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Partielle Differentialgleichungen

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Abstract. Invariants of topological spaces of dimension three play a major role in many areas, in particular . . .

Mathematics Subject Classification (2000): AMS-CLASSIFICATION.

Introduction by the Organisers

The workshop *Invariants of topological spaces of dimension three*, organised by Max Muster (München) and Bill E. Xample (New York) was held March 1st–March 6th, 2005. This meeting was well attended with over 30 participants with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds . . .

Workshop: Partielle Differentialgleichungen

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Abstracts

Convexity in Carnot groups

Valentino Magnani

We give an account of recent results and open questions related to the notion of convexity in Carnot groups. A Carnot group $\mathbb{G}$ is a connected, simply connected graded nilpotent Lie group equipped with a system of left invariant horizontal vector fields $X_1, X_2, \ldots, X_m$, spanning the first layer V_1 of the Lie algebra and satisfying the Lie bracket generating condition, [8]. These vector fields give the horizontal directions at each point of the space and define the so-called Carnot-Carathéodory distance, [9].

Let Ω be an open subset of $\mathbb{G}$. A function $u : \Omega \longrightarrow \mathbb{R}$ is H-convex if its restriction $t \longrightarrow u(x \exp(tX))$ is convex with respect to t, where $X \in V_1$, $x \in \Omega$ and $\gamma(t) = x \exp(tX)$ is the unique integral curve of X passing through x, namely, a horizontal line. This notion has been proposed by Caffarelli and Cabré and studied by Danielli, Garofalo and Nhieu, [5], see also [12]. Recall that horizontal lines constitute a special subset of (sub-Riemannian) geodesics. However, extending convexity of u to all geodesics of the group would yield a trivial notion, [13].

A first regularity property shows that H-convex functions which are locally bounded above are locally Lipschitz with respect to the Carnot-Carathéodory distance, [14]. Balogh and Rickly have shown that H-convex functions in the Heisenberg group are automatically locally bounded above, [2]. Recently, Rickly has shown that measurable H-convex functions are locally bounded above in any Carnot group and that the measurability assumption can be removed if the step is not greater than two, [17]. A detailed study of H-convex functions and H-convex sets in Carnot groups can be found in [16]. Here we mention that it is still not clear whether H-convex functions are locally bounded above in arbitrary Carnot groups.

The following estimates for continuous H-convex functions have been achieved in [5],

$$\sup_{y \in B_{\xi,r}} |u(y)| \leq C \fint_{B_{\xi,\lambda r}} |u(y)| \, dy \tag{1}$$

$$\|\nabla_H u\|_{L^\infty(B_{\xi,r})} \leq \frac{C}{r} \fint_{B_{\xi,\lambda r}} |u(y)| \, dy. \tag{2}$$

where $\lambda > 1$ is a fixed number, $C > 0$ depends on the group and $\nabla_H u = (X_1 u, \ldots, X_m u)$.

Convexity in Carnot groups can be also introduced in the viscosity sense, according to the following definition by Lu, Manfredi and Stroffolini, [12]. An upper semicontinuous function $u : \Omega \longrightarrow \mathbb{R}$ is said to be v-convex if for every $x \in \Omega$ and every $\varphi \in C^2(\Omega)$ being greater than or equal to u in a neighbourhood of x and such that $u(x) = \varphi(x)$, we have $\nabla_H^2 \varphi(x) \geq 0$. The horizontal Hessian $\nabla_H^2 \varphi(x)$ has elements $\frac{1}{2}\big(X_iX_j + X_jX_i\big)\big(\varphi(x)\big)$, for every $i, j = 1, \ldots, m$. Through comparison with subelliptic cones, whose existence and uniqueness in the Heisenberg group is provided by results of Bieske, [3], estimates (1) and (2) for v-convex functions have been obtained in [12]. By recent results of Wang, [22], these estimates have been further extended to Carnot groups, [11].

A natural question concerns the equality between v-convexity and H-convexity. A first positive answer has been achieved in the Heisenberg group, [2], then different proofs have been given in arbitrary Carnot groups, [11], [14], [16], [21]. Precisely, an upper semicontinuous function is v-convex if and only if it is H-convex.

Concerning second order regularity results, a natural question is extending the classical Aleksandrov-Busemann-Feller differentiability theorem to H-convex functions. A way to reach this result is showing that the second order distributional derivatives X_iX_ju, $i, j = 1, \ldots m$, of an H-convex function u are measures, namely $u \in BV_H^2(\Omega)$. In fact, as it is shown by Ambrosio and the author [1], if $u \in BV_H^2(\Omega)$, then for a.e. $x \in \Omega$ there exists a unique polynomial $P_{[x]}$ of homogeneous degree less than or equal to two satisfying

$$\frac{1}{r^2} \fint_{B_{x,r}} |u - P_{[x]}| \longrightarrow 0.$$

Here $P_{[x]}$ is the second order approximate differential of u at x. By a standard method, [7], it can be shown that functions in $BV_H^2(\Omega)$, satisfying (1) and (2) have a.e. pointwise second order differential. Then an important issue is studying whether H-convex functions belong to $BV_H^2(\Omega)$. The H-convexity easily implies that the symmetrized second order derivatives $(X_iX_ju + X_jX_iu)/2$ are measures, then proving that X_iX_ju are measures is equivalent to showing that so are $[X_i, X_j]u$ and we arrive at the following problem:

(3) *Is it true that $[X_i, X_j]u$ are measures when u is an H-convex function?*

This is an open question in arbitrary Carnot groups. A positive answer in Heisenberg groups has been given by Gutiérrez and Montanari, [10], and its extension to step two Carnot groups has been established by Danielli, Garofalo, Nhieu and Tournier, [6]. Trudinger has achieved a further extension to free divergence Hörmander vector fields of step two, [20]. The interesting feature of this approach is in finding a suitable subelliptic nonlinear operator satisfying a monotonicity property. In the Euclidean case, Trudinger and Wang obtained this property for k-Hessian operators applied to k-convex functions, [18], [19]. For a real symmetric matrix A we define

$$F_k(A) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} M_{i_1 i_2 i_3 \ldots i_k}(A)$$

where $M_{i_1 i_2 i_3 \ldots i_k}(A)$ are the k-minors on the diagonal of the matrix. A function $u \in C^2(\Omega)$ is k-convex if $F_j[u] := F_j(\nabla^2 u) \ge 0$ for every $j = 1, \ldots, k$. The monotonicity theorem, as proved in [18], shows that functions $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$

satisfying $u \le v$ on Ω, $u = v$ on $\partial\Omega$ and such that $u + v$ is 2-convex yield the inequality

$$\int_\Omega F_k[v] \le \int_\Omega F_k[u].$$

The proof follows immediately from both integration by parts and the free divergence formula $\sum_{j=1}^n \big(\partial_{x_j}\ \partial_{r_{ij}} F_k\big)(\nabla^2 u) = 0$. In Heisenberg groups the corresponding operator satisfying a monotonicity theorem has been found in [10] and it has the form

$$\mathcal{F}_2[u] = F_2(\nabla^2_H u) + 12n(\partial_t u)^2.$$

Then its form has been extended to two step Carnot groups in [6], obtaining

$$\mathcal{F}_2[u] = F_2(\nabla^2_H u) + \frac{3}{4}\sum_{i<j}\Big([X_i, X_j]u\Big)^2.$$

As observed in [20], these operators possess a free divergence formula. In fact, defining $G_2(A) = F_2(A) + \frac{1}{2}\sum_{i<j}(a_{ij} - a_{ji})^2$ and noting that $\mathcal{F}_2[u] = G_2(X^2u)$, where $X^2u = (X_iX_ju)_{ij}$ is the nonsymmetrized horizontal Hessian, one finds

$$\sum_{j=1}^m X_j\ \big(\big(\partial_{r_{ij}} G_2\big)(X^2 u)\big) = 0,$$

then the monotonicity theorem easily follows for more general free divergence, two step Hörmander vector fields, [20]. As a consequence of this theorem, the estimate

$$\int_{\Omega'} F_2(\nabla^2_H u) + \frac{3}{4}\sum_{i<j}\Big([X_i, X_j]u\Big)^2 \le C\ \left(\int_\Omega |u|\right)^2 \tag{4}$$

can be achieved for a function $u \in C^2(\Omega)$, satisfying $F_j(\nabla^2_H u) \ge 0$ for $j = 1, 2$, where Ω' is compactly contained in Ω and C depends on $\mathrm{dist}(\Omega', \partial\Omega)$. Now, to establish that $[X_i, X_j]u \in L^2_{loc}(\Omega)$ when u is H-convex, we introduce the following definition. According to [20], we say that a function $u \in C^2(\Omega)$ is k-convex with respect to the vector fields X_j (or simply k-convex) if $F_j(\nabla^2_H u) \ge 0$ for any $j = 1, \ldots, k$. The larger class of locally summable k-convex functions is obtained by closure of C^2 smooth k-convex functions with respect to L^1_{loc}-convergence. Hence, estimate (4) shows that locally summable 2-convex functions satisfy $[X_i, X_j]u \in L^2_{loc}(\Omega)$. Now we notice that in Carnot groups a function $u \in C^2(\Omega)$ is H-convex if and only if $\nabla^2_H u \ge 0$, therefore by a suitable smooth convolution it can be seen that the class of locally summable m-convex functions coincides with that of locally Lipschitz H-convex functions. As a result, in step two Carnot groups any H-convex function u has the property $[X_i, X_j]u \in L^2_{loc}(\Omega)$.

Now, it would be desirable having a characterization of the L^1_{loc}-limits of k-convex functions analogous to the case $k = m$. Here it is helpful the following distributional characterization of H-convex functions in step two Carnot groups. A Radon measure μ such that $\nabla^2_H \mu \ge 0$ is defined by an L^1_{loc}-limit of smooth H-convex functions, [14]. The problem of extending this characterization to higher step Carnot groups relies on the validity of the key identity

$$X_iX_j\theta(x) = X_iX_j\theta(x^{-1}) \tag{5}$$

for mollifiers θ such that $\theta(x) = \theta(x^{-1})$. Presently, this identity holds in step two Carnot groups, whereas it is not known in groups of higher step. The same approach of [14] and equality (5) easily imply that locally summable k-convex functions can be characterized in distributional sense, as in Lemma 2.2 of [19].

Second order differentiability can be extended to k-convex functions. In fact, among the gradient estimates obtained in [20] for k-convex functions, it is shown that

$$\sup_{\Omega'} \frac{|u(x)-u(y)|}{d(x,y)^\alpha} \le C\,\|u\|_{L^1(\Omega)}\,,$$

under the condition $k > (Q-1)m/(Q+m-2)$. As a consequence, arguing as in [4], from the fact that $[X_i, X_j]u \in L^2_{loc}(\Omega)$ and the approximate second order differentiability of functions in $BV^2_H(\Omega)$, the classical Aleksandrov-Busemann-Feller's theorem extends to k-convex functions in step two Carnot groups, when $k > (Q-1)m/(Q+m-2)$, [20].

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