Arithmetics of the real exponential field

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Let A denote a discrete ordered unitary ring; Shepherdson proved that A^+ satisfies **Open Induction** OI iff A is **integral part** of some real closed field. Where an integral part is a discrete subring that has an element at distance < 1 from each point of the field. We prove six versions of this result where the exponential function x^y is added to the language. They axiomatize six kinds of rings with x^y, that can be characterized as integral parts of six kinds of real fields with exponentiation : 1. exponential fields; 2. exponential fields satisfying the intermediate value scheme for 2^x-polynomials; 3. exponential fields with a uniform bound on the number of roots of an x^y-polynomial; 4. exponential fields first order equivalent to $(\mathbb{R}, 2^x)$; 5. fields of (4) subject to be in addition the field of fraction of some integral part; 6. elementary pairs of a field of (4) and a field of (5). In case (6) the notion of integral part has to be modified to the benefit of suitable non unitary rings. This strange case is perhaps the most interesting one. **Notations :**

1. an **exponential field** is a real field R with a function 2^x_R that satisfies the axioms below (denoted EXP); in any such field, x^y denotes 2^{ylogx}

   - $2^0 = 1$; 2^x is a homomorphism of $+$ on $\times$ and an ordermorphism faster than the power functions
   - $x > 0$ implies $logx$ exists (that is $\exists y \; 2^y = x$)

2. let exp denote 2^x or x^y; $\mathcal{L}(exp)$ denotes the language of ordered rings plus the function symbol exp and we call exp-polynomials the terms of this language. We say that (A, exp_A) is an integral part of the exponential field $(R, 2^x)$, or that A is an exp-integral part of $(R, 2^x)$ iff A is integral part of R, A^+ is closed under exp_R and $exp_A = exp_R | A^+$.