Kauffman brackets on surfaces

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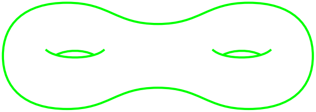
Geometric Topology in Cortona, June 2013

Joint work with Helen Wong

Joint work with Helen Wong (busy with another project)

Grace Tsapsie Hibbard, born March 22, 2013

S = closed oriented surface of genus $g \geqslant 0$



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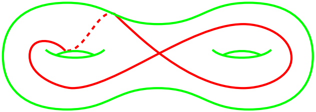

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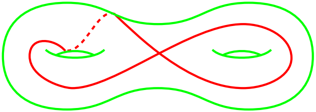

A group homomorphism $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ defines its *character*

$$\begin{aligned}\mathcal{K}_\rho\colon \{\text{closed curves in } S\} &\longrightarrow \mathbb{C}\\ K \qquad\qquad &\longmapsto \operatorname{Tr}\rho(K)\end{aligned}$$

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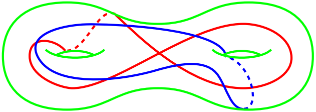

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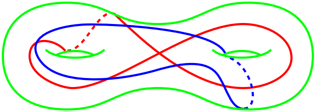
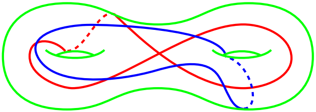
$$\mathcal{K}_\rho\colon \{\text{closed multicurves in } S\} \longrightarrow \mathbb{C}$$

$$K = \bigcup_{i=1}^{n} K_i \qquad \longmapsto (-1)^n \prod_{i=1}^{n} \mathrm{Tr}\,\rho(K_i)$$



Theorem (Helling 1967)

A function $\mathcal{K}\colon \{$closed multicurves in $S\} \longrightarrow \mathbb{C}$ is the character of a group homomorphism $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$ if and only if:



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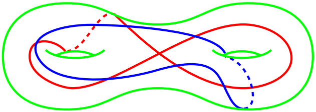

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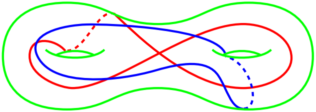

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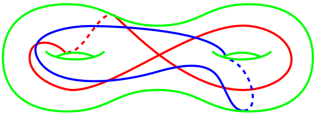

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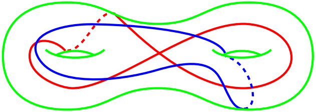
The Skein Relation just rephrases the classical trace relation of $\mathrm{SL}_2(\mathbb{C})$:

$$\mathrm{Tr}\, M\ \mathrm{Tr}\, N = \mathrm{Tr}\, MN + \mathrm{Tr}\, MN^{-1}, \quad \forall M, N \in \mathrm{SL}_2(\mathbb{C})$$



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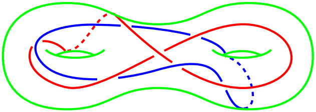

Definition

For $q = \mathrm{e}^{2\pi \mathrm{i} \hbar} \in \mathbb{C} - \{0\}$, *a Kauffman q–bracket is a function*

$$\mathcal{K} \colon \{\textit{framed links in } S \times [0,1]\} \longrightarrow \mathrm{End}(E)$$

for a finite-dimensional vector space E*, such that:*

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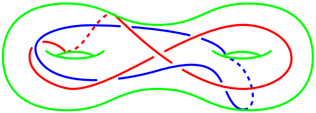

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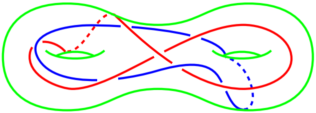

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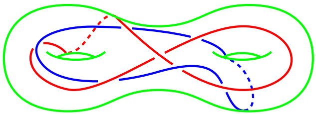

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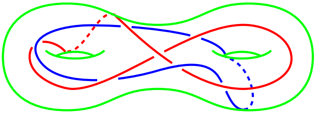

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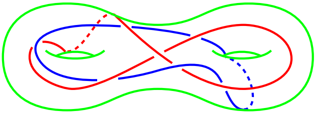

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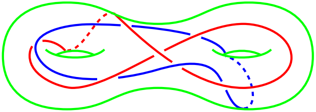

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Goal of this talk: Construct other examples of Kauffman brackets

When $q = 1$ and $q^{\frac{1}{2}} = -1$, a Kauffman 1–bracket is the same thing as an $\mathrm{SL}_2(\mathbb{C})$–character, namely as a point of the *character variety*

$$\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S) = \{\text{homomorphisms } \rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})\} /\!\!/ \mathrm{SL}_2(\mathbb{C})$$

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Turaev (1987), Frohman, Bullock, Kania-Bartoszýnska, Przytycki, Sikora (around 2000), Charles, Marché:
Interpretation of a Kauffman q–bracket as a "point" in a quantization of the character variety $\mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$, namely as a quantum $\mathrm{SL}_2(\mathbb{C})$–character.

How to construct a group homomorphism $\rho\colon \pi_1(S) \to \mathrm{SL}_2(\mathbb{C})$?

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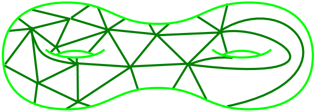

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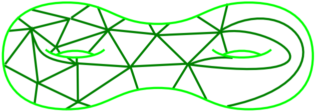

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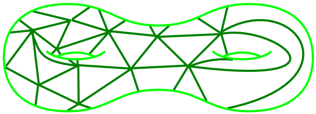

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This defines a pleated surface with shear-bend coordinates x_i, and with monodromy $\rho\colon \pi_1(S - \mathcal{V}_\Gamma) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\pm\mathrm{Id}$

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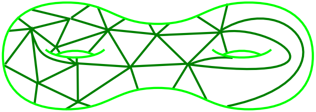

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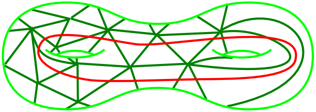

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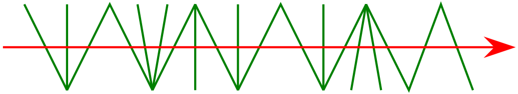
Main Point: The construction is classical and, for a curve $K \subset S - \mathcal{V}_\Gamma$, gives a very explicit formula for $\mathrm{Tr}\,\rho(K)$

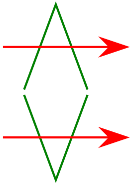
More precisely, if K crosses the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_n}$,



$$\begin{aligned}\mathrm{Tr}\,\rho(K) &= \pm\mathrm{Tr}\left[M_1\begin{pmatrix} x_{i_1}^{\frac12} & 0 \\ 0 & x_{i_1}^{-\frac12}\end{pmatrix} M_2\begin{pmatrix} x_{i_2}^{\frac12} & 0 \\ 0 & x_{i_2}^{-\frac12}\end{pmatrix}\ldots M_n\begin{pmatrix} x_{i_n}^{\frac12} & 0 \\ 0 & x_{i_n}^{-\frac12}\end{pmatrix}\right] \\ &= \pm\sum_{\pm\pm\cdots\pm}(0 \text{ or } 1)\, x_{i_1}^{\pm\frac12} x_{i_2}^{\pm\frac12}\ldots x_{i_n}^{\pm\frac12}\end{aligned}$$

where

$$M_k = \begin{cases}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} & \text{if} \\ \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} & \text{if}\end{cases}$$



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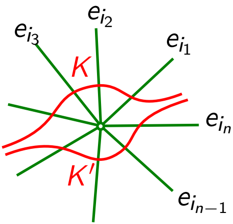

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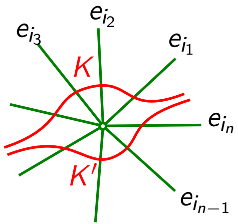

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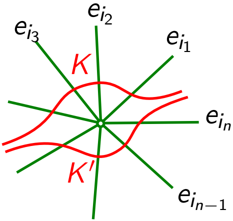

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Fact

The edge weights x_i define an $\mathrm{SL}_2(\mathbb{C})$–character on the closed surface S if and only if, for every vertex,

$$\begin{cases} x_{i_1}^{\frac{1}{2}} x_{i_1}^{\frac{1}{2}} \dots x_{i_1}^{\frac{1}{2}} = -1 \\ \\ 1 + x_{i_1} + x_{i_1}x_{i_2} + x_{i_1}x_{i_2}x_{i_3} + \cdots + x_{i_1}x_{i_2}\dots x_{i_{n-1}} = 0 \end{cases}$$

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namely such that:

- (Isotopy Invariance) $\mathcal{K}(K)$ depends only on the isotopy class of K in $S \times [0,1]$
- (Superposition Rule) $\mathcal{K}(K_1 \cup K_2) = \mathcal{K}(K_1) \circ \mathcal{K}(K_2)$ whenever $K = K_1 \cup K_2$ with $K_1 \subset S \times [0, \frac{1}{2}]$ and $K_2 \subset S \times [\frac{1}{2}, 1]$
- (Skein Relation) $\mathcal{K}(K_1) = q^{\frac{1}{2}} \mathcal{K}(K_0) + q^{-\frac{1}{2}} \mathcal{K}(K_\infty)$ if K_1, K_0, K_∞ are the same everywhere, except in a small box where $K_1 =$ , $K_0 =$  and $K_\infty =$ 

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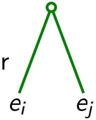

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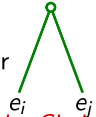

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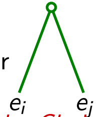

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Proposition (FB + Xiaobo Liu, 2007, relatively easy)

If $q^N = 1$ with N odd, smallest dimensional choices of such operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$ are classified by

- *edge weights $x_i \in \mathbb{C}^*$ such that $X_i^{\frac{N}{2}} = x_i^{\frac{1}{2}} \,\mathrm{Id}_E$*
- *choices of N–roots for numbers $x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \dots x_{i_n}^{\frac{1}{2}} \in \mathbb{C}^*$ associated to the vertices*

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Theorem (FB + Helen Wong, 2011)

Given operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$ *associated to the edges of the triangulation* Γ *as in Step 1, there is an explicit formula*

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that defines a Kauffman q–bracket

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FB + Qingtao Chen, 2013 More conceptual approach based on the representation theory of the quantum group $\mathrm{U}_q(\mathfrak{sl}_2)$

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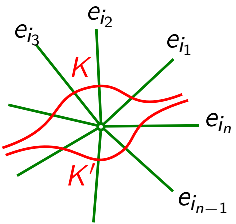

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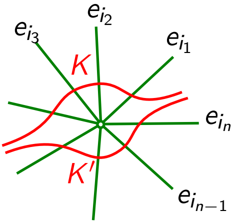

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Step 3a. If $x_{i_1}^{\frac{1}{2}} x_{i_2}^{\frac{1}{2}} \dots x_{i_n}^{\frac{1}{2}} = -1$ at a vertex, the corresponding operators $X_i^{\frac{1}{2}} \in \mathrm{End}(E)$ can be chosen so that

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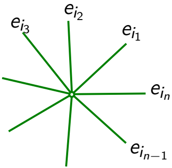
Step 3b. For a vertex $v =$ 

of the triangulation Γ for the operators $X_{i_j}^{\frac{1}{2}} \in \mathrm{End}(E)$ associated to the edges, consider

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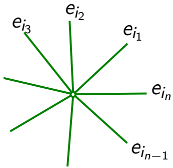
Step 3b. For a vertex $v =$ 

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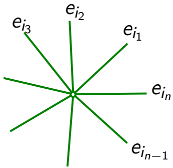
Step 3b. For a vertex $v =$ 

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$$F = \bigcap_{\text{vertices v}} F_v \subset E$$

Theorem

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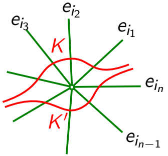

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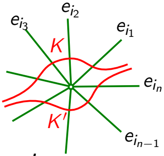

Corollary

$\mathcal{K}$ induces a Kauffman q–bracket
$$\bar{\mathcal{K}}\colon \{\textit{framed links in } S \times [0,1]\} \longrightarrow \mathrm{End}(F)$$
for the closed surface S

Proposition

$$\dim F \geqslant \begin{cases} N^{3(g-1)} & \textit{if } g \geqslant 2 \\ N & \textit{if } g = 1 \\ 1 & \textit{if } g = 0 \end{cases}$$

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depends only on the (classical) $\mathrm{SL}_2(\mathbb{C})$*–character* $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{SL}_2(\mathbb{C})}(S)$ *associated to the same edge weights* $x_i \in \mathbb{C}^*$*. In particular, it is independent of the triangulation* Γ

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\begin{speculations}

 

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- a vector space Z_S^{KBB} to each surface S endowed with a triangulation Γ and edge weights $x_i \in \mathbb{C}^*$ satisfying

$$\begin{cases} x_{i_1} x_{i_2} \dots x_{i_n} = 1 \\ 1 + x_{i_1} + x_{i_1} x_{i_2} + x_{i_1} x_{i_2} x_{i_3} + \dots + x_{i_1} x_{i_2} \dots x_{i_{n-1}} = 0 \end{cases}$$

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- a linear map $Z_{(M,L)}^{\mathrm{KBB}} \colon Z_{S_1}^{\mathrm{KBB}} \to Z_{S_2}^{\mathrm{KBB}}$ to each 3–dimensional cobordism M from S_1 to S_2, endowed with a framed link $L \subset M$ and an $\mathrm{PSL}_2(\mathbb{C})$–character $\mathcal{K}_\rho \in \mathcal{R}_{\mathrm{PSL}_2(\mathbb{C})}(M)$ compatible with the boundary data (plus a little more topological data)

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Good point Well behaved under cut-and-paste

Fact In our construction of a Kauffman q–bracket, the operators X_i associated to the edges of the triangulation can be chosen to to be in $\mathrm{End}(Z_S^{\mathrm{KBB}})$

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Proposal Define $F_S \subset Z_S^{\mathrm{KBB}}$ as the set of vectors $v \in Z_S^{\mathrm{KBB}}$ such that, for every vertex of Γ,

$$\begin{cases} X_{i_1}^{\frac{1}{2}} X_{i_2}^{\frac{1}{2}} \dots X_{i_n}^{\frac{1}{2}}(v) = -q^{\frac{n+2}{4}} v \\ \left(1+qX_{i_1}+q^2X_{i_1}X_{i_2}+q^3X_{i_1}X_{i_2}X_{i_3}+\cdots+q^{n-1}X_{i_1}X_{i_2}\dots X_{i_{n-1}}\right)(v) = 0 \end{cases}$$

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This would improve the Kashaev-Baseilhac-Benedetti TQFT, by making it more independent of choices (while maintaining the good behavior under cut-and-paste)

But we cannot improve Riccardo!

But we cannot improve Riccardo! He's perfect

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Happy Birthday!!