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Embeddings of submanifolds and normal bundles

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Received 11 July 2007; accepted 3 October 2008

Communicated by Gang Tian

Abstract

This paper studies the embeddings of a complex submanifold S inside a complex manifold M; in particular, we are interested in comparing the embedding of S in M with the embedding of S as the zero section in the total space of the normal bundle N_S of S in M. We explicitly describe some cohomological classes allowing to measure the difference between the two embeddings, in the spirit of the work by Grauert, Griffiths, and Camacho, Movasati and Sad; we are also able to explain the geometrical meaning of the separate vanishing of these classes. Our results hold for any codimension, but even for curves in a surface we generalize previous results due to Laufert and Camacho, Movasati and Sad.

Keywords: Embeddings; Normal bundle; Formal neighborhood; Splitting submanifolds

0. Introduction

This paper is devoted to the study of the embeddings of a complex submanifold S inside a larger complex manifold M; in particular, we are interested in comparing the embedding of S in M with the embedding of S as the zero section in the total space of the normal bundle N_S of S in M. We explicitly describe some cohomological classes allowing to measure the difference between the two embeddings, in the spirit of [8,11,12]. Our hope is that it may be a step towards a classification of foliations on M transverse to a submanifold S; see [8].

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0001-8708/$ – see front matter
doi:10.1016/j.aim.2008.10.001

Our interest in this topic originated in our previous papers [1,2], where we studied index theorems for holomorphic self-maps and foliations. We had a complex submanifold S of a complex manifold M and a holomorphic object $\mathcal{F}$ (either a holomorphic self-map of M fixing S pointwise, or a possibly singular holomorphic foliation of M); along the lines of the original Camacho–Sad index theorem [6], we wanted to recover Chern classes of the normal bundle N_S of S in M by means of local invariants associated to singular points of either S or of the holomorphic object $\mathcal{F}$. It turned out that to get index theorems of this kind one needs either hypotheses on the relative position of S and $\mathcal{F}$ (e.g., the holomorphic foliation should be tangent to S), or on the embedding of S into M: it should be close enough to the embedding of S in N_S as zero section.

We found two ways to express the geometrical conditions on the embedding we needed; either in terms of the existence of local coordinates with suitable properties (in a way similar to what was done in [8], a main source of inspiration for the present paper), or in a more intrinsic way, as splittings of suitable exact sequence of sheaves, thus allowing us to rephrase the conditions in terms of vanishing of cohomology classes. Furthermore, it turned out that we were actually working only with the first two of a list of more and more stringent conditions on the embedding, and that it might be interesting to study the whole list of conditions.

The first (well-known) condition on the embedding is the splitting condition. We say that S *splits* into M if the exact sequence

$$O \longrightarrow TS \longrightarrow TM|_S \longrightarrow N_S \longrightarrow O$$

splits as sequence of vector bundles over S, where TS (respectively, $TM|_S$) is the holomorphic tangent bundle of S (respectively, of M restricted to S). It turns out (see Section 1) that S splits into M if and only if the exact sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow \mathcal{O}_M/\mathcal{I}_S^2 \longrightarrow \mathcal{O}_S = \mathcal{O}_M/\mathcal{I}_S \longrightarrow O$$

splits as sequence of sheaves of rings, where $\mathcal{O}_M$ (respectively, $\mathcal{O}_S$) is the structure sheaf of M (respectively, of S), and $\mathcal{I}_S$ is the ideal sheaf of S.

Thus if S splits we have a way to extend germs of holomorphic functions on S to germs of holomorphic functions defined on M up to the first order. It is then natural to say that S is *k-splitting* into M (for some $k \geqslant 1$) if the exact sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_M/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_S \longrightarrow O$$

splits as sequence of sheaves of rings. If this happens, it turns out (see Section 3) that we can introduce a structure of $\mathcal{O}_S$-module on $\mathcal{I}_S/\mathcal{I}_S^{h+1}$ for $2 \leqslant h \leqslant k+1$ in such a way that the sequences

$$O \longrightarrow \mathcal{I}_S^h/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{h+1} \longrightarrow \mathcal{I}_S/\mathcal{I}_S^h \longrightarrow O$$

become exact sequences of $\mathcal{O}_S$-modules. If these sequences split, we say that S is *k-comfortably embedded* in M. (In [1,2] we introduced split, 2-split and 1-comfortably embedded submanifolds only.)

We can characterize these conditions in terms of local coordinates. Indeed, in Section 2 we prove that S is k-splitting into M if and only if there is an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of M adapted to S

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(that is, such that $U_\alpha \cap S \neq \emptyset$ implies $U_\alpha \cap S = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$, where m is the codimension of S) such that

$$\left.\frac{\partial^k z_\beta^p}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_k}}\right|_S \equiv 0,$$

for all $r_1, \dots, r_k = 1, \dots, m$, all $p = m+1, \dots, n = \dim M$, and all indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. Furthermore, S is k-comfortably embedded in M if and only if (Section 3) there is an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of M adapted to S such that

$$\left.\frac{\partial^k z_\beta^p}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_k}}\right|_S \equiv 0, \quad \text{and} \quad \left.\frac{\partial^{k+1} z_\beta^s}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_{k+1}}}\right|_S \equiv 0,$$

for all $r_1, \dots, r_{k+1}, s = 1, \dots, m$, all $p = m+1, \dots, n = \dim M$, and all indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. In particular, we see that if S is k-splitting and $(k-1)$-comfortably embedded then we have an atlas $\mathfrak{U}$ such that the changes of coordinates are of the form

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s + R_{k+1}, & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z_\alpha'') + R_{k+1}, & \text{for } p = m+1, \dots, n, \end{cases}$$

where $z_\alpha'' = (z_\alpha^{m+1}, \dots, z_\alpha^n)$ are local coordinates on S, and R_{k+1} denotes a term belonging to $\mathcal{I}_S^{k+1}$, that is vanishing of order at least $k+1$ along S. We remark that in the total space of the normal bundle N_S we can always find an atlas (the natural one induced by any adapted atlas of M) with changes of coordinates of the form

$$\begin{cases} z_\beta^r = \sum_{s=1}^m (a_{\beta\alpha})_s^r (z_\alpha'') z_\alpha^s, & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\beta\alpha}^p (z_\alpha''), & \text{for } p = m+1, \dots, n; \end{cases}$$

therefore we may say that the embedding of a k-splitting and $(k-1)$-comfortably embedded submanifold looks, up to order k, like its embedding into the normal bundle.

This is the reason why we were led to study the classical problem of comparing the embedding of a submanifold S in a complex manifold M with the embedding of S in the normal bundle N_S as zero section. In particular, one would like to know when these two embeddings are holomorphically equivalent, that is when there is a neighbourhood U of S in M biholomorphic to a neighbourhood of the zero section in N_S (in some sense, such a U would be a holomorphic tubular neighbourhood of S in M). In most approaches to this problem (see, e.g., [3,7–9,11,12, 14,16–18,22] and references therein) the first step consists in showing that the two embeddings are biholomorphic up to a finite order k (and we shall say that S is *k-linearizable*) if a suitable cohomology class vanishes. Then one gives geometrical conditions ensuring the vanishing of all the involved cohomology groups, and that if the two embeddings are formally isomorphic (that is, biholomorphic up to any finite order) then they are actually biholomorphic.

The main result of this paper is a direct proof (see Theorem 4.1) of the fact that a complex submanifold is k-linearizable in this sense if and only if it is k-splitting and $(k-1)$-comfortably

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embedded. Combining this with our results on the existence of suitable local coordinates, we are then able to explicitly write two cohomology classes providing the obstructions from k-linearizable to $(k+1)$-linearizable. It should be remarked that most other authors (see, e.g., [11]) first gives this obstruction as a single cohomology class, and then use a formal argument to split this class in two; our approach explains instead the geometrical meaning of the independent vanishing of any of the two classes. Furthermore, our results hold for any codimension, and not only in codimension one.

In Section 5 we exemplify our results in the case of a compact Riemann surface S embedded in a complex surface M. In particular, we are able to recover results originally proved in [8] under the slightly stronger assumption that S is fibered embedded into M (which implies, in particular, that S is k-splitting for any $k \geqslant 1$).

Finally, Section 6 is devoted to a slightly different characterization of 1-comfortably embedded submanifolds. In [1,2] we showed that the 1-comfortably embedded condition can be used to define holomorphic connections on suitable vector bundles; here we show that a possible justification for this phenomenon is that 1-comfortably embedded is exactly equivalent to the existence of an infinitesimal holomorphic connection on N_S.

1. Holomorphic splitting

Let us begin by recalling some general terminology on exact sequences of sheaves. We say that an exact sequence of sheaves (of abelian groups, rings, modules...)

$$O \longrightarrow \mathcal{R} \xrightarrow{\iota} \mathcal{S} \xrightarrow{p} \mathcal{T} \longrightarrow O \tag{1.1}$$

on a variety S *splits* if there is a morphism $\sigma : \mathcal{T} \to \mathcal{S}$ of sheaves (of abelian groups, rings, modules...) such that $p \circ \sigma = \mathrm{id}$. Any such morphism is called a *splitting morphism*. It is easy to see that (1.1) splits (as sequence of sheaves of modules) if and only if there exists a *left splitting morphism*, that is a morphism of sheaves of modules $\tau : \mathcal{S} \to \mathcal{R}$ such that $\tau \circ \iota = \mathrm{id}$. Furthermore, for every splitting morphism σ there exists a unique left splitting morphism τ such that

$$\iota \circ \tau + \sigma \circ p = \mathrm{id}. \tag{1.2}$$

Following Grothendieck and Atiyah (see [4]), one can give a cohomological characterization of splitting for sequences of locally free $\mathcal{O}_S$-modules defined over a complex manifold S.

Let $\mathcal{E}'$ and $\mathcal{E}''$ be two sheaves of locally free $\mathcal{O}_S$-modules over the same complex manifold S. An *extension* of $\mathcal{E}''$ by $\mathcal{E}'$ is an exact sequence of locally free $\mathcal{O}_S$-modules

$$O \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow O. \tag{1.3}$$

If $O \to \mathcal{E}' \to \tilde{\mathcal{E}} \to \mathcal{E}'' \to O$ is another extension of $\mathcal{E}''$ by $\mathcal{E}'$, one says that the two extensions are *equivalent* if there is an isomorphism $\chi : \mathcal{E} \to \tilde{\mathcal{E}}$ of $\mathcal{O}_S$-modules such that

$$\begin{tikzcd} O \arrow[r] & \mathcal{E}' \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d, "\chi"] & \mathcal{E}'' \arrow[r] \arrow[d, equal] & O \\ O \arrow[r] & \mathcal{E}' \arrow[r] & \tilde{\mathcal{E}} \arrow[r] & \mathcal{E}'' \arrow[r] & O \end{tikzcd}$$

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commutes. Almost by definition, an extension of $\mathcal{E}''$ by $\mathcal{E}'$ splits if and only if it is equivalent to the trivial extension $O \to \mathcal{E}' \to \mathcal{E}' \oplus \mathcal{E}'' \to \mathcal{E}'' \to O$.

Applying the functor $\mathrm{Hom}(\mathcal{E}'', \cdot)$ to the sequence (1.3) one gets the exact sequence

$$O \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}) \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'') \longrightarrow O. \tag{1.4}$$

Let $\delta : H^0(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'')) \to H^1(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'))$ be the connecting homomorphism in the long exact cohomology sequence of (1.4). Then one can associate to the exact sequence (1.3) the cohomology class

$$\delta(\mathrm{id}_{\mathcal{E}''}) \in H^1\big(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}')\big).$$

This procedure gives a 1-to-1 correspondence between the cohomology group $H^1(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'))$ and isomorphism classes of extensions of $\mathcal{E}''$ by $\mathcal{E}'$ (see [4, Proposition 1.2]):

Proposition 1.1. *Let S be a complex manifold, and $\mathcal{E}'$ and $\mathcal{E}''$ two locally free $\mathcal{O}_S$-modules. Then two extensions of $\mathcal{E}''$ by $\mathcal{E}'$ are equivalent if and only if they correspond to the same cohomology class in $H^1(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'))$. In particular, the exact sequence* (1.3) *splits if and only if it corresponds to the zero cohomology class.*

Let us now introduce the sheaves we are interested in. Let M be a complex manifold of dimension n, and let S be a reduced, globally irreducible subvariety of M of codimension $m \geqslant 1$. We denote: by $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on M; by $\mathcal{I}_S$ the subsheaf of $\mathcal{O}_M$ of germs vanishing on S; and by $\mathcal{O}_S$ the quotient sheaf $\mathcal{O}_M/\mathcal{I}_S$ of germs of holomorphic functions on S. Furthermore, let $\mathcal{T}_M$ denote the sheaf of germs of holomorphic sections of the holomorphic tangent bundle TM of M, and Ω_M the sheaf of germs of holomorphic 1-forms on M. Finally, we shall denote by $\mathcal{T}_{M,S}$ the sheaf of germs of holomorphic sections along S of the restriction $TM|_S$ of TM to S, and by $\Omega_{M,S}$ the sheaf of germs of holomorphic sections along S of $T^*M|_S$. It is easy to check that $\mathcal{T}_{M,S} = \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ and $\Omega_{M,S} = \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$.

For $k \geqslant 1$ we shall denote by $f \mapsto [f]_k$ the canonical projection of $\mathcal{O}_M$ onto $\mathcal{O}_M/\mathcal{I}_S^k$. The *cotangent sheaf* Ω_S of S is defined by

$$\Omega_S = \Omega_{M,S}/d_2\big(\mathcal{I}_S/\mathcal{I}_S^2\big),$$

where $d_2 : \mathcal{O}_M/\mathcal{I}_S^2 \to \Omega_{M,S}$ is given by $d_2[f]_2 = df \otimes [1]_1$. In particular, we have the *conormal sequence* of sheaves of $\mathcal{O}_S$-modules associated to S:

$$\mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow O. \tag{1.5}$$

Applying the functor $\mathrm{Hom}_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ to the conormal sequence we get the *normal sequence* of sheaves of $\mathcal{O}_S$-modules associated to S:

$$O \longrightarrow \mathcal{T}_S \xrightarrow{\iota} \mathcal{T}_{M,S} \xrightarrow{p_2} \mathcal{N}_S, \tag{1.6}$$

where $\mathcal{T}_S = \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ is the *tangent sheaf* of S, $\mathcal{N}_S = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_S/\mathcal{I}_S^2, \mathcal{O}_S)$ is the *normal sheaf* of S, and p_2 is the morphism dual to d_2.

The first condition we shall consider on the embedding of the variety S inside M is:

Definition 1.1. Let S be a reduced, globally irreducible subvariety of a complex manifold M. We say that S *splits* into M if there exists a morphism of sheaves of $\mathcal{O}_S$-modules $\sigma : \Omega_S \to \Omega_{M,S}$ such that $p \circ \sigma = \mathrm{id}$, where $p : \Omega_{M,S} \to \Omega_S$ is the canonical projection.

Remark 1.1. In the literature this notion has sometimes appeared under a different name; for instance, Morrow and Rossi in [19] say that the embedding $S \to M$ is direct.

It is not difficult to see that splitting subvarieties must be smooth:

Proposition 1.2. *Let S be a reduced, globally irreducible subvariety of a complex manifold M. Assume that S splits in M. Then S is non-singular, and the morphism $d_2 : \mathcal{I}_S/\mathcal{I}_S^2 \to \Omega_{M,S}$ is injective.*

Proof. Let us consider the sequence

$$O \longrightarrow \mathcal{K} \longrightarrow \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow O,$$

where $\mathcal{K} = d_2(\mathcal{I}_S/\mathcal{I}_S^2) = \mathrm{Ker}(p)$. This is a splitting exact sequence of $\mathcal{O}_S$-modules; let $\sigma : \Omega_S \to \Omega_{M,S}$ be a splitting morphism, and choose any $x \in S$. Then $\sigma((\Omega_S)_x)$ is a direct addend of the projective module $(\Omega_{M,S})_x$; therefore $\sigma((\Omega_S)_x)$ is itself projective and thus, being $\mathcal{O}_{S,x}$ Noetherian, it is $\mathcal{O}_{S,x}$-free. But σ is an injective $\mathcal{O}_S$-morphism; therefore $(\Omega_S)_x$ itself is $\mathcal{O}_{S,x}$-free for any $x \in S$.

Now, both $\Omega_{M,S}$ and Ω_S are coherent sheaves of $\mathcal{O}_S$-modules, and S is globally irreducible; therefore Ω_S is locally $\mathcal{O}_S$-free of constant rank r. Checking the rank at a point in the regular part of S we see that r must be equal to the dimension of S, and hence [13, Theorem II.8.15] S is non-singular, and [13, Theorem II.8.17] d_2 is injective. □

In particular, when S splits into M the sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow O \tag{1.7}$$

is a splitting exact sequence of $\mathcal{O}_S$-modules, and we also have a left splitting morphism $\tau : \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$.

Remark 1.2. If S splits into M then the normal sequence

$$O \longrightarrow \mathcal{T}_S \xrightarrow{\iota} \mathcal{T}_{M,S} \xrightarrow{p_2} \mathcal{N}_S \longrightarrow O \tag{1.8}$$

is a splitting exact sequence of $\mathcal{O}_S$-modules too: a splitting morphism is the dual $\tau^* : \mathcal{N}_S \to \mathcal{T}_{M,S}$ of a left splitting morphism of (1.7). Conversely, if S is a (reduced) locally complete intersection and (1.8) is exact, then S is non-singular (see, e.g., [21]) and so if moreover (1.8) splits then S splits into M. There are examples of singular varieties for which (1.8) is exact (see again [21]); ~~we do not know whether there are singular varieties for which (1.8) is exact and splits.~~

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The aim of this section is to describe several equivalent characterizations of splitting subvarieties. Most of them were already present in the literature; we collect them here because they provide a template for the study of the more stringent conditions on the embedding of S into M we shall study starting from the next section.

Definition 1.2. Let S be a reduced, globally irreducible subvariety of a complex manifold M. For any $k \geqslant h \geqslant 0$ let $\theta_{k,h} : \mathcal{O}_M/\mathcal{I}_S^{k+1} \to \mathcal{O}_M/\mathcal{I}_S^{h+1}$ be the canonical projection given by $\theta_{k,h}[f]_{k+1} = [f]_{h+1}$; when $h = 0$ we shall write θ_k instead of $\theta_{k,0}$. The *kth infinitesimal neighbourhood* of S in M is the ringed space $S(k) = (S, \mathcal{O}_M/\mathcal{I}_S^{k+1})$ together with the canonical inclusion of ringed spaces $\iota_k : S = S(0) \to S(k)$ given by $\iota_k = (\mathrm{id}_S, \theta_k)$. We shall also set $\mathcal{O}_{S(k)} = \mathcal{O}_M/\mathcal{I}_S^{k+1}$.

Definition 1.3. A *kth order infinitesimal retraction* is a morphism of ringed spaces $r : S(k) \to S$ such that $r \circ \iota_k = \mathrm{id}$. A kth order infinitesimal retraction is given by a pair $r = (\mathrm{id}_S, \rho)$, where $\rho : \mathcal{O}_S \to \mathcal{O}_{S(k)}$ is a ring morphism such that $\theta_k \circ \rho = \mathrm{id}$. So, the existence of a kth order infinitesimal retraction is equivalent to the existence of a splitting morphism for the exact sequence of sheaves of rings

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \longrightarrow \mathcal{O}_{S(k)} \xrightarrow{\theta_k} \mathcal{O}_S \longrightarrow O. \tag{1.9}$$

Such a splitting morphism is called a *kth order lifting*. More generally, for $k \geqslant h \geqslant 0$ we shall say that $S(k)$ *retracts onto* $S(h)$ if there is a morphism of ringed spaces $r : S(k) \to S(h)$ such that $r \circ \iota_{h,k} = \mathrm{id}$, where $\iota_{h,k} = (\mathrm{id}_S, \theta_{k,h}) : S(h) \to S(k)$ is the natural inclusion.

Remark 1.3. It is easy to see that $\rho([1]_1) = [1]_{k+1}$ for any kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$. This is not an automatic consequence of ρ being a morphism of sheaves of rings but can be proved as follows: from $\theta_k \circ \rho = \mathrm{id}$ we get $[1]_{k+1} - \rho([1]_1) \in \mathcal{I}_S/\mathcal{I}_S^{k+1}$, that is $\rho([1]_1) = [1+h]_{k+1}$ for a suitable $h \in \mathcal{I}_S$. Now

$$[1+h]_{k+1} = \rho\bigl([1]_1\bigr) = \rho\bigl([1]_1\bigr)\rho\bigl([1]_1\bigr) = \bigl[(1+h)^2\bigr]_{k+1} = \bigl[1+2h+h^2\bigr]_{k+1},$$

and so $[h+h^2]_{k+1} = O$. But $[1+h]_{k+1}$ is a unit in $\mathcal{O}_{S(k)}$; therefore $[h]_{k+1} = O$, and $\rho([1]_1) = [1]_{k+1}$.

Definition 1.4. Let $\mathcal{O}$, $\mathcal{R}$ be sheaves of rings, $\theta : \mathcal{R} \to \mathcal{O}$ a morphism of sheaves of rings, and $\mathcal{M}$ a sheaf of $\mathcal{O}$-modules. A θ*-derivation* of $\mathcal{R}$ in $\mathcal{M}$ is a morphism of sheaves of abelian groups $D : \mathcal{R} \to \mathcal{M}$ such that

$$D(r_1 r_2) = \theta(r_1) \cdot D(r_2) + \theta(r_2) \cdot D(r_1)$$

for any r_1, $r_2 \in \mathcal{R}$. In other words, D is a derivation with respect to the $\mathcal{R}$-module structure induced via restriction of scalars by θ.

We can now state a first list of properties equivalent to splitting (see [20, Lemma 1.1] and [10, Proposition 16.12] for proofs) including the existence of first order infinitesimal retractions:

Proposition 1.3. *Let S be a reduced, globally irreducible subvariety of a complex manifold M. Then there is a 1-to-1 correspondence among the following classes of morphisms:*

(a) *morphisms $\sigma : \Omega_S \to \Omega_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p \circ \sigma = \mathrm{id}$;*
(b) *morphisms $\tau : \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$ of sheaves of $\mathcal{O}_S$-modules such that $\tau \circ d_2 = \mathrm{id}$;*
(c) *derivations $D : \mathcal{O}_M \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $D|_{\mathcal{I}_S} = \pi_2|_{\mathcal{I}_S}$;*
(d) *morphisms $\tau_M : \Omega_M \to \mathcal{I}_S/\mathcal{I}_S^2$ of sheaves of $\mathcal{O}_M$-modules such that $d_2 \circ \tau_M = \pi$, where $\pi : \Omega_M \to \Omega_{M,S}$ is the canonical projection;*
(e) *θ_1-derivations $\tilde{\rho} : \mathcal{O}_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $\tilde{\rho} \circ i_1 = \mathrm{id}$, where $i_1 : \mathcal{I}_S/\mathcal{I}_S^2 \hookrightarrow \mathcal{O}_{S(1)}$ is the canonical inclusion and $\theta_1 : \mathcal{O}_{S(1)} \to \mathcal{O}_M/\mathcal{I}_S$ is the canonical projection;*
(f) *morphisms $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ of sheaves of rings such that $\theta_1 \circ \rho = \mathrm{id}$.*

In particular, S splits into M if and only if it admits a first order infinitesimal retraction. Finally, if any (and hence all) of the classes (a)–(f) *is not empty, then it is in 1-to-1 correspondence with the following classes of morphisms too:*

(g) *morphisms $\tau^* : \mathcal{N}_S \to \mathcal{T}_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p_2 \circ \tau^* = \mathrm{id}$;*
(h) *morphisms $\sigma^* : \mathcal{T}_{M,S} \to \mathcal{T}_S$ of sheaves of $\mathcal{O}_S$-modules such that $\iota \circ \sigma^* = \mathrm{id}$.*

We have already noticed that a splitting subvariety is necessarily non-singular; therefore we can use differential geometric techniques to get another couple of characterizations of splitting submanifolds.

Definition 1.5. Let S be a complex submanifold (not necessarily closed) of codimension $m \geqslant 1$ in an n-dimensional complex manifold M, and let (U_α, z_α) a chart of M. We shall systematically write $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n) = (z'_\alpha, z''_\alpha)$, with $z'_\alpha = (z_\alpha^1, \ldots, z_\alpha^m)$ and $z''_\alpha = (z_\alpha^{m+1}, \ldots, z_\alpha^n)$. We shall say that (U_α, z_α) is *adapted* to S if either $U_\alpha \cap S = \emptyset$ or $U_\alpha \cap S = U_\alpha \cap \bar{S} = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}$. In particular, if (U_α, z_α) is adapted to S then $\{z_\alpha^1, \ldots, z_\alpha^m\}$ is a set of generators of $\mathcal{I}_{S,x}$ for all $x \in U_\alpha \cap S$. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of M is *adapted* to S if all its charts are; then $\mathfrak{U}_S = \{(U_\alpha \cap S, z''_\alpha) \mid U_\alpha \cap S \neq \emptyset\}$ is an atlas for S. We shall say that an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S is *projectable* if $z_\alpha \in U_\alpha$ implies $(O', z''_\alpha) \in U_\alpha \cap S$ for any U_α such that $U_\alpha \cap S \neq \emptyset$. Clearly, every atlas adapted to S can be refined to a projectable adapted atlas.

Definition 1.6. Let S be a complex submanifold (not necessarily closed) of codimension $m \geqslant 1$ in an n-dimensional complex manifold M. The *normal bundle* N_S of S in M is the quotient bundle $TM|_S/TS$; its dual is the *conormal bundle* N_S^*. If (U_α, z_α) is a chart adapted to S, for $r = 1, \ldots, m$ we shall denote by $\partial_{r,\alpha}$ the projection of $\partial/\partial z_\alpha^r|_{U_\alpha \cap S}$ in N_S, and by ω_α^r the local section of N_S^* induced by $dz_\alpha^r|_{U_\alpha \cap S}$. Then $\{\partial_{1,\alpha}, \ldots, \partial_{m,\alpha}\}$ and $\{\omega_\alpha^1, \ldots, \omega_\alpha^m\}$ are local frames over $U_\alpha \cap S$ for N_S and N_S^* respectively, dual to each other.

Remark 1.4. From now on, every chart and atlas we consider on M will be adapted to S. Furthermore, we shall use Einstein convention on the sum over repeated indices. Indices like j, h, k will run from 1 to n; indices like r, s, t, u, v will run from 1 to m; and indices like p, q will run from $m+1$ to n.

Remark 1.5. If (U_α, z_α) and (U_β, z_β) are two adapted charts with $U_\alpha \cap U_\beta \cap S \neq \emptyset$, then it is easy to check that

$$\left.\frac{\partial z_\beta^r}{\partial z_\alpha^p}\right|_S \equiv O \tag{1.10}$$

for all $r = 1, \dots, m$ and $p = m+1, \dots, n$.

Definition 1.7. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an adapted atlas for a complex submanifold S of codimension $m \geqslant 1$ of a complex n-dimensional manifold M. We say that $\mathfrak{U}$ is a *splitting atlas* (see [1,2]) if

$$\left.\frac{\partial z_\beta^p}{\partial z_\alpha^r}\right|_S \equiv O$$

for all $r = 1, \dots, m$, $p = m+1, \dots, n$ and indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. In other words, and recalling (1.10), the jacobian matrices of the changes of coordinates become block-diagonal when restricted to S.

Definition 1.8. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an atlas adapted to S. If $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ is a first order lifting for S, we say $\mathfrak{U}$ is *adapted* to ρ if

$$\rho\big([f]_1\big) = [f]_2 - \left[\frac{\partial f}{\partial z_\alpha^r} z_\alpha^r\right]_2 \tag{1.11}$$

for all $f \in \mathcal{O}(U_\alpha)$ and all indices α such that $U_\alpha \cap S \neq \emptyset$.

In [2] we proved the following characterization of splitting submanifolds:

Proposition 1.4. *Let S be a complex submanifold of codimension $m \geqslant 1$ of a n-dimensional complex manifold M. Then:*

(i) *the cohomology class $\mathfrak{s} \in H^1(S, \mathrm{Hom}(\Omega_S, \mathcal{N}_S^*))$ associated to the conormal exact sequence is represented by the 1-cocycle $\{\mathfrak{s}_{\beta\alpha}\} \in H^1(\mathfrak{U}_S, \mathrm{Hom}(\Omega_S, \mathcal{N}_S^*))$ given by*

$$\mathfrak{s}_{\beta\alpha} = -\left.\frac{\partial z_\beta^r}{\partial z_\alpha^s}\frac{\partial z_\alpha^p}{\partial z_\beta^r}\right|_S \omega_\alpha^s \otimes \frac{\partial}{\partial z_\alpha^p} \in H^0\big(U_\alpha \cap U_\beta \cap S, \mathcal{N}_S^* \otimes \mathcal{T}_S\big),$$

where $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ is an atlas adapted to S. In particular, S splits into M if and only if $\mathfrak{s} = O$;

(ii) *S splits into M if and only if there exists a splitting atlas for S in M;*

(iii) *an atlas adapted to S is splitting if and only if it is adapted to a first order lifting;*

(iv) *if S splits into M, then for any first order lifting there exists an atlas adapted to it.*

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Remark 1.6. Assume that $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ is a projectable atlas adapted to S. Then if $f \in \mathcal{O}(U_\alpha)$ we can write

$$f(z_\alpha) = f\big(O', z''_\alpha\big) + \frac{\partial f}{\partial z^r_\alpha}\big(O', z''_\alpha\big) z^r_\alpha + R_2 = f\big(O', z''_\alpha\big) + \frac{\partial f}{\partial z^r_\alpha}(z_\alpha) z^r_\alpha + R_2,$$

where R_2 denotes an element of $\mathcal{I}^2_S(U_\alpha)$, possibly changing from one occurrence to the next. From this formula it follows that $\mathfrak{U}$ is adapted to a first order lifting ρ if and only if

$$\rho\big([f]_1\big) = f\big(O', z''_\alpha\big) + R_2$$

for every $f \in \mathcal{O}(U_\alpha)$. In other words, $\mathfrak{U}$ is a splitting atlas if and only if we can patch together the trivial local liftings $[f]_1 \mapsto [f(O', z''_\alpha)]_2$ so to get a global first order lifting.

Remark 1.7. Given a first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M / \mathcal{I}^2_S$ and an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to S, it is not difficult to check that $\mathfrak{U}$ is adapted to ρ if and only if for every $(U_\alpha, z_\alpha) \in \mathfrak{U}$ with $U_\alpha \cap S \neq \emptyset$ and every $f \in \mathcal{O}_M|_{U_\alpha}$ one has

$$\tilde{\rho}\big([f]_2\big) = \left[\frac{\partial f}{\partial z^r_\alpha} z^r_\alpha\right]_2,$$

where $\tilde{\rho} : \mathcal{O}_{S(2)} \to \mathcal{I}_S / \mathcal{I}^2_S$ is the θ_1-derivation associated to ρ by Proposition 1.3.

As mentioned in the introduction, one of the aims of our constructions will be the comparison of the embedding of S into M with its embedding (as zero section) in the normal bundle. The first result of this kind is our last characterization of splitting submanifolds:

Proposition 1.5. *Let S be a submanifold of a complex manifold M. Then S splits into M if and only if its first infinitesimal neighbourhood $S(1)$ in M is isomorphic to its first infinitesimal neighbourhood $S_N(1)$ in N_S, where we are identifying S with the zero section of N_S.*

Proof. The main observation here is that if E is any vector bundle over S, then $TE|_S$ is canonically isomorphic to $TS \oplus E$. When $E = N_S$ this implies that the projection $TN_S|_S \to N_S$ on the second direct summand induces an isomorphism $N_{O_S} \to N_S$, where N_{O_S} is the normal bundle of S (or, more precisely, of the zero section of N_S) in N_S; in particular, then, S always splits in N_S (see also Example 1.1 below). Furthermore, this isomorphism induces an isomorphism between $\mathcal{N}^*_S$ and $\mathcal{N}^*_{O_S}$, and thus an isomorphism of sheaves of $\mathcal{O}_S$-modules $\chi : \mathcal{I}_{S,N_S}/\mathcal{I}^2_{S,N_S} \to \mathcal{I}_S/\mathcal{I}^2_S$, where $\mathcal{I}_{S,N_S}$ is the ideal sheaf of S in N_S.

By definition, an isomorphism between $S_N(1)$ and $S(1)$ is given by an isomorphism of sheaves of rings $\psi : \mathcal{O}_{N_S}/\mathcal{I}^2_{S,N_S} \to \mathcal{O}_M/\mathcal{I}^2_S$ such that $\theta_1 \circ \psi = \theta^N_1$, where $\theta^N_1 : \mathcal{O}_{N_S}/\mathcal{I}^2_{S,N_S} \to \mathcal{O}_S$ is the canonical projection.

If $S_N(1)$ and $S(1)$ are isomorphic, we can define a morphism of sheaves of rings $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}^2_S$ by setting $\rho = \psi \circ \rho^N$, where ρ^N is the first order lifting induced by the splitting of S in N_S described above. Then it is easy to see that $\theta_1 \circ \rho = \mathrm{id}$, and thus S splits in M by Proposition 1.3.

Conversely, assume that S splits in M, and let $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^2$ be a first order lifting. Then we can define a morphism $\psi : \mathcal{O}_{N_S}/\mathcal{I}_{S,N_S}^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ by setting

$$\psi = \rho \circ \theta_1^N + i_1 \circ \chi \circ \tilde{\rho}^N,$$

where $\tilde{\rho}^N$ is the θ_1^N-derivation associated to the first order lifting ρ^N and $i_1 : \mathcal{I}_S/\mathcal{I}_S^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ is the canonical inclusion. Then it is not difficult to check that ψ is an isomorphism of sheaves of rings such that $\theta_1 \circ \psi = \theta_1^N$, and thus $S_N(1)$ and $S(1)$ are isomorphic. □

Example 1.1. A local holomorphic retract is always split in the ambient manifold (and thus it is necessarily non-singular). Indeed, if $p : U \to S$ is a local holomorphic retraction, then a first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^2$ is given by $\rho(f) = [f \circ p]_2$. In particular, the zero section of a vector bundle always splits, as well as any slice $S \times \{x\}$ in a product $M = S \times X$ (with both S and X non-singular, of course).

Example 1.2. If S is a Stein submanifold of a complex manifold M (e.g., if S is an open Riemann surface), then S splits into M. Indeed, we have $H^1(S, \mathcal{T}_S \otimes \mathcal{N}_S^*) = (O)$ by Cartan's Theorem B, and the assertion follows from Proposition 1.4.(i). In particular, if S is a *singular* curve in M then the non-singular part of S always splits in M.

Example 1.3. Let S be a non-singular, compact, irreducible curve of genus g on a surface M. If $S \cdot S < 4 - 4g$ then S splits into M. In fact, the Serre duality for Riemann surfaces implies that

$$H^1\big(S, \mathcal{T}_S \otimes \mathcal{N}_S^*\big) \cong H^0(S, \Omega_S \otimes \Omega_S \otimes \mathcal{N}_S),$$

and the latter group vanishes because the line bundle $T^*S \otimes T^*S \otimes N_S$ has negative degree by assumption. The bound $S \cdot S < 4 - 4g$ is sharp: for instance, a non-singular compact projective plane conic S has genus $g = 0$ and self-intersection $S \cdot S = 4$, but it does not split in the projective plane (see [19,20,23]).

Example 1.4. Let M be an algebraic surface embedded in $\mathbf{P}^n$ and let S be a section of M with an hyperplane H, with the property that there exists a point $P \notin H$ belonging to each plane tangent to M in points of S. Then S splits in M. In [5], ~~the authors~~ show a partial converse: if S splits in M and the natural morphism $H^0(S, \Omega_S) \otimes H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) \to H^0(S, \Omega_S(1))$ is injective, then there exists a point $P \notin H$ belonging to each plane tangent to M in a point of S.

Example 1.5. Let S be a compact Riemann surface of genus $g > 0$, and $\phi : \pi_1(S) \to \mathrm{Diff}_0(\mathbb{C}^n)$ be a representation of the fundamental group of S into the group of germs of biholomorphisms of $\mathbb{C}^n$ fixing the origin; assume that all the elements of the image of ϕ are convergent on some polydisk $\Delta \subseteq \mathbb{C}^n$ centered at the origin. If $\tilde{S}$ is the universal covering space of S, we shall also identify $\pi_1(S)$ with the group of the automorphisms of the covering. The *suspension* M of the representation ϕ is by definition the quotient of $\Delta \times \tilde{S}$ obtained identifying $(z, \tilde{p})$ and $(w, \tilde{q})$ if and only if there exists $\gamma \in \pi_1(S)$ such that $(w, \tilde{q}) = (\rho(\gamma)(z), \gamma \cdot \tilde{p})$. Then S embeds into M as the 0-slice, that splits into M.

Other examples of splitting submanifolds are discussed in [2] and [15].

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JID:YAIMA AID:3134 /FLA [m1+; v 1.104; Prn:21/10/2008; 10:38] P.12 (1-37)

2. k-splitting submanifolds

In the previous section we have seen that a complex submanifold S of a complex manifold M splits into M if and only if the sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{\iota_1} \mathcal{O}_M/\mathcal{I}_S^2 \xrightarrow{\theta_1} \mathcal{O}_M/\mathcal{I}_S \longrightarrow O \tag{2.1}$$

splits as a sequence of sheaves of rings. This suggests a natural generalization:

Definition 2.1. Let S be a submanifold of a complex manifold M, and $k \geqslant 1$. We shall say that S *k-splits* (or is *k-splitting*) into M if there is an infinitesimal retraction of $S(k)$ onto S, that is if there is a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$, or, in still other words, if the exact sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \hookrightarrow \mathcal{O}_M/\mathcal{I}_S^{k+1} \xrightarrow{\theta_k} \mathcal{O}_M/\mathcal{I}_S \longrightarrow O \tag{2.2}$$

splits as sequence of sheaves of rings.

Remark 2.1. In [12, p. 373] a k-splitting submanifold is called k-transversely foliated.

The main result of this section is a characterization of k-splitting submanifolds along the lines of Proposition 1.4. To state it, we need the analogue of Definitions 1.7 and 1.8:

Definition 2.2. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an adapted atlas for a complex submanifold S of codimension $m \geqslant 1$ of a complex n-dimensional manifold M, and let $k \geqslant 1$. We say that $\mathfrak{U}$ is a *k-splitting atlas* (see [1,2]) if

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^k \tag{2.3}$$

for all $r = 1, \dots, m$, $p = m+1, \dots, n$ and indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$.

Definition 2.3. We shall say that an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S is *adapted* to a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ if

$$\rho[f]_1 = \sum_{l=0}^{k} (-1)^l \frac{1}{l!} \left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l} \right]_{k+1}, \tag{2.4}$$

for every $f \in \mathcal{O}(U_\alpha)$ and all indices α such that $U_\alpha \cap S \neq \emptyset$.

Then:

Theorem 2.1. *Let S be an m-codimensional submanifold of an n-dimensional complex manifold M. Then:*

(i) *S is k-splitting into M if and only if there exists a k-splitting atlas;*
(ii) *an atlas adapted to S is k-splitting if and only if it is adapted to a kth order lifting;*

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(iii) *a projectable atlas adapted to S is k-splitting if and only if the local kth order liftings*

$$\rho_\alpha\big([f]_1\big) = f\big(O', z''_\alpha\big) + \mathcal{I}_S^{k+1} \tag{2.5}$$

patch together to define a global kth order lifting;

(iv) *if S is k-splitting into M then every kth order lifting admits an atlas adapted to it.*

Proof. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a projectable adapted atlas, and $f \in \mathcal{O}(U_\alpha)$; first of all we would like to prove that

$$\rho_\alpha\big([f]_1\big) = \sum_{l=0}^{k} (-1)^l \frac{1}{l!}\left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l}\right]_{k+1}. \tag{2.6}$$

Let us proceed by induction on k. For $k = 1$ we have already proved this in Remark 1.7; so assume that (2.6) holds for $k - 1$. Then we can write

$$\begin{aligned}
\rho_\alpha\big([f]\big)_1 &= [f]_{k+1} - \sum_{j=1}^{k} \frac{1}{j!}\left[\frac{\partial^j f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_j}}\big(O', z''_\alpha\big) z_\alpha^{r_1} \cdots z_\alpha^{r_j}\right]_{k+1} \\
&= [f]_{k+1} - \sum_{j=1}^{k} \frac{1}{j!} \sum_{h=0}^{k-j} (-1)^h \frac{1}{h!}\left[\frac{\partial^{j+h} f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_{j+h}}} z_\alpha^{r_1} \cdots z_\alpha^{r_{j+h}}\right]_{k+1} \\
&= [f]_{k+1} - \sum_{l=1}^{k} \left(\sum_{h=0}^{l-1} (-1)^h \frac{l!}{h!(l-h)!}\right) \frac{1}{l!}\left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l}\right]_{k+1} \\
&= \sum_{l=0}^{k} (-1)^l \frac{1}{l!}\left[\frac{\partial^l f}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} z_\alpha^{r_1} \cdots z_\alpha^{r_l}\right]_{k+1},
\end{aligned}$$

as claimed. In particular, the right-hand side of (2.6) is a ring morphism, and to get (ii) it suffices to prove (iii).

Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a projectable atlas adapted to S, and assume there is $0 \leqslant l \leqslant k$ such that

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^l$$

for all $r = 1, \ldots, m$ and $p = m + 1, \ldots, n$, which is equivalent to assuming that

$$\frac{\partial^l z_\beta^p}{\partial z_\alpha^{r_1} \cdots \partial z_\alpha^{r_l}} \in \mathcal{I}_S$$

for all $r_1, \ldots, r_l = 1, \ldots, m$ and all $p = m + 1, \ldots, n$. Then it easy to prove by induction on l that we can write

$$z_\alpha^p(z_\beta) = \phi_{\alpha\beta}^p\big(z''_\beta\big) + h_{r_1 \cdots r_{l+1}}^p(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}} \tag{2.7}$$

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for suitable $\phi^p_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta \cap S)$ and $h^p_{r_a \ldots r_{l+1}} \in \mathcal{O}(U_\alpha \cap U_\beta)$, symmetric in the lower indices; clearly, $\phi_{\beta\alpha} \circ \phi_{\alpha\beta} = \mathrm{id}$.

To simplify the understanding of the subsequent computations, we shall explicitly use the local chart $\varphi_\alpha : U_\alpha \to \mathbb{C}^n$ associated to (U_α, z_α). Now let (U_α, z_α) and $(U_\beta, z_\beta) \in \mathfrak{U}$ such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$; we need to evaluate $\rho_\beta([f]_1) - \rho_\alpha([f]_1)$. First of all we have

$$\begin{aligned}\rho_\beta\big([f]_1\big) - \rho_\alpha\big([f]_1\big) &= f \circ \varphi_\beta^{-1}\big(O', z''_\beta\big) - f \circ \varphi_\alpha^{-1}\big(O', z''_\alpha\big) + \mathcal{I}_S^{k+1} \\ &= f \circ \varphi_\beta^{-1}\big(O', z''_\beta\big) - f \circ \varphi_\beta^{-1}\big(\varphi_\beta \circ \varphi_\alpha^{-1}\big(O', \big(\varphi_\alpha \circ \varphi_\beta^{-1}\big)''(z_\beta)\big)\big) + \mathcal{I}_S^{k+1}.\end{aligned}$$

Now, (2.7) yields

$$\varphi_\beta \circ \varphi_\alpha^{-1}\big(O', \big(\varphi_\alpha \circ \varphi_\beta^{-1}\big)''(z_\beta)\big) = \big(O', \phi_{\beta\alpha}\big(\phi_{\alpha\beta}\big(z''_\beta\big) + h_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}}\big)\big)$$

and

$$\begin{aligned}\phi_{\beta\alpha}\big(\phi_{\alpha\beta}\big(z''_\beta\big) + h_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}}\big) &= z''_\beta + \frac{\partial \phi_{\beta\alpha}}{\partial z^p_\alpha}\big(\phi_{\alpha\beta}\big(z''_\beta\big)\big) h^p_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}} + R_{l+2} \\ &= z''_\beta + \frac{\partial z''_\beta}{\partial z^p_\alpha}\big(O', z''_\alpha\big) h^p_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}} + R_{l+2},\end{aligned}$$

where, here and elsewhere, R_j denotes a term with elements in $\mathcal{I}_S^j$. Therefore we get

$$\begin{aligned}\rho_\beta\big([f]_1\big) - \rho_\alpha\big([f]_1\big) &= -\frac{\partial z^q_\beta}{\partial z^p_\alpha} h^p_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}} \frac{\partial f}{\partial z^q_\beta} + R_{l+2} + \mathcal{I}_S^{k+1} \\ &= -h^p_{r_1 \cdots r_{l+1}}(z_\beta) z_\beta^{r_1} \cdots z_\beta^{r_{l+1}} \frac{\partial f}{\partial z^p_\alpha} + R_{l+2} + \mathcal{I}_S^{k+1}.\end{aligned}$$

In particular, if $l = k$ we get $\rho_\alpha \equiv \rho_\beta$, and thus if $\mathfrak{U}$ is a k-splitting atlas we get a global kth order lifting, proving one direction in (i), (ii) and (iii). Conversely, if $l < k$ then $\rho_\alpha \not\equiv \rho_\beta$, and thus we obtain the other direction in (ii) and (iii).

For later use, we explicitly remark that if $l = k - 1$ then

$$\rho_\beta - \rho_\alpha = -h^p_{r_1 \cdots r_k}\big|_S \frac{\partial}{\partial z^p_\alpha} \otimes \big[z_\beta^{r_1} \cdots z_\beta^{r_k}\big]_{k+1} \in H^0\big(U_\alpha \cap U_\beta \cap S, \mathcal{T}_S \otimes \mathcal{I}_S^k / \mathcal{I}_S^{k+1}\big). \tag{2.8}$$

Furthermore, it is easy to see that

$$h^p_{r_1 \cdots r_k}\big|_S = \frac{1}{k!} \frac{\partial z^p_\alpha}{\partial z_\beta^{r_1} \cdots \partial z_\beta^{r_k}}\bigg|_S. \tag{2.9}$$

Now, let us assume that we have a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M / \mathcal{I}_S^{k+1}$; we claim that there exists an atlas adapted to ρ. This will yield (iv) and the missing direction in (i), completing the proof.

We shall argue by induction on k. For $k = 1$ the assertion follows from Proposition 1.4. Now let $k > 1$. Then $\rho_1 = \theta_{k,k-1} \circ \rho$ is a $(k-1)$th order lifting; let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a (necessarily

$(k-1)$-splitting) projectable atlas adapted to ρ_1. Define local kth order liftings ρ_α as in (2.5), and set $\sigma_\alpha = \rho - \rho_\alpha$. Now

$$\theta_{k,k-1} \circ \sigma_\alpha = \rho_1 - \theta_{k,k-1} \circ \rho_\alpha \equiv O,$$

because the atlas is adapted to ρ_1; therefore the image of σ_α is contained in $\mathcal{I}_S^k/\mathcal{I}_S^{k+1}$. The latter is an $\mathcal{O}_S$-module; we claim that $\sigma_\alpha : \mathcal{O}_S|_{U_\alpha \cap S} \to \mathcal{I}_S^k/\mathcal{I}_S^{k+1}|_{U_\alpha \cap S}$ is a derivation. Indeed,

$$\begin{aligned}\sigma_\alpha(fg) &= \rho(f)\rho(g) - \rho_\alpha(f)\rho_\alpha(g) = \rho(f)(\rho - \rho_\alpha)(g) + \rho_\alpha(g)(\rho - \rho_\alpha)(f)\\ &= f \cdot \sigma_\alpha(g) + g \cdot \sigma_\alpha(f),\end{aligned}$$

because $\sigma_\alpha(f), \sigma_\alpha(g) \in \mathcal{I}_S^k/\mathcal{I}_S^{k+1}$, and $uv = \theta_k(u) \cdot v$ for all $u \in \mathcal{O}_M/\mathcal{I}_S^{k+1}$ and $v \in \mathcal{I}_S^k/\mathcal{I}_S^{k+1}$.

Hence we can find $(s_\alpha)^p_{r_1 \cdots r_k} \in \mathcal{O}(U_\alpha \cap S)$, symmetric in the lower indices, such that

$$\sigma_\alpha = (s_\alpha)^p_{r_1 \cdots r_k} \frac{\partial}{\partial z_\alpha^p} \otimes \left[z_\alpha^{r_1} \cdots z_\alpha^{r_k}\right]_{k+1}.$$

Now, by construction $\sigma_\alpha - \sigma_\beta = \rho_\beta - \rho_\alpha$; therefore (2.8) yields

$$h^p_{s_1 \cdots s_k} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_k}}{\partial z_\alpha^{r_k}} + (s_\alpha)^p_{r_1 \cdots r_k} - \frac{\partial z_\alpha^p}{\partial z_\beta^q} (s_\beta)^q_{s_1 \cdots s_k} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_k}}{\partial z_\alpha^{r_k}} \in \mathcal{I}_S,$$

and then

$$h^p_{s_1 \cdots s_{k-1} r} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_{k-1}}}{\partial z_\alpha^{r_{k-1}}} + (s_\alpha)^p_{r_1 \cdots r_k} \frac{\partial z_\alpha^{r_k}}{\partial z_\beta^r} - \frac{\partial z_\alpha^p}{\partial z_\beta^q} (s_\beta)^q_{s_1 \cdots s_{k-1} r} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_{k-1}}}{\partial z_\alpha^{r_{k-1}}} \in \mathcal{I}_S. \tag{2.10}$$

Let us then consider the change of coordinates

$$\begin{cases} \hat{z}_\alpha^r = z_\alpha^r, \\ \hat{z}_\alpha^p = z_\alpha^p + (s_\alpha)^p_{r_1 \cdots r_k}(z_\alpha'') z_\alpha^{r_1} \cdots z_\alpha^{r_k}, \end{cases}$$

defined in suitable open sets $\hat{U}_\alpha \subseteq U_\alpha$; we claim that $\{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ is the atlas we are looking for. Indeed, we have

$$\begin{aligned}\frac{\partial \hat{z}_\alpha^p}{\partial \hat{z}_\beta^r} &= \frac{\partial \hat{z}_\alpha^p}{\partial z_\beta^s} \frac{\partial z_\beta^s}{\partial \hat{z}_\beta^r} + \frac{\partial \hat{z}_\alpha^p}{\partial z_\beta^q} \frac{\partial z_\beta^q}{\partial \hat{z}_\beta^r} = \frac{\partial \hat{z}_\alpha^p}{\partial z_\beta^r} + \frac{\partial \hat{z}_\alpha^p}{\partial z_\beta^q} \frac{\partial z_\beta^q}{\partial \hat{z}_\beta^r}\\ &= \frac{\partial z_\alpha^p}{\partial z_\beta^r} + k\left[(s_\alpha)^p_{r_1 \cdots r_k} z_\alpha^{r_1} \cdots z_\alpha^{r_{k-1}} \frac{\partial z_\alpha^{r_k}}{\partial z_\beta^r} - \frac{\partial z_\alpha^p}{\partial z_\beta^q} (s_\beta)^q_{s_1 \cdots s_{k-1} r} z_\beta^{s_1} \cdots z_\beta^{s_{k-1}}\right] + R_k.\end{aligned}$$

Now, (2.7) with $l = k-1$ yields

$$\frac{\partial z_\alpha^p}{\partial z_\beta^r} = k h^p_{s_1 \cdots s_{k-1} r} z_\beta^{s_1} \cdots z_\beta^{s_{k-1}} + R_k,$$

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and so

$$\begin{aligned}
\frac{\partial \hat z_\alpha^p}{\partial \hat z_\beta^r} &= k\biggl[h^p_{s_1\cdots s_{k-1}r} z_\beta^{s_1}\cdots z_\beta^{s_{k-1}} + (s_\alpha)^p_{r_1\cdots r_k} z_\alpha^{r_1}\cdots z_\alpha^{r_{k-1}}\frac{\partial z_\alpha^{r_k}}{\partial z_\beta^r}\\
&\quad - \frac{\partial z_\alpha^p}{\partial z_\beta^q}(s_\beta)^q_{s_1\cdots s_{k-1}r} z_\beta^{s_1}\cdots z_\beta^{s_{k-1}}\biggr] + R_k\\
&= k\biggl[h^p_{s_1\cdots s_{k-1}r}\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{k-1}}}{\partial z_\alpha^{r_{k-1}}} + (s_\alpha)^p_{r_1\cdots r_k}\frac{\partial z_\alpha^{r_k}}{\partial z_\beta^r}\\
&\quad - \frac{\partial z_\alpha^p}{\partial z_\beta^q}(s_\beta)^q_{s_1\cdots s_{k-1}r}\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{k-1}}}{\partial z_\alpha^{r_{k-1}}}\biggr] z_\alpha^{r_1}\cdots z_\alpha^{r_{k-1}} + R_k\\
&= R_k \in \mathcal{I}_S^k
\end{aligned}$$

thanks to (2.10), where we used the fact that $z_\alpha^{r_k} = (\partial z_\alpha^{r_k}/\partial z_\beta^r) z_\beta^r + R_2$.

Finally, we should check that $\{(\hat U_\alpha, \hat z_\alpha)\}$ is adapted to ρ. But indeed (2.8) applied with $\hat z_\alpha$ instead of z_β yields

$$f\bigl(O', \hat z_\alpha''\bigr) - f\bigl(O', z_\alpha''\bigr) = (s_\alpha)^p_{r_1\cdots r_k} z_\alpha^{r_1}\cdots z_\alpha^{r_k}\frac{\partial f}{\partial z_\alpha^p} + R_{k+1} = \sigma_\alpha(f) + R_{k+1};$$

hence

$$\rho\bigl([f]_1\bigr) = \rho_\alpha\bigl([f]_1\bigr) + \sigma_\alpha\bigl([f]_1\bigr) = f\bigl(O', \hat z_\alpha''\bigr) + \mathcal{I}_S^{k+1},$$

and the assertion follows from (2.6). □

Remark 2.2. In particular, there is an infinitesimal retraction of $S(k)$ onto S if and only if there is an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S whose coordinates changes are of the form

$$\begin{cases} z_\beta^r = (a_{\beta\alpha})^r_s(z_\alpha) z_\alpha^s & \text{for } r = 1, \ldots, m,\\ z_\beta^p = \phi^p_{\alpha\beta}\bigl(z_\alpha''\bigr) + R_{k+1} & \text{for } p = m+1, \ldots, n, \end{cases}$$

which, roughly speaking, says that a neighbourhood of M is a fiber bundle over S up to order k. The jets of the vector fields $\frac{\partial}{\partial z_\alpha^r}$ in $\mathcal{T}_M \otimes \mathcal{O}_{S(k)}$, for $r = 1, \ldots, m$, generate an infinitesimal foliation $\mathcal{F}_k$, i.e., an involutive submodule of $\mathcal{T}_M \otimes \mathcal{O}_{S(k)}$.

We explicitly compute the obstruction, predicted by [12, Proposition 1.6], for passing from $(k-1)$-split to k-split:

Proposition 2.2. *Let S be an m-codimensional submanifold of an n-dimensional complex manifold M. Assume that S is $(k-1)$-splitting in M; let $\rho_{k-1} : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^k$ be a $(k-1)$th order lifting, and $\{(U_\alpha, \varphi_\alpha)\}$ a $(k-1)$-splitting atlas adapted to ρ_{k-1}. Let $\mathfrak{g}_k \in$*

$H^1(S, \mathrm{Hom}(\Omega_S, \mathcal{I}_S^k/\mathcal{I}_S^{k+1}))$ be the cohomology class represented by the 1-cocycle $\{(\mathfrak{g}_k)_{\beta\alpha}\} \in H^1(\mathfrak{U}_S, \mathrm{Hom}(\Omega_S, \mathcal{I}_S^k/\mathcal{I}_S^{k+1}))$ given by

$$(\mathfrak{g}_k)_{\beta\alpha} = -\frac{1}{k!}\frac{\partial^k z_\alpha^p}{\partial z_\beta^{r_1}\cdots\partial z_\beta^{r_k}}\bigg|_S \frac{\partial}{\partial z_\alpha^p}\otimes\left[z_\beta^{r_1}\cdots z_\beta^{r_k}\right]_{k+1} \in H^0\big(U_\alpha\cap U_\beta\cap S, \mathcal{T}_S\otimes\mathcal{I}_S^k/\mathcal{I}_S^{k+1}\big).$$

Then there exists a kth order lifting $\rho_k : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ such that $\rho_{k-1} = \theta_{k-\rho_k,k}\circ\rho_k$ if and only if $\mathfrak{g}_k = O$.

Proof. One direction follows from the previous theorem, (2.8) and (2.9). Conversely, if $\mathfrak{g}_k = O$ up to shrinking the U_α we can find $(s_\alpha)^p_{r_1\cdots r_k} \in \mathcal{O}(U_\alpha \cap S)$ such that setting

$$\sigma_\alpha = (s_\alpha)^p_{r_1\cdots r_k}\frac{\partial}{\partial z_\alpha^p}\otimes\left[z_\alpha^{r_1}\cdots z_\alpha^{r_k}\right]_{k+1}$$

we get $(\mathfrak{g}_k)_{\beta\alpha} = \sigma_\alpha - \sigma_\beta$. Then arguing as in the last part of the proof of the previous theorem we find a k-splitting atlas, and we are done. □

3. Comfortably embedded submanifolds

The sequence (2.2) is only one of the possible natural generalizations of (2.1). Another, apparently as natural, generalization is the sequence

$$O \longrightarrow \mathcal{I}_S^2/\mathcal{I}_S^3 \hookrightarrow \mathcal{O}_M/\mathcal{I}_S^3 \xrightarrow{\theta_{2,1}} \mathcal{O}_M/\mathcal{I}_S^2 \longrightarrow O; \tag{3.1}$$

the splitting (as sequence of sheaves of rings) of this exact sequence is equivalent to the existence of an infinitesimal retraction of $S(2)$ onto $S(1)$. Surprisingly enough, this cannot ever happen:

Proposition 3.1. *Let S be a reduced, globally irreducible subvariety of a complex manifold M, and take $k > h \geqslant 1$. Assume there is an infinitesimal retraction of $S(k)$ onto $S(h)$; then*

$$\lceil (k+1)/2\rceil < h+1$$

(where $\lceil x\rceil$ is the smallest integer greater than or equal to x). In particular, there are no infinitesimal retractions of $S(k)$ onto $S(1)$ for any $k \geqslant 2$.

Proof. For any $1 \leqslant l \leqslant h$ consider the following commutative diagram with exact rows and columns

$$
\begin{tikzcd}
 & & O \arrow[d] & O \arrow[d] & \\
O \arrow[r] & \mathcal{I}_S^{h+1}/\mathcal{I}_S^{k+1} \arrow[r, hook] \arrow[d, equal] & \mathcal{I}_S^{l}/\mathcal{I}_S^{k+1} \arrow[r, "\theta_{k,h}"] \arrow[d] & \mathcal{I}_S^{l}/\mathcal{I}_S^{h+1} \arrow[r] \arrow[d] & O \\
O \arrow[r] & \mathcal{I}_S^{h+1}/\mathcal{I}_S^{k+1} \arrow[r, hook] & \mathcal{O}_M/\mathcal{I}_S^{k+1} \arrow[r, "\theta_{k,h}"] \arrow[d, "\theta_{k,l-1}"'] & \mathcal{O}_M/\mathcal{I}_S^{h+1} \arrow[r] \arrow[d, "\theta_{h,l-1}"'] & O \\
 & & \mathcal{O}_M/\mathcal{I}_S^{l} \arrow[r, equal] \arrow[d] & \mathcal{O}_M/\mathcal{I}_S^{l} \arrow[d] & \\
 & & O & O &
\end{tikzcd}
.
$$

By assumption, we have a morphism of sheaves of rings $\rho : \mathcal{O}_M/\mathcal{I}_S^{h+1} \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ such that $\theta_{k,h} \circ \rho = \mathrm{id}$. Composing with $\theta_{h,l-1}$ on the left we get

$$\theta_{h,l-1} = \theta_{h,l-1} \circ \theta_{k,h} \circ \rho = \theta_{k,l-1} \circ \rho.$$

This implies that $\rho(\mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}) \subseteq \mathcal{I}_S^{l}/\mathcal{I}_S^{k+1}$: indeed, if $u \in \mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}$ we have $\theta_{k,l-1}(\rho(u)) = \theta_{h,l-1}(u) = O$, and hence $\rho(u) \in \mathcal{I}_S^{l}/\mathcal{I}_S^{k+1}$.

Now, if $u \in \mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}$ we have $u^r = O$ as soon as $r \geqslant (h+1)/l$. Therefore if $r \geqslant (h+1)/l$ we have

$$O = \rho(u^r) = \rho(u)^r \in \mathcal{I}_S^{l}/\mathcal{I}_S^{k+1}$$

for all $u \in \mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}$. But since S is reduced, we have $v^r = O$ in $\mathcal{I}_S^{l}/\mathcal{I}_S^{k+1}$ if and only if $v \in \mathcal{I}_S^{p}/\mathcal{I}_S^{k+1}$ with $p \geqslant (k+1)/r$. Therefore if $\lceil (k+1)/r \rceil \geqslant h+1$ we have $\rho(\mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}) \subseteq \mathcal{I}_S^{h+1}/\mathcal{I}_S^{k+1}$, and thus $\theta_{k,h} \circ \rho|_{\mathcal{I}_S^{l}/\mathcal{I}_S^{h+1}} \equiv O$, impossible; therefore, $\lceil (k+1)/r \rceil < h+1$.

Now, the largest value of $\lceil (k+1)/r \rceil$ is attained for the lowest value of r; and since $r \geqslant (h+1)/l$, the lowest value of r is 2, attained taking $l = h$. Therefore we get $\lceil (k+1)/2 \rceil < h+1$, as claimed. $\square$

The lesson suggested by the previous proof is that if one would like to study the splitting of sequences of sheaves of rings like (3.1), it is important to take care of what happens in the nilpotent part of the rings, that is in the sheaves $\mathcal{I}_S^{h}/\mathcal{I}_S^{k}$. We observe that the sheaf $\mathcal{I}_S^{k}/\mathcal{I}_S^{k+1}$ is isomorphic to the symmetric power $\mathrm{Sym}^k(\mathcal{N}_S^*)$ of the conormal sheaf, and thus it naturally is an $\mathcal{O}_S$-module. The main new idea of this section is that when S is k-splitting then the sheaf $\mathcal{I}_S/\mathcal{I}_S^{k+1}$ too has a canonical structure of $\mathcal{O}_S$-module:

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Proposition 3.2. *Let S be a complex submanifold of codimension m of a complex manifold M, and let $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ be a kth order lifting, with $k \geqslant 0$. Then for any $1 \leqslant h \leqslant k+1$ the lifting ρ induces a structure of locally $\mathcal{O}_S$-free module on $\mathcal{I}_S/\mathcal{I}_S^{h+1}$ so that the sequence*

$$O \longrightarrow \mathcal{I}_S^h/\mathcal{I}_S^{h+1} \hookrightarrow \mathcal{I}_S/\mathcal{I}_S^{h+1} \xrightarrow{\theta_{h,h-1}} \mathcal{I}_S/\mathcal{I}_S^h \longrightarrow O \tag{3.2$_h$}$$

becomes an exact sequence of locally $\mathcal{O}_S$-free modules.

Proof. We shall work by induction on k. For $k = 0$ there is nothing to prove; so let us assume that the assertion holds for $k - 1$. As we already remarked, for any $h \geqslant 1$ the sheaf $\mathcal{I}_S^h/\mathcal{I}_S^{h+1}$ has a natural structure of locally free $\mathcal{O}_S$-module. The kth order lifting ρ induces a $(k-1)$-order lifting $\rho_1 = \theta_{k,k-1} \circ \rho$; therefore by induction for $1 \leqslant h \leqslant k$ we have a structure of locally free $\mathcal{O}_S$-module on $\mathcal{I}_S/\mathcal{I}_S^{h+1}$ so that all the $(3.2)_h$ become exact sequences of locally free $\mathcal{O}_S$-modules. Now, $\mathcal{I}_S/\mathcal{I}_S^{k+2}$ naturally is a $\mathcal{O}_M/\mathcal{I}_S^{k+1}$-module; we can then endow it with the $\mathcal{O}_S$-module structure obtained by restriction of the scalars via ρ:

$$v \cdot [h]_{k+2} = \rho(v) \cdot [h]_{k+2},$$

for all $v \in \mathcal{O}_S$ and $h \in \mathcal{I}_S$, where in the right-hand side we are using the $\mathcal{O}_M/\mathcal{I}_S^{k+1}$-module operation. Since ρ is a ring morphism and (by Remark 1.4) $\rho[1]_1 = [1]_{k+1}$, we get a well-defined structure of $\mathcal{O}_S$-module on $\mathcal{I}_S/\mathcal{I}_S^{k+2}$. We must verify that the inclusion $\iota : \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2} \hookrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+2}$ and the projection $\theta_{k+1,k}$ are $\mathcal{O}_S$-module morphisms when $\mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}$ has its own $\mathcal{O}_S$-structure and $\mathcal{I}_S/\mathcal{I}_S^{k+1}$ has the $\mathcal{O}_S$-structure induced by ρ_1 by induction.

Given $v \in \mathcal{O}_S$, choose $f \in \mathcal{O}_M$ such that $v = [f]_1$, and $f^{\mathcal{I}} \in \mathcal{I}_S$ so that $\rho(v) = [f + f^{\mathcal{I}}]_{k+1}$. Then for all $g \in \mathcal{I}_S^{k+1}$ we have

$$\iota\big(v \cdot [g]_{k+2}\big) = \iota\big([fg]_{k+2}\big) = [fg]_{k+2} = \big[\big(f + f^{\mathcal{I}}\big)g\big]_{k+2} = \rho(v) \cdot [g]_{k+2} = v \cdot \iota[g]_{k+2},$$

and ι is an $\mathcal{O}_S$-morphism.

Analogously, if $g \in \mathcal{I}_S$ we have

$$\begin{aligned}\theta_{k+1,k}\big(v \cdot [g]_{k+2}\big) = \theta_{k+1,k}\big[fg + f^{\mathcal{I}}g\big]_{k+2} = \big[fg + f^{\mathcal{I}}g\big]_{k+1} &= \rho_1(v) \cdot [g]_{k+1} \\ &= v \cdot \theta_{k+1,k}[g]_{k+2},\end{aligned}$$

and $\theta_{k+1,k}$ is an $\mathcal{O}_S$-morphism.

Finally, since (by induction) $\mathcal{I}_S/\mathcal{I}_S^{k+1}$ and $\mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}$ are locally $\mathcal{O}_S$-free, $\mathcal{I}_S/\mathcal{I}_S^{k+2}$ is locally $\mathcal{O}_S$-free too, and we are done. □

Remark 3.1. If $\{(U_\alpha, z_\alpha)\}$ is an atlas adapted to a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$, it is easy to see that $\{[z_\alpha^r]_{h+1}, [z_\alpha^{r_1} z_\alpha^{r_2}]_{h+1}, \ldots, [z_\alpha^{r_1} \cdots z_\alpha^{r_h}]_{h+1}\}$ is a local free set of generators of $\mathcal{I}_S/\mathcal{I}_S^{h+1}$ over $\mathcal{O}_S$ for $h = 1, \ldots, k+1$.

We are thus led to the following generalization of the notion of comfortably embedded submanifolds introduced in [1,2]:

Definition 3.1. Let S be a (not necessarily closed) submanifold of a complex manifold M, and let $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ be a kth order lifting, with $k \geqslant 1$. A *comfortable splitting sequence* $\boldsymbol{\nu}$ *associated to* ρ is a $(k+1)$-uple $\boldsymbol{\nu} = (\nu_{0,1}, \dots, \nu_{k,k+1})$, where for $1 \leqslant h \leqslant k+1$ each $\nu_{h-1,h} : \mathcal{I}_S/\mathcal{I}_S^h \to \mathcal{I}_S/\mathcal{I}_S^{h+1}$ is a splitting $\mathcal{O}_S$-morphism of the sequence $(3.2)_h$ with respect to the $\mathcal{O}_S$-module structures induced by ρ. A pair $(\rho, \boldsymbol{\nu})$, where ρ is a kth order lifting and $\boldsymbol{\nu}$ is a comfortable splitting sequence associated to ρ, is called a *k-comfortable pair* for S in M. We say that S is *k-comfortably embedded* in M with respect to ρ if it exists a k-comfortable pair $(\rho, \boldsymbol{\nu})$ for S in M.

Remark 3.2. The choice of a k-comfortable pair $(\rho, \boldsymbol{\nu})$ fixes an isomorphism of $\mathcal{O}_S$-modules

$$\mathcal{O}_M/\mathcal{I}_S^{k+1} \cong \mathcal{O}_S \oplus \mathcal{I}_S/\mathcal{I}_S^2 \oplus \mathcal{I}_S^2/\mathcal{I}_S^3 \oplus \cdots \oplus \mathcal{I}_S^k/\mathcal{I}_S^{k+1}.$$

The computation of the cohomology class associated to the exact sequence $(3.2)_h$ is not too difficult:

Proposition 3.3. *Let S be a complex submanifold of codimension m of a complex manifold M, and let $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ be a kth order lifting. Choose a projectable atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to ρ. Then the 1-cocycle $\{\mathfrak{h}^\rho_{\beta\alpha}\} \in H^1(\mathfrak{U}_S, \operatorname{Hom}(\mathcal{I}_S/\mathcal{I}_S^{k+1}, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}))$ given by*

$$\mathfrak{h}^\rho_{\beta\alpha}\left(\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1}\right) = -\frac{1}{(k+1)!} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_{k+1}}}{\partial z_\alpha^{r_{k+1}}} \left.\frac{\partial^{k+1}(z_\alpha^{t_1} \cdots z_\alpha^{t_h})}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_{k+1}}}\right|_S \left[z_\alpha^{r_1} \cdots z_\alpha^{r_{k+1}}\right]_{k+2} \tag{3.3}$$

for $1 \leqslant t_1, \dots, t_h \leqslant m$ and $1 \leqslant h \leqslant k$, represents the class $\mathfrak{h}^\rho \in H^1(S, \operatorname{Hom}(\mathcal{I}_S/\mathcal{I}_S^{k+1}, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}))$ associated to the exact sequence

$$O \to \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2} \to \mathcal{I}_S/\mathcal{I}_S^{k+2} \to \mathcal{I}_S/\mathcal{I}_S^{k+1} \to O, \tag{3.4}$$

where $\mathcal{I}_S/\mathcal{I}_S^{k+1}$ and $\mathcal{I}_S/\mathcal{I}_S^{k+2}$ have the $\mathcal{O}_S$-module structure induced by ρ.

Proof. We can define local splittings $\nu_\alpha : \mathcal{I}_S/\mathcal{I}_S^{k+1}|_{U_\alpha} \to \mathcal{I}_S/\mathcal{I}_S^{k+2}|_{U_\alpha}$ by setting

$$\nu_\alpha\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1} = \left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+2}$$

and extending by $\mathcal{O}_S$-linearity; since $\mathfrak{U}$ is adapted to ρ, Theorem 2.1 implies that each ν_α is a well-defined morphism of $\mathcal{O}_S$-modules.

Now, for any $f \in \mathcal{O}_M$ we can write

$$f\left(z'_\beta, z''_\beta\right) = f\left(O', z''_\beta\right) + \sum_{j=1}^{k+1} \frac{1}{j!} \frac{\partial^j f}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_j}}\left(O', z''_\beta\right) z_\beta^{s_1} \cdots z_\beta^{s_j} + R_{k+2}, \tag{3.5}$$

where $R_{k+2} \in \mathcal{I}_S^{k+2}$. In particular, Theorem 2.1(iii) implies

$$[h]_{k+2} = \sum_{j=1}^{k+1} \frac{1}{j!} \rho\left(\left[\frac{\partial^j h}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_j}}\right]_1\right) \cdot \left[z_\beta^{s_1} \cdots z_\beta^{s_j}\right]_{k+2},$$

for all $h \in \mathcal{I}_S$, and

$$[f]_{k+1} = \rho([f]_1) + \sum_{j=1}^{k} \frac{1}{j!}\rho\left(\left[\frac{\partial^j f}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_j}}\right]_1\right)\left[z_\beta^{s_1} \cdots z_\beta^{s_j}\right]_{k+1} \tag{3.6}$$

for all $f \in \mathcal{O}_M$.

Using these formulas we easily see that

$$\begin{aligned}
\mathfrak{h}^\rho_{\beta\alpha}\left(\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1}\right) &= \nu_\beta\left(\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1}\right) - \nu_\alpha\left(\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1}\right) \\
&= \nu_\beta\left(\left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+1}\right) - \left[z_\alpha^{t_1} \cdots z_\alpha^{t_h}\right]_{k+2} \\
&= -\frac{1}{(k+1)!}\left[\frac{\partial^{k+1}(z_\alpha^{t_1} \cdots z_\alpha^{t_h})}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_{k+1}}}\right]_1 \left[z_\beta^{s_1} \cdots z_\beta^{s_{k+1}}\right]_{k+2} \\
&= -\frac{1}{(k+1)!}\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \cdots \frac{\partial z_\beta^{s_{k+1}}}{\partial z_\alpha^{r_{k+1}}} \frac{\partial^{k+1}(z_\alpha^{t_1} \cdots z_\alpha^{t_h})}{\partial z_\beta^{s_1} \cdots \partial z_\beta^{s_{k+1}}}\bigg|_S \left[z_\alpha^{r_1} \cdots z_\alpha^{r_{k+1}}\right]_{k+2},
\end{aligned}$$

as claimed. □

Corollary 3.4. *Let S be a $(k-1)$-comfortably embedded submanifold of a complex manifold M, and let $(\rho_1, \boldsymbol{\nu}_1)$ be a $(k-1)$-comfortable pair. Assume that we have a kth order lifting ρ such that $\theta_{k,k-1} \circ \rho = \rho_1$. Then the sequence $\boldsymbol{\nu}_1$ extends to a comfortable splitting sequence $\boldsymbol{\nu}$ associated to ρ if and only if the class $\mathfrak{h}^\rho \in H^1(S, \mathrm{Hom}(\mathcal{I}_S/\mathcal{I}_S^{k+1}, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}))$ vanishes.*

We can characterize k-comfortably embedded submanifolds using adapted atlases.

Definition 3.2. Let S be a complex submanifold of codimension m in a complex n-dimensional manifold M, and let $k \geqslant 1$. A *k-comfortable atlas* is an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S such that

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^k \quad \text{and} \quad \frac{\partial^2 z_\beta^r}{\partial z_\alpha^{s_1} \partial z_\alpha^{s_2}} \in \mathcal{I}_S^k \tag{3.7}$$

for all $r, s_1, s_2 = 1, \ldots, m$, all $p = m+1, \ldots, n$ and all indices α, β such that $U_\alpha \cap U_\beta \cap S \neq \emptyset$. In particular, a k-comfortable atlas is always k-splitting.

Definition 3.3. Let S be a k-comfortably embedded submanifold of codimension m of a complex manifold M, and $(\rho, \boldsymbol{\nu})$ a k-comfortable pair for S in M. We shall say that an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S is *adapted* to $(\rho, \boldsymbol{\nu})$ if it is adapted to ρ and

$$\nu_{h-1,h}\left(\left[z_\alpha^r\right]_h\right) = \left[z_\alpha^r\right]_{h+1}$$

for all $1 \leqslant h \leqslant k+1$, $1 \leqslant j \leqslant h-1$ and $1 \leqslant r \leqslant m$.

The following result, in particular, recovers the original definition of 1-comfortably embedded submanifold introduced in [1,2]:

Theorem 3.5. *Let S be an m-codimensional submanifold of an n-dimensional complex manifold M. Then:*

(i) *S is k-comfortably embedded into M if and only if there exists a k-comfortable atlas;*
(ii) *an atlas adapted to S is k-comfortable if and only if it is adapted to a k-comfortable pair;*
(iii) *if S is k-comfortably embedded into M then every k-comfortable pair admits a k-comfortable atlas adapted to it.*

Proof. First of all, notice that (3.7) implies that

$$\frac{\partial^l z_\beta^p}{\partial z_\alpha^{r_1}\cdots\partial z_\alpha^{r_l}} \in \mathcal{I}_S^{k-l+1} \quad \text{and} \quad \frac{\partial^l z_\beta^r}{\partial z_\alpha^{r_1}\cdots\partial z_\alpha^{r_l}} \in \mathcal{I}_S^{k-l+2} \tag{3.8}$$

for all $2\leqslant l\leqslant k+1$, $r, r_1,\dots,r_l=1,\dots,m$, $p=m+1,\dots,n$ and indices α,β such that $U_\alpha\cap U_\beta\cap S\neq\emptyset$. In particular, if there exists a k-comfortable atlas then Theorem 2.1, Proposition 3.3 and Proposition 1.1 imply that S is k-comfortably embedded, and that the atlas is adapted to a k-comfortable pair. Q1

To prove the rest of the theorem, we shall work by induction on k. For $k=0$ there is nothing to prove; so we assume that the statement holds for $k-1$, and that there exists a k-comfortable pair $(\rho,\boldsymbol{\nu})$ for S. We must prove that it exists an atlas adapted to $(\rho,\boldsymbol{\nu})$, and that this atlas is necessarily k-comfortable.

Let $\rho_1=\theta_{k,k-1}\circ\rho$ and $\boldsymbol{\nu}_1=(\nu_{0,1},\dots,\nu_{k-1,k})$; clearly, $(\rho_1,\boldsymbol{\nu}_1)$ is a $(k-1)$-comfortable pair for S in M. The induction hypothesis then provides us with a $(k-1)$-comfortable atlas $\mathfrak{U}$ adapted to $(\rho_1,\boldsymbol{\nu}_1)$; arguing as in the proof of Theorem 2.1 we can moreover modify this atlas to get a new projectable k-splitting and $(k-1)$-comfortable atlas (still denoted by $\mathfrak{U}$) adapted to ρ and to $(\rho_1,\boldsymbol{\nu}_1)$. We must now show how to modify $\mathfrak{U}$ so to get an atlas adapted to $(\rho,\boldsymbol{\nu})$, and to prove that such an atlas is necessarily k-comfortable.

The first observation is that from $\partial^2 z_\beta^r/\partial z_\alpha^s\partial z_\alpha^t\in\mathcal{I}_S^{k-1}$ we get

$$z_\beta^r = \left.\frac{\partial z_\beta^r}{\partial z_\alpha^s}\right|_S z_\alpha^s + h^r_{s_1\cdots s_{k+1}} z_\alpha^{s_1}\cdots z_\alpha^{s_{k+1}} + R_{k+2} \tag{3.9}$$

for suitable functions $h^r_{s_1\cdots s_{k+1}}\in\mathcal{O}(U_\alpha\cap U_\beta\cap S)$ symmetric in the lower indices. Notice that $h^r_{s_1\cdots s_{k+1}}\equiv 0$ if and only if $\mathfrak{U}$ is k-comfortable.

From (3.9) we derive three identities that will be useful later:

$$\begin{aligned}
\frac{\partial z_\beta^r}{\partial z_\alpha^s} &= \left.\frac{\partial z_\beta^r}{\partial z_\alpha^s}\right|_S + (k+1)h^r_{s_1\cdots s_k t} z_\alpha^{s_1}\cdots z_\alpha^{s_k} + R_{k+1},\\
\frac{\partial z_\beta^r}{\partial z_\alpha^{s_1}\partial z_\alpha^{s_2}} &= k(k+1)h^r_{r_1\cdots r_{k-1}s_1s_2} z_\alpha^{r_1}\cdots z_\alpha^{r_{k-1}} + R_k,\\
z_\alpha^s &= \left.\frac{\partial z_\alpha^s}{\partial z_\beta^r}\right|_S z_\beta^r - \frac{\partial z_\alpha^s}{\partial z_\beta^r} h^r_{s_1\cdots s_{k+1}} \left.\frac{\partial z_\alpha^{s_1}}{\partial z_\beta^{r_1}}\cdots\frac{\partial z_\alpha^{s_{k+1}}}{\partial z_\beta^{r_{k+1}}}\right|_S z_\beta^{r_1}\cdots z_\beta^{r_{k+1}} + R_{k+2}.
\end{aligned}\tag{3.10}$$

In particular it follows that

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$$\left[z_\alpha^{s_1}\cdots z_\alpha^{s_j}\right]_h=\begin{cases}\frac{\partial z_\alpha^{s_1}}{\partial z_\beta^{r_1}}\cdots\frac{\partial z_\alpha^{s_j}}{\partial z_\beta^{r_j}}\Big|_S\cdot[z_\beta^{r_1}\cdots z_\beta^{r_j}]_h\\ \quad\text{if } 1\leqslant h\leqslant k+2,\ 1\leqslant j\leqslant h-1,\ \text{and } (j,h)\neq(1,k+2),\\ \frac{\partial z_\alpha^{s_1}}{\partial z_\beta^{r_1}}\Big|_S\cdot[z_\beta^{r_1}]_{k+2}-\frac{\partial z_\alpha^{s_1}}{\partial z_\beta^{r}}h^r_{t_1\ldots t_{k+1}}\frac{\partial z_\alpha^{t_1}}{\partial z_\beta^{r_1}}\cdots\frac{\partial z_\alpha^{t_{k+1}}}{\partial z_\beta^{r_{k+1}}}\Big|_S\cdot[z_\beta^{r_1}\cdots z_\beta^{r_{k+1}}]_{k+2},\\ \quad\text{if } j=1 \text{ and } h=k+2.\end{cases}$$

Now for every index α such that $U_\alpha\cap S\neq\emptyset$ define $\nu_{k,k+1;\alpha}:\mathcal{I}_S/\mathcal{I}_S^{k+1}|_{U_\alpha}\to\mathcal{I}_S/\mathcal{I}_S^{k+2}|_{U_\alpha}$ by setting

$$\nu_{k,k+1;\alpha}\left(\left[z_\alpha^{s_1}\cdots z_\alpha^{s_j}\right]_{k+1}\right)=\left[z_\alpha^{s_1}\cdots z_\alpha^{s_j}\right]_{k+2}$$

and then extending by $\mathcal{O}_S$-linearity. The previous computations imply that

$$\begin{aligned}&\nu_{k,k+1;\beta}\left(\left[z_\alpha^{s_1}\cdots z_\alpha^{s_j}\right]_{k+1}\right)-\nu_{k,k+1;\alpha}\left(\left[z_\alpha^{s_1}\cdots z_\alpha^{s_j}\right]_{k+1}\right)\\ &\quad=\begin{cases}\frac{\partial z_\alpha^{s_1}}{\partial z_\beta^{r}}\Big|_S h^r_{r_1\cdots r_{k+1}}\cdot[z_\alpha^{r_1}\cdots z_\alpha^{r_{k+1}}]_{k+2} & \text{if } j=1,\\ 0 & \text{if } 2\leqslant j\leqslant k.\end{cases}\end{aligned}$$

In particular, $\mathfrak{U}$ is adapted to $(\rho,\boldsymbol{\nu})$ if and only if $h^r_{r_1\cdots r_{k+1}}\equiv 0$, if and only if $\mathfrak{U}$ is k-comfortable.

Now set $\sigma_\alpha=\nu_{k,k+1}-\nu_{k,k+1;\alpha}$; since $\theta_{k+1,k}\circ\sigma_\alpha=O$, it follows that $\operatorname{Im}\sigma_\alpha\subseteq\mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}$. In particular, there are $(c_\alpha)^s_{r_1\cdots r_{k+1}}\in\mathcal{O}(U_\alpha\cap S)$, symmetric in the lower indices, such that

$$\sigma_\alpha\left(\left[z_\alpha^s\right]_{k+1}\right)=(c_\alpha)^s_{r_1\cdots r_{k+1}}\cdot\left[z_\alpha^{r_1}\cdots z_\alpha^{r_{k+1}}\right]_{k+2}.$$

Since $\sigma_\alpha-\sigma_\beta=\nu_{k,k+1;\beta}-\nu_{k,k+1;\alpha}$, we get

$$\begin{aligned}&\frac{\partial z_\alpha^s}{\partial z_\beta^r}\bigg|_S h^r_{r_1\cdots r_{k+1}}\cdot\left[z_\alpha^{r_1}\cdots z_\alpha^{r_{k+1}}\right]_{k+2}\\ &\quad=\left[(c_\alpha)^s_{r_1\cdots r_{k+1}}-\frac{\partial z_\alpha^s}{\partial z_\beta^r}(c_\beta)^r_{s_1\cdots s_{k+1}}\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{k+1}}}{\partial z_\alpha^{r_{k+1}}}\bigg|_S\right]\cdot\left[z_\alpha^{r_1}\cdots z_\alpha^{r_{k+1}}\right]_{k+2},\end{aligned}$$

that is

$$h^r_{r_1\cdots r_{k+1}}+(c_\beta)^r_{t_1\ldots t_{k+1}}\frac{\partial z_\beta^{t_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{t_{k+1}}}{\partial z_\alpha^{r_{k+1}}}-\frac{\partial z_\beta^r}{\partial z_\alpha^t}(c_\alpha)^t_{r_1\cdots r_{k+1}}\in\mathcal{I}_S. \tag{3.11}$$

We are finally ready to modify $\mathfrak{U}$. We define new coordinates $\hat{z}_\alpha$ by setting

$$\begin{cases}\hat{z}_\alpha^r=z_\alpha^r+(c_\alpha)^r_{s_1\cdots s_{k+1}}\left(z''_\alpha\right)z_\alpha^{s_1}\cdots z_\alpha^{s_{k+1}} & \text{for } r=1,\ldots,m,\\ \hat{z}_\alpha^p=z_\alpha^p & \text{for } p=m+1,\ldots,n,\end{cases}$$

on suitable $\hat{U}_\alpha\subseteq U_\alpha$. We claim that $\hat{\mathfrak{U}}=\{(\hat{U}_\alpha,\hat{z}_\alpha)\}$ is as desired. First of all, it is easy to see that

$$\frac{\partial\hat{z}_\beta^p}{\partial\hat{z}_\alpha^r}=\frac{\partial z_\beta^p}{\partial z_\alpha^r}+R_k,$$

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and so $\hat{\mathfrak{U}}$ is still k-splitting and adapted to ρ. A quick computation shows that

$$\frac{\partial^2 \hat{z}^r_\beta}{\partial \hat{z}^{s_1}_\alpha \partial \hat{z}^{s_2}_\alpha} = \frac{\partial^2 z^t_\alpha}{\partial \hat{z}^{s_1}_\alpha \partial \hat{z}^{s_2}_\alpha} \frac{\partial \hat{z}^r_\beta}{\partial z^t_\alpha} + \frac{\partial z^{t_1}_\alpha}{\partial \hat{z}^{s_1}_\alpha} \frac{\partial z^{t_2}_\alpha}{\partial \hat{z}^{s_2}_\alpha} \frac{\partial^2 \hat{z}^r_\beta}{\partial z^{t_1}_\alpha \partial z^{t_2}_\alpha},$$

and hence

$$\begin{aligned}\frac{\partial^2 \hat{z}^r_\beta}{\partial \hat{z}^{s_1}_\alpha \partial \hat{z}^{s_2}_\alpha} = \frac{\partial^2 z^r_\beta}{\partial z^{s_1}_\alpha \partial z^{s_2}_\alpha} + k(k+1)\biggl[(c_\beta)^r_{t_1\dots t_{k+1}} \frac{\partial z^{t_1}_\beta}{\partial z^{r_1}_\alpha} \cdots \frac{\partial z^{t_{k-1}}_\beta}{\partial z^{r_{t-1}}_\alpha} \frac{\partial z^{t_k}_\beta}{\partial z^{s_1}_\alpha} \frac{\partial z^{t_{k+1}}_\beta}{\partial z^{s_2}_\alpha} \\ - \frac{\partial z^r_\beta}{\partial z^t_\alpha} (c_\alpha)^t_{r_1\cdots r_{k-1}s_1s_2}\biggr] z^{r_1}_\alpha \cdots z^{r_{k-1}}_\alpha + R_k \in \mathcal{I}^k_S\end{aligned}$$

as desired, thanks to (3.10) and (3.11).

Finally, it is easy to check that $\hat{\mathfrak{U}}$ is adapted to $(\rho, \boldsymbol{\nu})$. Indeed, if we define $\hat{\sigma}_\alpha$ by using $\hat{\mathfrak{U}}$ instead of $\mathfrak{U}$, the previous calculations can be used to show that $\sigma_\alpha - \hat{\sigma}_\alpha = \sigma_\alpha$, and thus $\hat{\sigma}_\alpha = O$, which means exactly (recalling the induction hypothesis) that $\hat{\mathfrak{U}}$ is adapted to $(\rho, \boldsymbol{\nu})$. □

Remark 3.3. In other words, S is k-comfortably embedded into M if and only if there is an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S whose changes of coordinates are of the form

$$\begin{cases} z^r_\beta = (a_{\beta\alpha})^r_s(z''_\alpha) z^s_\alpha + R_{k+2} & \text{for } r = 1, \dots, m, \\ z^p_\beta = \phi^p_{\alpha\beta}(z''_\alpha) + R_{k+1} & \text{for } p = m+1, \dots, n. \end{cases} \tag{3.12}$$

As a corollary of the previous theorem, we are able to characterize the obstruction for passing from $(k-1)$-comfortably embedded to k-comfortably embedded:

Corollary 3.6. *Let S be an m-codimensional k split submanifold of an n-dimensional complex manifold M and assume that there exists a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}^{k+1}_S$ such that S is $(k-1)$-comfortably embedded in M with respect to $\rho_1 = \theta_{k,k-1} \circ \rho$. Fix a $(k-1)$-comfortable pair $(\rho_1, \boldsymbol{\nu}_1)$, and let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a projectable atlas adapted to ρ and $(\rho_1, \boldsymbol{\nu}_1)$. Then the cohomology class $\mathfrak{h}^\rho$ associated to the exact sequence* (3.4) *is represented by the* 1*-cocycle $\{\tilde{\mathfrak{h}}^\rho_{\beta\alpha}\} \in H^1(\mathfrak{U}_S, \mathcal{N}_S \otimes \mathcal{I}^{k+1}_S/\mathcal{I}^{k+2}_S)$ given by*

$$\tilde{\mathfrak{h}}^\rho_{\beta\alpha} = -\frac{1}{(k+1)!} \frac{\partial z^{s_1}_\beta}{\partial z^{r_1}_\alpha} \cdots \frac{\partial z^{s_{k+1}}_\beta}{\partial z^{r_{k+1}}_\alpha} \frac{\partial^{k+1} z^t_\alpha}{\partial z^{s_1}_\beta \cdots \partial z^{s_{k+1}}_\beta}\bigg|_S \partial_{t,\alpha} \otimes \bigl[z^{r_1}_\alpha \cdots z^{r_{k+1}}_\alpha\bigr]_{k+2}. \tag{3.13}$$

Thus S is k-comfortably embedded (with respect to ρ) if and only if $\mathfrak{h}^\rho = O$ in $H^1(S, \mathcal{N}_S \otimes$ $\mathrm{Sym}^{k+1}(\mathcal{N}^*_S))$.

Proof. The $(k-1)$-comfortable pair $(\rho_1, \boldsymbol{\nu}_1)$ induces a canonical splitting

$$\mathcal{I}_S/\mathcal{I}^{k+1}_S \cong \bigoplus_{h=1}^{k} \mathcal{I}^h_S/\mathcal{I}^{h+1}_S;$$

therefore the class $\mathfrak{h}^\rho$ associated to the sequence (3.4) and computed in Proposition 3.3 lives in

$$H^1\big(S, \operatorname{Hom}\big(\mathcal{I}_S/\mathcal{I}_S^{k+1}, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}\big)\big) \cong \bigoplus_{h=1}^{k} H^1\big(S, \operatorname{Hom}\big(\mathcal{I}_S^h/\mathcal{I}_S^{h+1}, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}\big)\big).$$

The expression of $\mathfrak{h}^\rho$ given in (3.3) clearly reflects this decomposition. Now, (3.8) implies that

$$\frac{\partial^l z_\alpha^r}{\partial z_\beta^{r_1} \cdots \partial z_\beta^{r_l}} \in \mathcal{I}_S$$

for all $2 \leqslant l \leqslant k$. Therefore (3.3) shows that the only non-zero component of $\mathfrak{h}^\rho$ is the one contained in

$$H^1\big(S, \operatorname{Hom}\big(\mathcal{I}_S/\mathcal{I}_S^2, \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}\big)\big) \cong H^1\big(S, \mathcal{N}_S \otimes \mathcal{I}_S^{k+1}/\mathcal{I}_S^{k+2}\big) \cong H^1\big(S, \mathcal{N}_S \otimes \operatorname{Sym}^{k+1}\big(\mathcal{N}_S^*\big)\big),$$

and its expression is given by (3.13). □

Recalling Proposition 2.2, we then see that the obstruction for passing from $(k-1)$-split to k-split lives in $H^1(S, \mathcal{T}_S \otimes \operatorname{Sym}^k(\mathcal{N}_S^*))$, while the obstruction for passing from $(k-1)$-comfortably embedded to k-comfortably embedded lives in $H^1(S, \mathcal{N}_S \otimes \operatorname{Sym}^{k+1}(\mathcal{N}_S^*))$. Now, a vanishing theorem due to Grauert ([11, Hilfssatz 1]; see also [7]) says that if N_S is negative in the sense of Grauert (that is, the zero section of N_S can be blown down to a point) then these groups vanish for k large enough. We thus obtain the following

Corollary 3.7. *Let S be an m-codimensional compact complex submanifold of an n-dimensional manifold M, and assume that N_S is negative in the sense of Grauert. Then there exists a $k_0 \geqslant 1$ such that if S is k_0-splitting (respectively, k_0-comfortably embedded) in M then it is k-splitting (respectively, k-comfortably embedded) for all $k \geqslant k_0$.*

A similar result can also be obtained assuming instead that N_S is positive in a suitable sense; see [9,12,16,17,22].

Remark 3.4. At present we do not know whether a submanifold which is k-comfortably embedded with respect to a given kth order lifting is k-comfortably embedded with respect to any kth order lifting.

We end this section with some examples of k-split and k-comfortably embedded submanifolds. We refer to Section 5 for a more detailed study of k-split and k-comfortably embedded curves in a surface.

Example 3.1. The zero section of a vector bundle is always k-split and k-comfortably embedded in the total space of the bundle for any $k \geqslant 1$: indeed, any atlas trivializing the bundle satisfies

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \equiv \frac{\partial^2 z_\beta^r}{\partial z_\alpha^s \partial z_\alpha^t} \equiv 0.$$

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Example 3.2. A local holomorphic retract is always k-split in the ambient manifold. Indeed, if $p : U \to S$ is a local holomorphic retraction, then a kth order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^{k+1}$ is given by $\rho(f) = [f \circ p]_{k+1}$.

Example 3.3. If S is a Stein submanifold of a complex manifold M (e.g., if S is an open Riemann surface), then S is k-split and k-comfortably embedded into M for any $k \geqslant 1$. Indeed, by Cartan's Theorem B the first cohomology group of S with coefficients in any coherent sheaf vanishes, and the assertion follows from Propositions 1.1, 2.2 and 3.3. In particular, if S is a *singular* curve in M then the non-singular part of S is always comfortably embedded in M.

Example 3.4. Let $\tilde{M}$ be the blow-up of a submanifold X in a complex manifold M. Then the exceptional divisor $E \subset \tilde{M}$ is k-split and k-comfortably embedded in $\tilde{M}$ for any $k \geqslant 1$: indeed, it is easy to check that the atlas of $\tilde{M}$ induced by an atlas of M adapted to X is a k-comfortable atlas for any $k \geqslant 1$.

4. Embeddings in the normal bundle and k-linearizable submanifolds

Proposition 1.5 suggests a third way of generalizing the notion of splitting submanifold:

Definition 4.1. Let S be a complex submanifold of a complex manifold M. We shall say that S is *k-linearizable* if its kth infinitesimal neighbourhood $S(k)$ in M is isomorphic to its kth infinitesimal neighbourhood $S_N(k)$ in N_S, where we are identifying S with the zero section of N_S.

We have seen that S splits into M if and only if it is 1-linearizable (Proposition 1.5). In general, however, k-split does not imply k-linearizable (while the converse hold). The missing link is provided by the notion of $(k-1)$-comfortably embedded:

Theorem 4.1. *Let S be a complex submanifold of a complex manifold M, and $k \geqslant 2$. Then S is k-linearizable if and only if it is k-split and $(k-1)$-comfortably embedded (with respect to the $(k-1)$th order lifting induced by the k-splitting).*

Proof. We shall denote by $\mathcal{I}_{S,N}$ the ideal sheaf of S in N_S, and by $\theta^N_{h,k} : \mathcal{O}_{N_S}/\mathcal{I}^{h+1}_{S,N} \to \mathcal{O}_{N_S}/\mathcal{I}^{k+1}_{S,N}$ the canonical projections. Notice that S is k-split and k-comfortably embedded in N_S for any $k \geqslant 1$, by, for instance, Example 3.1; we shall denote by $\rho^N_k : \mathcal{O}_S \to \mathcal{O}_{N_S}/\mathcal{I}^{k+1}_{S,N}$ and $\nu^N_{k-1,k} : \mathcal{I}_{S,N}/\mathcal{I}^k_{S,N} \to \mathcal{I}_{S,N}/\mathcal{I}^{k+1}_{S,N}$ the corresponding morphisms.

We shall work by induction on k. We have already seen that $S_N(1) \cong S(1)$ implies that S is 1-split. Suppose now that $S_N(k-1) \cong S(k-1)$ implies that S is $(k-1)$-split and $(k-2)$-comfortably embedded, and assume that $S_N(k) \cong S(k)$. Let $\psi : \mathcal{O}_{N_S}/\mathcal{I}^{k+1}_{S,N} \to \mathcal{O}_M/\mathcal{I}^{k+1}_S$ be a ring isomorphism such that $\theta_k \circ \psi = \theta^N_k$. The gist of the proof is contained in the following commutative diagrams:

$$\begin{array}{ccccccccc}
O & \longrightarrow & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{k+1} & \hookrightarrow & \mathcal{O}_{N_S}/\mathcal{I}_{S,N}^{k+1} & \underset{\rho_k^N}{\overset{\theta_k^N}{\rightleftarrows}} & \mathcal{O}_S & \longrightarrow & O \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \| & & \\
O & \longrightarrow & \mathcal{I}_S/\mathcal{I}_S^{k+1} & \hookrightarrow & \mathcal{O}_M/\mathcal{I}_S^{k+1} & \underset{\rho_k}{\overset{\theta_k}{\rightleftarrows}} & \mathcal{O}_S & \longrightarrow & O,
\end{array}$$

$$\begin{array}{ccccccccc}
O & \longrightarrow & \mathcal{I}_{S,N}^k/\mathcal{I}_{S,N}^{k+1} & \hookrightarrow & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{k+1} & \underset{\nu_{k-1,k}^N}{\overset{\theta_{k,k-1}^N}{\rightleftarrows}} & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^k & \longrightarrow & O \\
 & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\hat{\psi}} & & \\
O & \longrightarrow & \mathcal{I}_S^k/\mathcal{I}_S^{k+1} & \hookrightarrow & \mathcal{I}_S/\mathcal{I}_S^{k+1} & \underset{\nu_{k-1,k}}{\overset{\theta_{k,k-1}}{\rightleftarrows}} & \mathcal{I}_S/\mathcal{I}_S^k & \longrightarrow & O.
\end{array}$$

First of all, we define $\rho_k = \psi \circ \rho_k^N$. As in the proof of Proposition 1.5, we see that this is a ring morphism such that $\theta_k \circ \rho_k = \mathrm{id}$, and so S is k-split in M.

Now, $\theta_k \circ \psi = \theta_k^N$ implies that $\psi(\operatorname{Ker}\theta_k^N) \subseteq \operatorname{Ker}\theta_k$, that is $\psi(\mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{k+1}) \subseteq \mathcal{I}_S/\mathcal{I}_S^{k+1}$. Since ψ is a ring isomorphism, it induces a ring isomorphism (still denoted by ψ) between the nilpotent parts of the two rings, $\mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{k+1}$ and $\mathcal{I}_S/\mathcal{I}_S^{k+1}$, and thus, by restriction, a ring isomorphism between $\mathcal{I}_{S,N}^k/\mathcal{I}_{S,N}^{k+1}$ and $\mathcal{I}_S^k/\mathcal{I}_S^{k+1}$. Therefore it also induces a quotient ring isomorphism $\hat{\psi}$ between $\mathcal{O}_{N_S}/\mathcal{I}_{S,N}^k$ and $\mathcal{O}_M/\mathcal{I}_S^k$ such that $\theta_{k-1} \circ \hat{\psi} = \theta_{k-1}^N$, and thus $S_N(k-1) \cong S(k-1)$. Furthermore, $\hat{\psi}$ sends $\mathcal{I}_{S,N}/\mathcal{I}_{S,N}^k$ into $\mathcal{I}_S/\mathcal{I}_S^k$ so that $\hat{\psi} \circ \theta_{k,k-1}^N = \theta_{k,k-1} \circ \psi$. This restriction of $\hat{\psi}$ is an isomorphism of $\mathcal{O}_S$-modules: indeed

$$\begin{aligned}
\hat{\psi}\big(u \cdot [h]_k\big) &= \hat{\psi}\big(\rho_{k-1}^N(u)[h]_k\big) = \hat{\psi}\big(\theta_{k,k-1}^N\big(\rho_k^N(u)[h]_{k+1}\big)\big) = \big[\psi\big(\rho_k^N(u)\big)\psi[h]_{k+1}\big]_k \\
&= \big[\rho_k(u)\psi[h]_{k+1}\big]_k = \rho_{k-1}(u)\hat{\psi}\big([h]_k\big) = u \cdot \hat{\psi}\big([h]_k\big)
\end{aligned}$$

for all $u \in \mathcal{O}_S$ and $h \in \mathcal{I}_{S,N}$, where $\rho_{k-1}^N = \theta_{k,k-1}^N \circ \rho_k^N$ and $\rho_{k-1} = \theta_{k,k-1} \circ \rho_k$.

We then define $\nu_{k-1,k} = \psi \circ \nu_{k-1,k}^N \circ \hat{\psi}^{-1}$; we claim that $\nu_{k-1,k}$ is a morphism of $\mathcal{O}_S$-modules such that $\theta_{k,k-1} \circ \nu_{k-1,k} = \mathrm{id}$. Indeed,

$$\begin{aligned}
\nu_{k-1,k}\big(u \cdot [h]_k\big) &= \psi \circ \nu_{k-1,k}^N \circ \hat{\psi}^{-1}\big(u \cdot [h]_k\big) = \psi \circ \nu_{k-1,k}^N\big(u \cdot \hat{\psi}^{-1}\big([h]_k\big)\big) \\
&= \psi\big(u \cdot \big(\nu_{k-1,k}^N \circ \hat{\psi}^{-1}\big)\big([h]_k\big)\big) = \psi\big(\rho_k^N(u)\big(\nu_{k-1,k}^N \circ \hat{\psi}^{-1}\big)\big([h]_k\big)\big) \\
&= \psi\big(\rho_k^N(u)\big)\big(\psi \circ \nu_{k-1,k}^N \circ \hat{\psi}^{-1}\big)\big([h]_k\big) = \rho_k(u)\nu_{k-1,k}\big([h]_k\big) \\
&= u \cdot \nu_{k-1,k}\big([h]_k\big)
\end{aligned}$$

for all $u \in \mathcal{O}_S$ and $h \in \mathcal{I}_S$. Finally, $\theta_{k,k-1} \circ \nu_{k-1,k} = \hat{\psi} \circ \theta_{k,k-1}^N \circ \nu_{k-1,k}^N \circ \hat{\psi}^{-1} = \mathrm{id}$, and hence S is $(k-1)$-comfortably embedded in M, as claimed.

Conversely, assume that S is k-split and $(k-1)$-comfortably embedded. Since we shall use different maps, let us write the involved commutative diagrams:

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$$\begin{tikzcd}
O \arrow[r] & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{k+1} \arrow[r, hook] \arrow[d, "\psi^k"'] & \mathcal{O}_{N_S}/\mathcal{I}_{S,N}^{k+1} \arrow[r, shift left, "\theta_k^N"] \arrow[d, "\psi"'] & \mathcal{O}_S \arrow[r] \arrow[l, shift left, "\rho_k^N"] \arrow[d, equal] & O \\
O \arrow[r] & \mathcal{I}_S/\mathcal{I}_S^{k+1} \arrow[r, hook] & \mathcal{O}_M/\mathcal{I}_S^{k+1} \arrow[r, shift left, "\theta_k"] & \mathcal{O}_S \arrow[r] \arrow[l, shift left, "\rho_k"] & O,
\end{tikzcd}$$

$$\begin{tikzcd}
O \arrow[r] & \mathcal{I}_{S,N}^l/\mathcal{I}_{S,N}^{l+1} \arrow[r, hook, shift left] \arrow[d, "\chi^l"'] & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{l+1} \arrow[l, shift left, "\tilde{\nu}_{l-1,l}^N"] \arrow[r, shift left, "\theta_{l,l-1}^N"] \arrow[d, "\psi^l"'] & \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^l \arrow[l, shift left, "\nu_{l-1,l}^N"] \arrow[r] \arrow[d, "\psi^{l-1}"'] & O \\
O \arrow[r] & \mathcal{I}_S^l/\mathcal{I}_S^{l+1} \arrow[r, hook, shift left] & \mathcal{I}_S/\mathcal{I}_S^{l+1} \arrow[l, shift left, "\tilde{\nu}_{l-1,l}^N"] \arrow[r, shift left, "\theta_{l,l-1}"] & \mathcal{I}_S/\mathcal{I}_S^l \arrow[l, shift left, "\nu_{l-1,l}"] \arrow[r] & O.
\end{tikzcd}$$

In the proof of Proposition 1.5 we defined an $\mathcal{O}_S$-module isomorphism $\chi : \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^2 \to \mathcal{I}_S/\mathcal{I}_S^2$; since $\mathcal{I}_S^l/\mathcal{I}_S^{l+1} = \mathrm{Sym}^l(\mathcal{I}_S/\mathcal{I}_S^2)$, and likewise for $\mathcal{I}_{S,N}^l/\mathcal{I}_{S,N}^{l+1}$, we get for all $l \geqslant 1$ an $\mathcal{O}_S$-module isomorphism $\chi^l : \mathcal{I}_{S,N}^l/\mathcal{I}_{S,N}^{l+1} \to \mathcal{I}_S^l/\mathcal{I}_S^{l+1}$. We claim that for all $1 \leqslant l \leqslant k$ we can define a ring and $\mathcal{O}_S$-module isomorphism $\psi^l : \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^{l+1} \to \mathcal{I}_S/\mathcal{I}_S^{l+1}$ so that the above diagram commutes.

We argue by induction on l. For $l = 1$, it suffices to take $\psi^1 = \chi$. Assume now that we have defined ψ^{l-1}, and let

$$\psi^l = \nu_{l-1,l} \circ \psi^{l-1} \circ \theta_{l,l-1}^N + \chi^l \circ \tilde{\nu}_{l-1,l}^N,$$

where $\tilde{\nu}_{l-1,l}^N$ is the left splitting morphism associated to $\nu_{l-1,l}^N$. It is easy to check that ψ^l is invertible, with inverse given by

$$\left(\psi^l\right)^{-1} = \nu_{l-1,l}^N \circ \left(\psi^{l-1}\right)^{-1} \circ \theta_{l,l-1} + \left(\chi^l\right)^{-1} \circ \tilde{\nu}_{l-1,l},$$

where $\tilde{\nu}_{l-1,l}$ is the left splitting morphism associated to $\nu_{l-1,l}$. Since all the maps involved are $\mathcal{O}_S$-module morphisms, ψ^l is a $\mathcal{O}_S$-module morphism; we are left to show that ψ^l is a ring morphism. Using the definition, it is easy to see that ψ^l is a ring morphism if and only if

$$\tilde{\nu}_{l-1,l}\left(\left(\nu_{l-1,l} \circ \psi^{l-1}\right)(u)\left(\nu_{l-1,l} \circ \psi^{l-1}\right)(v)\right) = \chi^l \circ \tilde{\nu}_{l-1,l}^N\left(\nu_{l-1,l}^N(u)\nu_{l-1,l}^N(v)\right), \tag{4.1}$$

for all $u, v \in \mathcal{I}_{S,N}/\mathcal{I}_{S,N}^l$.

To prove (4.1), we work in local coordinates. Let (U, z) be a chart in a $(k-1)$-comfortable atlas, so that we have $\nu_{i-1,i}[z^{r_1} \cdots z^{r_a}]_i = [z^{r_1} \cdots z^{r_a}]_{i+1}$ for all $i = 1, \dots, l$, $a = 1, \dots, l-1$ and $r_1, \dots, r_a = 1, \dots, m$. The chart (U, z) induces a chart $(\tilde{U}, \tilde{z})$ in a $(k-1)$-comfortable atlas of N_S, with $\tilde{z} = (v, z'')$, where $v = (v^1, \dots, v^m)$ are the fiber coordinates. Then we have $\chi[v^r]_2 = [z^r]_2$, and thus

$$\chi^i\left[v^{r_1} \cdots v^{r_i}\right]_{i+1} = \left[z^{r_1} \cdots z^{r_i}\right]_{i+1}$$

for all $i=1,\dots,k$. From this and the fact that $\tilde{\nu}^N_{i-1,i}([v^r]_{i+1})=O$ for all $i=1,\dots,k$ and $r=1,\dots,m$, it follows easily that $\psi^i([v^r]_{i+1})=[z^r]_{i+1}$ for all $i=1,\dots,l-1$ and $r=1,\dots,m$.

Now take $u,v\in\mathcal{I}_{S,N}/\mathcal{I}^l_{S,N}$; we can write

$$u=\sum_{a=1}^{l-1}\alpha_{r_1\cdots r_a}\cdot\left[v^{r_1}\cdots v^{r_a}\right]_l\quad\text{and}\quad u=\sum_{b=1}^{l-1}\beta_{s_1\cdots s_b}\cdot\left[v^{s_1}\cdots v^{s_b}\right]_l$$

for suitable $\alpha_{r_1\cdots r_a},\beta_{s_1\cdots s_b}\in\mathcal{O}_S$. Then

$$\chi^l\circ\tilde{\nu}^N_{l-1,l}\bigl(\nu^N_{l-1,l}(u)\nu^N_{l-1,l}(v)\bigr)=\sum_{a+b=l}\alpha_{r_1\cdots r_a}\beta_{s_1\cdots s_b}\cdot\left[z^{r_1}\cdots z^{r_a}z^{s_1}\cdots z^{s_b}\right]_{l+1}.$$

Using the fact that ψ^{l-1} is a ring morphism and an $\mathcal{O}_S$-morphism we also find

$$\bigl(\nu_{l-1,l}\circ\psi^{l-1}\bigr)(u)=\sum_{a=1}^{l-1}\alpha_{r_1\cdots r_a}\cdot\left[z^{r_1}\cdots z^{r_a}\right]_{l+1}\quad\text{and}$$
$$\bigl(\nu_{l-1,l}\circ\psi^{l-1}\bigr)(v)=\sum_{b=1}^{l-1}\beta_{s_1\cdots s_b}\cdot\left[z^{s_1}\cdots z^{s_b}\right]_{l+1},$$

and hence

$$\tilde{\nu}_{l-1,l}\bigl(\bigl(\nu_{l-1,l}\circ\psi^{l-1}\bigr)(u)\bigl(\nu_{l-1,l}\circ\psi^{l-1}\bigr)(v)\bigr)=\sum_{a+b=l}\alpha_{r_1\cdots r_a}\beta_{s_1\cdots s_b}\cdot\left[z^{r_1}\cdots z^{r_a}z^{s_1}\cdots z^{s_b}\right]_{l+1},$$

as claimed.

So in particular we have proved that $\psi^k:\mathcal{I}_{N,S}/\mathcal{I}^{k+1}_{N,S}\to\mathcal{I}_S/\mathcal{I}^{k+1}_S$ is a ring and $\mathcal{O}_S$-module isomorphism. Let us then define $\psi:\mathcal{O}_{N_S}/\mathcal{I}^{k+1}_{S,N}\to\mathcal{O}_M/\mathcal{I}^{k+1}_S$ by

$$\psi=\rho_k\circ\theta^N_k+\psi^k\circ\tilde{\rho}^N_k,$$

where as usual $\tilde{\rho}^N_k$ is the derivation associated to ρ^N_k. It is easy to check that $\theta_k\circ\psi=\theta^N_k$, and that ψ is invertible; we are left to show that it is a ring morphism.

If $u,v\in\mathcal{O}_{N_S}/\mathcal{I}^{k+1}_{S,N}$ we can write $u=\rho^N_k(u_o)+\tilde{\rho}^N_k(u)$, with $u_o=\theta^N_k(u)$; so $\psi(u)=\rho_k(u_o)+\psi^k\bigl(\tilde{\rho}^N_k(u)\bigr)$; and analogously for v. Therefore

$$\begin{aligned}uv&=\rho^N_k(u_o)\rho^N_k(v_o)+\left[\rho^N_k(u_o)\tilde{\rho}^N_k(v)+\rho^N_k(v_o)\tilde{\rho}^N_k(u)+\tilde{\rho}^N_k(u)\tilde{\rho}^N_k(v)\right]\\&=\rho^N_k(u_o)\rho^N_k(v_o)+\left[u_o\cdot\tilde{\rho}^N_k(v)+v_o\cdot\tilde{\rho}^N_k(u)+\tilde{\rho}^N_k(u)\tilde{\rho}^N_k(v)\right],\end{aligned}$$

and thus

$$\begin{aligned}\psi(uv)&=\rho_k(u_ov_o)+\psi^k\bigl(u_o\cdot\tilde{\rho}^N_k(v)+v_o\cdot\tilde{\rho}^N_k(u)+\tilde{\rho}^N_k(u)\tilde{\rho}^N_k(v)\bigr)\\&=\rho_k(u_o)\rho_k(v_o)+u_o\cdot\psi^k\bigl(\tilde{\rho}^N_k(v)\bigr)+v_o\cdot\psi^k\bigl(\tilde{\rho}^N_k(u)\bigr)+\psi^k\bigl(\tilde{\rho}^N_k(u)\bigr)\psi^k\bigl(\tilde{\rho}^N_k(v)\bigr)\\&=\psi(u)\psi(v).\qquad\square\end{aligned}$$

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Thus a submanifold S is k-linearizable if and only if there is an atlas $\{(U_\alpha, z_\alpha)\}$ adapted to S whose changes of coordinates are of the form

$$\begin{cases} z_\beta^r = (a_{\beta\alpha})_s^r\big(z_\alpha''\big)z_\alpha^s + R_{k+1} & \text{for } r = 1, \dots, m, \\ z_\beta^p = \phi_{\alpha\beta}^p\big(z_\alpha''\big) + R_{k+1} & \text{for } p = m+1, \dots, n. \end{cases}$$

Remark 4.1. Camacho, Movasati and Sad in [8] defined a k-linearizable curve as a complex curve S in a complex manifold M for which there exists an atlas adapted to S whose changes of coordinates are of the form

$$\begin{cases} z_\beta^r = (a_{\beta\alpha})_s^r\big(z_\alpha''\big)z_\alpha^s + R_{k+1} & \text{for } r = 1, \\ z_\beta^p = \phi_{\alpha\beta}^p\big(z_\alpha''\big) & \text{for } p = 2, \dots, n; \end{cases}$$

they dropped the remainder term in the z_α'' variables only because they were interested in curves with a neighbourhood fibered by $(n-1)$-dimensional disks. As a consequence, our notion of 2-linearizable curves (or submanifolds) is strictly weaker than the notion of 2-linearizable curves used in [8].

Recalling Proposition 2.2 and Corollary 3.6 we see that the obstructions for passing from $(k-1)$-linearizable to k-linearizable live in $H^1(S, \mathcal{T}_S \otimes \operatorname{Sym}^k(\mathcal{N}_S^*))$ and, for $k \geqslant 2$, in $H^1(S, \mathcal{N}_S \otimes \operatorname{Sym}^k(\mathcal{N}_S^*))$. Using again Grauert's vanishing theorem [11, Hilfssatz 1, p. 344] we get

Corollary 4.2. *Let S be an m-codimensional compact complex submanifold of an n-dimensional manifold M, and assume that N_S is negative in the sense of Grauert. Then there exists a $k_0 \geqslant 1$ such that if S is k_0-linearizable then it is k-linearizable for all $k \geqslant k_0$.*

Again, we can get similar results also assuming suitable positivity conditions on N_S; see [9,12,16,17,22]. Furthermore, in the next section we shall be able to compute the number k_0 for curves in a complex surface.

Remark 4.2. When S is a hypersurface in M, and thus N_S is a line bundle, we actually have found that the obstructions to k-linearizability live in $H^1(S, \mathcal{T}_S \otimes (\mathcal{N}_S^*)^{\otimes k})$ and in $H^1(S, (\mathcal{N}_S^*)^{\otimes k-1})$, in accord with Grauert's theory (see, again, [11] and [7]).

Remark 4.3. It is important to remark that the isomorphisms between $S(k)$ and $S_N(k)$ obtained in the previous corollary are compatible in the sense that if $k' > k$ then the restriction to $S(k)$ of the isomorphism between $S(k')$ and $S_N(k')$ induces the given isomorphism between $S(k)$ and $S_N(k)$. In some sense, we have obtained an isomorphism between the formal neighbourhood of S in M and the formal neighbourhood of S in N_S. Grauert [11] and others have given conditions ensuring that such a formal isomorphism extends to a biholomorphism between an actual neighbourhood of S in M to an actual neighbourhood of S in N_S. In particular, applying Grauert's formal principle (see [7, Theorem 4.3]) we recover Grauert's result:

Corollary 4.3. *(See Grauert [11].) Let S be a compact complex hypersurface of an n-dimensional manifold M. Assume that N_S is negative in the sense of Grauert, and that S is*

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exceptional in M (that is, it can be blown down to a point). Then there exists a $k_0 \geqslant 1$ such that if S is k_0-linearizable then a neighbourhood of S in M is biholomorphic to a neighbourhood of S in N_S.

Remark 4.4. A compact Riemann surface S in a complex surface M is exceptional if and only if its self-intersection $S \cdot S$ is negative, if and only if its normal bundle is negative in the sense of Grauert; see [7].

Remark 4.5. The formal principle holds in several other instances too (but not always). For instance, using [9,12,16,17,22] we can get a statement analogous to Corollary 4.3 assuming suitable positivity conditions on N_S (and arbitrary codimension).

Remark 4.6. Let S be a submanifold of a complex manifold M and denote by $g : S \to M$ the inclusion. Then one of these cases hold:

(a) S is k-linearizable in M for all $k \geqslant 1$.
(b) S is k-split in M for all $k \geqslant 1$, but there exist $m_g \geqslant 1$ and a m_g-splitting ρ such that S is not m_g-comfortably embedded in M but it is $(m_g - 1)$-comfortably embedded. In this case, we can associate to the embedding $S \to M$ a non-zero cohomology class $\mathfrak{h}^\rho \in H^1(S, \mathcal{N}_S \otimes \mathcal{I}_S^{m_g+1}/\mathcal{I}_S^{m_g+2})$.
(c) there exists an integer $k_g \geqslant 1$ such that S is not k-split in M, but it is $(k-1)$-splitting. In this case, given any fixed $(k_g - 1)$-splitting ρ, we can associate to the embedding $S \to M$ a non-zero cohomology class $\mathfrak{g}_{k_g} \in H^1(S, \mathcal{T}_S \otimes \mathcal{I}_S^{k_g}/\mathcal{I}_S^{k_g+1})$. Furthermore, we can choose $1 \leqslant m_g \leqslant k_g - 1$ so that S is $(m_g - 1)$-comfortably embedded (with respect to the lifting induced by ρ) in M but not m_g-comfortably embedded in M, and hence we get a non-zero cohomology class $\mathfrak{h}^\rho \in H^1(S, \mathcal{N}_S \otimes \mathcal{I}_S^{m_g+1}/\mathcal{I}_S^{m_g+2})$.

It is clear by the construction that if two different embeddings of the same submanifold S have biholomorphic neighbourhoods then the integers and the cohomology classes constructed above must be the same in both cases. It would be interesting to know other invariants. For instance, a consequence of Corollary 4.3 and Remark 4.4 is that two infinitely linearizable (in the sense of Remark 4.3) embeddings of a compact Riemann surface with negative self-intersection in a complex surface always have biholomorphic neighbourhoods. However, as far as we know, even for curves with negative self-intersection in case (b) other invariants beside m_g and $\mathfrak{h}^\rho$ are not yet known.

5. Embeddings of a smooth curve

In this section we shall use Serre duality to describe sufficient conditions for a compact curve in a complex surface to be k-split, k-comfortably embedded and/or k-linearizable.

Let S be a non-singular, compact, irreducible curve of genus g on a surface M. In particular, N_S is a line bundle; therefore $\mathrm{Sym}^k(\mathcal{N}_S^*) \cong (\mathcal{N}_S^*)^{\otimes k}$ for all $k \geqslant 1$, and the obstruction for passing from $(k-1)$-split to k-split lives in $H^1(S, \mathcal{T}_S \otimes (\mathcal{N}_S^*)^{\otimes k})$. The Serre duality for Riemann surfaces implies that

$$H^1\big(S, \mathcal{T}_S \otimes \big(\mathcal{N}_S^*\big)^{\otimes k}\big) \cong H^0\big(S, \Omega_S \otimes \Omega_S \otimes \mathcal{N}_S^{\otimes k}\big).$$

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Now,

$$\deg\left(\Omega_S \otimes \Omega_S \otimes \mathcal{N}_S^{\otimes k}\right) = 4g - 4 + k(S \cdot S);$$

therefore

$$k(S \cdot S) < 4 - 4g \quad \Rightarrow \quad H^1\left(S, \mathcal{T}_S \otimes \left(\mathcal{N}_S^*\right)^{\otimes k}\right) = (O). \tag{5.1}$$

It follows in particular that if $g \geqslant 1$ and $S \cdot S < 4 - 4g$, or $g = 0$ and $S \cdot S \leqslant 0$, then S is k-splitting in M for every $k \geqslant 1$.

The obstruction for a split curve to be 1-comfortably embedded is in $H^1(S, \mathcal{N}_S \otimes (\mathcal{N}_S^*)^{\otimes 2}) \cong H^1(S, \mathcal{N}_S^*)$. Serre duality yields

$$H^1\left(S, \mathcal{N}_S^*\right) \cong H^0(S, \Omega_S \otimes \mathcal{N}_S);$$

so, since $\deg(\Omega_S \otimes \mathcal{N}_S) = 2g - 2 + S \cdot S$, we get

$$S \cdot S < 2 - 2g \quad \Rightarrow \quad H^1\left(S, \mathcal{N}_S \otimes \left(\mathcal{N}_S^*\right)^{\otimes 2}\right) = (O). \tag{5.2}$$

In particular, if $g \geqslant 1$ and $S \cdot S < 4 - 4g$ or $g = 0$ and $S \cdot S < 2$ then S is (splitting and) 1-comfortably embedded.

More generally, assume that S is k-split and $(k-1)$-comfortably embedded. The obstruction for S to be k-comfortably embedded lives in $H^1(S, \mathcal{N}_S \otimes (\mathcal{N}_S^*)^{\otimes k+1}) \cong H^1(S, (\mathcal{N}_S^*)^{\otimes k})$. Then using Serre duality as before we find

$$k(S \cdot S) < 2 - 2g \quad \Rightarrow \quad H^1\left(S, \mathcal{N}_S \otimes \left(\mathcal{N}_S^*\right)^{\otimes k+1}\right) = (O). \tag{5.3}$$

In particular, if $g \geqslant 1$ and $S \cdot S < 2 - 2g$ then k-splitting implies k-comfortably embedded, while if $g = 0$ and $S \cdot S \leqslant 0$ then S is k-comfortably embedded for all $k \geqslant 1$.

We can summarize the content of our computations in the following

Proposition 5.1. *Let S be a non-singular, compact, irreducible curve of genus g in a surface M. Then:*

(i) *if $g \geqslant 1$ and $S \cdot S < 4 - 4g$ then S is k-split into M for all $k \geqslant 1$;*
(ii) *if $g \geqslant 1$ and $S \cdot S < 2 - 2g$ then S k-split implies S k-comfortably embedded into M for any $k \geqslant 1$; in particular, if $g \geqslant 1$ and $S \cdot S < 4 - 4g$ then S is k-linearizable for all $k \geqslant 1$;*
(iii) *if $g = 0$ and $S \cdot S \leqslant 0$ then S is k-linearizable for all $k \geqslant 1$;*
(iv) *if $g = 0$ and $S \cdot S \leqslant 1$ then S is 3-split and 1-comfortably embedded into M;*
(v) *if $g = 0$ and $S \cdot S \leqslant 3$ then S splits into M.*

Remark 5.1. Proposition 5.1(ii) has been proved in a slightly different way in [8], where S was assumed to be fibered imbedded into M (and thus, in particular, k-split for all $k \geqslant 1$).

Another way of looking at (5.1) and (5.3) yields the following

Proposition 5.2. *Let S be a non-singular, compact, irreducible curve of genus $g \geqslant 1$ in a surface M. Then:*

(i) *if* $S \cdot S < 0$ *and* S *is* k_0*-splitting for some* $k_0 > (4g-4)/|S \cdot S|$ *then* S *is* k*-splitting for all* $k \geqslant k_0$*;*
(ii) *if* $S \cdot S < 0$ *and* S *is* k_0*-comfortably embedded for some* $k_0 > (2g-2)/|S \cdot S|$ *then* k*-splitting implies* k*-comfortably embedded for any* $k \geqslant k_0$*;*
(iii) *if* $S \cdot S < 0$ *and* S *is* k_0*-linearizable for some* $k_0 > (4g-4)/|S \cdot S|$ *then* S *is* k*-linearizable for all* $k \geqslant k_0$.

Recalling Remarks 4.3 and 4.4, we can apply Grauert's formal principle [7, Theorem 4.3] to recover, among other things, results due to Laufer and Camacho, Movasati and Sad:

Corollary 5.3. *Let* S *be a non-singular, compact, irreducible curve of genus* g *in a surface* M *with negative self-intersection* $S \cdot S < 0$. *If*

(a) $g = 0$, *or*
(b) $g \geqslant 1$, S *is* k_0*-split and* k_1*-comfortably embedded for some* $k_0 > (4g-4)/|S \cdot S|$ *and* $k_1 > (2g-2)/|S \cdot S|$, *or*
(c) *(see* ~~*Laufer*~~ [18, *Chapter* VI]*)* $g \geqslant 1$ *and* $S \cdot S < 4 - 4g$, *or*
(d) *(see* [8]*)* $g \geqslant 1$, $S \cdot S < 2 - 2g$ *and* S *is* k_0*-split for some* $k_0 > (4g-4)/|S \cdot S|$,

then a neighbourhood of S *in* M *is biholomorphic to a neighbourhood of the zero section of* N_S.

6. Another characterization of split and comfortably embedded submanifolds

In [2] we used the 1-comfortably embedded condition to build partial holomorphic connections on the normal bundle, and we wondered why this condition appeared to be the right one for such constructions. In this section we give an answer of sort to this question, showing that a submanifold is 1-comfortably embedded if and only if it exists an infinitesimal holomorphic connection on the normal bundle.

Let us begin with a definition.

Definition 6.1. Let S be a complex subvariety of a complex manifold M. The *sheaf of holomorphic differentials* on $S(1)$ is given by

$$\Omega_{S(1)} = \Omega_M / \big(\mathcal{I}_S^2 \Omega_M + d\mathcal{I}_S^2\big);$$

its dual $\mathcal{T}_{S(1)} = \mathrm{Hom}_{\mathcal{O}_{S(1)}}(\Omega_{S(1)}, \mathcal{O}_{S(1)})$, where $\mathcal{O}_{S(1)} = \mathcal{O}_M / \mathcal{I}_S^2$ as usual, is the *holomorphic tangent sheaf* of $S(1)$. The map $d_{(1)} : \mathcal{O}_{S(1)} \to \Omega_{S(1)}$ given by $d_{(1)}([f]_2) = \pi_1(df)$, where $\pi_1 : \Omega_M \to \Omega_{S(1)}$ is the natural projection, is the *canonical differential*. We refer to [18, Chapter VI] for properties of differentials on an analytic space with nilpotents.

Q2 Proposition 1.3 yields a characterization of splitting manifolds in terms of $\Omega_{S(1)}$:

Proposition 6.1. *Let* S *be a submanifold of a complex manifold* M. *Then* S *splits into* M *if and only if there exists a surjective* $\mathcal{O}_{S(1)}$*-morphism*

$$X_{(1)} : \Omega_{S(1)} \to \mathcal{I}_S / \mathcal{I}_S^2$$

such that $X_{(1)} \circ d_{(1)} \circ i_1 = \mathrm{id}$, *where* $i_1 : \mathcal{I}_S/\mathcal{I}_S^2 \to \mathcal{O}_M/\mathcal{I}_S^2$ *is the natural inclusion. Furthermore, if* $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ *is a first order lifting, and* $\tilde{\rho} : \mathcal{O}_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ *is the associated left splitting morphism, then* $\tilde{\rho} = X_{(1)} \circ d_{(1)}$.

Q3 **Proof.** By Proposition 1.3, S splits in M if and only if there exists a θ_1-derivation $\tilde{\rho} : \mathcal{O}_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $\tilde{\rho} \circ i_1 = \mathrm{id}$.

Assuming $\tilde{\rho}$ given, the universal property of differentials yields a $\mathcal{O}_{S(1)}$-morphism $X_{(1)} : \Omega_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $X_{(1)} \circ d_{(1)} = \tilde{\rho}$; in particular, $X_{(1)} \circ d_{(1)} \circ i_1 = \mathrm{id}$.

Conversely, given $X_{(1)}$ then $\tilde{\rho} = X_{(1)} \circ d_{(1)}$ is a θ_1-derivation such that $\tilde{\rho} \circ i_1 = \mathrm{id}$, and thus S splits into M. □

To give the announced characterization of 1-comfortably embedded submanifolds we need a last definition and a last proposition.

Definition 6.2. Let S be a submanifold of a complex manifold M. Assume that S splits in M and denote by $X_{(1)} : \Omega_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ the $\mathcal{O}_{S(1)}$-morphism associated to the choice of a first order lifting by Proposition 6.1. An *infinitesimal normal connection along* $X_{(1)}$ on a $\mathcal{O}_{S(1)}$-module $\mathcal{E}$ on S is a $\mathbb{C}$-linear map $\tilde{X}_{(1)} : \mathcal{E} \to \mathcal{I}_S/\mathcal{I}_S^2 \otimes_{\mathcal{O}_{S(1)}} \mathcal{E}$ satisfying the Leibniz rule

$$\tilde{X}_{(1)}(gs) = X_{(1)}\big(d_{(1)}(g)\big) \otimes s + g\tilde{X}_{(1)}(s)$$

for all local sections g of $\mathcal{O}_{S(1)}$ and s of $\mathcal{E}$.

Remark 6.1. Any locally free $\mathcal{O}_S$-module $\mathcal{E}$ can be considered as a locally free $\mathcal{O}_{S(1)}$-module endowing it with the structure obtained by restriction of the scalars via the first order lifting ρ. However, for the application we have in mind we shall need a locally free $\mathcal{O}_{S(1)}$-module which is not obtained in this way.

Remark 6.2. In this section, indices like a, b, c, d will run from 1 to $\mathrm{rk}(\mathcal{E})$.

Proposition 6.2. *Let S be a submanifold of a complex manifold M. Assume that S splits in M, with first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated $\mathcal{O}_{S(1)}$-morphism $X_{(1)} : \Omega_{S(1)} \to \mathcal{N}_S^*$. Let $\mathcal{E}$ be a locally free $\mathcal{O}_{S(1)}$-module on S. Then the obstruction to the existence of an infinitesimal normal connection on $\mathcal{E}$ along $X_{(1)}$ is the class $\delta_\rho(\mathcal{E}) \in H^1(S, \mathcal{I}_S/\mathcal{I}_S^2 \otimes \mathrm{End}(\mathcal{E}))$ represented, in an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to ρ and trivializing $\mathcal{E}$, by the 1-cocycle*

$$\big[(\Phi_{\beta\alpha})_a^c\big]_2 \left[\frac{\partial (\Phi_{\alpha\beta})_c^d}{\partial z_\alpha^r} z_\alpha^r\right]_2 \otimes e_\beta^{*a} \otimes e_{d,\beta},$$

where $e_{b,\alpha}$ (for $b = 1, \ldots, \mathrm{rk}\,\mathcal{E}$) is a local frame for $\mathcal{E}$ over $U_\alpha \cap S$, e_α^{*b} is the dual frame, and $[(\Phi_{\alpha\beta})_c^b]_2 \in \mathcal{O}_{S(1)}$ are the transition functions of $\mathcal{E}$.

Proof. Let $\tilde{X}_{(1)} : \mathcal{E} \to \mathcal{I}_S/\mathcal{I}_S^2 \otimes_{\mathcal{O}_S} \mathcal{E}$ be an infinitesimal normal connection along $X_{(1)}$, and define an element $\eta_{c,\alpha}^b \in H^0(U_\alpha \cap S, \mathcal{I}_S/\mathcal{I}_S^2)$ by the formula

$$\tilde{X}_{(1)}(e_{c,\alpha}) = \eta_{c,\alpha}^b \otimes e_{b,\alpha}.$$

Now, if $U_\alpha \cap U_\beta \cap S \neq \emptyset$ we have $e_{b,\alpha} = [(\Phi_{\alpha\beta})_b^d]_2 e_{d,\beta}$; so

$$\eta_{c,\alpha}^b \otimes \left[(\Phi_{\alpha\beta})_b^d\right]_2 e_{d,\beta} = \tilde{X}_{(1)}(e_{c,\alpha}) = \tilde{X}_{(1)}\left(\left[(\Phi_{\alpha\beta})_c^d\right]_2 e_{d,\beta}\right)$$
$$= X_{(1)}\left(d_{(1)}\left[(\Phi_{\alpha\beta})_c^d\right]_2\right) \otimes e_{d,\beta} + \left[(\Phi_{\alpha\beta})_c^b\right]_2 \cdot \eta_{b,\beta}^d \otimes e_{d,\beta}.$$

But

$$X_{(1)}\left(d_{(1)}\left(\left[(\Phi_{\alpha\beta})_c^d\right]_2\right)\right) = \tilde{\rho}\left(\left[(\Phi_{\alpha\beta})_c^d\right]_2\right) = \left[\frac{\partial(\Phi_{\alpha\beta})_c^d}{\partial z_\alpha^r} z_\alpha^r\right]_2,$$

by Remark 1.8, and hence

$$\left[(\Phi_{\alpha\beta})_b^d\right]_2 \cdot \eta_{c,\alpha}^b = \left[\frac{\partial(\Phi_{\alpha\beta})_c^d}{\partial z_\alpha^r} z_\alpha^r\right]_2 + \left[(\Phi_{\alpha\beta})_c^b\right]_2 \cdot \eta_{b,\beta}^d.$$

If we define the 0-cocycle $\mathfrak{k} = \{\mathfrak{k}_\alpha\} \in H^0(\mathfrak{U}_S, \mathcal{I}_S/\mathcal{I}_S^2 \otimes \mathcal{E}^* \otimes \mathcal{E})$ by setting

$$\mathfrak{k}_\alpha = \eta_{c,\alpha}^b \otimes e_\alpha^{*c} \otimes e_{b,\alpha},$$

we get

$$\begin{aligned}
\mathfrak{k}_\alpha - \mathfrak{k}_\beta &= \eta_{c,\alpha}^b \otimes e_\alpha^{*c} \otimes e_{b,\alpha} - \eta_{c,\beta}^b \otimes e_\beta^{*c} \otimes e_{b,\beta} \\
&= \left[(\Phi_{\beta\alpha})_d^b\right]_2 \left(\left[\frac{\partial(\Phi_{\alpha\beta})_c^d}{\partial z_\alpha^r} z_\alpha^r\right]_2 + \left[(\Phi_{\alpha\beta})_c^a\right]_2 \cdot \eta_{a,\beta}^d\right) \otimes \left[(\Phi_{\beta\alpha})_d^c\right]_2 e_\beta^{*d} \otimes \left[(\Phi_{\alpha\beta})_b^d\right]_2 e_{d,\beta} \\
&\quad - \eta_{c,\beta}^b \otimes e_\beta^{*c} \otimes e_{b,\beta} \\
&= \left[\frac{\partial(\Phi_{\alpha\beta})_c^d}{\partial z_\alpha^r} z_\alpha^r\right]_2 \otimes \left[(\Phi_{\beta\alpha})_a^c\right]_2 e_\beta^{*a} \otimes e_{d,\beta},
\end{aligned}$$

and thus $\delta_\rho(\mathcal{E}) = O$.

Conversely, assume that $[(\Phi_{\beta\alpha})_a^c]_2[\frac{\partial(\Phi_{\beta\alpha})_c^d}{\partial z_\alpha^r}]_2[z_\alpha^r]_2 \otimes e_\beta^{*a} \otimes e_{d,\beta} = \mathfrak{k}_\alpha - \mathfrak{k}_\beta$ with $\mathfrak{k}_\alpha \in H^0(\mathfrak{U}_S, \mathcal{I}_S/\mathcal{I}_S^2 \otimes \mathcal{E}^* \otimes \mathcal{E})$. Writing $\mathfrak{k}_\alpha = \eta_{c,\alpha}^b \otimes e_\alpha^{*,c} \otimes e_{b,\alpha}$, it is easy to check that setting

$$\tilde{X}_{(1)}(e_{c,\alpha}) = \eta_{c,\alpha}^b \otimes e_{b,\alpha}$$

we define an infinitesimal normal connection on $\mathcal{E}$. □

If S splits into M, and $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ is a splitting atlas, then it is easy to check that the position

$$(\Phi_{\alpha\beta})_s^r = \left[\frac{\partial z_\beta^r}{\partial z_\alpha^s}\right]_2$$

defines a 1-cocycle with coefficients in $GL(m, \mathcal{O}_{S(1)})$, and hence a locally free $\mathcal{O}_{S(1)}$-module on S that, with a slight abuse of notations, we shall denote by $\mathcal{N}_S$. One of the reasons justifying this notation is that the 1-cocycle of the locally free $\mathcal{O}_{S(1)}$-module $\mathcal{I}_S/\mathcal{I}_S^2$ is the inverse transposed of the 1-cocycle of $\mathcal{N}_S$, and thus $\mathcal{N}_S^* \cong \mathcal{I}_S/\mathcal{I}_S^2$ as $\mathcal{O}_{S(1)}$-modules too. Notice, however, that (1.11) implies that this $\mathcal{N}_S$ is not the locally free $\mathcal{O}_{S(1)}$-module obtained by restriction of the scalars via ρ starting from the usual normal sheaf on S (as described in Remark 6.1) *unless* S is 1-comfortably embedded in M.

We finally have the promised characterization of 1-comfortably embedded submanifolds:

Proposition 6.3. *Let S be a submanifold of a complex manifold M. Assume that S splits in M, with first order lifting $\rho : \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated $\mathcal{O}_{S(1)}$-morphism $X_{(1)} : \Omega_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$. Then S is 1-comfortably embedded into M if and only if there exists an infinitesimal normal connection on $\mathcal{N}_S$.*

Proof. Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an atlas adapted to ρ, and denote by $\{\partial_{r,\alpha}\}$ and $\{[z_\alpha^r]_2\}$ the induced local frames on $\mathcal{N}_S$ and $\mathcal{I}_S/\mathcal{I}_S^2$ as locally free $\mathcal{O}_{S(1)}$-modules. Proposition 6.2 says that there exists an infinitesimal holomorphic connection on $\mathcal{N}_S$ along $X_{(1)}$ if and only if the 1-cocycle $\delta_\rho(\mathcal{N}_S)$ in $H^1(S, (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes 2} \otimes \mathcal{N}_S)$ given by

$$\left[\frac{\partial z_\alpha^r}{\partial z_\beta^s}\right]_2 \left[\frac{\partial^2 z_\beta^t}{\partial z_\alpha^u \partial z_\alpha^r}\right]_2 \left[z_\alpha^u\right]_2 \otimes \left[z_\beta^s\right]_2 \otimes \partial_{t,\beta} = \left[\frac{\partial^2 z_\beta^t}{\partial z_\alpha^u \partial z_\alpha^v}\right]_1 \left[z_\alpha^u\right]_2 \otimes \left[z_\alpha^v\right]_2 \otimes \partial_{t,\beta}$$

vanishes.

Now, $\delta_\rho(\mathcal{N}_S)$ clearly belongs to $H^1(S, \mathrm{Sym}^2(\mathcal{I}_S/\mathcal{I}_S^2) \otimes \mathcal{N}_S)$. Since $\mathrm{Sym}^2(\mathcal{I}_S/\mathcal{I}_S^2)$ is a direct summand of $(\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes 2}$, a 1-cocycle in $H^1(S, \mathrm{Sym}^2(\mathcal{I}_S/\mathcal{I}_S^2) \otimes \mathcal{N}_S)$ vanishes in $H^1(S, (\mathcal{I}_S/\mathcal{I}_S^2)^{\otimes 2} \otimes \mathcal{N}_S)$ if and only if it vanishes in $H^1(S, \mathrm{Sym}^2(\mathcal{I}_S/\mathcal{I}_S^2) \otimes \mathcal{N}_S)$. Since $\mathrm{Sym}^2(\mathcal{I}_S/\mathcal{I}_S^2) \cong \mathcal{I}_S^2/\mathcal{I}_S^3$, the assertion follows from Corollary 3.6. □

Remark 6.3. Assume S splits into M and let $X_{(1)} : \Omega_{S(1)} \to \mathcal{N}_S^*$ be the corresponding $\mathcal{O}_{S(1)}$-morphism. Then one can adapt the notion (see [4]) of first jet bundle and associate to any $\mathcal{O}_{S(1)}$-module $\mathcal{E}$ an $\mathcal{O}_{S(1)}$-module $J^1_{\mathcal{N}_S}\mathcal{E}$, the *sheaf of normal first jets* of $\mathcal{E}$, and an exact sequence of $\mathcal{O}_{S(1)}$-modules

$$O \to \mathcal{N}_S^* \otimes \mathcal{E} \to J^1_{\mathcal{N}_S}\mathcal{E} \to \mathcal{E} \to O \tag{6.1}$$

in such a way that if $\mathcal{E}$ is locally free than $J^1_{\mathcal{N}_S}\mathcal{E}$ is locally free too, and the class $\delta_\rho(\mathcal{E})$ introduced in Proposition 6.2 is exactly the class associated to the extension (6.1). In particular, Proposition 6.3 implies that the sequence (6.1) splits if and only if S is 1-comfortably embedded.

Acknowledgments

We would like to thank Francesco Russo for pointing out reference [20], Jorge Vitorio Pereira for several useful conversations, and IMPA for the warm hospitality during part of the preparation of this paper. This work has been partially supported by INdAM and FIRB.

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Please cite this article in press as: M. Abate et al., Embeddings of submanifolds and normal bundles, Adv. Math. (2008), doi:10.1016/j.aim.2008.10.001