mathbf{N}$. Thus

$$c_\nu^r = \sum_{j=1}^{d} \int_{\sigma_r} c_\nu^j \omega_j = \int_{\sigma_r} (f^*)^{k_\nu}\omega_1 = \int_{\sigma_r} (f^{k_\nu})^*\omega_1.$$

Now, since σ_r is represented by a compactly supported cycle, we may take the limit as $\nu \to +\infty$ obtaining

$$\left| \int_{\sigma_r} \rho^* \omega_1 \right| \geq 1,$$

because $|c_\nu^r| \geq 1$ for every $\nu \in \mathbf{N}$. Hence $\rho(X)$ should have dimension at least q, **q.e.d.**

Then

Theorem 2.1.34: *Let $D \subset\subset \mathbf{C}^n$ be a bounded C^1 domain with simple boundary. Assume that the n-th cohomology group $H^n(D, \mathbf{C})$ is non-trivial and finite dimensional. Then $f \in \mathrm{Hol}(D, D)$ is an automorphism iff $f^*: H^n(D, \mathbf{C}) \to H^n(D, \mathbf{C})$ is not nilpotent.*

Proof: If $f \in \mathrm{Aut}(D)$ then f^* is not nilpotent, of course. Conversely, assume that f^* is not nilpotent. Since a bounded domain of $\mathbf{C}^n$ with simple boundary is taut, there are only two possibilities: either $\{f^k\}$ is compactly divergent, or $\{f^k\}$ admits a subsequence converging to a holomorphic retraction $\rho: D \to M$ (by Theorem 2.1.29). In the latter case, by Proposition 2.1.33, ρ is the identity and, by Lemma 2.1.20, f is an automorphism.

If $\{f^k\}$ is compactly divergent, there is a subsequence $\{f^{k_\nu}\}$ converging to a constant map $x_0 \in \partial D$, for D has simple boundary. Since ∂D is an imbedded manifold, x_0 has a fundamental system of contractible neighbourhoods in $\overline{D}$; hence, arguing as in the proof of Corollary 1.3.21, we find that f_* (and hence f^*) is nilpotent, contradiction, **q.e.d.**

In particular, this theorem holds for strongly pseudoconvex domains of $\mathbf{C}^n$, thanks to Corollary 2.1.14.

Notes

The systematic investigation of normal families of holomorphic maps of several complex variables began only recently, with the papers by Grauert and Reckziegel [1965], Wu [1967] and Kaup [1968]; before them, the only general result was Proposition 2.1.8, an easy consequence of the one-variable Montel theorem (see Montel [1927]).

The theorems of general topology we used in our discussion of the one-point compactification can be found, e.g., in Kelley [1955] and in Dugundji [1966].

The concept of taut manifold was first introduced by Wu [1967] and Kaup [1968] (Kaup used the term *hyperbolic* that we have reserved, following the common usage, to another concept to be introduced in chapter 2.3). Their definition stated that a complex manifold X is taut iff the family $\mathrm{Hol}(Y, X)$ is normal for any complex manifold Y; only later Barth [1970] proved Proposition 2.1.1 and Theorem 2.1.2, establishing the equivalence of our definition with the original one.

Tautly imbedded domains have been introduced by Kiernan [1973], who also proved Proposition 2.1.4.

The relevance of conditions like the one described in Proposition 2.1.6 has been already pointed out by Hervé [1951], who studied iteration theory in what he called D_0 domains in $\mathbf{C}^2$, which are exactly bounded domains satisfying the characterization of tautness given in Proposition 2.1.6.

Proposition 2.1.9 is taken from Wu [1967]. Its best converse (improving the fairly standard Corollary 2.1.14) is due to Demailly [1987], who proved that every bounded pseudoconvex domain with Lipschitz boundary is taut. On the other hand, Kerzman and Rosay [1981] (see also Barth [1983]) have given an example of a bounded weakly pseudoconvex domain in $\mathbf{C}^2$ which is not taut. This is somehow related to the fact that weakly pseudoconvex domains are not necessarily locally convex; an example is in Kohn and Niremberg [1973].

A large part of Wu [1967] is devoted to show that large classes of hermitian manifolds (far beyond our rough Lemma 2.1.15) are taut; for instance, every complete hermitian manifold with holomorphic sectional curvature bounded above by a negative constant is taut (see also Grauert and Reckziegel [1965]). In chapter 2.3 we shall introduce on every complex manifold an invariant pseudodistance — the multidimensional version of the Poincaré distance of Riemann surfaces — and we shall see that if this distance is complete then the manifold is taut, thus providing the examples we mentioned at the end of section 2.1.1.

The notion of relatively taut manifold has been introduced here only to avoid repeating two times every statement, once for taut manifolds and once for tautly imbedded domains. At present, it does not seem to have any other significance.

Section 2.1.2 is a short account in the setting of taut manifolds of the classical theory of holomorphic mappings of bounded domains of $\mathbf{C}^n$ as developed mainly by Carathéodory [1932] and H. Cartan [1930a, b, 1932]. Corollary 2.1.17, Propositions 2.1.18, 2.1.24 and 2.1.25 are easy extensions of their one-variable counterparts. Theorem 2.1.21 was first proved by H. Cartan [1930a, b] for domains in $\mathbf{C}^2$, and subsequently generalized by Carathéodory [1932] to domains in $\mathbf{C}^n$. In the present version was first proved by Wu [1967], who also showed that the automorphism group of a taut manifold is a Lie group, generalizing the classical theorem of H. Cartan [1935] (for a modern proof see Narasimhan [1971]).

Corollary 2.1.22 is somehow akin to the É. Cartan uniqueness theorem of differential geometry: if X is a Riemannian manifold, $z_0 \in X$ and f, $g\colon X \to X$ are two isometries with fixed point z_0, then $df_{z_0} = dg_{z_0}$ iff $f \equiv g$. The proof (see, for instance, Kobayashi and Nomizu [1968]) is completely different, relying on the exponential map of Riemannian manifolds, but the phenomenon is absolutely the same.

Corollary 2.1.23 is again due to H. Cartan [1930a]. Finally, Theorem 2.1.26 is in H. Cartan [1932], as well as Lemma 2.1.20.

Lemma 2.1.28 is due to Rossi [1963]; the proof we presented is taken from H. Cartan [1986].

Theorem 2.1.29 (see Abate [1988c]) is inspired by Bedford [1983b]; his result, slightly less complete, was in turn inspired by H. Cartan [1932], and forecast by Hervé [1951] for domains in $\mathbf{C}^2$. An analogous statement for convex domains has been proved by Suzuki [1987]. A different approach to the existence of the limit retraction can be obtained via topological semigroup theory; see Wallace [1955] and Shields [1964].

Corollary 2.1.32 was first proved by Wavre [1926]. Our proof is very similar to the one given by Hervé [1963b]. A different approach can be found in Earle and Hamilton [1969].

Looking at the proofs, it is clear that Proposition 2.1.33 and Theorem 2.1.34 hold for generic taut Stein manifolds (respectively, tautly imbedded Stein C^1 domains with simple boundary), which is the original statement due to Bedford [1983b]. Again, it seems that the holomorphic structure tries to get rid of topological obstructions in a rather decise way. The influence of topology on function theory of several complex variables is, at present, not well understood, and only few papers on the argument have appeared (we quote, for instance, Bedford [1983a, b] and Mok [1983]).