A general approach to Camacho-Sad-like index theorems

Marco Abate

Dipartimento di Matematica, Università di Pisa
Largo Pontecorvo 5, 56127 Pisa, Italy
abate@dm.unipi.it

December 2007

0. Introduction

In 1982, C. Camacho and P. Sad [CS] proved the existence of separatrices for singular holomorphic foliations in dimension 2. One of the main tools in their proof was the following index theorem:

Theorem 0.1: (Camacho-Sad) *Let S be a compact Riemann surface embedded in a smooth complex surface M. Let $\mathcal{F}$ be a one-dimensional singular holomorphic foliation defined in a neighbourhood of S and such that S is a leaf of $\mathcal{F}$. Then it is possible to associate to any singularity $q \in S$ of $\mathcal{F}$ a complex number $\iota_q(\mathcal{F}, S) \in \mathbb{C}$, the* index of the foliation along S at q, *depending only on the local behavior of $\mathcal{F}$ near q, such that*

$$\sum_{q \in \mathrm{Sing}(\mathcal{F})} \iota_q(\mathcal{F}, S) = \int_S c_1(N_S), \tag{0.1}$$

where $c_1(N_S)$ is the first Chern class of the normal bundle N_S of S in M.

The index can be defined as follows. Let $\mathcal{A} = \{(U_\alpha, z_\alpha)\}$ be an atlas of M *adapted* to S, that is such that $U_\alpha \cap S = \{z^1_\alpha = 0\}$ for all indices α such that $U_\alpha \cap S \neq \varnothing$. Locally, the foliation $\mathcal{F}$ is generated by a local vector field

$$X_\alpha = X^1_\alpha \frac{\partial}{\partial z^1_\alpha} + X^2_\alpha \frac{\partial}{\partial z^2_\alpha},$$

with $X^1_\alpha|_S \equiv 0$ because S is a leaf of $\mathcal{F}$; in particular, $q \in S$ is a singularity of $\mathcal{F}$ if and only if $X^2_\alpha(q) = 0$. Then the index is defined by

$$\iota_q(\mathcal{F}, S) = \mathrm{Res}_q \left(\left. \frac{\partial (X^1_\alpha / X^2_\alpha)}{\partial z^1_\alpha} \right|_S dz^2_\alpha \right); \tag{0.2}$$

it is not difficult to check that it is independent of the adapted chart chosen.

Thus Theorem 0.1 gives a quantitative connection between the global way S sits in M (the first Chern class of N_S is also equal to the self-intersection number $S \cdot S$ of S) and the local behavior of singular foliations tangent to S.

This theorem has been subsequently generalized in several ways, first allowing the existence of singularities of the Riemann surface S (Lins-Neto [Li], Suwa [S1]), and then to singular holomorphic foliations defined in a neighbourhood of a possibly singular subvariety S of a complex manifold M, without restrictions on the dimension of M or the codimension either of $\mathcal{F}$ or of S (see Lehmann [Le], Lehmann-Suwa [LS1, 2] and references therein). In particular, using Čech-de Rham cohomology Lehmann and Suwa (see [S2] for a systematic exposition) developed a very complete theory of indices for holomorphic foliations along a possibly singular leaf. One of their results is the following generalization of Theorem 0.1:

Theorem 0.2: (Lehmann-Suwa) *Let S be a compact connected subvariety of codimension m in a complex manifold M of dimension n. Assume that S is a locally complete intersection, and that there is a smooth vector bundle N over M extending the normal bundle N_S of (the regular part of) S in M (such a vector bundle N always exists if S is an hypersurface). Let $\mathcal{F}$ be a singular holomorphic foliation of rank p defined in a neighbourhood of S and leaving S invariant. Let $\{\Sigma_\lambda\}$ be the decomposition in connected components of the singular set $\mathrm{Sing}(S) \cup \big(\mathrm{Sing}(\mathcal{F}) \cap S\big)$. Finally, let φ be a polynomial of degree $d > n - p - m$. Then we can associate to each connected component Σ_λ a residue*

$$\mathrm{Res}_\varphi(\mathcal{F}, N_S; \Sigma_\lambda) \in H_{2n-2(m+d)}(\Sigma_\lambda; \mathbb{C}),$$

depending only on the local behavior of $\mathcal{F}$ near Σ_λ, such that

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\mathcal{F}, N_S; \Sigma_\lambda) = [S] \frown \varphi(N) \quad \textit{in } H_{2n-2(m+d)}(S;\mathbb{C}), \tag{0.3}$$

where $i_\lambda\colon \Sigma_\lambda \hookrightarrow S$ is the inclusion, and $\varphi(N)$ denotes the cohomology class obtained evaluating φ in the Chern classes of N.

In particular, when $n = 2$ and $m = p = d = 1$, we recover Theorem 0.1. Furthermore, if the singular component Σ_λ is an isolated point (and $d = n - m$), we can often compute $\mathrm{Res}_\varphi(\mathcal{F}, N_S; \Sigma_\lambda)$ using Grothendieck residues; see [S2] for details.

In these theorems, the subvariety S is always supposed to be invariant by the foliation. Recent results by Camacho [C], Camacho and Lehmann [CL], and by Camacho-Movasati-Sad [CMS] have opened a different, and somewhat unexpected, venue of research: it is possible to get index theorems also when S is transversal to the foliation $\mathcal{F}$, if one is willing to assume that S sits into the ambient manifold M in a particularly nice way. For instance, one might assume that M is the total space of a line bundle over S (identifying S with the zero section of M); see [C, CL]. For more general subvariety S, Camacho, Movasati and Sad proved the following

Theorem 0.3: (Camacho-Movasati-Sad) *Let S be a compact Riemann surface embedded in a smooth complex surface M. Assume that S is fibered embedded and 2-linearizable in M, that is that there exists an atlas $\mathcal{A} = \{(U_\alpha, z_\alpha)\}$ adapted to S such that the changes of coordinates satisfy*

$$\forall i \geq 1 \qquad \frac{\partial^i z^2_\beta}{\partial (z^1_\alpha)^i} \equiv 0, \tag{0.4}$$

and

$$\left.\frac{\partial^2 z^1_\beta}{\partial (z^1_\alpha)^2}\right|_S \equiv 0. \tag{0.5}$$

Let $\mathcal{F}$ be a singular holomorphic foliation of rank one, and let $\mathrm{Sing}_S(\mathcal{F})$ denote the set of points of S where $\mathcal{F}$ is (singular or) tangent to the normal direction spanned by $\partial/\partial z^1_\alpha$, which is well-defined because of (0.4)*. Assume that $\mathrm{Sing}_S(\mathcal{F}) \neq S$. Then*

$$\sum_{q\in \mathrm{Sing}_S(\mathcal{F})} \iota_q(\mathcal{F}, S) = \int_S c_1(N_S),$$

where $\iota_q(\mathcal{F}, S)$ is again defined by (0.2).

Thus if S is fibered embedded and 2-linearizable into M we recover the Camacho-Sad index formula without assuming that the foliation is tangent to S; the informations on the way S sits into M replace the informations on the way S is placed with respect to $\mathcal{F}$.

A further indication of the fact that these index theorems are not inside standard foliation theory only is given by what happens for holomorphic self-maps of a complex manifold.

In 2001, studying the local dynamics of holomorphic maps f tangent to the identity (that is defined in a neighbourhood of the origin in $\mathbb{C}^n$ and such that $f(O) = O$ and $df_O = \mathrm{id}$), we discovered an unexpected analogy between the problem of the existence of separatrices for singular holomorphic foliations and the problem of existence of invariant curves for maps tangent to the identity. One component of this analogy, instrumental in the proof given in [A] of the Leau-Fatou flower theorem in dimension 2, is an index theorem very similar to Theorem 0.1:

Theorem 0.4: ([A]) *Let S be a compact Riemann surface embedded in a smooth complex surface M, and let f be a germ about S of a holomorphic self-map of M such that $f|_S = \mathrm{id}_S$. Assume that f is tangential to S (see below). Then it is possible to associate to any point $q \in S$ a complex number $\iota_q(f, S) \in \mathbb{C}$, the* *index of f along S at q, depending only on the local behavior of f near q, and vanishing at all but a finite number of points, such that*

$$\sum_{q\in\mathrm{Sing}(\mathcal{F})} \iota_q(\mathcal{F},S) = \int_S c_1(N_S).$$

This time the index is defined as follows. Let $\mathcal{A} = \{(U_\alpha, z_\alpha)\}$ be an atlas adapted to S. Then if we write $f = (f^1_\alpha, f^2_\alpha)$ in local coordinates, we can consider the local meromorphic function

$$k_\alpha(z^2_\alpha) = \lim_{z^1_\alpha\to 0} \frac{f^1_\alpha(z_\alpha) - z^1_\alpha}{z^1_\alpha\bigl(f^2_\alpha(z_\alpha) - z^2_\alpha\bigr)};$$

we shall say that f is *tangential* to S if $k_\alpha \not\equiv \infty$ (and it is not difficult to check that this condition is independent of α). Then the index $\iota_q(f,S)$ is defined by

$$\iota_q(f,S) = \mathrm{Res}_q\left(k_\alpha\, dz^2_\alpha\right), \tag{0.6}$$

and again it does not depend on the adapted chart chosen.

It should be remarked that, as first avatar of things to come, the same proof yielded a version of Theorem 0.4 for any self-map f, not necessarily tangential, if M was the total space of a line bundle over S.

Soon after [A] was completed, F. Bracci and F. Tovena in [BT] found a version of Theorem 0.4 for possibly singular compact Riemann surfaces; thus it became natural to try and generalize Theorem 0.4 in a way similar to what Lehmann and Suwa did starting from Theorem 0.1.

Let M be an n-dimensional complex manifold, S a (reduced, irreducible, possibly singular) subvariety of M of codimension m, and f a germ about S of holomorphic self-map of M fixing S pointwise. Then in [ABT1] we defined the following two main objects:

– the *order of contact* $\nu_f \in \mathbb{N}^*$ of f with S;
– the *canonical section* X_f, a section over S of the coherent sheaf $\mathcal{T}_{M,S}\otimes(\mathcal{N}^*_S)^{\otimes\nu_f}$, where $\mathcal{N}^*_S$ is the conormal sheaf of S and $\mathcal{T}_{M,S}$ is the sheaf of germs of holomorphic sections over S of the restriction $TM|_S$ to S of the tangent bundle of M.

Roughly speaking, the order of contact measures how close to the identity in a neighbourhood of S the map f is; and the canonical section indicates how f would move S if it were allowed to move it (in Subsection 1.4 we shall recall the definitions of ν_f and X_f, showing that these statements actually make sense).

The canonical section can be thought of as a section of the sheaf $\mathrm{Hom}((\mathcal{N}_S)^{\otimes\nu_f}, \mathcal{T}_{M,S})$, that is as a morphism $X_f\colon \mathcal{N}_S^{\otimes\nu_f} \to \mathcal{T}_{M,S}$, and thus it determines a *canonical distribution* $F_f \subset TM|_S$, the image of $N_S^{\otimes\nu_f}$ through X_f. It turns out that, from a dynamical point of view, the most interesting case is when f is *tangential*, that is when the canonical distribution is tangent to (the regular part of) S (and when S is a compact Riemann surface in a smooth complex surface M this definition of tangential reduces to the one recalled above). In this case, it turns out that the dynamics of f is concentrated around the singular points of S and X_f: indeed, it is possible to prove that if $p \in S$ is a smooth point of S not singular for X_f, then there is no infinite orbit of f arbitrarily close to p (see [ABT1]).

For the sake of simplicity, let us now assume that S is a hypersurface of M; in [ABT1] are described results with S of codimension greater than 1, but in a slightly different spirit and of a more technical character. When S is a hypersurface and f is tangential to S, we can use the canonical section to define a *partial holomorphic connection* on $\mathcal{N}_S$ along F_f (or, using the terminology of [ABT1], a *holomorphic action* of $\mathcal{N}_S^{\otimes\nu_f}$ on N_S) outside the singular points of f. Then following the ideas introduced by Lehmann and Suwa it is possible to prove the following generalization of Theorem 0.4:

Theorem 0.5: ([ABT1]) *Let S be a compact, complex, reduced, irreducible, possibly singular, hypersurface in an n-dimensional complex manifold M, and let f be a germ about S of a holomorphic self-map of M fixing S pointwise. Assume that f is tangential to S, and let $\{\Sigma_\lambda\}$ be the decomposition in connected components of the singular set* $\mathrm{Sing}(X_f)\cup\mathrm{Sing}(S)$. *Then there exist complex numbers* $\mathrm{Res}(X_f,S,\Sigma_\lambda)\in\mathbb{C}$, *depending only on the local behavior of f near Σ_λ, such that*

$$\sum_\lambda \mathrm{Res}(X_f,S,\Sigma_\lambda) = \int_S c_1^{n-1}(N_S),$$

where $c_1(N_S)$ *is the first Chern class of the normal bundle* N_S *of* S *in* M*. Furthermore, when* Σ_λ *is an isolated point there is an explicit formula for computing* $\mathrm{Res}(X_f, S, \Sigma_\lambda)$ *using a Grothendieck residue.*

Now, while in Theorem 0.2 the hypothesis that $\mathcal{F}$ is tangential to S looks completely natural, the corresponding hypothesis here that f is tangential to S might look somewhat artificial. Actually, this is not the case: in [ABT1] we showed that from a dynamical point of view this is the most interesting situation (if f is transversal to S the dynamics of f nearby S is much easier to determine), and furthermore f is naturally tangential to S in important applications (for instance when blowing-up a non-dicritical map tangent to the identity).

However, already in [ABT1] we noticed that we do not need f to be tangential to S if we ask something more on the way the regular part of S sits into M. To describe the exact hypotheses we need, which are (of course) much weaker than asking M to be the total space of a line bundle over S, let us recall a couple of definitions.

Let S be a (smooth) complex submanifold of a complex manifold M. We shall say that S *splits* into M if the exact sequence

$$O \longrightarrow TS \longrightarrow TM|_S \longrightarrow N_S \longrightarrow O$$

splits as sequence of vector bundles over S, that is if there is a projection $\sigma: TM|_S \to TS$ which is the identity on TS. It turns out (see [ABT2, 3]) that S splits into M if and only if the exact sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow \mathcal{O}_M/\mathcal{I}_S^2 \longrightarrow \mathcal{O}_S = \mathcal{O}_M/\mathcal{I}_S \longrightarrow O$$

splits as sequence of $\mathcal{O}_S$-modules, where $\mathcal{O}_M$ is the sheaf of germs of holomorphic functions on M, and $\mathcal{I}_S$ is the ideal sheaf of S. Furthermore, if S splits into M it is possible to introduce a structure of $\mathcal{O}_S$-module on $\mathcal{I}_S/\mathcal{I}_S^3$ such that

$$O \longrightarrow \mathcal{I}_S^2/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow O$$

becomes an exact sequence of $\mathcal{O}_S$-modules (see again [ABT2, 3] for details). Then we shall say that S is *comfortably embedded* in M if the latter sequence splits too.

In terms of adapted atlases, it turns out that S splits into M if there is an atlas $\mathcal{A} = \{(U_\alpha, z_\alpha)\}$ adapted to S (that is, such that $U_\alpha \cap S = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$, where m is the codimension of S) such that

$$\left.\frac{\partial z_\beta^p}{\partial z_\alpha^r}\right|_S \equiv 0, \tag{0.7}$$

for all $r = 1, \ldots, m$ and $p = m+1, \ldots, n$, where $n = \dim M$; compare with (0.4). Furthermore, S is comfortably embedded in M if there is an adapted atlas satisfying (0.7) and

$$\left.\frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t}\right|_S \equiv 0, \tag{0.8}$$

for all r, s, $t = 1, \ldots, m$; compare with (0.5). Clearly, the zero section of a vector bundle is comfortably embedded in the total space of the bundle. Furthermore, both splitting and being comfortably embedded can be characterized by the vanishing of a suitable sheaf cohomology class; in particular, Stein submanifolds always split and are comfortably embedded.

Then the proof of Theorem 0.5 can be adapted to yield the following index theorem:

Theorem 0.6: ([ABT1]) *Let* S *be a compact, complex, reduced, irreducible, possibly singular, hypersurface in an* n*-dimensional complex manifold* M*, such that the regular part* S' *of* S *is comfortably embedded in* M*, and let* f *be a germ about* S *of holomorphic self-map of* M *fixing* S *pointwise. If* $\sigma: TM|_{S'} \to TS'$ *is the splitting projection, set* $X = (\sigma \otimes \mathrm{id}) \circ X_f$*; roughly speaking, the image of* X *is the projection of the canonical distribution on* TS'*. [Actually, if* df *does not act as the identity on the normal bundle of* N_S *we should slightly change the definition of* X *to take into account the action of* df*; but after this technical adjustement, everything works.] Assume that* $X \not\equiv O$*, and let* $\{\Sigma_\lambda\}$ *be the decomposition in connected components of the* *singular set* $\mathrm{Sing}(X) \cup \mathrm{Sing}(S)$. *Then there exist complex numbers* $\mathrm{Res}(X, S, \Sigma_\lambda) \in \mathbb{C}$, *depending only on the local behavior of* f *near* Σ_λ, *such that*

$$\sum_\lambda \mathrm{Res}(X, S, \Sigma_\lambda) = \int_S c_1^{n-1}(N_S).$$

Moreover, when Σ_λ *is an isolated smooth point of* S *there is an explicit formula for computing* $\mathrm{Res}(X, S, \Sigma_\lambda)$ *using a Grothendieck residue.*

Thus our investigations in the discrete setting yielded Camacho-Sad-like index theorems both for subvarieties tangential to holomorphic self-maps and for subvarieties transversal to holomorphic self-maps. The tantalizing aspect of the index theorems we presented is that even though the statements are clearly very similar, the original proofs are all slightly different; none of these theorems is a direct consequence of any other. (There are ways to deduce Theorem 0.4 from Theorem 0.1 using formal or semi-formal vector fields — see [BCL] and [Di] — but they are by no means direct, and they do not seem to apply to the transversal case or to higher-dimensional settings.) This suggests that some deeper phenomenon is at work here; there must be a unified way to obtain all these theorems.

Indeed, this is exactly what we found out in [ABT2]: there is a general procedure (built starting from ideas due to Atiyah, Baum, Bott, Lehmann, Suwa and others) to obtain Camacho-Sad-like index theorems, giving not only all the theorems mentioned before but a couple of new ones too. For instance, we got the following generalization of Theorem 0.3:

Theorem 0.7: ([ABT2]) *Let* S *be a compact, complex, reduced, irreducible, possibly singular, subvariety of codimension* m *of an* n*-dimensional complex manifold* M, *such that the regular part* S' *of* S *is comfortably embedded in* M. *Assume that there exists a coherent sheaf of* $\mathcal{O}_M$*-modules* $\mathcal{N}$ *on* M *such that* $\mathcal{N} \otimes \mathcal{O}_{S'} = \mathcal{N}_{S'}$ *(this can always be arranged if* S *is smooth, or it is an hypersurface). Let* $\mathcal{F}$ *be a singular holomorphic foliation of rank* $n - m$ *on* M, *and let* $\mathcal{F}^\sigma = \sigma(\mathcal{F} \otimes \mathcal{O}_S)$, *where* $\sigma\colon TM|_{S'} \to TS'$ *is the splitting projection. Assume that* $\mathrm{Sing}(\mathcal{F}^\sigma) \neq S$, *and that* $S^o = S' \setminus \mathrm{Sing}(\mathcal{F}^\sigma)$ *is (comfortably embedded and) 2-linearizable, that is satisfies* (0.4) *for* $i = 1, 2$. *Let* $\{\Sigma_\lambda\}$ *be the decomposition in connected components of the singular set* $\mathrm{Sing}(\mathcal{F}^\sigma) \cup \mathrm{Sing}(S)$. *Finally, let* φ *be a polynomial of degree* $d > 0$. *Then we can associate to each connected component* Σ_λ *a residue*

$$\mathrm{Res}_\varphi(\mathcal{F}^\sigma, \mathcal{N}; \Sigma_\lambda) \in H_{2n-2(m+d)}(\Sigma_\lambda; \mathbb{C}),$$

depending only on the local behavior of $\mathcal{F}$ *near* Σ_λ, *such that*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\mathcal{F}^\sigma, \mathcal{N}; \Sigma_\lambda) = [S] \frown \varphi(\mathcal{N}) \quad \text{in } H_{2n-2(m+d)}(S; \mathbb{C}), \tag{0.9}$$

where $i_\lambda\colon \Sigma_\lambda \hookrightarrow S$ *is the inclusion, and* $\varphi(N)$ *denotes the class obtained evaluating* φ *in the Chern classes of* $\mathcal{N}$.

The paper [ABT2] contains several other results of this kind, some very general. The proofs in [ABT2] are hard and very technical; however, the general ideas can be explained and appreciated at a non-technical level. The aim of this survey is exactly to expose these ideas without entering in the technical details, referring the readers to [ABT2] and [ABT3] for proofs and more insight.

Acknowledgments. First of all, I am honored to thank my two fellow coworkers, Francesca Tovena and Filippo Bracci; without them none of this would ever have existed. A heartful thank you also goes to Leandro Arosio, Cesar Camacho, Daniel Lehmann, Jasmin Raissy, Paulo Sad and Tatsuo Suwa, for several interesting and stimulating discussions. Finally, it is a pleasure to thank Eugene Mihailescu for inviting me to Bucarest.

1. The strategy

Our general approach to Camacho-Sad-like index theorems can be summarized in four steps:

(a) vanishing theorems yield index theorems;
(b) partial holomorphic connections yield vanishing theorems;

(c) splitting of suitable sequences of sheaves yield partial holomorphic connections;
(d) the construction of a splitting.

As we shall see, steps (b), (c) and the main body of step (a) have nothing to do with either maps or foliations; they are the backbone of our general strategy, and work in the same way in all cases described in the introduction (and conceivably in other cases too). The holomorphic map or foliation is only needed to provide the geometrical data used in the construction of the splitting in the last step (and in the choice of the indeces in the first step; see Remark 1.1 below). Furthermore, the last step in the tangential case is very easy and natural (the transversal case is considerably harder, though). In this way, we get a sensible explanation of both the similarities and the differences among statements and proofs of all known instances of Camacho-Sad-like index theorems.

Let us then describe these steps one at the time.

1.1. Vanishing theorems yield index theorems (inspired by Lehmann and Suwa)

The first step is essentially cohomological, and it is based on ideas developed by Lehmann and Suwa (see, e.g., [S2] and [S3]).

Roughly speaking, let us say that a *characteristic class* is a cohomology class φ canonically associated to a geometrical situation identically vanishing as soon as a specific kind of object θ exists; in other words, φ is an obstruction to the existence of θ. The most famous example of characteristic class is the top Chern class of a complex vector bundle. If $\pi\colon E \to S$ is a complex vector bundle of rank r over a manifold S, the top Chern class $c_r(E)$ is a cohomology class of degree $2r$ on S — to be precise, it lives in $H^{2r}(S,\mathbb{Z})$ — vanishing as soon as E admits a never vanishing section θ. So in this case the geometrical situation is the complex vector bundle E and the object θ is a never vanishing section. As we shall see later, the geometrical situation we shall be interested in will be a complex vector bundle over a complex manifold S, and the specific kind of object will be a partial holomorphic connection on E.

In this context, a *vanishing theorem* is just the statement that the existence of the object θ implies the vanishing of the characteristic class φ; the first step in our strategy consists then in seeing the index theorem as a *localization* of the characteristic class to the singular set of the object θ.

Given the geometric situation on the manifold S, and hence the characteristic class $\varphi \in H^*(S)$, suppose that the object θ exists outside a closed set $\Sigma \subset S$ (the *singular set* of θ); in particular, by the vanishing theorem (and suitable naturality properties of the characteristic class), the restriction of φ to $S \setminus \Sigma$ vanishes: $\varphi|_{S\setminus\Sigma} \equiv O$. The long exact cohomology sequence of the pair $(S, S \setminus \Sigma)$

$$H^*(S, S\setminus\Sigma) \xrightarrow{p} H^*(S) \xrightarrow{|_{S\setminus\Sigma}} H^*(S\setminus\Sigma)$$

then yields a relative class $\eta \in H^*(S, S\setminus\Sigma)$ such that $p(\eta) = \varphi$.

Now assume that S is compact, and that Σ is a subvariety (more general closed sets are possible too, but this is enough for our aims). We then have the Poincaré isomorphism $P\colon H^*(S) \to H_{d-*}(S)$ and the Alexander isomorphism $A\colon H^*(S, S\setminus\Sigma) \to H_{d-*}(\Sigma)$, where $d = \dim S$. From the commutativity of the diagram

$$\begin{array}{ccc} H^*(S, S\setminus\Sigma) & \xrightarrow{p} & H^*(S) \\ \big\downarrow A & & \big\downarrow P \\ H_{d-*}(\Sigma) & \xrightarrow{i_*} & H_{d-*}(S) \end{array} ,$$

where $i_*\colon H_{d-*}(\Sigma) \to H_{d-*}(S)$ is induced by the inclusion $i\colon \Sigma \hookrightarrow S$, we then get

$$i_*\big(A(\eta)\big) = P(\varphi) . \tag{1.1}$$

Now let $\Sigma = \bigcup_\lambda \Sigma_\lambda$ be the decomposition of Σ in connected components. The homology group $H_{d-*}(\Sigma)$ decomposes in the direct sum of the homology groups of the components Σ_λ; accordingly, $A(\eta)$ decomposes in a sum of elements in $H_{d-*}(\Sigma_\lambda)$. If we denote by $\mathrm{Res}(\varphi,\theta;\Sigma_\lambda)$ the component of $A(\eta)$ living in $H_{d-*}(\Sigma_\lambda)$, then (1.1) becomes an *index theorem:*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}(\varphi,\theta;\Sigma_\lambda) = P(\varphi) , \tag{1.2}$$

where $i_\lambda = i|_{\Sigma_\lambda}$. A moment of thought shows that this formula is the exact analogue of formulas like (0.1) and (0.3). Indeed, in the left-hand side we have a sum of homology classes (of numbers when $* = d$, as happens for instance in Theorem 0.1), each one depending only on what happens in a neighbourhood of the singular set (because they come from a relative cohomology class). In the right-hand side we have the Poincaré dual of the characteristic class; and the Poincaré dual is exactly given by cap multiplication by the fundamental class $[S]$ (or by integration over S when $* = d$). Thus in this way we get a Camacho-Sad-like index theorem starting from a vanishing theorem for a characteristic class.

Remark 1.1: An important point here is that, in general, the class $\eta \in H^*(S, S\setminus\Sigma)$ such that $p(\eta) = \varphi$ is *not* unique. Any choice of η gives rise to an index theorem, but clearly some choices make more sense than others. For instance, one would like to be able to read in η some properties of the object θ whose existence ensured the vanishing of the characteristic class outside the singular set Σ. Since, ultimately, the construction of θ will depend on the data (maps or foliations) at hand, in this way the residues in the left-hand side of (1.2) will keep memory of the data, whereas the right-hand side is independent of them (which is one of the main reasons an index theorem is useful).

As Lehmann and Suwa noticed, Čech-de Rham cohomology theory is an efficient tool to satisfy this requirement. In particular, when the object θ is a partial holomorphic connection (which is the case we are interested in), the Čech-de Rham theory yields a pretty explicit expression for the residues $\mathrm{Res}(\varphi, \theta; \Sigma_\lambda)$. For instance, when Σ_λ is an isolated point (and $* = d$) then it is possible to express $\mathrm{Res}(\varphi, \theta; \Sigma_\lambda)$ as a Grothendieck residue, with a formula very similar to (0.2).

For details on this subsection see [S2], [S3] and [ABT1], [ABT2].

1.2. Partial holomorphic connections yield vanishing theorems (inspired by Baum and Bott)

To apply the scheme described in the previous subsection we need to choose the geometrical situation we would like to work in and the objects we would like to look for.

The geometrical situation is easily explained: it is given by a holomorphic vector bundle $p: E \to S$ on a complex manifold S. Thus as characteristic classes we can use the Chern classes of E (or, more generally, polynomials in the Chern classes).

The objects we are looking for are partial holomorphic connections on E. A *partial holomorphic connection* is given by a holomorphic subbundle $F \subseteq TS$ and a map $\nabla: \Gamma(F) \times \Gamma(E) \to \Gamma(E)$, where $\Gamma(E)$ (resp., $\Gamma(F)$) denotes the space of C^∞-sections of E (resp., F), such that ∇ is C^∞-linear in the first argument, satisfies the usual Leibniz condition

$$\forall f \in C^\infty(S),\, \forall v \in \Gamma(F),\, \forall s \in \Gamma(s) \qquad \nabla_v(fs) = v(f)s + f\nabla_v s\ ,$$

and so that $\nabla_v s$ is holomorphic for every holomorphic section v of F and s of E. In other words, a partial holomorphic connection gives a way to differentiate sections of E only along some tangent directions (the ones contained in F) but preserving holomorphicity.

Remark 1.2: Since it will be useful in the sequel, let me restate this definition in terms of sheafs. Let $\mathcal{E}$ (resp., $\mathcal{F}$) denote the (locally free) sheaf of germs of holomorphic sections of E (resp., F). Then a partial holomorphic connection on E along F is given by a $\mathbb{C}$-linear map $\nabla: \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$, where $\mathcal{F}^*$ is the dual of $\mathcal{F}$, satisfying the Leibniz condition

$$\forall f \in \mathcal{O}_S,\, \forall s \in \mathcal{E} \qquad \nabla(fs) = df|_{\mathcal{F}} \otimes s + f\nabla s\ ,$$

where $\mathcal{O}_S$ is the structure sheaf of S.

When $F = TS$, a partial holomorphic connection ∇ is just a honest holomorphic connection. In this case (or, more generally, when F is involutive) we shall say that ∇ is *flat* if

$$\forall u, v \in \Gamma(F) \qquad \nabla_u \circ \nabla_v - \nabla_v \circ \nabla_u - \nabla_{[u,v]} = O\ .$$

Baum and Bott in [BB] proved that the existence of a flat holomorphic connection on E implies the vanishing of all the Chern classes of E, and thus a vanishing theorem of the kind we need. It is not difficult (see [ABT2] and [CC]) to extend Baum-Bott's argument to get a vanishing theorem for not necessarily flat partial holomorphic connections:

Theorem 1.1: *Let S be a complex manifold, F a holomorphic sub-bundle of TS of rank ℓ, and E a holomorphic vector bundle on S. Assume we have a partial holomorphic connection on E along F. Then:*

(i) *every polynomial in the Chern classes of E of degree larger than $\dim S - \ell + \lfloor \ell/2 \rfloor$ vanishes, where $\lfloor x \rfloor$ denotes the integer part of x.*

(ii) *Furthermore, if F is involutive and the partial holomorphic connection is flat then every polynomial in the Chern classes of E of degree larger than $\dim S - \ell$ vanishes.*

The idea of the proof consists in extending ∇ to a C^∞ real connection on E, defining it arbitrarily on a C^∞-complement of F in $TS = T^{(1,0)}S$ and by using $\overline{\partial}$ on $T^{(0,1)}S$. Then the holomorphicity of ∇ on F (and the flatness, if any) implies that the curvature matrix is expressed using only 2-forms of a particular kind, and the external product of a sufficiently high number of those forms must vanish. Since polynomials in the Chern classes are represented by taking external products of forms in the curvature matrix, Theorem 1.1 follows.

So the existence of a partial holomorphic connection outside of a singular set yields a vanishing theorem, and thus an index theorem. The next step consists then in finding conditions ensuring the existence of partial holomorphic connections.

For details on this subsection see [BB] and [ABT2].

1.3. Splitting of suitable sequences of sheaves yield partial holomorphic connections (inspired by Atiyah) The problem of finding conditions ensuring the existence of a holomorphic connection on a given holomorphic vector bundle has been considered by Atiyah in [At]. Let us briefly recall his construction.

Let E be a holomorphic vector bundle of rank d over a complex manifold S; we shall denote by P_E the principal bundle associated to E, with structure group $GL(d,\mathbb{C})$. The group $GL(d,\mathbb{C})$ acts on the tangent vector bundle TP_E of the total space of P_E, and the quotient $A_E = TP_E/GL(d,\mathbb{C})$ can be identified with the vector bundle on S of rank d^2-1 composed by the fields of tangent vectors to P_E defined along one of its fibres and invariant under the action of $GL(d,\mathbb{C})$. Since the action of $GL(d,\mathbb{C})$ on P_E preserves the fibers of the canonical projection $\pi_0\colon P_E \to S$, the differential of π_0 defines a vector bundle morphism, still denoted by π_0, from A_E onto TS. Atiyah has shown ([At, Theorem 1 and Proposition 9]) that there is a canonical exact sequence

$$O \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{A}_E \xrightarrow{\pi_0} \mathcal{T}_S \longrightarrow O\ , \tag{1.3}$$

of locally free $\mathcal{O}_S$-modules, where $\mathcal{T}_S$ (resp., $\mathcal{A}_E$) is the sheaf of germs of holomorphic sections of TS (resp., A_E). Furthermore, this sequence splits if and only if there is a holomorphic connection on E ([At, Theorem 2]; see also [GR], where part of this theory is extended to subvarieties S having normal crossing singularities).

It is easy to adapt this construction to the case of partial holomorphic connections. If F is a holomorphic sub-bundle of TS, we can consider the restriction to F of the sequence (1.3)

$$O \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{A}_{E,F} \xrightarrow{\pi_0} \mathcal{F} \longrightarrow O\ , \tag{1.4}$$

where $\mathcal{A}_{E,F} = \pi_0^{-1}(\mathcal{F}) \subseteq \mathcal{A}_E$. Then arguing as in [At] it is easy to prove the following

Proposition 1.2: *Let F be a holomorphic sub-bundle of the tangent bundle TS of a complex manifold S, and let E be a holomorphic vector bundle over S. Then there is a partial holomorphic connection on E along F if and only if the sequence* (1.4) *splits, and this happens if and only if there is an $\mathcal{O}_S$-morphism $\psi_0\colon \mathcal{F} \to \mathcal{A}_E$ such that $\pi_0 \circ \psi_0 = \mathrm{id}$.*

Now, in this paper we are interested in studying Camacho-Sad-like index theorems, and not index theorems in general. A Camacho-Sad-like index theorem says something on the normal bundle N_S of a complex submanifold S of a complex manifold M. Thus, to proceed we need a more explicit representation of $\mathcal{A}_E$ when $E = N_S$.

So let S be a complex submanifold of a complex manifold M. Set $\mathcal{O}_{S(1)} = \mathcal{O}_M/\mathcal{I}_S^2$; the sheaf $\mathcal{O}_{S(1)}$ can be thought of as the structure sheaf of the first infinitesimal neighbourhood of S in M. Let $\mathcal{T}_M$ (resp., $\mathcal{T}_{M,S}$) be the sheaf of germs of holomorphic sections of TM (resp., of $TM|_S$ along S), then $\mathcal{N}_S = \mathcal{T}_{M,S}/\mathcal{T}_S$ is the sheaf of germs of holomorphic sections of the normal bundle $N_S = TM|_S/TS$. We are not done yet:

set $\mathcal{T}_{M,S(1)} = \mathcal{T}_M \otimes \mathcal{O}_{S(1)}$, let $p_2: \mathcal{T}_{M,S} \to \mathcal{N}_S$ and $\theta_1: \mathcal{T}_{M,S(1)} \to \mathcal{T}_{M,S}$ be the natural projections, and define $\mathcal{T}^S_{M,S} = \operatorname{Ker} p_2 \subset \mathcal{T}_{M,S}$ and $\mathcal{T}^S_{M,S(1)} = \operatorname{Ker}(p_2 \circ \theta_1) \subset \mathcal{T}_{M,S(1)}$.

Roughly speaking, the elements of $\mathcal{T}^S_{M,S}$ are germs at S of vector fields of M that are tangent to S when restricted to it, that is germs that in a chart adapted to S are of the form

$$\sum_{r=1}^{m} f^r \frac{\partial}{\partial z^r} + \sum_{p=m+1}^{n} f^p \frac{\partial}{\partial z^p} \tag{1.5}$$

with $m = \operatorname{codim} S$, $n = \dim M$, $f^1, \ldots, f^m \in \mathcal{I}_S$ and $f^{m+1}, \ldots, f^n \in \mathcal{O}_M$. The elements of $\mathcal{T}^S_{M,S(1)}$ are of the same form, but with $f^1, \ldots, f^m \in \mathcal{I}_S/\mathcal{I}_S^2$ and $f^{m+1}, \ldots, f^n \in \mathcal{O}_{S(1)}$.

Inside $\mathcal{T}^S_{M,S}$ (resp., $\mathcal{T}^S_{M,S(1)}$) we can consider $\mathcal{I}_S \cdot \mathcal{T}^S_{M,S}$ (resp., $\mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)}$), whose elements are of the form (1.5) with $f^1, \ldots, f^m \in \mathcal{I}_S^2$ and $f^{m+1}, \ldots, f^n \in \mathcal{I}_S$ (resp., with $f^1, \ldots, f^m = 0$ in $\mathcal{O}_{S(1)}$ and $f^{m+1}, \ldots, f^n \in \mathcal{I}_S/\mathcal{I}_S^2$). Then in [ABT2] we proved

Proposition 1.3: *Let S be a complex submanifold of a complex manifold M. Then*

$$\mathcal{A}_{N_S} = \mathcal{T}^S_{M,S}/\mathcal{I}_S \cdot \mathcal{T}^S_{M,S} = \mathcal{T}^S_{M,S(1)}/\mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)} \; .$$

The elements of $\mathcal{A}_{N_S}$ are locally of the form (1.5) with $f^1, \ldots, f^m \in \mathcal{I}_S/\mathcal{I}_S^2$ and $f^{m+1}, \ldots, f^n \in \mathcal{O}_S$. In particular, there is a natural projection $\pi_0: \mathcal{A}_{N,S} \to \mathcal{T}_S$ obtained by dropping the first sum in (1.5); and it is not difficult to see that the elements of the first sum in (1.5) generate a sheaf isomorphic to $\operatorname{End}(\mathcal{N}_S, \mathcal{N}_S)$. In this way we get the exact sequence

$$O \longrightarrow \operatorname{Hom}(\mathcal{N}_S, \mathcal{N}_S) \longrightarrow \mathcal{A}_{N_S} \xrightarrow{\pi_0} \mathcal{T}_S \longrightarrow O \; , \tag{1.6}$$

which exactly is (1.3) in this setting. Therefore to get a partial holomorphic connection on N_S (and ultimately a Camacho-Sad-like index theorem) it suffices (and it is necessary) to build a morphism $\psi: \mathcal{F} \to \mathcal{A}_{N_S}$ such that $\pi_0 \circ \psi = \mathrm{id}$. As we shall see in the next subsection, the explicit description of $\mathcal{A}_{N_S}$ given in Proposition 1.3 will come in very handy.

For details on this subsection see [At] and [ABT2].

1.4. The construction of a splitting.

We have at last reached the final step of our construction, the only part of our strategy explicitely depending on the data at hand: how to build a splitting of (1.6) starting from a holomorphic foliation or from a holomorphic self-map of M. We shall describe the procedure in the tangential case only; the transversal case is similar but technically (much) more involved.

Let us begin with the foliation tangential case, which is now very easy. A possibly singular holomorphic foliation on M is given by an involutive subsheaf $\mathcal{F} \subset \mathcal{T}_M$; the singular set of $\mathcal{F}$ is the locus of points in M where $\mathcal{T}_M/\mathcal{F}$ is not locally free. Let $S \subset M$ be a complex subvariety of M, and assume that $\mathcal{F}$ is tangent to the regular part of S. In particular, if we set $\Sigma = \operatorname{Sing}(S) \cup \big(\operatorname{Sing}(\mathcal{F}) \cap S\big)$ and $S^o = S \setminus \Sigma$, then $\mathcal{F}_{S^o} = \mathcal{F} \otimes \mathcal{O}_{S^o}$ is a locally free involutive sub-bundle of $\mathcal{T}_{S^o}$, and $\mathcal{F}|_{S^o}$ can be thought of as a subsheaf of $\mathcal{T}^{S^o}_{M,S^o}$. To get a splitting $\psi: \mathcal{F}_{S^o} \to \mathcal{A}_{N_{S^o}}$ is then very easy: the lack of singularities gives a natural map $\iota: \mathcal{F}_{S^o} \hookrightarrow \mathcal{F}|_{S^o} \subset \mathcal{T}^{S^o}_{M,S^o}$. Then we set $\psi = \pi \circ \iota$, where $\pi: \mathcal{T}^{S^o}_{M,S^o} \to \mathcal{A}_{N_S}$ is the canonical projection given by Proposition 1.3. The map ψ clearly is a splitting morphism — and thus we got a Camacho-Sad-like index theorem for foliations tangent to S (to be precise, in this way we recovered Theorem 0.2).

To describe the construction in the map case we need to recall a few notions introduced in [ABT1]. Let $f: M \to M$ be a holomorphic self-map of a complex n-dimensional manifold M, and assume that f leaves a complex irreducible possibly singular hypersurface $S \subset M$ pointwise fixed; we shall write $f \in \operatorname{End}(M, S)$, and always assume that $f \not\equiv \mathrm{id}_M$. Take $p \in S$; then for every $h \in \mathcal{O}_{M,p}$ the germ $h \circ f$ is well-defined, and we have $h \circ f - h \in \mathcal{I}_{S,p}$. The *$f$-order of vanishing* at p of $h \in \mathcal{O}_{M,p}$ is

$$\nu_f(h; p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathcal{I}^\mu_{S,p}\},$$

and the *order of contact* ν_f of f with S is

$$\nu_f = \min\{\nu_f(h;p) \mid h \in \mathcal{O}_{M,p}\}.$$

In [ABT1] we proved that ν_f does not depend on p, and that

$$\nu_f(p) = \min_{j=1,\ldots,n}\{\nu_f(z^j;p)\},$$

where (U,z) is any local chart centered at $p \in S$. In particular, if p is a smooth point of S, and the local chart (U,z) is adapted to S, that is $S \cap U = \{z^1 = 0\}$, then setting $f^j = z^j \circ f$ we can write

$$f^j(z) = z^j + (z^1)^{\nu_f} g^j(z) \;, \tag{1.7}$$

where z^1 does not divide g^j for at least some $1 \le j \le n$.

In [ABT1] we also said that a map $f \in \mathrm{End}(M,S)$ is *tangential* to S if

$$\min\bigl\{\nu_f(h;p) \mid h \in \mathcal{I}_{S,p}\bigr\} > \nu_f$$

for some (and hence any) point $p \in S$. If p is a smooth point of S and we choose a local chart (U,z) adapted to S so that we can express the coordinates of f in the form (1.7), it turns out that f is tangential if and only if $z^1|g^1$, that is if and only if

$$g^1|_S \equiv O \;.$$

The coefficients g^j in (1.7) clearly depend on the chosen chart. However, in [ABT1] we proved that setting

$$\mathcal{X}_f = \sum_{j=1}^n g^j \frac{\partial}{\partial z^j} \otimes (dz^1)^{\otimes \nu_f} \tag{1.8}$$

then $\mathcal{X}_f|_{U\cap S}$ defines a *global* section X_f of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ out of the singularities of S. The section X_f is the *canonical section* associated to f. Actually, the sheaf of germs of holomorphic sections of N_S^* can be canonically identified with $\mathcal{I}_S/\mathcal{I}_S^2$; then it is not difficult to see that X_f actually is a global section on the whole of S of the sheaf $\mathcal{T}_{M,S} \otimes (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes \nu_f}$.

The bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ is canonically isomorphic to the bundle $\mathrm{Hom}(N_S^{\otimes \nu_f}, TM|_S)$. Therefore the section X_f induces a morphism, still denoted by $X_f\colon N_S^{\otimes \nu_f} \to TM|_S$, out of the singularities of S, and it is easy to check (see [ABT1]) that f is tangential if and only if the image of X_f is contained in TS.

We shall say that $p \in S \setminus \mathrm{Sing}(S)$ is a *singular point for f* if X_f vanishes at p; we shall denote by $\mathrm{Sing}(f)$ the set of singular points for f, and by $S^o = S \setminus \bigl(\mathrm{Sing}(S) \cup \mathrm{Sing}(f)\bigr)$ the subset of regular (both for S and for f) points of S. By definition, $p \in S \setminus \mathrm{Sing}(S)$ is a singular point for f if and only if

$$g^1(p) = \cdots = g^n(p) = 0$$

for any local chart adapted to S; so singular points are generically isolated.

Now let $\mathcal{F}_{S^o}$ be the image of $\mathcal{N}_S^{\otimes \nu_f}$ via X_f over S^o. Since we removed the singularities, if f is tangential then $\mathcal{F}_{S^o}$ is a rank 1 locally free subsheaf of $\mathcal{T}_{S^o}$, and thus it determines a rank 1 sub-bundle $F \subset TS^o$. Furthermore, in [ABT1, 2] we showed that when f is tangential we can use $\mathcal{X}_f$ to define a canonical injection ι of $\mathcal{F}_{S^o}$ into $\mathcal{T}_{M,S^o(1)}^{S^o}$. It is important to notice that in general we cannot embed $\mathcal{F}_{S^o}$ into $\mathcal{T}_{M,S^o}^{S^o}$; therefore to deal with maps we have to work with infinitesimal neighbourhoods.

We are now done. Indeed, denoting by $\pi\colon \mathcal{T}_{M,S^o(1)}^{S^o} \to \mathcal{A}_{N_{S^o}}$ the canonical projection given by Proposition 1.3, we get a splitting morphism just by setting $\psi = \pi \circ \iota$ — and thus a Camacho-Sad-like index theorem for maps (more precisely, we recovered Theorem 0.5).

In the transversal case, $\mathcal{F}|_{S^o}$ lives in $\mathcal{T}_{M,S^o}$ and not in $\mathcal{T}_{M,S^o}^{S^o}$; and analogously $\mathcal{X}_f$ gives an injection into $\mathcal{T}_{M,S^o(1)}$ and not into $\mathcal{T}_{M,S^o(1)}^{S^o}$. To deal with these cases we need a morphism $\tilde\pi\colon \mathcal{T}_{M,S^o(1)} \to \mathcal{A}_{N_{S^o}}$ extending π; and such a morphism exists when S is comfortably embedded in M. There are a few more details to be worked out, but essentially using $\tilde\pi$ instead of π it is possible to follow the previous scheme to get a splitting morphism in the transversal case too — and thus Camacho-Sad-like index theorems of the kind exemplified by Theorem 0.7.

For details on this subsection see [ABT1] and [ABT2].

References

[A] M. Abate: *The residual index and the dynamics of holomorphic maps tangent to the identity*. Duke Math. J. **107** (2001), 173–207.

[ABT1] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic self-maps*. Ann. of Math. **159** (2004), 819–864.

[ABT2] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic maps and foliations*. Preprint, arXiv:math/0601602v1, 2006.

[ABT3] M. Abate, F. Bracci, F. Tovena: *Embeddings of submanifolds and normal bundles*. Preprint, arXiv:math/0612449v1, 2007.

[At] M. F. Atiyah: *Complex analytic connections in fibre bundles*. Trans. Amer. Math. Soc. **85** (1957), 181–207.

[BB] P. Baum, R. Bott: *Singularities of holomorphic foliations*. J. Differential Geometry **7** (1972), 279–342.

[BT] F. Bracci, F. Tovena: *Residual indices of holomorphic maps relative to singular curves of fixed points on surfaces*. Math. Z. **242** (2002), 481–490.

[BCL] F.E. Brochero Martinez, F. Cano, L. Lopez Hernanz: *Parabolic curves for diffeomorphisms in* $(\mathbb{C}^2, 0)$. Preprint, arXiv:math/0611019v1, 2006.

[C] C. Camacho: *Dicritical singularities of holomorphic vector fields*. Contemp. Math. **269** (2001), 39–45.

[CS] C. Camacho, P. Sad: *Invariant varieties through singularities of holomorphic vector fields*. Ann. of Math. **115** (1982), 579–595.

[CL] C. Camacho, D. Lehmann: *Residues of holomorphic foliations relative to a general submanifold*. Bull. London Math. Soc. **37** (2005), 435–445.

[CMS] C. Camacho, H. Movasati, P. Sad: *Fibered neighborhoods of curves in surfaces*. J. Geom. Anal. **13** (2003), 55–66.

[CC] J.B. Carrell, D.I. Lieberman: *Vector fields and Chern numbers*. Math. Ann. **225** (1977), 263–273.

[Di] F. Degli Innocenti: *On the relations between discrete and continuous dynamics in* $\mathbb{C}^2$. Ph.D. Thesis, Università di Pisa, 2007.

[GR] R. Gérard, J.-P. Ramis: *Résidu d'une connexion holomorphe*. In **Equations différentielles et systèmes de Pfaff dans le champ complexe, II,** R. Gérard, J.-P. Ramis (eds.), Lect. Notes in Math. 1015, Springer, Berlin, 1983, pp. 243–306.

[Le] D. Lehmann: *Résidus des sous-variétés invariants d'un feuilletage singulier*. Ann. Inst. Fourier, Grenoble **41** (1991), 211–258.

[LS1] D. Lehmann, T. Suwa: *Residues of holomorphic vector fields relative to singular invariant subvarieties*. J. Diff. Geom. **42** (1995), 165–192.

[LS2] D. Lehmann, T. Suwa: *Generalizations of variations and Baum-Bott residues for holomorphic foliations on singular varieties*. International J. Math. **10** (1999), 367–384.

[Li] A. Lins Neto: *Algebraic solutions of polynomial differential equations and foliations in dimension two*. In **Holomorphic dynamics (Mexico, 1986),** Lect. Notes in Math. 1345, Springer, Berlin, 1988, pp. 192–232.

[S1] T. Suwa: *Indices of holomorphic vector fields relative to invariant curves on surfaces*. Proc. Amer. Math. Soc. **123** (1995), 2989–2997.

[S2] T. Suwa: **Indices of vector fields and residues of singular holomorphic foliations.** Hermann, Paris, 1998.

[S3] T. Suwa: *Residues of Chern classes on singular varieties*. In **Singularités Franco-Japonaises,** Sémin. Congr., 10, Soc. Math. France, Paris, 2005, pp. 265–285.