Non-Standard Analysis and J.F. Colombeau's New Generalized Functions

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Abstract

We present a particular c^+-saturated non-standard model of the field of complex numbers $\mathbb{C}$ (where $c = \text{card}(\mathbb{R})$) based on the ultrapower $\mathbb{C}^{\mathcal{D}}$ and a particularly chosen ultrafilter $\mathcal{U}$ on the space of test-functions $\mathcal{D} \stackrel{\text{def}}{=} \mathcal{C}_0^\infty(\mathbb{R}^d)$. The ultrafilter $\mathcal{U}$ is closely related to J.F. Colombeau's theory of new generalized functions. Next, we show that there exists a canonical (explicit) embedding of the space of Schwartz distributions $\mathcal{D}'(\mathbb{R}^d)$ in ${}^*\mathcal{C}^\infty(\mathbb{R}^d)$ which preserves the usual multiplication in $\mathcal{C}^\infty(\mathbb{R}^d)$ up to ρ-null numbers for a given positive infinitesimal ρ in ${}^*\mathbb{R}$ ($h \in {}^*\mathbb{R}$ is a ρ-null if $|h| < \rho^n$ for all $n \in \mathbb{N}$). This embedding presents a solution of the problem of multiplication of Schwartz distributions similar to (but different from) J.F. Colombeau's solution of the same problem. We also present an improvement of the original J.F. Colombeau's theory where the counterpart of the field ${}^*\mathbb{C}$ is a ring $\overline{\mathbb{C}}$ with zero divisors.