Title to be announced

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Cortona, Italy, September 2004

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- **Pseudo-noise vs Hadamard-Sylvester matrices** [BOS2004, to be submitted].
- **Order-one quasiseparable matrices** [EGO2004].

I. Bezoutains and the classical Kharitonov thm[OO2004]

Stability of interval polynomials

- **A single polynomial**
  - A polynomial
$$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n \tag{1}$$
    is called stable if all its roots are in the LHP.
  - The Routh-Hurwitz test checks using only $O(n^2)$ operations if a polynomial is stable.

- **A family of polynomials**
  - Let we are given an infinite set of **interval polynomials** of the form (1)

$$IP = \{F(z) \text{ of the form } (1)\} \quad \text{where} \quad \overbrace{\underline{p_i} \leq p_i \leq \overline{p_i}}^{\textbf{intervals}}$$

- **A Question:** Is there any way to check if all the polynomials in IP are stable?

The classical Kharitonov's theorem

- Let we are given an interval polynomial

$$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n x^n \qquad \text{where} \qquad \underline{p_i} \leq p_i \leq \overline{p_i} \tag{2}$$

- Kharitonov (1978): The infinite set of polynomials of the form (5) is stable if only the following four "boundary" polynomials are stable:

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

where

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \ldots,$$
$$F_{e,max}(z) = \overline{p}_0 + \underline{p}_2 z^2 + \overline{p}_4 z^4 + \underline{p}_6 z^6 + \ldots,$$
$$F_{o,min}(z) = \underline{p}_1 z + \overline{p}_3 z^3 + \underline{p}_5 z^5 + \overline{p}_7 z^7 + \ldots,$$
$$F_{o,max}(z) = \overline{p}_1 z + \underline{p}_3 z^3 + \overline{p}_5 z^5 + \underline{p}_7 z^7 + \ldots,$$

A connection to structured matrices?

The Hermite criterion

Stability of a polynomial $\Longleftrightarrow$ P.D. of the Bezoutian

The classical Hermite theorem. Bezoutians

- All the roots of $F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n z^n$ are in the UHP if and only if the Bezoutian matrix $B = \left[\ r_{k,l}\ \right]$ is positive definite, where

$$-\frac{i}{2} \cdot \frac{F(x)\breve{F}(y) - \breve{F}(x)F(y)}{x - y} = \sum_{k,l=0}^{n-1} r_{k,l} x^k y^l$$

where $\breve{F}(z) = {p_0}^* + {p_1}^* z + {p_2}^* z^2 + \cdots + {p_n}^* z^n$.

  – C.Hermite, *Extrait d'une lettre de Mr. Ch. Hermite de Paris à Mr. Borchardt de Berlin, sur le nombre des racines d'une èquation algèbrique comprises entre des limits donèes*, J. Reine Angew. Math., **52** (1856), 39-51.

Kharitonov's Theorem and Structured Matrices

- Kharitonov's theorem is equivalent to the following: $Bez(F)$ is positive definite if and only if $Bez(F_{max,max})$, $Bez(F_{max,min})$, $Bez(F_{min,max})$, $Bez(F_{min,min})$ are all positive definite.

- Willems and Tempo [WT99] asked if a direct Bezoutian proof of this fact is possible. A brute-force approach does not work here because examples show that $B(F) - B(F_{m??,m??})$ are not necessarily positive definite.

- [OO2004] gives a proof based only on the properties of Bezoutians.

- The proof is universal, i.e. it carries over to the **discrete-time** case (it proves The Vaidyanathan/Schur-Fujivara Theorem.

discrete-time sense = the roots are inside the unit circle.

An open question

- **Kharitonov for matrix polynomials? Is the (block) Anderson-Jury Bezoutian of help?**

II. Kharitonov-like theorem for quasipolynomials and entire functions [OS2004a]

Example I. Stability of Quasi-polynomials

- Control engineering: **retarded feedback time delay system**

$$\frac{dy}{dt} = Ay(t) + \sum_{r=1}^{p} By(\overbrace{t - \tau_r}^{\textbf{delays}}) \tag{3}$$

- After Laplace transformation one gets

$$F(s) = \det(sI - A - \sum_{r=1}^{p} B_r e^{-\tau_r s}) = \underbrace{f_0(s) + e^{-sT_1} f_1(s) + \cdots + e^{-sT_m} f_m(s)}_{\textbf{a quasi-polynomial}} \tag{4}$$

where $f_k(s)$ are polynomials.

- Stability of (3) $\Leftrightarrow$ all the roots of $F(s)$ in (4) are in the left half plane.

Example II. Stability of entire functions

$$\frac{dy}{dt} = zy(t), \qquad y(t) + \int_0^T \beta(\tau) y(t-\tau) d\tau = 0.$$

where T is fixed and $\beta(\tau)$ is given.

This system is stable if and only if the roots of the entire function

$$F(z) = 1 + \int_0^T \beta(\tau) e^{-z\tau} d\tau$$

are in the LHP.

Stability of entire functions

- **Some history**:
  - L.Pontryagin, *On the zeros of some transcedental functions,* IAN USSR, Math. series, vol. 6, 115-134, 1942.
  - N.Chebotarev, N.Meiman, *The Routh-Hurwitz probelm for polynomials and entire functions,* Trudy MIAN, 1949, vol. 26.

- **Some relevant literature**:
  - B.Ya. Levin , *Lectures on Entire Functions* , AMS, 1996.
  - B.Ya.Levin. Distribution of zeros of entire functions. AMS,1980.
  - J.K. Hale and S.Verdun Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, Applied Mathematical Sciences Vol. 99, 1993.
  - S.I. Nuculescu

- **Some applications**:
  - L.Dugard and E.Verriest (eds), *Stability and control of time-delay systems*, Springert Verlag 1998.
  - S.P. Bhattacharyya, H. Chapellat, L.H. Keel, *Robust Control - The Parametric Approach*, Prentice Hall, 1995.
  - A.Datta, M.-T. Ho and S.P. Bhattacharyya, *Structure and Synthesis of PID Controllers*, Springer Verlag, 2003.

Recall the classical Kharitonov's theorem

- Let we are given an interval polynomial

$$F(z) = p_0 + p_1 z + p_2 z^2 + \cdots + p_n x^n \quad \text{where} \quad \underline{p_i} \leq p_i \leq \overline{p_i} \tag{5}$$

- Kharitonov (1978): The infinite set of polynomials of the form (5) is stable if only the following four "boundary" polynomials are stable:

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

where

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \ldots,$$
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$$F_{o,max}(z) = \overline{p}_1 z + \underline{p}_3 z^3 + \overline{p}_5 z^5 + \underline{p}_7 z^7 + \ldots,$$

The Kharitonov theorem revisited

- The meaning of max and min.

$$F_{e,min}(z) = \underline{p}_0 + \overline{p}_2 z^2 + \underline{p}_4 z^4 + \overline{p}_6 z^6 + \ldots,$$
$$F_{e,min}(iz) = \underline{p}_0 - \overline{p}_2 z^2 + \underline{p}_4 z^4 - \overline{p}_6 z^6 \pm \ldots,$$

- Kharitonov (1978): If only four polynomialls

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

are stable then all the polynomials

$$F(z) = \underbrace{F_e(z)}_{even} + \underbrace{F_o(z)}_{odd}$$

are stable provided that (for $z = \overline{z}$)

$$\frac{F_{o,min}(iz)}{iz} \leq \frac{F_o(iz)}{iz} \leq \frac{F_{o,max}(iz)}{iz}.$$

$$F_{e,min}(iz) \leq F_e(iz) \leq F_{e,max}(iz)$$

A generalization of Kharitonov for (scalar) entire functions

- THM. If only four entire functions of exponential type

$$F_{min,min}(z) = F_{e,min}(z) + F_{o,min}(z), \qquad F_{min,max}(z) = F_{e,min}(z) + F_{o,max}(z)$$

$$F_{max,min}(z) = F_{e,max}(z) + F_{o,min}(z), \qquad F_{max,max}(z) = F_{e,max}(z) + F_{o,max}(z)$$

belong to the class HP then all the functions

$$F(z) = F_e(z) + F_o(z)$$

belong to the class HP as well provided that

$$\frac{F_{o,min}(iz)}{iz} \leq \frac{F_o(iz)}{iz} \leq \frac{F_{o,max}(iz)}{iz}.$$

$$F_{e,min}(iz) \leq F_e(iz) \leq F_{e,max}(iz)$$

for $z = \overline{z}$.

Conditions

- $0 < m_o \leq \mid \frac{F_{o,min}(z)}{F_{o,max}(z)} \mid \leq M_o < \infty \quad \text{for} \quad z = \overline{z}$
- $h_{F_o}(\theta) = h_{F_{o,min}}(\theta)$.
- $\frac{F_o(z)}{F_{o,max}(z)} = O(1) \quad \text{for} \quad z = \overline{z}$.
- $0 < m_e \leq \mid \frac{F_{e,min}(z)}{F_{e,max}(z)} \mid \leq M_e < \infty \quad \text{for} \quad z = \overline{z}$
- $h_{F_e}(\theta) = h_{F_{e,min}}(\theta)$.
- $\frac{F_e(z)}{F_{e,max}(z)} = O(1) \quad \text{for} \quad z = \overline{z}$.

(Classical) Kharitonov via Hermite-Biehler. I

- THM (Hermite-Biehler). Let

$$F(z) = \underbrace{F_e(z)}_{even} + \underbrace{F_o(z)}_{odd}$$

Then the polynomial $F(z)$ is stable if and only if the following two conditions hold true.

1. The **roots** of the polynomials $F_e(iz)$ and $F_o(iz)$ are all real and they interlace.
2. There is at least one point $z_0 \in \mathbb{R}$ such that

$$F_e(iz_0)F_o'(iz_0) - F_e'(iz_0)F_o(iz_0) > 0.$$

Kharitonov via Hermite-Biehler. II

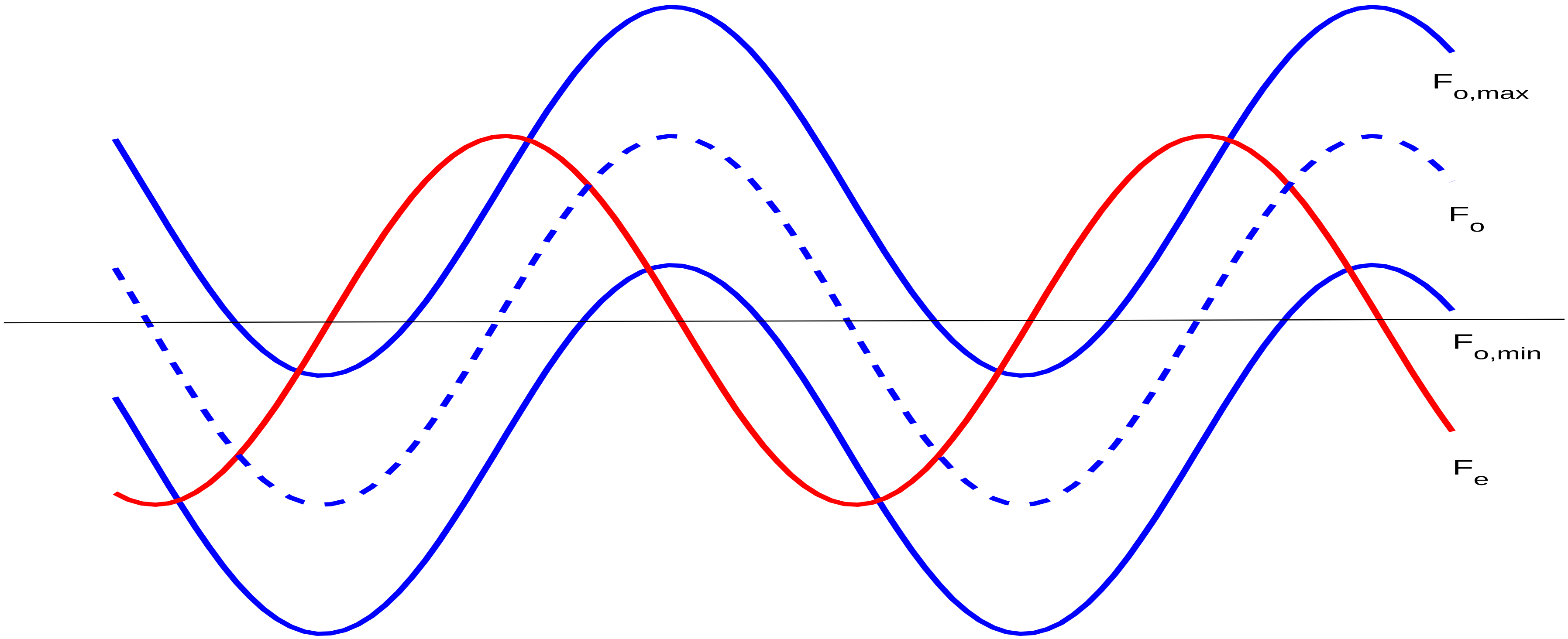


Illustration for the Proof of the classical Kharitonov theorem for polynomials via the Hermite-Biehler.

Two difficulties

- 1) The Hermite-Biehler theorem (interlacing of the roots) cannot be carried over to entire functions.
  - Remedy: The class HP.

- 2) New roots can occur.
  - Remedy: We need the fixed-degree property.

Remedy for the first difficulty

- Krein(????)/Levin (1950) considered class P. We consider its slight modification: the class HP:
  - $F(z)$ is
    1. **stable**;
    2. $\underbrace{d_F = h_F(0) - h_F(\pi) \geq 0}_{\text{HP-defect}}$, where $\underbrace{h_F(\theta) = \overline{\lim_{r\to\infty}} \frac{|F(re^{i\theta})|}{r}}_{\text{indicator function}}$, $\theta = \overline{\theta}$.

- Example: If $F(z)$ is a polynomial then $d_F^{(HP)} = 0$.

- Example:

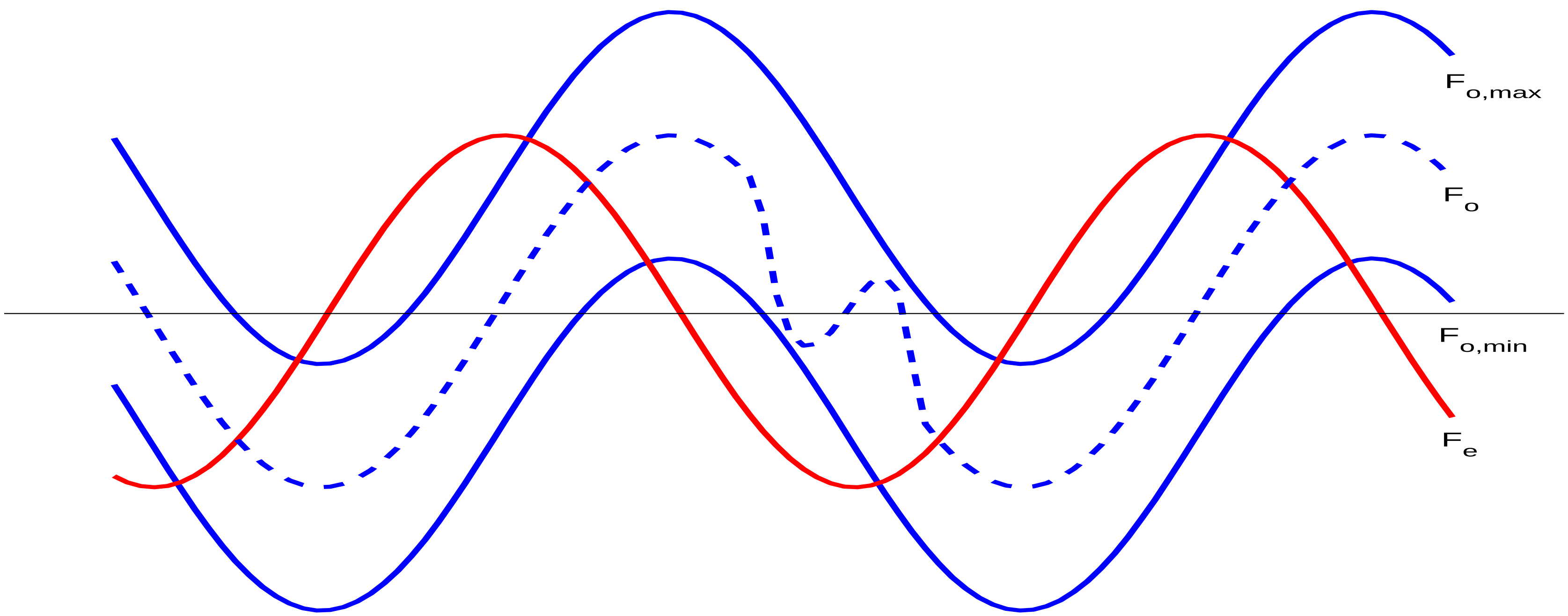
$$F(z) = \sum_{1}^{m} e^{\lambda_k z} f_k(z),$$

where $f_k(z)$ are real polynomials, and $\lambda_1 < \lambda_2 < \ldots < \lambda_n$.

If we assume $|\lambda_1| < \lambda_n$ then

$$d_F^{(HP)} = \lambda_n - \lambda_1 > 0.$$

Remedy for the second difficulty



The fixed-degree property $F_o(z)/F_{o,max}(z) = O(1)$, $F_e(z)/F_{e,max}(z) = O(1)$ can prevent this.

III. Generalized Bezoutains[OS2004b]

The definition.

- Bezoutians were used by

  L.Euler, 1748, É.Bezout, 1764, I.Sylvester, 1853.

- **1857** The definition we all know is due to
  - A.Cayley, *Note sur la méthode d'élimination de Bezout*, J. Reine Angew. Math., **53** (1857), 366-367.

- Let $\deg a(x) \leq n$, and $\deg b(x) \leq n$.

  The matrix $B = \left[\ r_{kl} \ \right]$ is called the Bezoutian of $a(x)$, and $b(x)$ if

$$\sum_{k,l=0}^{n-1} r_{kl} x^k y^l = \frac{a(x)b(y) - b(x)a(y)}{x - y}$$

Basic facts about Bezoutians?

Two basic theorems on Bezoutians.

- **1) The Jacobi(1836)-Darboux(1876) theorem** Let B be the Bezoutian matrix of two scalar polynomials $a(z)$ and $b(z)$. Then

$$\dim \mathrm{Ker} B = \text{the number of common zeros of } a(z) \text{ and } b(z) \text{ (with multiplicities).}$$

- **2) The Hermite(1856) theorem** All the roots of $P(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n$ are in the UHP if and only if the matrix $B = \left[\ r_{k,l}\ \right]$ is positive definite, where

$$-\frac{i}{2} \cdot \frac{P(\lambda)\breve{P}(\mu) - \breve{P}(\lambda)P(\mu)}{\lambda - \mu} = \sum_{k,l=0}^{n-1} r_{k,l} \lambda^k \mu^l$$

where $\breve{P}(x) = {p_0}^* + {p_1}^* x + {p_2}^* x^2 + \cdots + {p_n}^* x^n$.

1976 Bezoutians

- Early generalizations of Bezoutians to entire functions:
  - Grommer (1920)
  - Krein (1933) in "Some Questions in the Theory of Moments."

- 1976 Bezoutians
  - Sakhnovich (1976)
    - A generalization of JD and H theorems to entire functions of the form $F(z) = 1 + iz\int_0^w e^{izt}\overline{\Phi(t)}dt$.
  - Gohberg-Heinig (1976)
    - considered entire functions of the form $F(z) = 1 + \int_0^w e^{izt}\overline{\Phi(t)}dt$.
  - Anderson-Jury (1976)
    - introduced Bezoutians for matrix polynomials.
    - cojectured that the H theorem holds true.
    - The JD and H theorems for matrix polynomials were proven by Lerer-Tysmenetsky (1982).

Further generalizations

- – Haimovichi-Lerer (1995)
    - ∗ gave a general definition for Bezoutians of two entire functions of the form

$$F(z) = I_m + zC(I - zA)^{-1}B,$$

      that includes Sakhnovich, Gohberg-Heinig and Anderson-Jury as special cases. In the general case the JD and H theorems were not proven.
  – Lerer-Rodman (1994,1996,1999)
    - ∗ introduced Bezoutians for rational matrix functions. Obtained the JD and H theorems.

Bezoutains and operator identities

- Exploiting the method of operator identities we obtained a number of properties of the Bezoutians of two functions of the form

$$F(z) = I_m - zQ^*(I - Az)^{-1}\Phi,$$

Special cases:

  - If $Af = i\int_0^x f(t)dt$, where $f \in L_m^2(0,a)$ then it can be shown that $F(z)$ is an matrix entire functions of the exponential type.
  - If A is a single Jordan block with the zero eigenvalue then $F(z)$ is a matrix polynomial.
  - If A is a matrix then $F(z)$ is a rational matrix function.
  - In general the operator A needs not to be finite dimensional.

- We obtained several results including the JD and H theorems in the above rather general situation.

A generalization of the Hermite's theorem

- A Function $F(z)$:

$$F(z) = I_m - zQ^*(I - Az)^{-1}\Phi,$$

- The Corresponding Bezoutian T:

$$TB - B^*T = iN_1\alpha N_1, \qquad \alpha > 0, N_1 = T\Phi, B = A + \Phi Q^*.$$

- THM If $T \geq \delta I > 0$ then $\det F(z) \neq 0$ in $Imz > 0$.

IV. Generalized filters via Gohberg-Semencul [OS2004c]

Classical definitions

- Classical stationary processes. $x(t)$ is stationary in the wide sense if $E[x(t)] = const$ and $E[x(t)\overline{x(s)}] = K_x(t-s)$.

- Classical Optimal Filter:

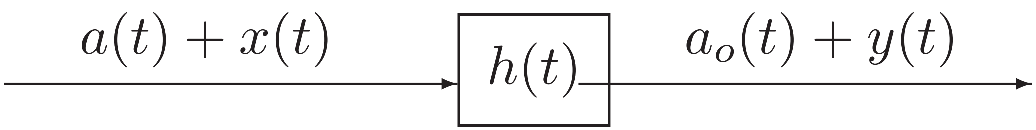


Figure 1. $a_o(t) + y(t) = \int_0^T h(\tau)[a(t-\tau) + x(t-\tau)]d\tau$

- Optimality:
  - Determenistic signals. Matched filter maximizes the SNR.
  - Random signal. Wiener filters minimizes the mean-square value of the difference between $a_o(t) + y(t)$ and $a(t)$.

Generalized processes

- Vilenkin and Gelfand (1961) noticed that any receiving device has a certain "inertia" and hence instead of actually measuring the classical stochastic process $\xi(t)$ it measures its averaged value

$$\Phi(\varphi) = \int \varphi(t)\xi(t)dt, \tag{6}$$

  where $\varphi(t)$ is a certain function characterizing the device.

- Small changes in φ yield small changes in $\Phi(\varphi)$, hence Φ is a continuous linear functional, i.e., a generalized stochastic process

Definitions (Vilenkin-Gelfand(1961))

- Let $\mathcal{K}$ be the set of all infinitely differentiable finite functions. A stochastic functional Φ assigns to any $\varphi(t) \in \mathcal{K}$ a stochastic value $\Phi(\varphi)$.

- Assume that all $\Phi(\varphi)$ have expectations $m(\varphi)$ given by

$$m(\varphi) = E[\Phi(\varphi)] = \int_{-\infty}^{\infty} x dF(x), \quad \text{where} \quad F(x) = P[\Phi(\varphi) \leq x].$$

- The bilinear functional

$$B(\varphi, \psi) = E[\Phi(\varphi)\overline{\Phi(\psi)}]$$

  is a correlation functional.

- Φ is called *generalized stationary in the wide sense* [VG61] if

$$m[\varphi(t)] = m[\varphi(t+h)], \tag{7}$$

$$B[\varphi(t), \psi(t)] = B[\varphi(t+h), \psi(t+h)] \tag{8}$$

S_J-generalized processes.

- S_J-generalized processes are those satisfying

$$B_J(\varphi, \psi) = (S_J\varphi, \psi)_{L^2}, \tag{9}$$

for such $\varphi(t), \psi(t)$ that $\varphi(t) = \psi(t) = 0$ when $t \notin J = [a, b]$. Here S_J is a bounded nonnegative operator acting in $L^2(a, b)$ and having the form

$$S_J\varphi = \frac{d}{dt}\int_a^b \varphi(u)s(t-u)du. \tag{10}$$

- Examples: white noise is not the classical but S_J-generalized process with S_J=I.

Solutions to the optimal filtering problems

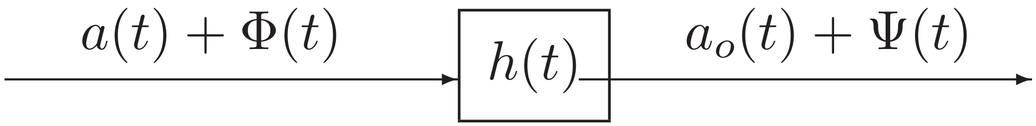


Figure 3. Generalized Optimal Filters.

- S_J-generalized Matched filters.

$$h_{opt} = \frac{S_J^{-1} a(t_0 - t)}{(a(t_0 - t), S_J^{-1} a(t_0 - t))_{L^2}},$$

Example. Matched filtering via Gohberg-Semencul

- Let

$$S_J f = f(x)\mu + \int_0^w f(t)K(x-t)dt.$$

with $K(x) \in L(-w, w)$. If there are two functions $\gamma_{\pm}(x) \in L(0, w)$ such that

$$S_J \gamma_+(x) = k(x), \qquad S_J \gamma_-(x) = k(x-w)$$

then

$$S_J^{-1} f = f(x) + \int_0^w f(t)\gamma(x,t)dt,$$

where $\gamma(x,t)$ is given by

$$\gamma(x,t) = \begin{cases} -\gamma_+(x-t) - \quad {}_{t}^{w+t-x} \quad [\gamma_-(w-s)\gamma_+(s+x-t) - \gamma_+(w-s)\gamma_-(s+x-t)]ds & x > t, \\ -\gamma_-(x-t) - \quad {}_{t}^{w} \quad [\gamma_-(w-s)\gamma_+(s+x-t) - \gamma_+(w-s)\gamma_-(s+x-t)]ds & x < t \end{cases}$$

Example. A specification: a colored noise

- As again, let

$$S_J f = f(x)\mu + \int_0^w f(t)K(x-t)dt.$$

where

$$K(x) = \sum_{m=1}^{N} \beta_m e^{-\alpha_m |x|}, \qquad \beta_j = \frac{\pi}{\alpha_m}\gamma_m$$

is the Fourier transform of

$$f(t) = \sum_{m=1}^{N} \gamma_m \frac{1}{t^2 + \alpha_m^2}, \quad \alpha_m > 0, \quad \gamma_m > 0.$$

Solution

- $$\gamma_+(x) = -\gamma(x,0), \quad \gamma_-(x) = -\gamma(w-x,0).$$

  Here
  $$\gamma(x,0) = G(x) \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}^{-1} B,$$
  where
  $$G(x) = \begin{bmatrix} e^{\nu_1 x} & e^{\nu_2 x} & \cdots & e^{\nu_{2N} x} \end{bmatrix}, \quad F_1 = \begin{bmatrix} \frac{1}{\alpha_i + \nu_k} \end{bmatrix}_{1 \le i \le N, 1 \le k \le 2N},$$
  $$F_2 = \begin{bmatrix} \frac{-e^{\nu_k w}}{\alpha_i - \nu_k} \end{bmatrix}_{1 \le i \le N, 1 \le k \le 2N}, \quad B = \underbrace{\begin{bmatrix} 1 & \cdots & 1}_{N} & \underbrace{0 & \cdots & 0 \end{bmatrix}}_{N}.$$

V. Hadamard-Sylvester vs Pseudo-Noise matrices [BOS2004]

Hadamard Matrices

Hadamard matrices of size n x n, are $(-1, 1)$ matrices such that

$$H_n^T H_n = n I_n$$

A special case: **Hadamard-Sylvester matrices**

$$H_1 = [1], \qquad H_{2n} = \begin{bmatrix} H_n & H_n \\ H_n & -H_n \end{bmatrix}$$

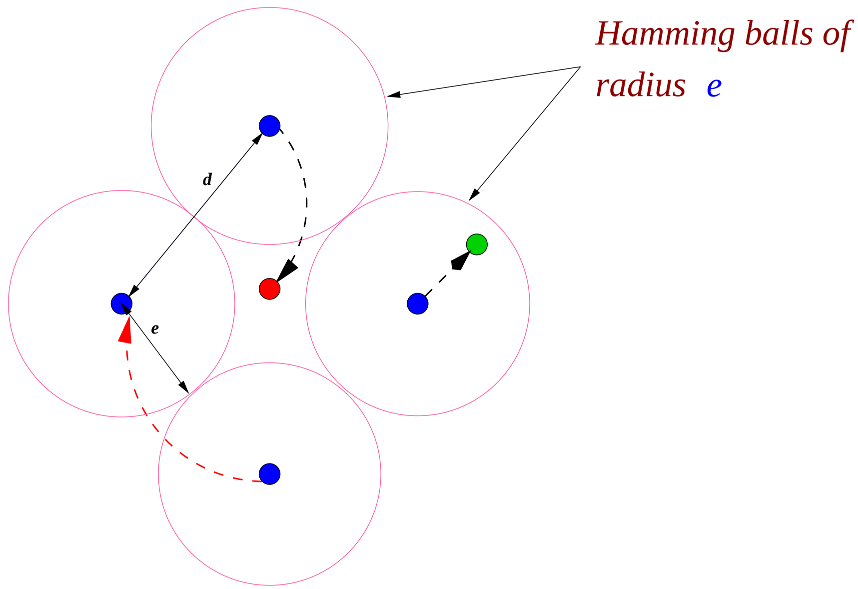
For example,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

What makes Hadamard-Sylvester Matrices to be Useful for Coding?

- **Rows & Columns Orthogonal** - Any two rows/columns of an $n \times n$ matrix agree in exactly $\frac{n}{2}$ places.

- The **minimum distance** between the columns is large: $\frac{n}{2}$



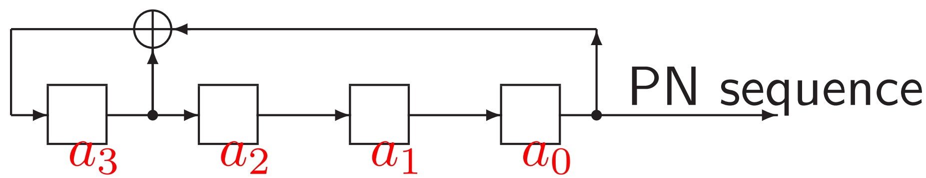
- This code is capable of **correcting** up to $\frac{n-2}{4}$ errors.

Another good code: the columns of Pseudo-Noise Matrices

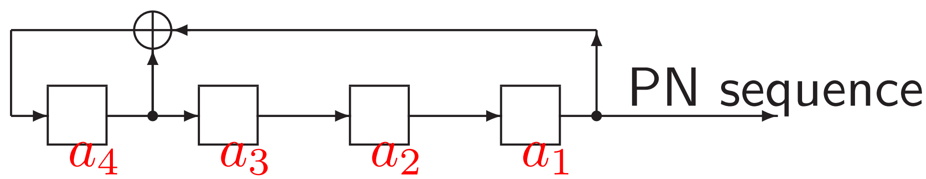
- # Primitive feedback registers. Example for $n = 4$

$$\mathbf{a_i} \quad = \mathbf{a_{i-1} h_3} \quad + \mathbf{a_{i-2} h_2} \quad + \mathbf{a_{i-3} h_1} \quad + \mathbf{a_{i-4} h_0}$$

- **Time moment zero**. The **initial** state $\{a_3, a_2, a_1, a_0\}$:



- **Time moment one**. The **next** state $\{a_4, a_3, a_2, a_1\}$:



- A register of length m can have **at most** $2^m - 1$ **different states** (**could be less**).

- A register (its characteristic polynomial) is called **primitive** if the corresponding register passes through **all possible** $2^m - 1$ **states**.

PN Sequences

- The output

$$\mathbf{a_0 a_1 a_2 \ldots}$$

  of a register corresponding to a primitive polynomial is called a PN sequence.

- **Fact**: $\forall\ m\ \exists$ primitive polynomials.

- **Fact**: A PN sequence generated by an **m**-degree primitive polynomial is **periodic** with period $2^m - 1$.

- For $h(x) = x^4 + x^3 + 1$ (i.e., $m = 4$), and the initial state $a_0 a_1 a_2 a_3 = 1000$, the resulting PN Sequence is given by

$$\underbrace{100011110101100}_{\text{period 15}}\underbrace{100011110101100}_{\text{period 15}}\underbrace{100011110101100}_{\text{period 15}}\ldots\ldots$$

PN Matrices

- A **Pseudo Noise Matrix** is one of the form

$$T = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & \widetilde{T} & \\ 0 & & & \end{bmatrix}$$

where $\widetilde{T}$ is a **circulant Hankel** matrix whose rows are **PN sequences**.

- **Theorem**

The $(0,1)$ Hadamard-Sylvester matrices and the $(0,1)$ PN matrices are equivalent, i.e., they can be obtained one from another via row and column permutations.

- **Sakhnovich(1998)** proved this result for $n = 16$ using combinatorial tricks.

VI. Order-one quasiseparable matrices

Order-one quasiseparable matrices

- R is called *quasiseparable* order (r_L, r_U) if

$$r_L = \max \operatorname{rank} R_{21}, \qquad r_U = \max \operatorname{rank} R_{12},$$

where the maximum is taken over all *symmetric* partitions of the form $R = \left[\begin{array}{c|c} * & R_{12} \\ \hline R_{21} & * \end{array}\right]$.

Example 1. Tridiagonal matrices and real orthogonal polynomials

- Let $\{\widetilde{\gamma}_k(x)\}$ be *real orthogonal polynomials* satisfying *three-term recurrence relations*:

$$\widetilde{\gamma}_k(x) = (\alpha_k \cdot x - \beta_k) \cdot \widetilde{\gamma}_{k-1}(x) - \gamma_k \cdot \widetilde{\gamma}_{k-2}(x), \tag{11}$$

- The relations (11) translate into the matrix form

$$\widetilde{\gamma}_k(x) = (\alpha_0 \cdot \ldots \cdot \alpha_k) \cdot \det(xI - R_{k\times k}) \quad (1 \le k \le N) \tag{12}$$

where

$$R = \begin{bmatrix} \frac{\beta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 & 0 \\ \frac{1}{\alpha_1} & \frac{\beta_2}{\alpha_2} & \frac{\gamma_3}{\alpha_3} & \ddots & \vdots & 0 \\ 0 & \frac{1}{\alpha_2} & \frac{\beta_3}{\alpha_3} & \ddots & 0 & \vdots \\ \vdots & 0 & \frac{1}{\alpha_3} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \ddots & \frac{\beta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\beta_n}{\alpha_n} \end{bmatrix} \tag{13}$$

Example 2. UH matrices and the Szego polynomials

- Let $\{\widetilde{\gamma}_k(x)\}$ be *the Szego polynomials* satisfying *two-term recurrence relations*

$$\left[\begin{array}{c} G_{k+1}(x) \\ \widetilde{\gamma}_{k+1}(x) \end{array}\right] = \frac{1}{\mu_{k+1}} \left[\begin{array}{cc} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{array}\right] \left[\begin{array}{cc} 1 & 0 \\ 0 & x \end{array}\right] \left[\begin{array}{c} G_k(x) \\ \widetilde{\gamma}_k(x) \end{array}\right]. \qquad (14)$$

- The relations (14) translate into the matrix form

$$\widetilde{\gamma}_k(x) = \frac{\det(xI - R_{k\times k})}{\mu_0 \cdot \ldots \cdot \mu_k} \quad (1 \le k \le N)$$

where

$$R = \left[\begin{array}{cccccc} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{array}\right] \qquad (15)$$

Observation. These two matrices are order-one

- $$\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{\gamma_4}{\alpha_4} & 0 & \cdots & 0 & 0 \end{bmatrix}}_{R_{12} \text{ for (13)}}, \underbrace{\begin{bmatrix} -\rho_4\mu_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ -\rho_4\mu_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ -\rho_4\mu_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \end{bmatrix}}_{R_{12} \text{ for (15)}}$$

Main results

- Three-term and two-term rr for the characteristic polynomials of submatrices of general order-one quasi-separable.
- These new set of polynomials includes real orthogonal and the Szego polynomials as special cases.
- Eigenstructure analysis, formulas for the eigenvectors. Simple and multiple eigenvalue cases are considered.