Geometrisation of three-manifolds

Bruno Martelli

17 november 2016

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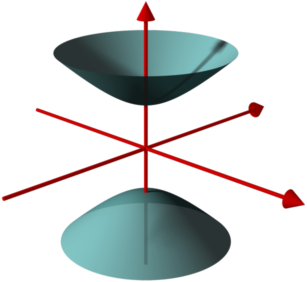

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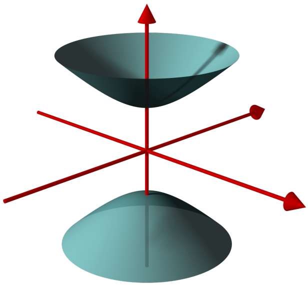

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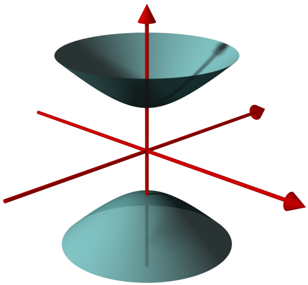

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More precisely: some finite-index subgroup of Γ.

More models of the hyperbolic space $\mathbb{H}^n$:

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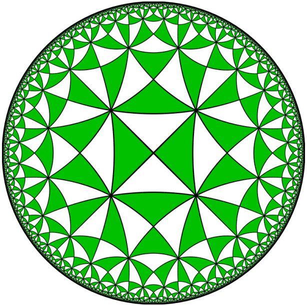

Disc model

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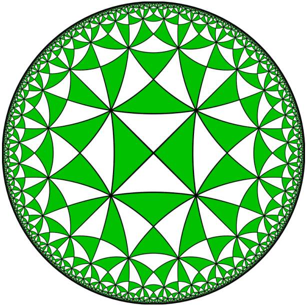

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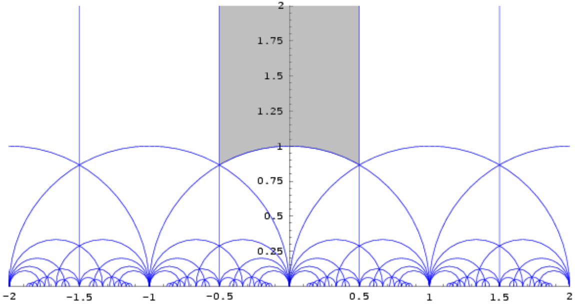
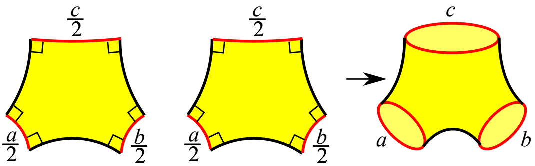


Half-space model

Pictures created by Claudio Rocchini and Kilom691

Hyperbolic pairs-of-pants:



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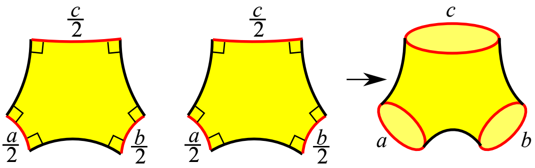
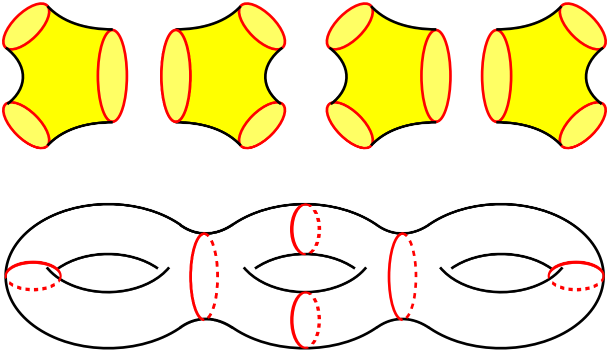
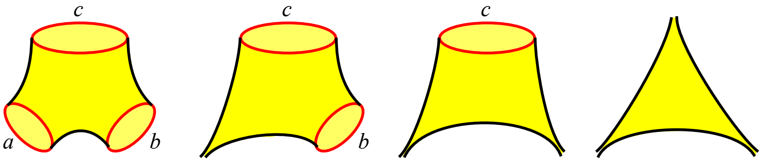


Hyperbolic surfaces:



Pants with cusps:



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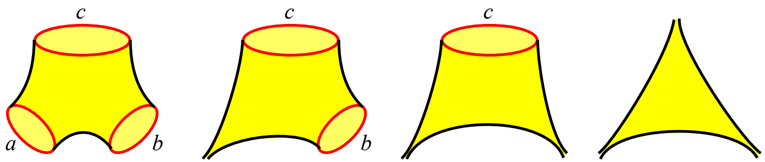
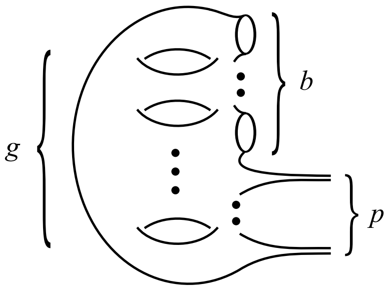
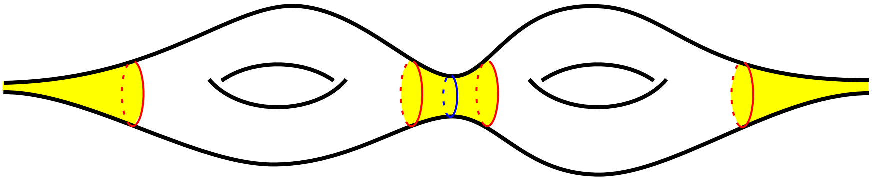


Surfaces of finite type, possibly with geodesic boundary and/or cusps:



Thick-thin decomposition for finite-volume hyperbolic manifolds:



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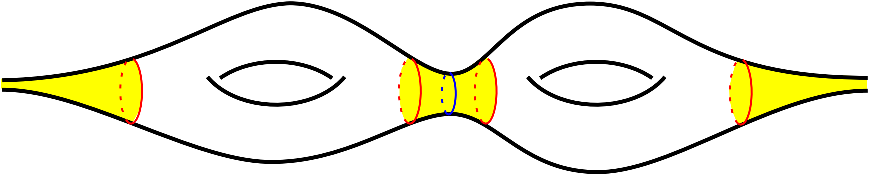

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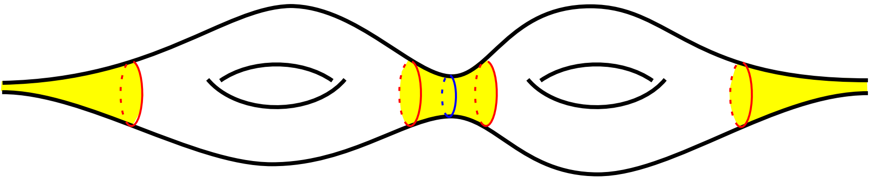

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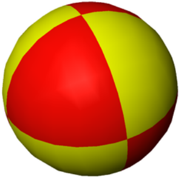
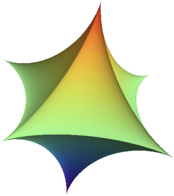
$$\mathrm{Vol}(\mathrm{cusp}) = \frac{\mathrm{Vol}(M)}{n-1}.$$

Regular polyhedra:





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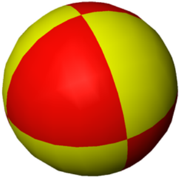
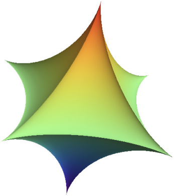
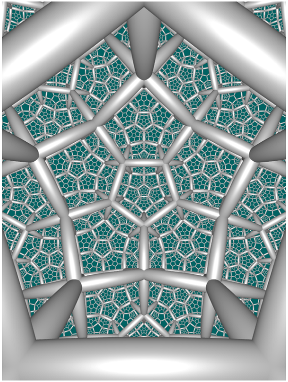
  

polyhedron	$\theta = \frac{\pi}{3}$	$\theta = \frac{2\pi}{5}$	$\theta = \frac{\pi}{2}$	$\theta = \frac{2\pi}{3}$
tetrahedron	ideal $\mathbb{H}^3$	S^3	S^3	S^3
cube	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{R}^3$	S^3
octahedron			ideal $\mathbb{H}^3$	S^3
icosahedron				$\mathbb{H}^3$
dodecahedron	ideal $\mathbb{H}^3$	$\mathbb{H}^3$	$\mathbb{H}^3$	S^3

The right picture was created by Win

Every regular polyhedron with dihedral angle $\theta = \frac{2\pi}{n}$ gives rise to a tessellation of $S^3, \mathbb{R}^3$, or $\mathbb{H}^3$:



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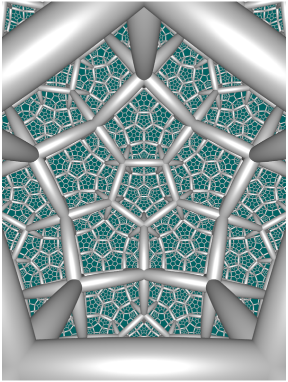

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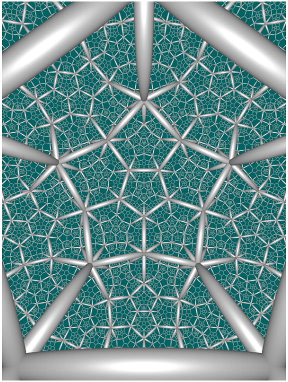

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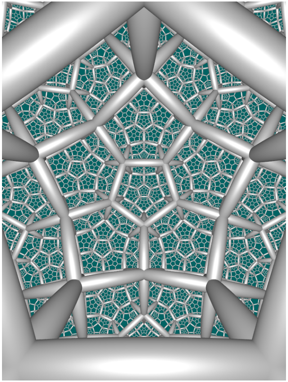
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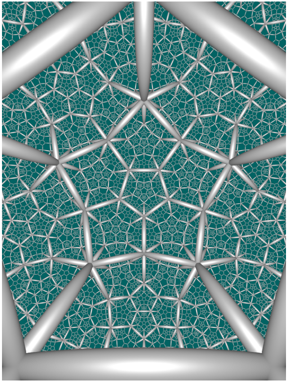
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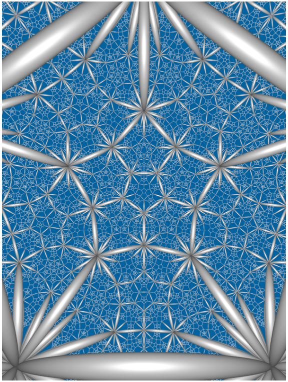
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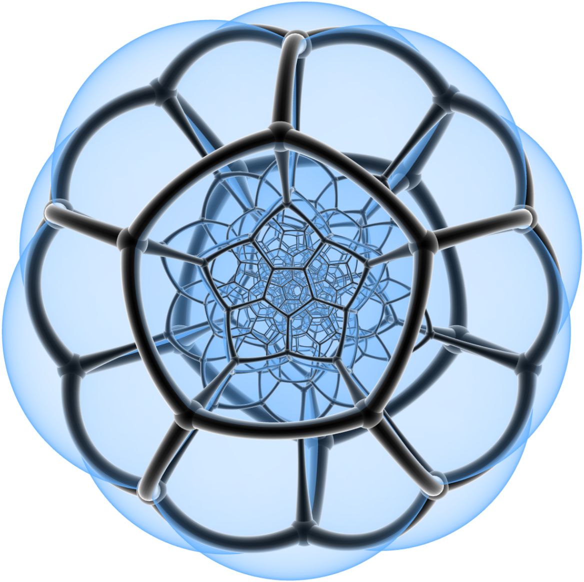
Pictures created by Roice3



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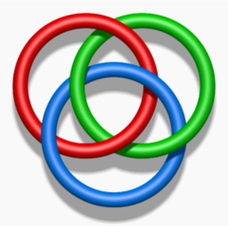
The complements in S^3 of the *figure-eight knot* and the *borromean link* are hyperbolic:



They decompose in regular ideal octahedra and tetrahedra, respectively.

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- There are 8 types of such metrics:

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- (Mostow rigidity) The hyperbolic metric is unique.

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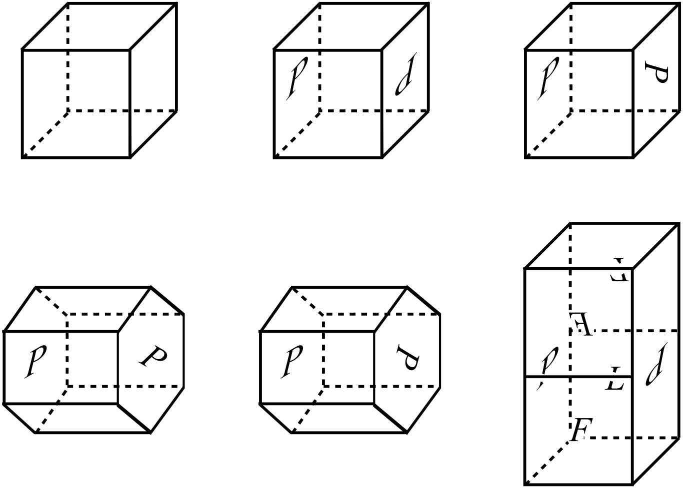

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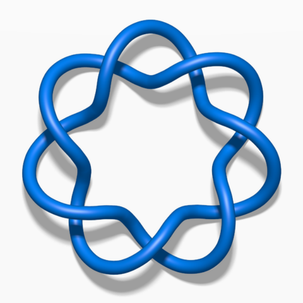

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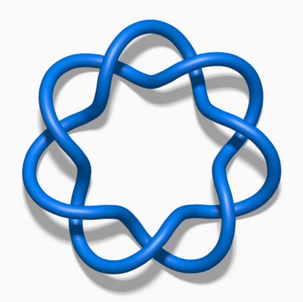

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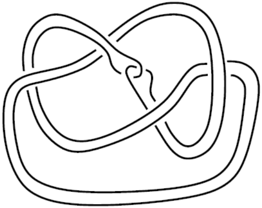

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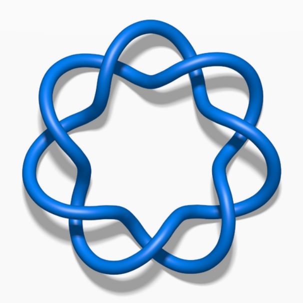

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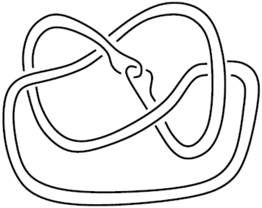

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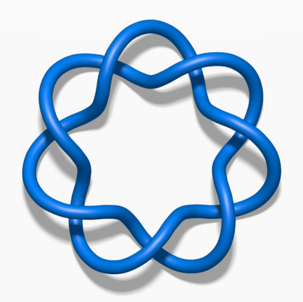

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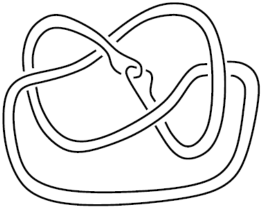

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crossings	3	4	5	6	7	8	9	10	11	12	13	14
toric	1	0	1	0	1	1	1	1	1	0	1	1
satellite	0	0	0	0	0	0	0	0	0	0	2	2
hyperbolic	0	1	1	3	6	20	48	164	551	2176	9985	46969

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- Hyperbolisation: $|\pi_1(M)| = \infty$, indecomposable and without $\mathbb{Z} \times \mathbb{Z}$ $\Longrightarrow M = \mathbb{H}^3/_\Gamma$ is hyperbolic.

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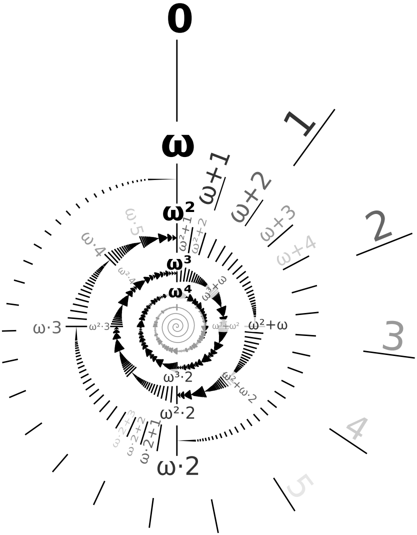

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- What is the role of hyperbolic geometry in dimension four?

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- There are many non-arithmetic hyperbolic manifolds [Gelander – Levit 14]

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