NON-HORIZONTAL SUBMANIFOLDS AND COAREA FORMULA

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Abstract. Let k be a positive integer and let m be the dimension of the horizontal subspace of a stratified group. Under the condition $k \leq m$, we show that all submanifolds of codimension k are generically non-horizontal. For these submanifolds we prove an area-type formula that allows us to compute their $Q-k$ dimensional spherical Hausdorff measure. Finally, we observe that a.e. level set of a sufficiently regular vector-valued mapping on a stratified group is a non-horizontal submanifold. This allows us to establish a sub-Riemannian coarea formula for vector-valued Riemannian Lipschitz mappings on stratified groups.

Contents

1. Introduction

In the last decade, Geometric Measure Theory with respect to sub-Riemannian metrics has been the object of many contributions spread into different research streams. We limit ourselves to mention just a few recent works for a sketchy overview of this subject and further references, [1], [3], [4], [7], [9], [16], [17], [27], [31], [32], [33].

Scope of our investigations is constituted by stratified groups, that are simply connected graded nilpotent Lie groups equipped with a homogeneous norm, [13], that is equivalent to the so-called Carnot-Carathéodory distance, [18]. In fact, these groups are best known as "Carnot groups". All notions of this introduction will be precisely introduced in Section 2. Aim of this paper is to show an area-type formula for a class of submanifolds in stratified groups, emphasizing its connection with the corresponding sub-Riemannian coarea formula for vector valued mappings.

We consider a C^1 smooth submanifold Σ of the stratified group $\mathbb{G}$ and denote by $H_x\mathbb{G}$ the horizontal subspace at x, that has dimension m at every point $x \in \mathbb{G}$. We say that $x \in \Sigma$ is a *non-horizontal* point when the subspaces $T_x\Sigma$ and $H_x\mathbb{G}$ are transversal, namely, they span all of $T_x\mathbb{G}$. If these subspaces are not transversal, then we say that x is *horizontal*. Notice that this definition is equivalent to Definition 2.10 of [27], where horizontal points are required to satisfy the condition

$$\dim H_x\mathbb{G} - \dim(T_x\Sigma \cap H_x\mathbb{G}) < k \tag{1}$$

and k is the codimension of Σ. In fact, due to Grassmann formula, the left hand side of (1) is equal to $\dim(T_x\Sigma + H_x\mathbb{G}) - \dim T_x\Sigma$ and this number is less than k if and only if $\dim(T_x\Sigma + H_x\mathbb{G})$ is less than $\dim T_x\mathbb{G}$.

A submanifold is called non-horizontal if it has at least one non-horizontal point and it is called horizontal otherwise. Notice that a non-horizontal submanifold may have horizontal points. These two classes of submanifolds have been recently introduced in the work [31], whose results can be applied especially to horizontal submanifolds.

Our main motivation for the study of k-codimensional non-horizontal submanifolds is that they naturally appear as "intrinsically regular" level sets of vector-valued mapping $f : \Omega \longrightarrow \mathbb{R}^k$, where Ω is an open set of a stratified group. There are in turn two primary reasons to investigate this connection. The first one is related to the validity of nontrivial coarea formulae for vector-valued mapping on stratified groups that involve the horizontal Jacobian, as we explain below and in Section 7. The second one is that non-horizontal submanifolds fit into the larger class of $(\mathbb{G}, \mathbb{R}^k)$-regular sets, that are locally defined as level sets of a P-differentiable vector-valued mapping with surjective P-differential, [27]. These class of sets naturally appeared in the study of intrinsic rectifiability of the reduced boundary of finite perimeter sets in the Heisenberg group, [12], and have been independently studied in codimension one for general stratified groups, [15]. Notice that one can find nontrivial examples of these sets that are highly irregular from the Eulidean viewpoint, [21].

In the present paper, we focus our attention on arbitrary $(\mathbb{G}, \mathbb{R}^k)$-regular sets that are in addition C^1 smooth as submanifolds of $\mathbb{G}$ and that precisely coincide with non-horizontal submanifolds. This follows from Lemma 2.11 of [27], taking into account that the horizontal differential used in the lemma coincides with the P-differential.

By Lie bracket generating condition, that is satisfied in stratified groups, all smooth hypersurfaces are always non-horizontal. On the other hand, in higher codimension it is not true that all k-codimensional submanifolds are non-horizontal. First of all, non-horizontal submanifolds exist if and only if $m \geq k$, where $\dim(V_1) = m$, see Section 4. Even if this condition is satisfied, as for curves in the Heisenberg group $\mathbb{H}^1$, where $m = 2$, then horizontal curves provides examples of horizontal submanifolds. On the other hand, in general we have that all k-codimensional submanifolds are "generically" non-horizontal. Precisely, under the condition $m \geq k$, Theorem 4.3 shows that k-codimensional non-horizontal C^1 smooth submanifolds are dense with respect to the topology introduced in Definition 4.2.

Our main result is an intrinsic blow-up at non-horizontal points of C^1 submanifolds, proved in Theorem 5.4. Let Q denote the Hausdorff dimension of a stratified group $\mathbb{G}$ with respect to a homogeneous distance ρ and let q be the topological dimension of $\mathbb{G}$. Due to Theorem 2.10.17(2) and Theorem 2.10.18(1) of [11], our blow-up (29) joined with $(Q-k)$-negligibility of horizontal points, [27], yields the following

Theorem 1.1. *Let $\Sigma \subset \mathbb{G}$ be a p-dimensional non-horizontal submanifold of class C^1 and define the codimension $k = q - p$. Then the following formula holds*

$$\int_\Sigma \theta^g_\rho\big(\mathbf{n}_H(x)\big)\, d\mathcal{S}^{Q-k}(x) \,= c(g,\tilde g)\, \int_\Sigma |\tilde{\mathbf{n}}_{g,H}(x)|\, d\tilde\mu_p(x), \tag{2}$$

where $\tilde g$ is an arbitrary Riemannian metric generating the Haar measure $\mathrm{vol}_{\tilde g}$ of the group, g is the fixed left invariant metric on $\mathbb{G}$, $c(g,\tilde g)$ satisfies the formula $c(g,\tilde g)\ \mathrm{vol}_{\tilde g} = \mathrm{vol}_g$, $\tilde\mu_p$ is the p-dimensional Riemannian measure on Σ with respect to $\tilde g$ and $|\cdot|$ denotes the Riemannian norm with respect to g.

The metric factor $\theta^g_\rho(\cdot)$ takes into account the fixed left invariant metric g and the homogeneous distance ρ, see Definition 5.3. The spherical Hausdorff measure $\mathcal{S}^{Q-k}$ defined in (7) only depends on ρ and the *horizontal k-normal* $\mathbf{n}_H(x)$ to Σ at x depends on g, according to Definition 3.2. This shows that the left hand side of (2) is surprisingly independent from the metric $\tilde g$ and the homogeneous distance ρ. In fact, the metric $\tilde g$ need not be left invariant and essentially plays an auxiliary role. The independence of ρ is discussed in Section 6.

Let us consider the first consequences of the area-type formula (2). As a first remark, usually it is convenient to work with homogeneous distances with constant metric factor. Section 6 will discuss existence of these distances in arbitrary graded groups. Under this condition, we immediately achieve the following

Corollary 1.1. *Under hypotheses of Theorem 1.1, if in addition the homogeneous distance ρ has constant metric factor $\theta^g_\rho(\cdot) = \alpha$, then*

$$\mathcal{S}^{Q-k}_{\mathbb{G}}(\Sigma) = c(g,\tilde g)\, \int_\Sigma |\tilde{\mathbf{n}}_{g,H}(x)|\, d\tilde\mu_p(x)\,, \tag{3}$$

where we have set $\mathcal{S}^{Q-k}_{\mathbb{G}} = \alpha\, \mathcal{S}^{Q-k}$.

This formula provides the "natural" intrinsic measure of non-horizontal submanifolds, according to [31] and then provides an integral formula for the $(Q-k)$-dimensional spherical Hausdorff measure of these C^1 smooth k-codimensional submanifolds. As a byproduct, formula (3) also shows in particular that the Hausdorff dimension of smooth non-horizontal submanifolds is equal to $Q-k$. This fits into the general formula for the Hausdorff dimension of smooth submanifolds stated in Section 0.6B of [18]. Finding all possible Hausdorff dimensions of submanifolds in a stratified group corresponds to solve the so-called "Gromov's dimension comparison problem", that has been recently raised in Problem 1.1 of [2].

If we restrict our attention to one codimensional submanifolds, then we first point out that a non-horizontal point x of a hypersurface Σ exactly coincides with the so-called non-characteristic point, since $H_x\mathbb{G} \nsubseteq T_x\Sigma$, see [27] for more information on characteristic points and related references. The intrinsic blow-up at non-characteristic points of hypersurfaces along with formula (3) have been obtained in [15]. Notice that this area-type formula coincides with ours assuming that the auxiliary metric $\tilde g$ equals the Euclidean metric. The corresponding formulae in [20], [24] and [27] refer to the case where $\tilde g$ equals the fixed left invariant metric g. In all previous cases, (3) can be interpreted as the natural surface measure with respect to the sub-Riemannian geometry of $\mathbb{G}$, since it naturally appears in isoperimetric inequalities in stratified groups and more general Carnot-Carathéodory spaces. In fact, by our formula (9), the surface measure in (3.1.b) of [14] and the perimeter measure in (3.2) of [8] fit into (3) where $\tilde g$ equals the Euclidean metric, under the condition that the corresponding vector fields define a stratified group.

This suggests that our area-type formula could be also studied in connection with possible extensions of the isoperimetric inequality to higher codimensional submanifolds. If it happens that either the boundary conditions or the algebraic structure of the group in an area minimization problem force the submanifold to be non-horizontal, then one could find the corresponding Euler-Lagrange equations of the area functional (3). In this perspective, our area-type formula could be also studied to investigate an "intrinsic" notion of mass for the associated non-horizontal integral current of codimension k. Incidentally, this was one of our initial motivations for this type of studies, that started in codimension one in [24].

Previous results for higher codimensional submanifolds have been recently obtained in [17], where an area-type formula for intrinsically regular submanifolds in Heisenberg groups have been achieved. In addition, Theorem 4.6 of [17] provides this formula for C^1 smooth submanifolds of low codimension. This area-type formula in Heisenberg groups fits into formula (3) with $\rho = d_\infty$. In fact, formula (9) with $\tilde g$ equal to the Euclidean metric proves that the density $|\tilde{\mathbf{n}}_{g,H}(x)|$ coincides with the density of formula (61) in [17].

By virtue of Proposition 3.2, according to which the length of the vertical tangent vector $\tau_{\Sigma,\mathcal{V}}$ is proportional to the length of the horizontal normal, also Theorem 1.2 of [28] follows from (3) with $\tilde g = g$, see Remark 3.3. As we show in Section 3, we also have another

Corollary 1.2. *Let U be an open subset of $\mathbb{R}^d$ and let $\Phi : U \longrightarrow \mathbb{G}$ be a C^1 embedding. Then*

$$\mathcal{S}^{Q-k}_{\mathbb{G}}\big(\Phi(U)\big) = \int_U \Big|\big(\Phi_{\xi_1}(\xi) \wedge \cdots \wedge \Phi_{\xi_p}(\xi)\big)_{Q-k}\Big| \, d\xi \,. \tag{4}$$

If v is a p-vector, then we denote by $(v)_{Q-k}$ its component of degree equal to $Q-k$. In Definition 3.3, we recall this notion, that have been introduced in [31] to study the intrinsic measure of higher codimensional submanifolds. In rough terms, the projection $(v)_{Q-k}$ represents the component of v having weight equal to $Q-k$ with respect to intrinsic dilations of the group. Taking into account (10) and Proposition 3.3, one easily notices that horizontal points coincide with those points of degree less than $Q-k$.

It is rather clear how (4) resembles an area-type formula, where the "weighted" Jacobian of the parametrization Φ takes into account only terms of degree $Q-k$. In addition, by Proposition 3.3, the previous corollary allows us to characterize all k-codimensional non-horizontal submanifolds Σ of class C^1 as those submanifolds where the restriction $\mathcal{S}^{Q-k}_{\mathbb{G}} \llcorner \Sigma$ does not vanish. In Section 6 we add an example of non-horizontal C^1 submanifold for which we explicitly compute its $\mathcal{S}_{\mathbb{G}}^{Q-k}$-measure. Other computations for more regular submanifolds can be found in [28] and [31].

In fact, if we assume $C^{1,1}$ regularity and $\mathcal{S}^d$-negligibility for points of degree less than d, then (4) can be extended to arbitrary submanifolds of degree d, [31]. We also point out that results of [29] show that $\mathcal{S}^d$-negligibility holds in two step groups for $C^{1,1}$ submanifolds of arbitrary degree. There is a sort of compensation between the regularity of Σ and the degree of the point where we perform the blow-up. In fact, in broad terms, C^1 regularity suffices for (4), since the Taylor expansion of the submanifold at a non-horizontal point has a non-vanishing first order term. This fact introduces us to the sub-Riemannian coarea formula, that is the main application of (3) in this paper.

We first notice that a.e. level set of a C^1 smooth vector-valued mapping on a stratified group has an $\mathcal{S}^{Q-k}$-negligible set of horizontal points. This follows by a weak Sard-type theorem, [23]. To achieve the sub-Riemannian coarea formula (5), we apply Corollary 1.1 to these level sets. Through the Whitney extension theorem, Riemannian Lipschitz mappings differ from C^1 mappings outside a set of small measure. This extends (5) to Lipschitz continuous mappings. Unfortunately, this C^1 approximation prevents us to use the general area-type formula of [31] that would require slightly more smoothness.

Theorem 1.2. *Let $f : A \longrightarrow \mathbb{R}^k$ be a Riemannian Lipschitz map, where $A \subset \mathbb{G}$ is a measurable subset and let $u : A \longrightarrow [0, +\infty]$ be a measurable function. We assume that the homogeneous distance ρ has constant metric factor $\theta^g_\rho(\cdot) = \alpha$. Then we have*

$$\int_A u(x)\, J_{g,H} f(x)\, d\mathrm{vol}_g(x) = \int_{\mathbb{R}^k} \left(\int_{f^{-1}(t)} u(y)\, d\mathcal{S}_{\mathbb{G}}^{Q-k}(y) \right) dt. \tag{5}$$

There are several results concerning sub-Riemannian coarea formulae for real-valued functions, see for instance [25] and [26] for relevant discussions and references. Here we just point out that (5) extends previous coarea formulae obtained in [20], [24], [26] and [28]. Notice that in this formula the auxiliary metric $\tilde{g}$ does not appear. The horizontal Jacobian $J_{g,H} f(x)$ corresponds to the restriction of the Riemannian Jacobian of f to the horizontal subspace, see Section 7. Let us point out that this restriction exactly corresponds to the Pansu differential of the mapping. As we have already pointed out in previous works focused on some special cases, the proof of (5) relies on the the key formula

$$J_{\tilde{g}} f(x)\ \left|\tilde{\mathbf{n}}_{g,H}(x)\right|_g = J_{g,H} f(x) \tag{6}$$

that relates the horizontal k-normal with the horizontal Jacobian. Notice that both sides of (6) vanish in the case $\dim(V_1) < k$, since there do not exists k-codimensional horizontal submanifolds and the horizontal Jacobian vanishes. For these special cases, it might be interesting to investigate novel notions of Jacobian that would go beyond the P-differential of the mapping, since the condition $\dim(V_1) < k$ implies that this differential of f is never surjective. We also wish to stress that in all cases where $k > 1$, the validity of (5) for a Lipschitz mapping with respect to the Carnot-Carathéodory distance is presently an open question.

Let us give a short overview of the paper. Section 2 recalls all notions about stratified groups that will be used throughout the paper. In Section 3 we introduce the horizontal k-normal and the vertical tangent vector with respect to the metrics $\tilde{g}$ and g. We recall the notion of degree of a simple p-vector and present a formula to compute the horizontal normal and its relationship with the vertical tangent vector. In Proposition 3.3 we show that non-horizontal points are characterized as those points where the vertical tangent vector $\tau_{\Sigma,\mathcal{V}}$ does not vanish. These results lead to a short proof of Corollary 1.2. In Section 4 we show that non-horizontal submanifolds exist if and only if the geometric condition $k \geq m$ is satisfied and in this case they form a dense family of k-codimensional submanifolds with respect to a suitable topology. Section 5 is devoted to the proof of the blow-up theorem at non-horizontal points. We introduce the notion of vertical subalgebra and of metric factor associated with a k-vector. In the blow-up theorem we also show that the intrinsically rescaled submanifold converges with respect to the Hausdorff convergence of sets to a normal subgroup. In Section 6 we introduce a class of homogeneous distances with constant metric factor and we discuss its role that strikingly makes the measure $\mathcal{S}_{\mathbb{G}}^{Q-k} \llcorner \Sigma$ independent of the homogeneous distance used to construct it. We also present an example where we compute $\mathcal{S}_{\mathbb{G}}^{Q-k} \llcorner \Sigma$ for a C^1 submanifold. In Section 7 we recall the definition of horizontal Jacobian, we show the validity of formula (6) and prove the coarea formula stated in Theorem 1.2.

2. Basic notions

Let us consider a simply connected *nilpotent Lie group* $\mathbb{G}$, whose Lie algebra $\mathcal{G}$ admits the grading $\mathcal{G} = V_1 \oplus \cdots \oplus V_\iota$, where $[V_i, V_j] \subset V_{i+j}$ and $V_i = \{0\}$ iff $i > \iota$. Then we say that $\mathbb{G}$ is a *graded group*. A graded group $\mathbb{G}$ where the stronger condition $[V_i, V_j] = V_{i+j}$ holds for every i, j is called *stratified group*, or *Carnot group*. The subspace V_1 of $\mathcal{G}$ defines at any point $\xi \in \mathbb{G}$ the horizontal subspace

$$H_\xi \mathbb{G} = \{X(\xi) \mid X \in V_1\}.$$

Left translations of the group are denoted by $l_\xi : \mathbb{G} \longrightarrow \mathbb{G}$, $l_\xi(y) = \xi y$. The graded structure defines a one parameter group of dilations $\delta_r : \mathcal{G} \longrightarrow \mathcal{G}$, where $r > 0$. Precisely, we have

$$\delta_r\Big(\sum_{j=1}^{\iota} v_j\Big) = \sum_{j=1}^{\iota} r^j v_j \,,$$

where $\sum_{j=1}^{\iota} v_j = v$ and $v_j \in V_j$ for each $j = 1, \dots \iota$. To any element of V_j we associate the integer j, which is called the *degree* of the vector. Since $\mathbb{G}$ is simply connected and nilpotent it follows that $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism, hence we have a canonical way to transpose dilations from $\mathcal{G}$ to $\mathbb{G}$. We will use the same symbol to denote dilations of the group. The following standard properties hold

(1) $\delta_r(x \cdot y) = \delta_r x \cdot \delta_r y \quad$ for any $x, y \in \mathbb{G}$ and $r > 0$,
(2) $\delta_r(\delta_s x) = \delta_{rs} x \quad$ for any $r, s > 0$ and $x \in \mathbb{G}$.

To provide a metric structure on the group we will fix a *graded metric* g on $\mathbb{G}$, namely a left invariant metric such that all the subspaces V_j of the Lie algebra $\mathcal{G}$ are orthogonal each other. This metric will be always understood throughout the paper.

A continuous distance $\rho : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{R}$ is said *homogeneous* if it satisfies the following properties

(1) $\rho(x, y) = \rho(ux, uy)$ for every $u, x, y \in \mathbb{G}$,
(2) $\rho(\delta_r x, \delta_r y) = r\, \rho(x, y)$ for every $r > 0$.

The Carnot-Carathèodory distance provides an important example of homogeneous distance in stratified groups. In Section 6 we will see an example of homogeneous distance that can be defined in arbitrary graded groups. Notice that all homogeneous distances are bi-Lipschitz equivalent and induce the topology of $\mathbb{G}$.

Definition 2.1 (Graded coordinates)**.** Let us set $n_j = \dim V_j$ for any $j = 1, \ldots, \iota$, $m_0 = 0$ and $m_i = \sum_{j=1}^{i} n_j$ for any $i = 1, \ldots \iota$. Let $(X_1, \ldots, X_q)$ of $\mathcal{G}$ be an orthonormal basis with respect to g and assume that

$$(X_{m_{j-1}+1}, X_{m_{j-1}+2}, \ldots, X_{m_j})$$

is a basis of V_j for any $j = 1, \ldots \iota$. We consider the mapping $F : \mathbb{R}^q \longrightarrow \mathbb{G}$, defined by

$$F(y) = \exp\Big(\sum_{i=1}^{q} y_i\, X_i\Big)\,.$$

We say that $(X_1, \ldots, X_q)$ is a *graded basis* and that F is a system of *graded coordinates*.

Definition 2.2 (Metric ball)**.** The open ball and the closed one with center at $x \in \mathbb{G}$ and radius $r > 0$ with respect to the homogeneous distance ρ are denoted by $B_{x,r}$ and $D_{x,r}$, respectively. We omit the center when it coincides with the unit element of the group.

Recall that the subspace associated with a simple p-vector τ is defined as $\{v \in \mathcal{G} \mid v \wedge \tau = 0\}$. In this paper, our definition of spherical Hausdorf measure does not consider any dimensional factor, namely, we set

$$\mathcal{S}^{Q-k}(E) = \lim_{\varepsilon \to 0^+} \inf \Big\{ \sum_{j=1}^{\infty} r_i^{Q-k} \mid E \subset \bigcup_{i=1}^{\infty} B_{x_i, r_i}\,,\; r_i \le \varepsilon \Big\}, \tag{7}$$

where a fixed homogeneous distance ρ is understood.

3. Horizontal k-normal, vertical tangent vector and applications

Throughout the paper, $\tilde{g}$ will denote a Riemannian metric whose corresponding Riemannian volume is the Haar measure of the group. We denote by vol_g and $\mathrm{vol}_{\tilde{g}}$ the Riemannian volume measures arising from g and $\tilde{g}$, respectively. The Riemannian norm of vectors with respect to the fixed graded metric g will be denoted by $|\cdot|$.

Definition 3.1 (Horizontal projections)**.** Let $\overline{g}$ be a Riemannian metric on $\mathbb{G}$. We introduce the *horizontal projection* with respect to $\overline{g}$ as the smooth mapping of bundles $\pi_{\overline{g},H} : \Lambda(T\mathbb{G}) \longrightarrow \Lambda(H\mathbb{G})$ such that for each $x \in \mathbb{G}$ and $1 \le k \le m$ it is defined by setting

$$\pi_{\overline{g},H}(x) : \Lambda_k(T_x\mathbb{G}) \longrightarrow \Lambda_k(H_x\mathbb{G}),$$

that is the orthogonal projection of $\Lambda_k(T_x\mathbb{G})$ onto $\Lambda_k(H_x\mathbb{G})$ with respect to the metric induce on $\Lambda_k(T_x\mathbb{G})$ by $\overline{g}$.

Definition 3.2 (Horizontal k-normal)**.** Let $\Sigma \subset \mathbb{G}$ be a k-codimensional submanifold of class C^1 and let $x \in \Sigma$. We denote by $\mathbf{n}(x)$ and $\tilde{\mathbf{n}}(x)$ the unit k-normals to Σ at x with respect to g and $\tilde{g}$, respectively. We define the k-vector

$$\tilde{\mathbf{n}}_g(x) = (g_k^*)^{-1} \tilde{g}_k^*\big(\tilde{\mathbf{n}}(x)\big) \tag{8}$$

where $g_k^*, \tilde{g}_k^* : \Lambda_k(T\mathbb{G}) \longrightarrow \Lambda^k(T\mathbb{G})$ are canonically induced by the Riemannian metrics $\tilde{g}^*, g^* : T\mathbb{G} \longrightarrow T^*\mathbb{G}$. The *horizontal k-normal* with respect to both metrics g and $\tilde{g}$ is defined by $\tilde{\mathbf{n}}_{g,H}(x) = \pi_{g,H}\left(\tilde{\mathbf{n}}_g(x)\right)$. The horizontal normal with respect to g is given by $\pi_{g,H}\big(\mathbf{n}(x)\big)$ and it is denoted by $\mathbf{n}_H(x)$.

Remark 3.1. We notice that $\tilde{\mathbf{n}}_g(x) = \lambda\, \mathbf{n}(x)$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Let $\mathbf{n}(x) = n_1 \wedge \cdots \wedge n_k$ and let $\tilde{\mathbf{n}}(x) = \tilde{n}_1 \wedge \cdots \wedge \tilde{n}_k$. We have

$$\tilde{\mathbf{n}}_g(x) = (g_k^*)^{-1} \tilde{g}_k^* \big(\tilde{\mathbf{n}}(x)\big) = \big((g_k^*)^{-1} \tilde{g}_k^*(n_1)\big) \wedge \cdots \wedge \big((g_k^*)^{-1} \tilde{g}_k^*(n_k)\big)\,.$$

All vectors $v_j = (g_k^*)^{-1} \tilde{g}_k^*(n_j)$ are normal to $T_x\Sigma$ with respect to g and they are linearly independent. Then both v_j and n_j span the orthogonal space to $T_x\Sigma$ with respect to g. It follows that $v_1 \wedge \cdots \wedge v_k = \lambda\, n_1 \wedge \cdots n_k$ for some $\lambda \in \mathbb{R} \setminus \{0\}$.

Proposition 3.1. *Let $\Sigma \subset \mathbb{G}$ be a k-codimensional submanifold of class C^1, let $x \in \Sigma$ and let $(X_1, X_2, \ldots, X_m)$ be an orthonormal basis of V_1 with respect to g. Then we have*

$$|\tilde{\mathbf{n}}_{g,H}(x)|^2 = \sum_{1 \le \alpha_1 < \cdots < \alpha_k \le m} \langle \tilde{\mathbf{n}}(x), X_{\alpha_1}(x) \wedge \cdots \wedge X_{\alpha_k}(x) \rangle_{\tilde{g}}^2 \tag{9}$$

Proof. Let us fix the set of multi-indexes

$$I = \{(\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k \mid 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_k \le m\}\,.$$

By definition of horizontal projection, we have

$$|\tilde{\mathbf{n}}_{g,H}(x)|^2 = \sum_{\alpha \in I} \langle \tilde{\mathbf{n}}_g(x), X_\alpha(x) \rangle^2$$

and the definition of $\tilde{\mathbf{n}}_g(x)$ exactly yields

$$\langle \tilde{\mathbf{n}}_g(x), X_\alpha(x) \rangle = \langle \tilde{\mathbf{n}}(x), X_\alpha(x) \rangle_{\tilde{g}},$$

leading us to formula (9). □

Definition 3.3 (Degree of p-vectors)**.** Let $(X_1, \ldots, X_q)$ be a graded basis of $\mathcal{G}$. The *degree* $d(j)$ of X_j is the unique integer k such that $X_j \in V_k$. Let

$$X_\alpha := X_{\alpha_1} \wedge \cdots \wedge X_{\alpha_p}$$

be a simple p-vector of $\Lambda_p\mathcal{G}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_p \le q$. The degree of X_α is the integer $d(\alpha)$ defined by the sum $d(\alpha_1) + \cdots + d(\alpha_p)$. If $\tau = \sum_\beta c_\beta\, X_\beta$ is a p-vector, then we define its component of degree $d \le Q$ as

$$(\tau)_d = \sum_{d(\beta) = d} c_\beta\, X_\beta\,.$$

Notice that the q-vector containing all directions X_j has clearly degree Q, corresponding to the Hausdorff dimension of $\mathbb{G}$.

Definition 3.4 (Vertical tangent vector)**.** Let $\Sigma \subset \mathbb{G}$ be a p-dimensional submanifold of class C^1, with $x \in \Sigma$, and let $\tau_\Sigma(x)$ be a unit p-tangent vector of Σ at x with respect to the metric $\tilde{g}$. We define the *vertical tangent p-vector* at x as

$$\tau_{\Sigma,\mathcal{V}}(x) = \big(\tau_\Sigma(x)\big)_{Q-k}, \tag{10}$$

where $k = q - p$ is the codimension of Σ.

Remark 3.2. Notice that $Q - k$ is exactly the highest degree that can have a p-vector, where $k = q - p$. We use the adjective "vertical", according to [28], since the subspace associated with $\tau_{\Sigma,\mathcal{V}}(x)$ contains all directions of the layers V_j with $j \ge 2$. Notice that this also fits the terminology of Definition 5.2.

Definition 3.5 (Hodge operator)**.** Let X be a q-dimensional, oriented Hilbert space with orientation $e \in \Lambda_q(X)$, $|e| = 1$ and let $1 \le k \le q$. The Hodge operator is the mapping $* : \Lambda_k(X) \longrightarrow \Lambda_{q-k}(X)$ defined on each $\eta \in \Lambda_k(X)$ as the unique vector $*\eta \in \Lambda_{q-k}(X)$ satisfying the relation

$$\xi \wedge *\eta = \langle \xi, \eta \rangle \; e \tag{11}$$

for every $\xi \in \Lambda_k(X)$.

Proposition 3.2. *Let Σ be a C^1 smooth k-codimensional submanifold of $\mathbb{G}$ and let $x \in \Sigma$. Then*

$$|\tau_{\Sigma,\mathcal{V}}(x)| = c(g, \tilde{g}) \; |\tilde{\mathbf{n}}_{g,H}(x)|, \tag{12}$$

where $c(g, \tilde{g}) \operatorname{vol}_{\tilde{g}} = \operatorname{vol}_g$.

Proof. We fix the graded basis $(X_1, \ldots, X_q)$ and consider

$$X_\alpha = X_{\alpha_1} \wedge \cdots \wedge X_{\alpha_k} \quad \text{and} \quad I = \{(\alpha_1, \ldots, \alpha_k) \mid 1 \le \alpha_1 < \cdots < \alpha_k \le m\},$$

where $m = \dim V_1$. Notice that $(X_\alpha)_{\alpha \in I}$ defines an arthonormal basis of $\Lambda_k(V_1)$ with respect to the metric induced by g. Now, we choose a basis $(B_1, \ldots, B_q)$ of $T_x\mathbb{G}$ that is orthonormal with respect to $\tilde{g}$ and has the same orientation of $(X_1(x), \ldots, X_q(x))$. Then the definition of Hodge operator along with Lemma 5.1 yields

$$X_\alpha \wedge \tau_\Sigma(x) = (-1)^{kp} \; \big\langle X_\alpha, \tilde{*}\big(\tau_\Sigma(x)\big)\big\rangle_{\tilde{g}} \; B = \pm \, (-1)^{kp} \; \langle X_\alpha, \tilde{\mathbf{n}}(x) \rangle_{\tilde{g}} \; B, \tag{13}$$

where $\tilde{*}$ is the Hodge operator with respect to the metric $\tilde{g}$ and the orientation

$$B = B_1 \wedge \cdots \wedge B_q.$$

Recall that $p = q - k$ is the dimension of Σ. We obtserve that p-vectors of degree $Q - k$ can be represented as linear combinations of elements $*X_\alpha$, where α varies in I and $*$ denotes the Hodge operator with respect to g and the orientation $X_1 \wedge \cdots \wedge X_q$. Thus, we can write

$$\tau_\Sigma(x) = \sum_{\alpha \in I} \gamma_\alpha \; *X_\alpha(x) + \sum_{d(\beta) < Q-k} c_\beta \, X_\beta(x), \tag{14}$$

observing that the definition of vertical tangent p-vector yields

$$\tau_{\Sigma,\mathcal{V}}(x) = \sum_{\alpha \in I} \gamma_\alpha \; *X_\alpha(x). \tag{15}$$

The relationship between the metrics g and $\tilde{g}$ is contained in the formula

$$B_1 \wedge \cdots \wedge B_q = \sqrt{\frac{\det(g_{ij}(x))}{\det(\tilde{g}_{ij}(x))}} \; X_1 \wedge \cdots \wedge X_q, \tag{16}$$

that holds for any coordinate system with respect to which we consider g_{ij} and $\tilde{g}_{ij}$. In fact, let $C = (c^i_j)$ be the matrix defined by the equations $B_i = \sum_{j=1}^q c^j_i \, X_j(x)$. We have

$$B_1 \wedge \cdots \wedge B_q = \det(C) \; X_1 \wedge \cdots \wedge X_q \tag{17}$$

where $\det(C) > 0$, since (B_j) and (X_j) have the same orientation. Let $(\partial/\partial y_j)$ be local vector fields around x with respect to a fixed system of coordinates. Then we have the relationships $\partial/\partial y_i(x) = \sum_{j=1}^q a_i^j\, B_j$. As a consequence, we get

$$\tilde g_{ij}(x) = \sum_{l,s=1}^q a_i^l\, a_j^s\, \tilde g(B_l, B_s) = (A^T A)_s^i \tag{18}$$

where $A = (a_i^j)$. Similar computations yield

$$g_{ij}(x) = \sum_{l,s,k,p=1}^q a_i^l\, a_j^s\, c_l^k\, c_s^p\;\, g\big(X_k(x), X_p(x)\big) = (A^T C^T C A)_i^j$$

that joined with (17) and (18) leads to (16). Left invariance of vol_g and $\mathrm{vol}_{\tilde g}$ imply that $\det\big(g_{ij}(x)\big)$ and $\det\big(\tilde g_{ij}(x)\big)$ are independent of x, therefore

$$B_1 \wedge \cdots \wedge B_q = c(g,\tilde g)\; X_1 \wedge \cdots \wedge X_q, \tag{19}$$

Inserting (14) in (13) and using (19), we get

$$\gamma_\alpha\, X_\alpha \wedge *X_\alpha = c(g,\tilde g)\,(-1)^{kp}\;\, \langle X_\alpha, \tilde{\mathbf{n}}(x)\rangle_{\tilde g}\;\, X_1 \wedge \cdots \wedge X_q\,. \tag{20}$$

According to the definition of *horizontal k-normal*, we have $\langle X_\alpha, \tilde{\mathbf{n}}(x)\rangle_{\tilde g} = \langle X_\alpha, \tilde{\mathbf{n}}_g(x)\rangle$ and

$$\tilde{\mathbf{n}}_{g,H}(x) = \sum_{\alpha\in I} \langle X_\alpha, \tilde{\mathbf{n}}_g(x)\rangle\, X_\alpha\,, \tag{21}$$

therefore (20) implies

$$|\tilde{\mathbf{n}}_{g,H}(x)|^2 = \frac{1}{c(g,\tilde g)^2}\;\sum_{\alpha\in I}\gamma_\alpha^2 \tag{22}$$

that along with (15) concludes the proof. $\Box$

Proposition 3.3. *Let Σ be a C^1 smooth k-codimensional submanifold of $\mathbb{G}$. Then $x\in\Sigma$ is a non-horizontal point if and only if $\tau_{\Sigma,\mathcal{V}}(x) \neq 0$.*

PROOF. We set $p = q-k$ and $s = q-m$, where $m = \dim V_1$ and notice that p is the dimension of Σ. The integer s is the dimension $V_2 \oplus \cdots \oplus V_\iota$. If $k > m$, then $H_x\mathbb{G} + T_x\Sigma \subsetneqq T_xG$, hence x is horizontal. In addition, we also have $p < s$, then

$$(t_1 \wedge \cdots \wedge t_p)_{Q-k} = 0$$

for every basis $(t_1, \ldots, t_p)$ of $T_x\Sigma$. Now we assume that $k \le m$, namely, $s \le p$. If x is non-horizontal, then by definition we have $H_x\mathbb{G} + T_x\Sigma = T_x\mathbb{G}$. Thus, we can find an orthonormal basis $(t_1, \ldots, t_p)$ of $T_x\Sigma$ such that

$$t_i = W_i(x) + \omega_i, \quad \omega_i \in H_x\mathbb{G} \quad \text{for} \quad i = 1, \ldots, s$$

and $t_i \in H_x\mathbb{G}$ for each $i = s+1, \ldots, p$, where $(W_1, \ldots, W_s)$ is a basis of $V_2 \oplus \cdots \oplus V_\iota$. Then

$$\begin{aligned}\tau_{\Sigma,\mathcal{V}}(x) &= \Big((W_1(x)+\omega_1) \wedge \cdots \wedge (W_s(x)+\omega_s) \wedge t_{s+1} \wedge \cdots \wedge t_p\Big)_{Q-k}\\ &= W_1(x) \wedge \cdots \wedge W_s(x) \wedge t_{s+1} \wedge \cdots \wedge t_p \neq 0.\end{aligned}$$

Conversely, we suppose that $\tau_{\Sigma,\mathcal{V}}(x) \neq 0$. Any orthonormal basis $(t_1, \ldots, t_p)$ of $T_x\Sigma$ can be written in the form $t_i = W_i(x) + \omega_i$, where $W_i \in V_2 \oplus \cdots \oplus V_\iota$ and $\omega_i \in H_x\mathbb{G}$. By contradiction, if we had

$$\mathrm{span}\{W_1, \ldots, W_p\} \subsetneqq V_2 \oplus \cdots \oplus V_\iota,$$

then $(t_1 \wedge \cdots \wedge t_p)_{Q-k} = 0$, that conflicts with our assumption. As a consequence, the vectors $W_1, \ldots, W_p$ must generate all of $V_2 \oplus \cdots \oplus V_\iota$. This implies that

$$\operatorname{span}\{W_1(x) + \omega_1, \ldots, W_p(x) + \omega_p\} + H_x\mathbb{G} = T_x\mathbb{G},$$

namely x is a non-horizontal point. □

Proof of Corollary 1.2. First, in the case $\Phi(U)$ is a horizontal submanifold, then Theorem 2.16 of [27] implies that $\mathcal{S}^{Q-k}(\Phi(U)) = 0$. In fact, horizontal submanifolds coincide with their subset of horizontal points. Furthermore, the vertical tangent vector $\tau_{\Sigma,\mathcal{V}}$ vanishes at horizontal points, due to Proposition 3.3. It follows that

$$(\Phi_{\xi_1}(\xi) \wedge \cdots \wedge \Phi_{\xi_d}(\xi))_{Q-k} = 0 \quad \text{for every} \quad \xi \in U.$$

This makes the validity of (4) trivial. Now we assume that $\Phi(U)$ is non horizontal. Then we can apply formula (3), obtaining

$$\mathcal{S}^{Q-k}_{\mathbb{G}}\big(\Phi(U)\big) = c(g, \tilde{g}) \int_{\Phi(U)} |\tilde{\mathbf{n}}_{g,H}(x)| \, d\tilde{\mu}_p(x).$$

Due to Proposition 3.2, we get

$$\mathcal{S}^{Q-k}_{\mathbb{G}}\big(\Phi(U)\big) = \int_{\Phi(U)} |\tau_{\Sigma,\mathcal{V}}(x)| \, \tilde{\mu}_p(x) = \int_U |\tau_{\Sigma,\mathcal{V}}(\Phi(\xi))| \, |\Phi_{\xi_1}(\xi) \wedge \cdots \wedge \Phi_{\xi_d}(\xi)|_{\tilde{g}} \, d\xi.$$

Observing that

$$\tau_{\Sigma,\mathcal{V}}(\Phi(\xi)) = \frac{\big(\Phi_{\xi_1}(\xi) \wedge \cdots \wedge \Phi_{\xi_d}(\xi)\big)_{Q-k}}{|\Phi_{\xi_1}(\xi) \wedge \cdots \wedge \Phi_{\xi_d}(\xi)|_{\tilde{g}}}$$

we are then lead to our claim. □

Remark 3.3. Notice that in [28], the vertical tangent vector $\tau_{\Sigma,\mathcal{V}}(x)$ has been defined as the orthogonal projection of $\tau_\Sigma(x)$ onto the ideal of p-vectors generated by the Heisenberg vertical direction Z. This orthogonal projection is considered with respect to the metric g. It is then immediate to check that this notion coincides with ours exactly in the case $\tilde{g} = g$.

4. Density of non-horizontal submanifolds

By definition of non-horizontal point, it is immediate to check that graded groups where $\dim(V_1) < k$ do not have non-horizontal submanifolds Σ of codimension k. In fact, we have

$$\dim(H_x\mathbb{G} + T_x\Sigma) < \dim(T_x\mathbb{G}) \quad \text{for every} \quad x \in \Sigma.$$

Then we consider the existence of non-horizontal k-codimensional submanifolds in graded groups, where $m = \dim(H_x\mathbb{G}) \geq k$. Under this condition, we show that *any k-codimensional submanifold can be modified around a horizontal point by a local perturbation that preserves the point and makes it non-horizontal.*

In fact, up to left translations, we can assume that the unit element e of Σ be horizontal and that the C^1 mapping $\Phi : U \longrightarrow \Sigma$ parametrizes Σ, with $\Phi(0) = e$. The open set U is contained in $\mathbb{R}^p$ and it can be chosen such that

$$\lambda = \inf_{\xi \in U} |\phi_{\xi_1}(\xi) \wedge \cdots \wedge \phi_{\xi_p}(\xi)| > 0, \tag{23}$$

where we have set $p + k = q = \dim \mathbb{G}$ and $\Phi(\xi) = \exp \phi(\xi)$ for every $\xi \in U$. Vectors $\phi_{\xi_j}(\xi)$ are thought of as elements of the Lie algebra $\mathcal{G}$ equipped with the norm induced by the graded metric g. In other words we identify $T_{\phi(\xi)}\mathbb{G}$ with $\mathcal{G}$. Now we define the canonical projection $\pi^2 : \mathcal{G} \longrightarrow S_2$, where

$$S_2 = V_2 \oplus \cdots \oplus V_\iota.$$

Then for every $j = 1, \ldots, p$, we define $w_j = \pi^2\big(\Phi_{\xi_j}(0)\big)$. Our assumption on k implies that $p \geq s$, where $\dim(S_2) = s$. By contradiction, if $\dim\big(\mathrm{span}\{w_1, \ldots, w_p\}\big) = s$, then there exists $\alpha \neq 0$ such that

$$\tau_{\Sigma,\mathcal{V}}(x) = \alpha\big((w_1 + \omega_1) \wedge \cdots \wedge (w_p + \omega_p)\big)_{Q-k},$$

with $\omega_i \in V_1$. Arguing as in the proof of Proposition 3.3, one gets $\tau_{\Sigma,\mathcal{V}}(e) \neq 0$ that conflicts with our assumption. Then we must have

$$s_0 = \dim\big(\mathrm{span}\{w_1, \ldots, w_p\}\big) < s.$$

We define $s_1 = s - s_0$ and $s_2 = p - s_0$, observing that $s_2 \geq s_1$. There exist integers $1 \leq \alpha_1 < \cdots < \alpha_{s_0} \leq p$ such that

$$\mathrm{span}\{w_{\alpha_1}, \ldots, w_{\alpha_{s_0}}\} = \mathrm{span}\{w_1, \ldots, w_p\}.$$

We consider the integers $1 \leq \beta_1 < \cdots < \beta_{s_2} \leq p$ such that

$$\{\alpha_1, \ldots, \alpha_{s_0}\} \cap \{\beta_1, \ldots, \beta_{s_2}\} = \emptyset$$

and choose linearly independent vectors $z_1, \ldots, z_{s_1} \in S_2$ such that

$$\mathrm{span}\{w_{\alpha_1}, \ldots, w_{\alpha_{s_0}}, z_1, \ldots, z_{s_1}\} = S_2.$$

Clearly, it follows that

$$\mathrm{span}\{w_{\alpha_1}, \ldots, w_{\alpha_{s_0}}, w_{\beta_1} + tz_1, \ldots, w_{\beta_{s_1}} + tz_{s_1}\} = S_2 \tag{24}$$

for every $t \neq 0$. We define $L : \mathbb{R}^p \longrightarrow S_2$ such that

$$L(\xi) = \xi_{\beta_1} z_1 + \cdots + \xi_{\beta_{s_1}} z_{s_1}$$

and choose $\chi \in C_c^\infty(B_\delta^{|\cdot|})$ such that $\chi(\xi) = 1$ whenever $\xi \in B_{\delta/2}^{|\cdot|}$. Here $B_\delta^{|\cdot|}$ denotes the Euclidean ball, where the radius $\delta > 0$ is chosen such that $B_\delta^{|\cdot|} \subset U$. We define the functions

$$\psi(\xi) = \chi(\xi)\; L(\xi) \quad \text{and} \quad \Phi_\varepsilon(\xi) = \exp\big(\phi(\xi) + \varepsilon\,\psi(\xi)\big).$$

We first observe that

$$\partial_{\xi_{\alpha_j}}\Phi_\varepsilon(0) = w_{\alpha_j} + \omega_j \quad \text{and} \quad \partial_{\xi_{\beta_l}}\Phi_\varepsilon(0) = w_{\beta_l} + \varepsilon\, z_{\beta_l}$$

for every $j = 1, \ldots, s_0$ and $l = 1, \ldots, s_1$, where $\omega_j \in V_1$. By (24), it follows that

$$\dim\big(\mathrm{span}\,\big\{\pi^2\big(\partial_{\xi_1}\Phi_\varepsilon(0)\big), \ldots, \pi^2\big(\partial_{\xi_p}\Phi_\varepsilon(0)\big)\big\}\big) = s\,.$$

Then for every $\varepsilon > 0$ we have

$$\big(\partial_{\xi_1}\Phi_\varepsilon(0) \wedge \cdots \wedge \partial_{\xi_p}\Phi_\varepsilon(0)\big)_{Q-k} \neq 0.$$

Now, we are left to check that for some $\varepsilon > 0$ suitably small the differential $d\Phi_\varepsilon(\xi)$ is injective for every $\xi \in U$. To see this, we notice that

$$|(\phi_{\xi_1} + \varepsilon\psi_{\xi_1}) \wedge \cdots \wedge (\phi_{\xi_p} + \varepsilon\psi_{\xi_p})| \geq |\phi_{\xi_1} \wedge \cdots \wedge \phi_{\xi_p}| - \varepsilon\, C \sum_{j=1}^{p} \binom{p}{j} \varepsilon^{j-1} |\nabla\psi|^j |\nabla\phi|^{p-j}$$

where $|\nabla\psi| = \sum_{j=1}^p |\psi_{\xi_j}|$, $|\nabla\phi| = \sum_{j=1}^p |\phi_{\xi_j}|$ and

$$|\gamma_1 \wedge \cdots \wedge \gamma_p| \le C\, |\gamma_1| \cdots |\gamma_p|$$

for every $\gamma_1, \dots, \gamma_p \in \mathcal{G}$, where we have identified $\mathcal{G}$ with $T_{\phi(\xi)+\varepsilon\psi(\xi)}\mathcal{G}$ by left translations. As a consequence, we can find $\varepsilon_0 > 0$ depending on C and on λ, which is defined in (23), such that for every $\varepsilon \in]0, \varepsilon_0[$ we have

$$|(\phi_{\xi_1} + \varepsilon\psi_{\xi_1}) \wedge \cdots \wedge (\phi_{\xi_p} + \varepsilon\psi_{\xi_p})| > 0$$

at every point of U. Taking into account that $\exp : \mathcal{G} \longrightarrow \mathbb{G}$ is a diffeomorphism, this implies that $d\Phi_\varepsilon(\xi) : \mathbb{R}^p \longrightarrow T_{\Phi(\xi)}\mathbb{G}$ is injective for every $\xi \in U$ and every $\varepsilon \in]0, \varepsilon_0[$. Now, we wish to prove that Φ_ε is also injective for $\varepsilon > 0$ suitably small. To do this, we can clearly assume that $\Phi : U \longrightarrow \Sigma$ is an embedding and that

$$|\phi(\xi) - \phi(\xi')| \ge \kappa\, |\xi - \xi'|$$

for a constant $\kappa > 0$ and every $\xi, \xi' \in U$. By contradiction, if we had an infinitesimal sequence $(\varepsilon_j) \subset]0, \varepsilon_0[$ and couples of distinct points $\xi_j, \eta_j \in U$ such that $\Phi_{\varepsilon_j}(\xi_j) = \Phi_{\varepsilon_j}(\eta_j)$, then we would get

$$|\phi(\xi_j) - \phi(\eta_j)| = \varepsilon_j\, |\psi(\xi_j) - \psi(\eta_j)|$$

that implies

$$\frac{\kappa}{\varepsilon_j} \le \frac{|\psi(\xi_j) - \psi(\eta_j)|}{|\xi_j - \eta_j|} \le \mathrm{Lip}(\psi)$$

leading to a contradiction as $\varepsilon_j \to 0$. Then we have shown the existence of some $\varepsilon_1 \in]0, \varepsilon_0]$ such that $\Phi_\varepsilon : U \longrightarrow \mathbb{G}$ is an immersion for every $\varepsilon \in]0, \varepsilon_1[$. In addition, we have

$$\Phi_\varepsilon(\xi) = \Phi(\xi) \quad \text{whenever} \quad \xi \in U \setminus B_\delta^{|\cdot|} \quad \text{and} \quad \Phi_\varepsilon(0) = e$$

namely $\Phi_\varepsilon(U)$ is a perturbation of $\Phi(U)$ around e.

Denote by $\mathcal{S}_k(\mathbb{G})$ be the family of C^1 smooth k-codimensional submanifolds of $\mathbb{G}$ without boundary. Then we have shown the following

Theorem 4.1. *Let $\Sigma \in \mathcal{S}_k(\mathbb{G})$ and let $x \in \Sigma$ be a horizontal point. Then for every open set $A \subset \mathbb{G}$ containing x there exists $\Sigma' \in \mathcal{S}_k(\mathbb{G})$ such that $x \in \Sigma'$ is a non-horizontal point of Σ' and $\Sigma' \setminus A = \Sigma \setminus A$.*

Definition 4.2 (Open sets of $\mathcal{S}_k(\mathbb{G})$)**.** Let A be an open set of $\mathbb{G}$ and denote by $\mathcal{O}_A$ the family of all submanifolds $\Sigma \in \mathcal{S}_k(\mathbb{G})$ such that $\Sigma \cap A \neq \emptyset$. We denote by $\mathcal{T}_k(\mathbb{G})$ the topology generated by these sets, namely, the family of submanifolds of $\mathcal{S}_k(\mathbb{G})$ formed by arbitrary unions of finite intersections of sets of the form $\mathcal{O}_A$.

Theorem 4.3. *The family of non-horizontal submanifolds of $\mathbb{G}$ forms a dense subset of $\mathcal{S}_k(\mathbb{G})$ with respect to the topology $\mathcal{T}_k(\mathbb{G})$.*

Proof. Let Σ be a horizontal submanifold of $\mathcal{S}_k(\mathbb{G})$ and consider an open subset $\mathcal{U}$ of $\mathcal{T}_k(\mathbb{G})$ containing Σ. It follows that $\Sigma \in \mathcal{O}_{A_1} \cap \cdots \cap \mathcal{O}_{A_n}$ for some open sets A_j of $\mathbb{G}$. We choose $x_j \in \Sigma \cap A_j$ for every $j = 1, \dots, n$. Let H be an open subset of $\mathbb{G}$ containing x_1 and not containing all x_j different from x_1. Then $x_j \in \Sigma \setminus H$ whenever $x_j \neq x_1$. The condition $m \ge k$ allows us to apply Theorem 4.1, hence there exists $\Sigma' \in \mathcal{S}_k(\mathbb{G})$ such that $\Sigma \setminus A_1 = \Sigma' \setminus A_1$ and x is a non-horizontal point of Σ'. It follows that $\Sigma' \cap A_j \neq \emptyset$ whenever $x_j \neq x_1$ and clearly $\Sigma' \cap A_i \neq \emptyset$ in the case $x_j = x_1$. We have proved that $\Sigma' \in \mathcal{O}_{A_1} \cap \cdots \cap \mathcal{O}_{A_n} \subset \mathcal{U}$ and that it is a non-horizontal submanifold. $\square$

Remark 4.1. The terminology "generically" we have used in the introduction usually refers to countable intersections of open dense subsets. Then for different scopes, it might be of interest either checking whether non-horizontal submanifolds have this property with respect to the topology $\mathcal{T}_k(\mathbb{G})$ or finding another topology with this property.

5. Blow-up at non-horizontal points

Proposition 5.1. *Let (X,d) be a metric space where closed bounded sets are compacts. Let (A_n) be a sequence of compact sets contained in a bounded set of X and let A be a compact set of X. Let us consider the following statements:*

1. $\max_{y\in A} \operatorname{dist}(y,A_n) \longrightarrow 0$ *as* $n\to\infty$,
2. *for every $y\in A$, there exists a sequence (x_n) such that $x_n\to y$ and* $\operatorname{dist}(x_n,A_n)\to 0$.
3. $\max_{x\in A_n} \operatorname{dist}(x,A) \longrightarrow 0$ *as* $n\to\infty$
4. *for every subsequence (A_ν) and every converging sequence (x_ν) such that $x_\nu\in A_\nu$, we have $\lim_\nu x_\nu \in A$.*

Then (1) is equivalent to (2) and (3) is equivalent to (4).

Proof. We show that statement 2 implies statement 1. By contradiction, if we have a subsequence such that

$$\max_{y\in A} \operatorname{dist}(y,A_\mu) = \operatorname{dist}(y_\mu,A_\mu) \geq \varepsilon > 0 \tag{25}$$

for every μ, then we get a further subsequence y_ν convering to y_0 belonging to A and a sequence (x_n) such that $x_n\to y_0$ and $\operatorname{dist}(x_n,A_n)\to 0$, then

$$\limsup_\nu \operatorname{dist}(y_\nu,A_\nu) \leq \limsup_\nu \left[\operatorname{dist}(y_\nu,y_0)+\operatorname{dist}(y_0,A_\nu)\right] \tag{26}$$

$$\leq \limsup_\nu \left[d(y_0,x_\nu)+\operatorname{dist}(x_\nu,A_\nu)\right] = 0, \tag{27}$$

which conflicts with (25). If statement 1 holds, then for an arbitrary element $y\in A$ we consider the sequence (x_n) such that $\operatorname{dist}(y,A_n)=d(y,x_n)$, where $x_n\in A_n$ and observe that

$$d(y,x_n) \leq \max_{z\in A} \operatorname{dist}(z,A_n) \longrightarrow 0.$$

This shows the validity of statement 2. Now, we wish to show that statement 3 implies statement 4. By contradiction, assume that there exists $\varepsilon>0$ such that

$$\varepsilon \leq \max_{x\in A_\mu} \operatorname{dist}(x,A) = \operatorname{dist}(x_\mu,A) \tag{28}$$

for some subsequence x_μ. Then we get a further subsequence x_ν converging to y. By hypothesis we have that $y\in A$. This conflicts with (28). If statement 3 holds, then for every subsequence (A_ν) and every sequence (x_ν) such that $x_\nu\in A_\nu$ and $x_\nu\to y$, we have

$$d(y,A) = \lim_\nu d(x_\nu,A) \leq \lim_\nu \max_{x\in A_\nu} \operatorname{dist}(x,A) = 0,$$

hence $y\in A$. This shows the validity of statement 4 and concludes the proof. □

Remark 5.1. Under notation of the previous proposition, by definition of Hausdorff distance bewteen sets, we have

$$d_H(A,B) = \max\left\{\sup_{y\in B} \operatorname{dist}(y,A), \sup_{x\in A} \operatorname{dist}(x,B)\right\}$$

where $A, B \subset X$. Thus, conditions (2) and (4) of Proposition 5.1 characterize the Hausdorff convergence.

Lemma 5.1. *Let N and S be orthogonal subspaces of a q-dimensional oriented Hilbert space and assume that $X = N \oplus S$. We set $\dim(N) = k$ and consider the simple multivectors $n \in \Lambda_k(N)$ and $\tau \in \Lambda_{q-k}(S)$. Then there exists $\lambda \in \mathbb{R}$ such that $*n = \lambda\tau$.*

Proof. It is clearly not restrictive assuming that that both $n \in \Lambda_k(N)$ and $\tau \in \Lambda_{q-k}(S)$ have unit norm. We define $d{=}q{-}k$ and assume that

$$n = n_1 \wedge n_2 \wedge \cdots \wedge n_k \quad \text{and} \quad \tau = t_1 \wedge t_2 \wedge \cdots \wedge t_d\,,$$

where $(n_1, n_2, \ldots, n_k)$ and $(t_1, t_2, \ldots, t_d)$ are orthonormal bases of N and S, respectively. Taking into account the possible orientations of n and τ we have

$$n_1 \wedge n_2 \wedge \cdots \wedge n_k \wedge t_1 \wedge t_2 \wedge \cdots \wedge t_d = \pm\eta,$$

where $\eta \in \Lambda_q(X)$ defines the orientation of X and the Hodge operator according to (11). Thus, by definition of $*$ we get $*n = \pm\tau$. □

Definition 5.2 (Vertical subalgebra and vertical subgroup)**.** Let $\mathcal{G}$ be a graded algebra equipped with a graded metric g and let ν be a simple k-vector of $\Lambda_k(V_1)$. We define the *vertical subalgebra* $\mathcal{L}(\nu)$ with respect to ν as the orthogonal complement to the subspace associated with ν. Its image through the exponential mapping is called *vertical subgroup* and it is denoted by $N(\nu)$.

Remark 5.2. The terminology of Definition 5.2 fits the fact that the orthogonal complement of any subspace of V_1 with respect to a graded metric is a subalgebra. Precisely, it is an ideal of $\mathcal{G}$. Then the vertical subgroup $N(\nu)$ is clearly a normal subgroup.

Definition 5.3 (Metric factor)**.** Let $\mathcal{G}$ be a stratified Lie algebra equipped with a graded metric g and let ν be a simple k-vector of $\Lambda_k(V_1)$. The *metric factor* of a homogeneous distance ρ with respect to ν is defined by

$$\theta^g_\rho(\nu) = \mathcal{H}^p_{|\cdot|}\big(F^{-1}\big(B_1 \cap N(\nu)\big)\big)\,,$$

where $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ is a system of graded coordinates with respect to g. The symbol $\mathcal{H}^p_{|\cdot|}$ denotes the p-dimensional Hausdorff measure with respect to the Euclidean distance of $\mathbb{R}^p$ and B_1 is the unit ball of $\mathbb{G}$ with respect to ρ.

In the case of subspaces $\mathcal{L}$ of codimension one, the notion of metric factor fits into the one introduced in [24]. As we have already pointed out in [24] the metric factor does not depend on the system of coordinates we are using, since $F_1^{-1} \circ F_2 : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ is an Euclidean isometry whenever $F_1, F_2 : \mathbb{R}^q \longrightarrow \mathbb{G}$ represent systems of graded coordinates with respect to the same graded metric.

Theorem 5.4 (Blow-up Theorem)**.** *Let Σ be a p-dimensional submanifold of $\mathbb{G}$ and let $x \in \Sigma$ be a non-horizontal point. Then the following limit holds*

$$\frac{\tilde{\mu}_p(\Sigma \cap B_{x,r})}{r^{Q-k}} \longrightarrow c(\tilde{g}, g)\,\frac{\theta^g_\rho\big(\mathbf{n}_H(x)\big)}{|\tilde{\mathbf{n}}_{g,H}(x)|} \quad \text{as} \quad r \to 0^+, \tag{29}$$

where the integer k is the codimension of Σ. Moreover, for every $R > 0$ we have

$$D_R \cap \delta_{1/r}\,(l_{x^{-1}}\Sigma) \longrightarrow D_R \cap N\big(\mathbf{n}_H(x)\big) \tag{30}$$

with respect to the Hausdorff distance of sets.

Proof. Due to the fact that x is a non-horizontal point, the Grassmann formula yields

$$\dim(T_x\Sigma \cap H_x\mathbb{G}) = m-k.$$

As a result, we can find an open neighbourhood $\Omega \subset \mathbb{G}$ of x and a mapping $f:\Omega \longrightarrow \mathbb{R}^k$ of class C^1 such that $\Sigma\cap\Omega = f^{-1}(0)$ and $df(s)_{|H_s\mathbb{G}} : H_s\mathbb{G} \longrightarrow \mathbb{R}^k$ is surjective for every $s\in\Omega$. We choose an orthonormal basis $(v_{k+1}, v_{k+2}, \dots, v_m)$ of $T_x\Sigma\cap H_x\mathbb{G}$ and consider the corresponding left invariant vector fields $(Y_{k+1}, Y_{k+2}, \dots, Y_m)$ such that $Y_j(x) = v_j$ for every $j = k+1,\dots,m$. Thus, we get

$$Y_j f(x) = 0 \qquad \text{whenever} \qquad k+1 \le j \le m \tag{31}$$

and we can find $(Y_1, Y_2, \dots, Y_k)$ such that $(Y_1,\dots,Y_k,Y_{k+1},\dots,Y_m)$ is a left invariant orthonormal frame such that

$$\operatorname{span}\{Y_1(s), Y_2(s), \dots, Y_m(s)\} = H_s\mathbb{G}$$

for every $s\in\mathbb{G}$. We complete this frame to the following orthonormal basis of $\mathcal{G}$

$$(Y_1, Y_2, \dots, Y_m, Y_{m+1}, \dots, Y_q).$$

Let $F:\mathbb{R}^q \longrightarrow \mathbb{G}$ be a system of graded coordinates defined by

$$F(y) = \exp\Big(\sum_{j=1}^q y^j\, Y_j\Big) \tag{32}$$

and let us center this system of coordinates at x defining

$$F_x(y) = l_x F(y) \quad \text{and} \quad \tilde f(y) = f(F_x(y)).$$

Then we have

$$\partial_{y_j}\tilde f = df\circ \partial_{y_j}F_x. \tag{33}$$

We consider the canonical vector fields e_j in $\mathbb{R}^q$ and define their images

$$Z_j = (F_x)_*(e_j), \quad \text{where} \quad Z_j(x) = Y_j(x). \tag{34}$$

Then in a neighbourhood $\Omega'\subset\Omega$ of x, we get

$$Z_1 f\wedge Z_2 f\wedge\cdots\wedge Z_k f \neq 0 \tag{35}$$

in a neighbourhood of x. By definition of image of vector field, we get

$$Z_j f(z) = (F_x)_*(e_j) f(z) = \frac{\partial}{\partial y_j}\big(f\circ F_x\big)(F_x^{-1}(z)) = \partial_{y_j}\tilde f(F_x^{-1}(z)).$$

It follows that $\partial_{y_1}\tilde f\wedge\partial_{y_2}\tilde f\wedge\cdots\wedge\partial_{y_k}\tilde f \neq 0$ on $F_x^{-1}(\Omega')\subset\mathbb{R}^q$. From the implicit function theorem there exist a bounded open neighbourhood $U\subset\mathbb{R}^p$ of the origin and a mapping

$$\psi(\xi) = (\varphi^1(\xi), \dots, \varphi^k(\xi), \xi_1, \dots, \xi_p) \tag{36}$$

defined on U such that $\tilde f(\varphi^1(\xi),\dots,\varphi^k(\xi),\xi_1,\dots,\xi_p) = 0$ for every $\xi\in U$. We define

$$\Phi(\xi) = F_x\big(\psi(\xi)\big), \quad \text{where} \quad f\circ\Phi = 0.$$

We have the formula

$$\frac{\tilde\mu_p(\Sigma\cap B_{x,r})}{r^{Q-k}} = r^{k-Q}\int_{\Phi^{-1}(B_{x,r})} \left|\frac{\partial\Phi}{\partial\xi_1}(\xi)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_d}(\xi)\right|_{\tilde g}\, d\xi.$$

Dilations with respect to graded coordinates become $\tilde\delta_r = F^{-1}\circ\delta_r\circ F$, then we restrict $\tilde\delta_r$ to the subspace $\{\xi\in\mathbb{R}^q \mid \xi_1=\xi_2=\cdots=\xi_k=0\}$ and perform the change of variable

$$\xi=\sum_{j=k+1}^{q} r^{d(j)}\xi'_{j-k}e_j=\tilde\delta_r\xi'. \tag{37}$$

Observing that $Q-k=\sum_{j=k+1}^{q} d(j)$, we get

$$\frac{\tilde\mu_p(\Sigma\cap B_{x,r})}{r^{Q-k}}=\int_{\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})}\left|\frac{\partial\Phi}{\partial\xi_1}(\tilde\delta_r\xi')\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}(\tilde\delta_r\xi')\right|_{\tilde g}d\xi',$$

where

$$\tilde\delta_{1/r}\Phi^{-1}(B_{x,r})=\left\{\xi\in\mathbb{R}^p\;\Big|\;\left(\frac{\varphi^1(\tilde\delta_r\xi)}{r},\ldots,\frac{\varphi^k(\tilde\delta_r\xi)}{r},\xi_1,\ldots,\xi_p\right)\in F^{-1}(B_1)\right\}.$$

For every $i=1,\ldots,k$, Taylor expansion yields

$$r^{-1}\varphi^i(\tilde\delta_r\xi)=\sum_{j=1}^{m-k}\varphi^i_{\xi_j}(0)\xi_j+o(1). \tag{38}$$

In view of (31) and (33), we obtain

$$\varphi_{\xi_j}(0)=-\left(\nabla_{y_1\cdots y_k}\tilde f(0)\right)^{-1}\tilde f_{y_{j+k}}(0)=-\left(\nabla_{y_1\cdots y_k}\tilde f(0)\right)^{-1}Y_{j+k}f(x)=0$$

for every $j=1,\ldots,m-k$. It follows that

$$\sup_{\xi\in A}\left|\frac{\varphi(\tilde\delta_r\xi)}{r}\right|\longrightarrow 0\quad\text{as}\quad r\to 0^+, \tag{39}$$

where A is a bounded subset of $\mathbb{R}^p$. As a result, we have proved that

$$\lim_{r\to0^+}\frac{\tilde\mu_p(\Sigma\cap B_{x,r})}{r^{Q-k}}=\mathcal{H}^p_{|\cdot|}\left(\Pi\cap F^{-1}(B_1)\right)\left|\frac{\partial\Phi}{\partial\xi_1}(0)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}(0)\right|_{\tilde g}, \tag{40}$$

where we have set $\Pi=\{(0,0,\ldots,0,\xi_1,\xi_2,\ldots,\xi_p)\in\mathbb{R}^q\}$. Now we wish to obtain a geometric characterization of the limit (40). Let us consider the Hodge operator $\tilde *$ with respect to $\tilde g$ and the orientation $e=T_1\wedge\cdots\wedge T_q$, where the frame of vector fields $(T_1,\ldots,T_q)$ is an orthonormal frame with respect to $\tilde g$ defined on a neighbourhood of x. We define

$$N=\operatorname{span}\left\{\nabla_{\tilde g}f^1,\nabla_{\tilde g}f^2,\ldots,\nabla_{\tilde g}f^k\right\}\quad\text{and}\quad S=\operatorname{span}\left\{\frac{\partial\Phi}{\partial\xi_1},\ldots,\frac{\partial\Phi}{\partial\xi_p}\right\},$$

observing that they are orthogonal with respect to $\tilde g$. Due to Lemma 5.1 and the identity $\tilde *\tilde *=(-1)^{k\,d}$, we get $\lambda\neq0$ such that

$$\nabla_{\tilde g}f^1\wedge\cdots\wedge\nabla_{\tilde g}f^k=\lambda\,\tilde *\left(\frac{\partial\Phi}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}\right), \tag{41}$$

where $\tilde{*}$ denotes the Hodge operator with respect to the metric $\tilde{g}$. The same identity and the defining property (11) yield

$$\Big\langle Z_1\wedge\cdots\wedge Z_k,\ \tilde{*}\Big(\frac{\partial\Phi}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial x_p}\Big)\Big\rangle_{\tilde g}\ T_1\wedge\cdots\wedge T_q \tag{42}$$
$$=(-1)^{kp}\ Z_1\wedge\cdots Z_k\wedge\frac{\partial\Phi}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}.$$

From equality

$$\frac{\partial\Phi}{\partial\xi_i}=dF_x\circ\frac{\partial\psi}{\partial\xi_i},$$

observing that $\psi_{\xi_i}=e_{k+i}+\sum_{s=1}^k\varphi^s_{x_i}e_s$ and taking into account (34), we achieve

$$\frac{\partial\Phi}{\partial\xi_i}=Z_{k+i}\circ\Phi+\sum_{s=1}^k\varphi^s_{\xi_i}\ Z_s\circ\Phi.$$

The last formula along with (42) implies that

$$\Big\langle Z_1\wedge\cdots\wedge Z_k,\ *\Big(\frac{\partial\Phi}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}\Big)\Big\rangle_{\tilde g}\ T_1\wedge\cdots\wedge T_q \tag{43}$$
$$=(-1)^{kp}\ Z_1\wedge\cdots Z_k\wedge Z_{k+1}\wedge\cdots\wedge Z_q.$$

Applying the relationships

$$T_1\wedge\cdots\wedge T_q=\frac{1}{\sqrt{\det(\tilde g_{ij})}}\frac{\partial}{\partial y_1}\wedge\cdots\wedge\frac{\partial}{\partial y_q},\qquad Y_1\wedge\cdots\wedge Y_q=\frac{1}{\sqrt{\det(g_{ij})}}\frac{\partial}{\partial y_1}\wedge\cdots\wedge\frac{\partial}{\partial y_q}$$

and evaluating (43) at x, we get

$$\Big\langle Y_1\wedge\cdots\wedge Y_k,\ *\Big(\frac{\partial\Phi}{\partial\xi_1}\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}\Big)\Big\rangle_{\tilde g}(x)=(-1)^{kp}\sqrt{\frac{\det(\tilde g_{ij}(x))}{\det(g_{ij}(x))}} \tag{44}$$

hence (41) yields

$$\lambda(x)=c(g,\tilde g)\,(-1)^{kp}\Big\langle Y_1\wedge\cdots\wedge Y_k,\nabla_{\tilde g}f^1\wedge\cdots\wedge\nabla_{\tilde g}f^k\Big\rangle_{\tilde g}(x), \tag{45}$$

where $c(g,\tilde g)$ is equal to $\sqrt{\det(g_{ij}(x))/\det(\tilde g_{ij}(x))}$ and it is a constant independent from x, due to the left invariance of both $\mathrm{vol}_{\tilde g}$ and vol_g. We precisely have

$$c(\tilde g,g)=\sqrt{\frac{\det(\tilde g_{ij}(x))}{\det(g_{ij}(x))}}=\frac{\mathrm{vol}_{\tilde g}(B_1)}{\mathrm{vol}_g(B_1)}.$$

From (41) and (45), it follows that

$$\left|\frac{\partial\Phi}{\partial\xi_1}(0)\wedge\cdots\wedge\frac{\partial\Phi}{\partial\xi_p}(0)\right|_{\tilde g}=c(\tilde g,g)\frac{|\nabla_{\tilde g}f^1(x)\wedge\cdots\wedge\nabla_{\tilde g}f^k(x)|_{\tilde g}}{\Big|\langle Y_1(x)\wedge\cdots\wedge Y_k(x),\nabla_{\tilde g}f^1(x)\wedge\cdots\wedge\nabla_{\tilde g}f^k(x)\rangle_{\tilde g}\Big|_{\tilde g}}. \tag{46}$$

We consider the following unit normal k-vector field

$$\tilde{\mathbf{n}}=\frac{\nabla_{\tilde g}f^1\wedge\cdots\wedge\nabla_{\tilde g}f^k}{|\nabla_{\tilde g}f^1\wedge\cdots\wedge\nabla_{\tilde g}f^k|_{\tilde g}}.$$

In view of Definition 3.2, we notice that

$$\tilde{\mathbf{n}}_g = (g_k^*)^{-1}\tilde{g}_k^*(\tilde{\mathbf{n}}) = \frac{\nabla_g f^1 \wedge \cdots \wedge \nabla_g f^k}{|\nabla_{\tilde{g}} f^1 \wedge \cdots \wedge \nabla_{\tilde{g}} f^k|_{\tilde{g}}}.$$

Thus, taking into account also Definition 3.1, we get

$$\tilde{\mathbf{n}}_{g,H} = \sum_{1\le\alpha_1<\cdots<\alpha_k\le m} \frac{\langle \nabla_g f^1 \wedge \cdots \wedge \nabla_g f^k, Y_{\alpha_1} \wedge \cdots \wedge Y_{\alpha_k}\rangle}{|\nabla_{\tilde{g}} f^1 \wedge \cdots \wedge \nabla_{\tilde{g}} f^k|_{\tilde{g}}}\; Y_{\alpha_1} \wedge \cdots \wedge Y_{\alpha_k}\,.$$

Applying (31), it follows that

$$\tilde{\mathbf{n}}_{g,H}(x) = \frac{\langle \nabla_g f^1(x) \wedge \cdots \wedge \nabla_g f^k(x), Y_1(x) \wedge \cdots \wedge Y_k(x)\rangle}{|\nabla_{\tilde{g}} f^1(x) \wedge \cdots \wedge \nabla_{\tilde{g}} f^k(x)|_{\tilde{g}}}\;\; Y_1(x) \wedge \cdots \wedge Y_k(x)\,. \tag{47}$$

We also observe that

$$\langle \nabla_g f^1 \wedge \cdots \wedge \nabla_g f^k, Y_1 \wedge \cdots \wedge Y_k\rangle = \langle \nabla_{\tilde{g}} f^1 \wedge \cdots \wedge \nabla_{\tilde{g}} f^k, Y_1 \wedge \cdots \wedge Y_k\rangle_{\tilde{g}}\,,$$

hence joining (46) with (47), we establish that

$$\left|\frac{\partial \Phi}{\partial \xi_1}(0) \wedge \cdots \wedge \frac{\partial \Phi}{\partial \xi_p}(0)\right|_{\tilde{g}} = c(\tilde{g}, g)\; \frac{1}{|\tilde{\mathbf{n}}_{g,H}(x)|}\,. \tag{48}$$

Next, we will give a geometric characterization of the number $\mathcal{H}^p_{|\cdot|}\big(\Pi \cap F^{-1}(B_1)\big)$. By virtue of (47), the subspace orthogonal to that associated with $\tilde{\mathbf{n}}_{g,H}(x)$ is exactly

$$\operatorname{span}\{Y_{k+1}(x), \ldots, Y_q(x)\} \subset \mathcal{G}.$$

Due to Remark 3.1 we have $\tilde{\mathbf{n}}_g = \lambda \mathbf{n}$ for some $\lambda \neq 0$, then $\tilde{\mathbf{n}}_{g,H}(x) = \lambda\, \mathbf{n}_H(x)$. Being $\tilde{\mathbf{n}}_{g,H}(x)$ a simple k-vector, it follows that $\mathbf{n}_H(x)$ is simple too. As a consequence, Definition 5.3 implies that

$$N(\mathbf{n}_H(x)) = \exp\big(\operatorname{span}\{Y_{k+1}(x), \ldots, Y_q(x)\}\big) = F(\Pi),$$

therefore we get

$$\theta^g_\rho\big(\mathbf{n}_H(x)\big) = \mathcal{H}^p_{|\cdot|}\left(\Pi \cap F^{-1}(B_1)\right). \tag{49}$$

Taking into account formulae (40), (48) and (49), the proof of (29) is achieved.

Now, we are left to prove the validity of the Hausdorff convergence (30). Let $R > 0$ be fixed. For every $r > 0$ suitably small, we observe that

$$D_R \cap \delta_{1/r}\big(l_{x^{-1}}\Sigma\big) = D_R \cap \delta_{1/r}\big(l_{x^{-1}}\Phi(U)\big) = D_R \cap \delta_{1/r} F\big(\psi(U)\big), \tag{50}$$

where $\Phi : U \longrightarrow \Sigma$ parametrizes a neighbourhood of x in Σ and ψ is defined in (36). In order to apply Proposition 5.1, we will prove its conditions (2) and (4). Let (r_n) be a positive sequence such that $r_n \to 0^+$ and let (x_n) be a converging sequence such that

$$x_n \in D_R \cap \delta_{1/r_n} F\big(\psi(U)\big).$$

We have

$$x_n = F\left(\frac{\varphi(\xi_n)}{r_n}, \tilde{\delta}_{1/r_n}\xi_n\right)$$

where the restriction $\tilde{\delta}_r$ is defined in (37). If we set $\tilde{\delta}_{1/r_n}\xi_n = \eta_n$, then the convergence of (x_n) implies the convergence of (η_n) to a vector $\eta \in \mathbb{R}^p$. Due to (39), it follows that $\varphi(\tilde{\delta}_{r_n}\eta_n)/r_n \to 0$, therefore

$$x_n \longrightarrow F\,(0, \ldots, 0, \eta) \in F(\Pi).$$

Thus, we have proved that

$$\lim_{n\to\infty} x_n \in D_R \cap F(\Pi) = D_R \cap N\big(\tilde{\mathbf{n}}_{g,H}(x)\big).$$

This shows condition (4) of Proposition 5.1. Now we choose

$$\overline{x} = F(0,\eta) \in D_R \cap N\big(\tilde{\mathbf{n}}_{g,H}(x)\big),$$

where $\eta \in \mathbb{R}^p$ and define

$$x_n = F\left(\frac{\varphi(\tilde{\delta}_{r_n}\eta)}{r_n}, \eta\right) \quad \text{and} \quad x_n(t) = F\left(\frac{\varphi\big(\tilde{\delta}_{r_n}(t\eta)\big)}{r_n}, t\eta\right)$$

where we have arbitrarily fixed $0 < t < 1$. For n sufficiently large, depending on t, we have

$$x_n(t) \in B_R \cap \delta_{1/r_n} F\big(\psi(U)\big),$$

since $x_n(t) \to F(0,t\eta) \in B_R$. In particular, by (50) we have

$$\text{dist}\Big(x_n(t), D_R \cap \delta_{1/r_n} F\big(\psi(U)\Big) = \text{dist}\Big(x_n(t), D_R \cap \delta_{1/r_n}\big(l_{x^{-1}}\Sigma\big)\Big) \longrightarrow 0. \tag{51}$$

We also notice that

$$\lim_{n\to\infty} \rho\big(x_n(t), x_n\big) = \rho\big(F(0,t\eta), F(0,\eta)\big) = \sigma(t) \longrightarrow 0 \quad \text{as} \quad t \to 1^-.$$

Triangle inequality yields

$$\limsup_{n\to\infty}\ \text{dist}\Big(x_n, D_R \cap \delta_{1/r_n}\big(l_{x^{-1}}\Sigma\big)\Big) \le \sigma(t) + \limsup_{n\to\infty}\ \text{dist}\Big(x_n(t), D_R \cap \delta_{1/r_n}\big(l_{x^{-1}}\Sigma\big)\Big),$$

then (51) gives $\limsup_{n\to\infty}\ \text{dist}\Big(x_n, D_R \cap \delta_{1/r_n}\big(l_{x^{-1}}\Sigma\big)\Big) \le \sigma(t)$. Letting $t \to 1^-$, we have shown that

$$\text{dist}\big(x_n, D_R \cap \delta_{1/r_n}\big(l_{x^{-1}}\Sigma\big)\big) \to 0.$$

This shows the validity of condition (2) of Proposition 5.1. As pointed out in Remark 5.1, conditions (2) and (4) of Proposition 5.1 characterize the Hausdorff convergence, hence the proof is finished. □

6. Remarks on the role of the metric factor

We wish first to present a class of homogeneous distances such that $\theta^g_\rho(\cdot)$ is constant.

Definition 6.1. Let $F : \mathbb{R}^q \longrightarrow \mathbb{G}$ be a system of graded coordinates and let $x = (y_1, \ldots, y_\iota)$ belong to $\mathbb{R}^q$, where $y_j \in \mathbb{R}^{n_j}$ and $n_j = \dim V_j$ for every $j = 1, \ldots, \iota$. We say that a homogeneous distance ρ on $\mathbb{G}$ is *symmetric on all layers* if there exists $\omega : \mathbb{R}^q \longrightarrow \mathbb{R}$ only depending on the Euclidean norms of y_j such that

$$\rho(x,e) = \omega\left(F^{-1}(x)\right),$$

where $x \in \mathbb{G}$ and e is the unit element of $\mathbb{G}$.

Proposition 6.1. *Let ν, μ be in $\Lambda_k(V_1)$ and let ρ be a homogeneous distance, which is symmetric on all layers. Then $\theta^g_\rho(\nu) = \theta^g_\rho(\mu)$.*

Proof. Assume that $\nu = U_1 \wedge U_2 \wedge \cdots \wedge U_k$ and $\mu = W_1 \wedge W_2 \wedge \cdots \wedge W_k$, where $U_i, W_i \in V_1$ for every $i = 1, \ldots, k$. Both orthogonal spaces $\mathcal{L}(\nu)$ and $\mathcal{L}(\mu)$ can be written as

$$\mathcal{L}(\nu) = T_1 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_\iota \qquad \text{and} \qquad \mathcal{L}(\mu) = T_2 \oplus V_2 \oplus V_3 \oplus \cdots \oplus V_\iota \,,$$

where the $(m-k)$-dimensional subspaces T_1 and T_2 are orthogonal to $\text{span}\{U_1, U_2, \ldots, U_k\}$ and to $\text{span}\{W_1, W_2, \ldots, W_k\}$ in V_1, respectively. Let us define $S_j = F^{-1}\left(\exp\left(T_j\right)\right) \subset \mathbb{R}^q$ and consider an Euclidean isometry $l : \mathbb{R}^q \longrightarrow \mathbb{R}^q$ such that $l(S_1) = S_2$, $l(S_1^\perp) = S_2^\perp$ and $F \circ l \circ F^{-1} : \exp(V_2 \oplus \cdots \oplus V_\iota) \longrightarrow \exp(V_2 \oplus \cdots \oplus V_\iota)$ equal to the identity mapping. Defining $L = F \circ l \circ F^{-1} : \mathbb{G} \longrightarrow \mathbb{G}$ we have $L\left(\exp\left(\mathcal{L}(\nu)\right)\right) = \exp\left(\mathcal{L}(\mu)\right)$ and

$$\rho\big(L(x), e\big) = \omega\left(F^{-1}\big(L(x)\big)\right) = \omega\left(l\big(F^{-1}(x)\big)\right) = \omega\big(F^{-1}(x)\big) = \rho\big(x, e\big).$$

This implies both inclusions $L(B_1) \subset B_1$ and $L^{-1}(B_1) \subset B_1$, then $L(B_1) = B_1$, where $B_1 = \{p \in \mathbb{G} \mid \rho(p, e) = \omega(F^{-1}(p)) < 1\}$. As a consequence, it follows that

$$F^{-1}\left(\exp \mathcal{L}(\mu) \cap B_1\right) = F^{-1} \circ L\left(\exp\left(\mathcal{L}(\nu)\right) \cap B_1\right)$$

and we notice that $F^{-1} \circ L \circ F = l$ is an Euclidean isometry. Then the definition of metric factor concludes the proof. □

Example 6.1. An interesting example of homogeneous distance which is symmetric on all layers is given in Theorem 5.1 of [16], where

$$\omega(x) = \max_{j=1,\ldots,\iota} \{\varepsilon_j \, |(x^{m_{j-1}+1}, \ldots, x^{m_j})|^{1/j}\}, \tag{52}$$

$y_j = (x^{m_{j-1}+1}, \ldots, x^{m_j})$, $m_j = \sum_{s=1}^{j} \dim V_s$ and ε_j are suitably small constants, depending only on the group. Then the corresponding distance $d_\infty(p, s) = \omega\big(F^{-1}(p^{-1}s)\big)$ has constant metric factor with respect to all horizontal k-vectors.

Other examples arise from some classes of H-type groups, including Heisenberg groups, where the metric factor on horizontal vectors with respect to the Carnot-Carathéodory distance is constant, [24].

Now we recall that in the introduction we have used the notation $\mathcal{S}_{\mathbb{G}}^{Q-k} = \alpha\, \mathcal{S}^{Q-k}$, where it has been assumed that ρ has constant metric factor and then

$$\alpha = \theta_\rho^g(\nu) \quad \text{for every} \quad \nu \in \Lambda_k(V_1).$$

Due to formula (4) it is easy to realize a rather surprising phenomenon. In fact, the measure $\mathcal{S}_{\mathbb{G}}^{Q-k} \llcorner \Sigma$ does not depend on the homogeneous distance ρ used to construct it, since in the right hand side of (4) the distance ρ does not appear. This means that the metric factor in a sense really makes the measure

$$\theta_\rho^g\big(\mathbf{n}_H(\cdot)\big)\, \mathcal{S}^{Q-k} \llcorner \Sigma$$

more intrinsic, compare with [31]. As an easy example to test this fact, we simply consider two homogeneous distances ρ and $\overline{\rho} = \lambda\, \rho$, where we have fixed $\lambda > 0$. Then it is easy to observe that

$$\mathcal{S}_{\overline{\rho}}^{Q-k} = \lambda^{Q-k} \mathcal{S}_\rho^{Q-k}. \tag{53}$$

We consider the formula

$$\theta_{\overline{\rho}}^g\big(\mathbf{n}_H(x)\big) = \mathcal{H}^p\Big(F^{-1}\big(B_1^{\overline{\rho}} \cap N(\mathbf{n}_H(x))\big)\Big)$$

and notice that $B_1^{\overline{\rho}} = \delta_{1/\lambda}\big(B_1^\rho\big)$. The key observation is that the restriction $\tilde\delta_r$ to the vertical subgroup $N\big(\mathbf{n}_H(x)\big)$ has Jacobian equal to r^{Q-k}. In other words, the simple p-vector associated with the vertical subalgebra $\mathcal{L}\big(\mathbf{n}_H(x)\big)$ has degree $Q-k$. Then we have

$$F^{-1}\big(B_1^{\overline{\rho}}\cap N(\mathbf{n}_H(x))\big) = \tilde\delta_{1/\lambda}\Big(F^{-1}\big(B_1^{\rho}\cap N(\mathbf{n}_H(x))\big)\Big)$$

that implies

$$\theta^g_{\overline{\rho}}(\mathbf{n}_H(x)) = \frac{1}{\lambda^{Q-k}}\,\theta^g_{\rho}(\mathbf{n}_H(x)). \tag{54}$$

Taking into account (53) and (54), we have shown that

$$\theta^g_{\overline{\rho}}(\mathbf{n}_H(x))\,\mathcal{S}^{Q-k}_{\overline{\rho}}\llcorner\Sigma = \theta^g_{\rho}(\mathbf{n}_H(x))\,\mathcal{S}^{Q-k}_{\rho}\llcorner\Sigma$$

for every non-horizontal submanifold Σ.

Example 6.2. Let us consider the stratified group $\mathbb{G}=\mathbb{H}^1\times\mathbb{H}^1$ that is the direct product of two Heisenberg groups. Let represent $\mathbb{G}$ through the system of graded coordinates (x_1,x_2,y_1,y_2,t,τ) with respect to the vector fields

$$\begin{array}{lll} X_1=\partial_{x_1}-x_2\partial_t, & X_2=\partial_{x_2}+x_1\partial_t, & T=\partial_t, \\ Y_1=\partial_{y_1}-y_2\partial_\tau, & Y_2=\partial_{y_2}+y_1\partial_\tau, & Z=\partial_\tau\,. \end{array}$$

Notice that here only 1-dimensional submanifolds are always horizontal. We consider a function $h\in C^1(\mathbb{R}^2)$ such that $h(0,0)=1$. Then we consider the surface parametrized by

$$\Phi(u,v)=\big(u,h(u,v),u,v,u,0\big)\,.$$

We compute the partial derivatives, getting

$$\left\{\begin{array}{l} \Phi_u(u,v)=\Big(\partial_{x_1}+h_u\,\partial_{x_2}+\partial_{y_1}+\partial_t\Big)_{|\Phi(u,v)} \\ \Phi_v(u,v)=\Big(h_v\partial_{x_2}+\partial_{y_2}\Big)_{|\Phi(u,v)} \end{array}\right.$$

therefore, using the expressions of the left invariant vector fields, we obtain

$$\left\{\begin{array}{l} \Phi_u(u,v)=X_1+Y_1+(h-uh_u+1)T+h_uX_2+vZ \\ \Phi_v(u,v)=h_vX_2+Y_2-uT-uZ \end{array}\right. . \tag{55}$$

The homogeneous dimension Q of $\mathbb{H}^1\times\mathbb{H}^1$ is 8 and the codimension k of $\Phi(\mathbb{R}^2)$ is 4. Then we consider

$$\big(\Phi_u\wedge\Phi_v\big)_4 = u\,(h+1-uh_u-v)\,Z\wedge T$$

Then our assumption on h implies that any neighbourhood in $\Phi(\mathbb{R}^2)$ of $\Phi(0)$ clearly contains a non-horizontal point. Then the C^1 submanifold $\Sigma=\Phi(\mathbb{R}^2)$ is non-horizontal and we have the formula

$$\mathcal{S}^4_{\mathbb{H}^1\times\mathbb{H}^1}\llcorner\Sigma = \Phi_\#\Big(|u\,(h+1-uh_u-v)|\,\mathcal{L}^2\Big)$$

where $\mathcal{L}^2$ denotes the Lebesgue measure of $\mathbb{R}^2$ and we have fixed a precise homogeneous distance ρ in $\mathbb{H}^1\times\mathbb{H}^1$ with constant metric factor.

7. Proof of the coarea formula

The proof of coarea formula (5) essentially follows the same steps of [28], where the domain of the mapping is an Heisenberg group. For the sake of the reader, in this section we wish to present its proof with the appropriate definitions extended to this more general context, where the domain is a stratified group and the auxiliary metric $\tilde{g}$ is also taken into account.

Definition 7.1 (Horizontal Jacobian)**.** Let $f : \Omega \longrightarrow \mathbb{R}^k$ be a mapping of class C^1 defined on an open subset $\Omega \subset \mathbb{G}$ and let $x \in \Omega$. The *horizontal Jacobian* of f at x is defined by

$$J_{g,H} f(x) = \left| \pi_{g,H} \left(\nabla f^1(x) \wedge \cdots \wedge \nabla f^k(x) \right) \right| .$$

Proposition 7.1. *Let Σ be a submanifold of $\mathbb{G}$ and let $f : U \longrightarrow \mathbb{R}^k$ be a C^1 mapping defined on a neighbourhood U of $x \in \Sigma$ such that $U \cap \Sigma = U \cap f^{-1}(0)$. We have the formula*

$$J_{\tilde{g}} f(x) \, |\tilde{\mathbf{n}}_{g,H}(x)| = J_{g,H} f(x). \tag{56}$$

Proof. One of the two possible horizontal k-normals $\tilde{\mathbf{n}}(x)$ with respect to $\tilde{g}$ can be defined as follows

$$\tilde{\mathbf{n}}(x) = \frac{\nabla_{\tilde{g}} f^1(x) \wedge \cdots \wedge \nabla_{\tilde{g}} f^k(x)}{|\nabla_{\tilde{g}} f^1(x) \wedge \cdots \wedge \nabla_{\tilde{g}} f^k(x)|_{\tilde{g}}}.$$

Then we have

$$|\tilde{\mathbf{n}}_{g,H}(x)| = \left| \pi_{g,H} \left((g_k^*)^{-1} \tilde{g}_k^* (\tilde{\mathbf{n}}(x)) \right) \right| = \frac{\left| \pi_{g,H} \left((g_k^*)^{-1} \left(df^1(p) \wedge \cdots \wedge df^k(x) \right) \right) \right|}{|\nabla_{\tilde{g}} f^1(x) \wedge \cdots \wedge \nabla_{\tilde{g}} f^k(x)|_{\tilde{g}}}$$

$$= \frac{\left| \pi_{g,H} \left(\nabla f^1(x) \wedge \cdots \wedge \nabla f^k(x) \right) \right|}{|\nabla_{\tilde{g}} f^1(x) \wedge \cdots \wedge \nabla_{\tilde{g}} f^k(x)|_{\tilde{g}}} = \frac{J_{g,H} f(x)}{J_{\tilde{g}} f(x)} .$$

This concludes the proof. □

Proof of Theorem 1.2. We first recall the following Riemannian coarea formula from Section 13.4 of [6]:

$$\int_{\mathbb{G}} u(x) \; J_{\tilde{g}} f(x) \, d\mathrm{vol}_{\tilde{g}}(x) = \int_{\mathbb{R}^k} \left(\int_{f^{-1}(t)} u(y) \, d\tilde{\mu}_p(y) \right) dt, \tag{57}$$

where $f : \mathbb{G} \longrightarrow \mathbb{R}^k$ is a Riemannian locally Lipschitz mapping, $u : \mathbb{G} \longrightarrow \mathbb{R}$ is a summable function and $\tilde{\mu}_p$ is the p-dimensional Riemannian surface measure with respect to $\tilde{g}$ restricted to $f^{-1}(t)$. Notice that for a.e. $t \in \mathbb{R}^k$ the measure $\tilde{\mu}_p$ is well defined on $f^{-1}(t)$, since the set of singular points in $f^{-1}(t)$ is $\tilde{\mu}_p$-negligible. We first suppose that $f : \mathbb{G} \longrightarrow \mathbb{R}^k$ is of class C^1. Let Ω be a bounded open set of $\mathbb{G}$ and consider the summable function

$$u_0(x) = \mathbf{1}_{\{y \in \Omega | J_{g,H} f(y) \neq 0\}}(x) \; \frac{J_{g,H} f(x)}{J_{\tilde{g}} f(x)}.$$

If the subset $f^{-1}(t) \cap \{y \in \Omega \mid J_{g,H} f(y) \neq 0\}$ is nonempty, then it is a C^1 submanifold. Thus, taking into account (56) and applying (57) to u_0, we get

$$\int_{\Omega} J_{g,H} f(x) \, d\mathrm{vol}_{\tilde{g}}(x) = \int_{\mathbb{R}^k} \left(\int_{f^{-1}(t) \cap \{y \in \Omega | J_{g,H} f(y) \neq 0\}} |\tilde{\mathbf{n}}_{g,H}(y)| \, d\tilde{\mu}_p(y) \right) dt.$$

Multiplying the previous equation by $c(g,\tilde g)$ and taking into account (3), we get

$$\int_\Omega J_{g,H}f(x)\,d\mathrm{vol}_g(x)=\int_{\mathbb{R}^k}\mathcal{S}^{Q-k}_{\mathbb{G}}\Big(f^{-1}(t)\cap\{y\in\Omega\mid J_{g,H}f(y)\neq0\}\Big)dt.$$

Notice that we can apply (3), since the set

$$f^{-1}(t)\cap\{y\in\Omega\mid J_{g,H}f(y)\neq0\},$$

when nonempty, is a non-horizontal submanifold. This follows from formula (56) and taking into account both Proposition 3.2 and Proposition 3.3. Due to the generallized Sard type theorem of [23], we get

$$\int_\Omega J_{g,H}f(x)\,d\mathrm{vol}_g(x)=\int_{\mathbb{R}^k}\mathcal{S}^{Q-k}_{\mathbb{G}}\big(f^{-1}(t)\cap\Omega\big)dt.$$

In the previous assertion, we have used the fact that the Pansu differential of f at x considered in [23] is surjective if and only if $J_{g,H}f(x)\neq0$. The previous formula can be extended to all measurable sets E of $\mathbb{G}$ as follows

$$\int_E J_{g,H}f(x)\,d\mathrm{vol}_g(x)=\int_{\mathbb{R}^k}\mathcal{S}^{Q-k}_{\mathbb{G}}\big(f^{-1}(t)\cap E\big)dt. \tag{58}$$

In fact, the Eilenberg inequality 2.10.25 of [11] and the outer approximation of bounded measurable sets by open sets show the validity of (58) for all bounded measurable sets. Its extension to all measurable sets is then obtained by the Beppo Levi convergence theorem.

Now, we consider the Lipschitz mapping f defined on a measurable set A. Then we extend it to a Lipschitz mapping $\tilde f$ defined on all of $\mathbb{G}$. Let $\varepsilon>0$ be arbitrarily fixed and apply the classical Whitney's extension theorem, see for instance 3.1.15 of [11], according to which there exists a C^1 mapping $\tilde f_1:\mathbb{G}\longrightarrow\mathbb{R}^k$ such that the open set

$$O=\{x\in\mathbb{G}\mid f_1(x)\neq\tilde f_1(x)\}$$

has volume measure vol_g less than or equal to ε. Notice that vol_g in a system of graded coordinates corresponds to the Lebesgue measure up to a factor. In view of definition of horizontal Jacobian, we have

$$J_{g,H}f(x)\le C\prod_{j=1}^k|\nabla f^j|\le C\,\mathrm{Lip}(f)^k.$$

for a.e. $x\in A$, where $\mathrm{Lip}(f)$ is the Riemannian Lipschitz constant of f. It follows that

$$\int_{A\cap O}J_{g,H}f(x)\,d\mathrm{vol}_g(x)\le C\,\mathrm{Lip}(f)^k\,\varepsilon. \tag{59}$$

By Eilenberg inequality 2.10.25 of [11], we have

$$\int_{\mathbb{R}^k}\mathcal{S}^{Q-k}(f^{-1}(t)\cap O)\,dt\le\overline{C}\,\mathrm{Lip}(f)^k\,\varepsilon \tag{60}$$

for some geometric constant $\overline{C}$. Applying (58) to the subset $E=A\setminus O$

$$\int_{A\setminus O}J_{g,H}f(x)\,d\mathrm{vol}_g(x)=\int_{\mathbb{R}^k}\mathcal{S}^{Q-k}_{\mathbb{G}}\big(f^{-1}(t)\setminus O\big)dt. \tag{61}$$

As a consequence, estimates (59), (60) along with equality (61) give

$$\left|\int_A J_{g,H}f(x)\,d\mathrm{vol}_g(x)-\int_{\mathbb{R}^k}\mathcal{S}_{\mathbb{G}}^{Q-k}\big(f^{-1}(t)\big)\,dt\right|\leq\big(C\,\mathrm{Lip}(f)^k+\overline{C}\,\mathrm{Lip}(f)^k\big)\,\varepsilon. \tag{62}$$

Arbitrary choice of ε proves the formula

$$\int_A J_{g,H}f(x)\,d\mathrm{vol}_g(x)=\int_{\mathbb{R}^k}\mathcal{S}_{\mathbb{G}}^{Q-k}\big(f^{-1}(t)\big)\,dt \tag{63}$$

It is clearly not restrictive considering $u\geq 0$ a.e. Thus, our claim is achieved taking an increasing sequence of step functions a.e. converging to let u, since one can apply the Beppo Levi convergence theorem. □

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