Morse theory for strongly indefinite functionals and Hamiltonian systems

TESI DI PERFEZIONAMENTO

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Introduction

Since its infinite dimensional generalizations, developed by Palais in [Pal63], Morse theory has become a powerful tool in the calculus of variations. Loosely speaking, the standard approach is the following: assume that the solutions of a differential problem can be seen as critical points of a suitable smooth functional f, defined on an infinite dimensional manifold. Then the number of such critical points can be estimated by studying the topology of the sub-levels of f.

More precisely, assume that f it is a Morse functional, meaning that the second order differential $d^2f(x)$ is non-degenerate at every critical point x, and that it satisfies some compactness condition (for example the Palais-Smale condition). The Morse index $m(x,f)$ of a critical point x is the dimension of the maximal subspace on which $d^2f(x)$ is negative definite. If $c \in]a,b[$ is the only critical level in $[a,b]$ and x_0 is the only critical point at level c, then the set $f^b = \{x \in M \mid f(x) \leq b\}$ can be continuously deformed onto $f^a \cup B_{m(x,f)}$, where $B_{m(x,f)}$ is a $m(x,f)$-dimensional closed ball, attached to f^a by its boundary. This local result is used to prove the Morse relations, which can be written in the form

$$\sum_{\substack{a<f(x)<b \\ x \text{ critical point}}} \lambda^{m(x,f)} = P(f^b,f^a) + (1+\lambda)Q(\lambda) \tag{1}$$

where Q is a polynomial with positive integer coefficients and P is the Poincaré polynomial of the pair (f^b,f^a):

$$P(f^b,f^a) = \sum_{q \geq 0} \dim H_q(f^b,f^a)\ \lambda^q.$$

Here H_* denotes the singular homology theory, with coefficients in a given field.

If a critical point has infinite Morse index, then the corresponding ball which has to be attached is infinite dimensional. Since every infinite dimensional ball can be continuously deformed onto its boundary (see, for example, [Bes66]), attaching such a ball does not change the homotopy type, and therefore the homology, of the sublevels. As a consequence, critical points with infinite Morse index are not detected by formula (1), which still makes sense with the convention that $\lambda^\infty = 0$. Notice that also other topological arguments, such as Ljusternik-Schnirelman theory, or Krasnoselskii genus, would fail in such a situation: all these methods detect critical points looking at changes in the homotopy type of the sublevels, and here there is no such change.

The problem is relevant, because there are many variational problems in which all the solutions are critical points with infinite Morse index of some functional. This happens, for example, in the study of Hamiltonian systems, of symplectic geometry, of wave equations, of minimal submanifolds in semi-Riemannian geometries, of Dirac-type equations. Such functionals are called strongly indefinite.

In this thesis we consider the case in which the domain of the functional is a Hilbert space E. In most applications one notes that, although infinite dimensional, the negative eigenspace of $d^2f(x)$ has a finite relative dimension with respect to a fixed subspace of E. To be more precise, there exists a fixed orthogonal splitting $E = E^+ \oplus E^-$ such that, for every critical point x, the positive eigenspace V_x^+ of $d^2f(x)$ is *commensurable* with E^+, while the negative eigenspace V_x^- is *commensurable* with E^- (two closed subspaces V and W are called *commensurable* if the quotient projections $V \mapsto E/W$ and $W \mapsto E/V$ are compact).

In this case it seems reasonable to define the relative dimension of V_x^-, which may be called E^+-dimension, as

$$E^+\text{-dim}\, V_x^- = \dim V_x^- \cap E^+ - \dim V_x^+ \cap E^-.$$

By the commensurability condition, this is a finite integer. The E^+-Morse index $E^+\text{-m}\,(x;f)$ of a critical point x can be defined as the E^+-dimension of the negative eigenspace of $d^2f(x)$.

The next step is to build a suitable generalized cohomology $H^*_{E^+}$, which may be called E^+-cohomology (a generalized cohomology is a functor which satisfies all the Eilenberg-Steenrod axioms, [ES52], except the dimension axiom; moreover the functoriality and the homotopy invariance may hold for a restricted class of continuous maps and homotopies). We require that $H^*_{E^+}$ detects infinite dimensional spheres and distinguishes between them, according to the E^+-dimension. More precisely, we require that $H^q_{E^+}(B_m, \partial B_m) = \mathcal{A}$, $\mathcal{A}$ being the coefficient ring, if $q = m$, and 0 otherwise; here B_m is a closed ball in a subspace of E^+-dimension m. The idea of such an approach is due to Szulkin, [Szu92], whose theory will be discussed in the last section of Chapter 2. The techniques involved in the construction go back to Gęba and Granas, [GG73].

Then our goal will be to prove that, for f in a certain class of functionals, the following generalized Morse relations hold

$$\sum_{\substack{a<f(x)<b \\ x \text{ critical point}}} \lambda^{E^+\text{-m}\,(x,f)} = P_{E^+}(f^b, f^a) + (1+\lambda)Q(\lambda) \tag{2}$$

where P_{E^+} is the E^+-Poincaré polynomial of the pair (f^b, f^a)

$$P_{E^+}(f^b, f^a) = \sum_{q\in\mathbf{Z}} \dim H^q_{E^+}(f^b, f^a)\ \lambda^q$$

We emphasize that, since the E^+-dimension can also be negative, P_{E^+} and Q are Laurent polynomials.

Actually, we will prove the Morse relations in a different form, generalizing the so called Morse-Conley relations. The advantage of such formulation is that degenerate critical points can be considered. As a particular case, we will get the Morse-Bott relations for functionals with non-degenerate critical manifolds. Developing this E^+-Conley theory will be the aim of Chapter 1.

In Chapter 2 we will construct the E^+-cohomology. Such a construction is quite technical and the reader, if not interested, may skip this chapter completely. In fact, as often happens with homology and cohomology, the theory can be used axiomatically, without knowing the details which stay behind. The axioms are given in Chapter 1, section 1.3, and are used immediately afterwards.

Then our aim will be to apply this general theory to asymptotically linear periodic Hamiltonian systems, and to find some new results about the existence of periodic orbits. In order to do this, we will devote Chapter 3 to the definition of the Maslov index, an important symplectic invariant which measures how much the Hamiltonian flow rotates near a given periodic solution. This is a review chapter and it contains no original results. However we have included some important topics, which are not easily found in the literature: a simple exposition of the two dimensional case, the computation of the Maslov index of an autonomous system.

Chapter 4 is devoted to the applications. First of all we prove that the Maslov index of a periodic solution equals its E^+-Morse index as a critical point of a strongly indefinite functional, the action functional. Then give a proof in our framework of the Morse relations for systems which are not resonant at infinity, a well known result due to Conley and Zehnder, see [CZ84].

Conley and Zehnder's theorem has the following consequence: a system with a periodic solution z_0, whose Maslov index is different from the Maslov index at infinity, must have at least another periodic solution, two in the non-degenerate case. Many authors have proved analogous statements for systems which are resonant at infinity. Technically, the difficulty arises from the fact that, in such cases, the functional may not satisfy the Palais-Smale condition. There are essentially two kind of hypotheses which guarantee the existence of another solution: strong resonance conditions and conditions of Landesman-Lazer type. Both these conditions are assumptions on the behavior at infinity of the non-quadratic part of the Hamiltonian function (see [CLL97] for the most general statements and for a complete bibliography). Here we prove the statement without such conditions, generalizing a result by Benci and Fortunato, [BF94], for second order Lagrangian systems.

Then we deal with sub-harmonics, in two dimensional systems, in the spirit of a recent theorem of Franks, [Fra92]. Franks theorem asserts that an area-preserving homeomorphism of the disc with two or more fixed points must have infinitely many periodic points (two fixed points are necessary: a rotation of an irrational angle has no periodic points, apart from the fixed point in the origin). Here we prove, using a variational approach, that a two-dimensional periodic asymptotically linear system, non resonant at infinity, with two or more periodic solutions, must have infinitely many sub-harmonics.

Finally we study autonomous Hamiltonian systems, giving a lower bound for the number $n(T)$ of non-constant T-periodic solutions. We find that

$$n(T) \geq \frac{1}{2}\Theta T - M$$

where the constants Θ and M can be explicitly computed, depending only on the critical points of the Hamilton function. This result generalizes an analogous estimate for second order Lagrangian systems, due to Benci and Fortunato, [BF97].

We would like to emphasize that the proof of each of these results makes use of Morse theory in a fundamental way, since it relies on the particular form $(1+\lambda)Q(\lambda)$ of the correction term in formula (2) (or, equivalently, on the fact that the Morse complex and the De Rahm complex have the same cohomology, see [Sch93]).

As not to unduly encumber the text, only those references which seemed strictly necessary have been included in the exposition: more complete bibliographical indications and further remarks have been gathered in the last section of each chapter.

I would like to express my gratitude to Prof. Joseph Bernstein and his group at Tel-Aviv University, whose suggestions have made the E^+-cohomology simpler, to Prof. Pietro Majer and Dr. Roberto Tauraso, for uncountable interesting discussions, to Prof. Helmut Hofer for his kind hospitality at E.T.H., where I first heard about Franks' theorem. Finally my gratitude goes to Prof. Vieri Benci, who inspired a large part of the material of this thesis.

Chapter 1

The E^+-Conley theory

The aim of this chapter is to study functionals, defined on a Hilbert space E, whose critical points may have infinite Morse index.

The basic assumption is that the negative eigenspaces of such critical points are *commensurable* with a given closed subspace E^- of E. Roughly speaking, commensurability means that we are adding or removing a finite number of dimensions from a subspace obtained from E^- by means of a transformation of the form Identity + Compact.

Calling E^+ the orthogonal complement of E^-, we will define the E^+-dimension of a subspace commensurable with E^-: E^- itself has E^+-dimension 0, adding a finite dimensional subspace increases the E^+-dimension, while by removing a subspace we can obtain spaces of negative E^+-dimension. The reason for the name E^+-dimension (instead of E^--dimension, as one could expect) comes from the fact that when no Hilbert structure is present and there is no privileged complementary space, the relevant space in the whole construction is E^+.

Then we will introduce a generalized cohomology which distinguishes infinite dimensional subspaces, according to their E^+-dimension. The construction of such cohomology will be the aim of the next chapter. Here we only give the equivalent of the Eilenberg-Steenrod axioms for this theory, which are about all one needs to know for practical purposes.

Then we will define the E^+-index pair for a compact isolated subset K_0 of the critical points of a functional. An E^+-index pair is a pair of suitable sets (X, A) such that X is a neighborhood of K_0 and A is the subset of X through which the gradient flow of f exits. The crucial fact is that the E^+-cohomology of an E^+-index pair depends only on the flow, and not on the choice of the E^+-index pair. The E^+-cohomology groups of an E^+-index pair of an isolated critical set can be seen as a generalization of the Morse index.

Finally we will prove the Morse-Conley relations for our cohomology, paying attention to the case of non-degenerate critical manifolds, whose E^+-index pairs can be easily constructed.

1.1 The E^+-dimension.

Let E be a real Hilbert space. We denote by $\|\cdot\|$ the Hilbert norm and by $< \cdot, \cdot >$ the scalar product on E.

Our assumption is that E is given with an orthogonal splitting into closed subspaces

$$E = E^+ \oplus E^-.$$

Both E^+ and E^- may be infinite dimensional. We would like to compare infinite dimensional subspaces with E^+ and E^- and to define a sort of relative dimension.

Definition 1.1.1 *Two closed subspaces V and V' of E are called commensurable if the quotient projections $V' \to E/V$ and $V \to E/V'$ are compact.*

Taking advantage of the Hilbert, structure we can reformulate the notion of commensurability in the following way: two closed subspaces V and V' of E are commensurable if and only if both $P_{V^\perp}$ restricted to V' and $P_{V'^\perp}$ restricted to V are compact. Here P_V denotes the orthogonal projection onto V and $V^\perp$ denotes the orthogonal complement of V.

Commensurability is an equivalence relation: we have only to check the transitivity property, so assume that V is commensurable with W and W is commensurable with Z. We must show that $P_{Z^\perp}(\overline{B_V})$ is relatively compact, $\overline{B_V}$ being the closed unit ball in V. Notice that

$$\overline{B_V} \subset P_W(\overline{B_V}) + P_{W^\perp}(\overline{B_V}) \subset \overline{B_W} + P_{W^\perp}(\overline{B_V})$$

and thus

$$P_{Z^\perp}(\overline{B_V}) \subset P_{Z^\perp}(\overline{B_W}) + P_{Z^\perp}(P_{W^\perp}(\overline{B_V})).$$

$P_{Z^\perp}(\overline{B_W})$ and $P_{W^\perp}(\overline{B_V})$ are relatively compact and so must be $P_{Z^\perp}(P_{W^\perp}(\overline{B_V}))$. Then $P_{Z^\perp}(\overline{B_V})$ is relatively compact.

If $V' = W \oplus Y$, where W is a finite codimensional subspace of V and Y is finite dimensional, then V and V' are commensurable: in fact the projections involved have finite rank.

Definition 1.1.2 *Let V be a closed subspace of E commensurable with E^-. The E^+-dimension of V is defined as*

$$\begin{aligned} E^+\text{-}\dim V &= \dim V \cap E^+ - \operatorname{codim}(V + E^+) \\ &= \dim V \cap E^+ \text{-} \dim V^\perp \cap E^-. \end{aligned}$$

Both the addenda in the above formula are finite because of the commensurability of V and E^-. In fact the quotient projection $V \to E/E^- \cong E^+$ is compact and it equals the identity on $V \cap E^+$: therefore $V \cap E^+$ must be finite dimensional. By the same argument $V^\perp \cap E^-$ is finite dimensional.

Some examples will clarify the meaning of the E^+-dimension. The subspace E^- has E^+-dimension zero. If Y is finite dimensional and $Y \cap E^- = \{0\}$, then

$$E^+\text{-}\dim E^- \oplus Y = \dim Y \geq 0.$$

If V is a finite codimensional subspace of E^-

$$E^+\text{-}\dim V = -\operatorname{codim}_{E^-} V \leq 0.$$

It is useful to give a second way to express the E^+-dimension of a subspace V. Let S denote the restriction of the orthogonal projection $P_V : E \to V$ to the subspace E^-:

$$S = P_V|_{E^-} : E^- \to V.$$

The kernel of S is $E^- \cap V^\perp$. The orthogonal complement of the image of S in V is $V \cap E^+$. Therefore if V is commensurable with E^-, S is a Fredholm operator, and the E^+-dimension of V equals the opposite of the index of S:

$$E^+\text{-}\dim V = -\operatorname{ind} S = \operatorname{codim}_V S(E^-) - \dim \operatorname{Ker} S. \tag{1.1}$$

The commensurability class of a subspace is invariant under the action of a linear operator of the form Identity + Compact, as the next proposition shows.

Proposition 1.1.1 *Let V be a closed subspace of E. Let $K : V \to E$ be a compact operator and let $T = I + K$. Then $T(V)$ is closed and commensurable with V. Moreover, if V is commensurable with E^-*

$$E^+\text{-}\dim V = \dim \operatorname{Ker} T \cap V + E^+\text{-}\dim T(V). \tag{1.2}$$

PROOF. A standard argument shows that $T(V)$ is closed: let W be the orthogonal complement of $\operatorname{Ker} T$ in V. It is enough to show that the one-to-one and onto operator $T|_W : W \to T(V)$ has a bounded inverse. Arguing by contradiction, we can find a sequence w_n in W of unitary vectors such that $\|Tw_n\| \to 0$. Considering a subsequence, we can assume that $Kw_n \to w \in W$, and thus $w_n = Tw_n - Kw_n \to w$. Therefore $\|w\| = 1$ and $Tw = 0$, which is impossible because $W \cap \operatorname{Ker} T = 0$.

Since K is compact, we can find a finite codimensional subspace Z of V such that

$$\|K|_Z\| < 1.$$

We will show that Z is commensurable with the closed subspace $T(Z)$. Since Z has finite codimension in V and $T(Z)$ has finite codimension in $T(V)$, the transitivity property of the commensurability relation implies that V and $T(V)$ are commensurable. Let

$$\pi : E \to E/Z, \quad \pi' : E \to E/T(Z)$$

be the quotient projections. If $z \in Z$

$$\pi'(z) = \pi'(z + Kz - Kz) = -\pi'(Kz).$$

Therefore

$$\pi'|_Z = -\pi' \circ K|_Z$$

and π' is compact on Z.

Denote by $\overline{B_Y}(r)$ the closed ball of radius r in the subspace Y. If $w = z + Kz \in \overline{B_{T(Z)}}(1)$, then

$$\|z\| \le \|w\| + \|Kz\| \le 1 + \|K|_Z\| \|z\|$$

and so

$$\|z\| \leq \frac{1}{1 - \|K|_Z\|}.$$

Since $\pi(w) = \pi(Kz)$ we deduce that

$$\pi(\overline{B_{T(Z)}}(1)) \subset \pi \circ K \left(\overline{B_Z} \left(\frac{1}{1 - \|K|_Z\|} \right) \right).$$

The above inclusion implies that π is compact on $T(Z)$. Therefore Z and $T(Z)$ are commensurable.

Consider T as an operator from V to $T(V)$. Since V is commensurable with E^-, $P_V|_{E^-}$ is a compact perturbation of the identity. Therefore $T \circ P_V|_{E^-}$ is an operator from E^- to $T(V)$ of the form Identity + Compact, being the composition of two operator of this kind.

Also $P_{T(V)}|_{E^-} : E^- \to T(V)$ has the form Identity + Compact and, since adding a compact operator does not change the Fredholm index, we have

$$\operatorname{ind} P_{T(V)}|_{E^-} = \operatorname{ind} T \circ P_V|_{E^-}.$$

Thus

$$\operatorname{ind} P_{T(V)}|_{E^-} = \operatorname{ind} T + \operatorname{ind} P_V|_{E^-} = \dim \operatorname{Ker} T \cap V + \operatorname{ind} P_V|_{E^-}$$

and (1.1) implies formula (1.2). □

Let $\mathcal{T}_{E^+}$ be the weakest topology on E such that all the bounded linear functionals and the quotient projection $\pi : E \to E/E^+$ are continuous. Equivalently, $\mathcal{T}_{E^+}$ is the product topology between the weak topology of E^+ and the strong topology of E^-.

The next proposition shows that the topology $\mathcal{T}_{E^+}$ depends only on the commensurability class of E^+.

Proposition 1.1.2 *If E^+ and F^+ are commensurable closed subspaces of E, the topologies $\mathcal{T}_{E^+}$ and $\mathcal{T}_{F^+}$ coincide.*

PROOF. Denote by E^- and F^- the orthogonal complements of E^+ and F^+. It is enough to prove that the quotient projection $\pi : E \to E/E^+$ is $\mathcal{T}_{F^+}$ continuous. But this is true because π is strongly continuous on F^- and, being compact, it is continuous from the weak topology of F^+ to the strong topology of E/E^+. □

1.2 The E^+-Morse index

Let f be a twice differentiable functional on E. Let $d^2 f(x)$ be the second order differential of f at x, thought as a symmetric bilinear form. Let $D^2 f(x)$ be the associated self-adjoint operator:

$$< D^2 f(x) u, v >= d^2 f(x)[u, v].$$

Definition 1.2.1 *A critical point x of f is called non-degenerate if $D^2 f(x)$ is invertible. A critical point x of f is called weakly non-degenerate if $D^2 f(x)$ is a Fredholm operator.*

By the spectral representation of self-adjoint operators

$$D^2 f(x) = \int_{-\infty}^{+\infty} \mu \, dP_\mu$$

where $\{P_\mu \mid \mu \in \mathbf{R}\}$ is a partition of the identity.

If x is a weakly non-degenerate critical point of f, E splits into three closed $D^2 f(x)$-invariant orthogonal subspaces

$$E = V^0 \oplus V^+ \oplus V^-, \quad V^0 = \operatorname{Ker} D^2 f(x),$$
$$V^+ = \int_0^{+\infty} dP_\mu(E), \quad V^- = \int_{-\infty}^0 dP_\mu(E).$$

Here V^0 is finite dimensional, while $D^2 f(x)$ is strictly positive on V^+ and strictly negative on V^-: there exists a positive number α such that

$$< D^2 f(x)u, u > \geq \alpha \|u\|^2 \quad \forall u \in V^+$$
$$< D^2 f(x)u, u > \leq -\alpha \|u\|^2 \quad \forall u \in V^-.$$

V^+ and V^- are the positive and the negative eigenspaces of $D^2 f(x)$.

Definition 1.2.2 *Let x be a weakly non-degenerate critical point of f. Assume moreover that the negative eigenspace V^- of $D^2 f(x)$ is commensurable with E^-. The E^+-Morse index is the finite relative integer*

$$E^+\text{-m}\,(x) = E^+\text{-m}\,(x; f) = E^+\text{-dim}\, V^-.$$

The large E^+-Morse index is the finite relative integer

$$E^+\text{-m}^*\,(x) = E^+\text{-m}^*\,(x; f) = E^+\text{-dim}\,(V^0 \oplus V^-).$$

The following propositions provide useful tools to compute the E^+-Morse index.

Proposition 1.2.1 *Let L be a Fredholm operator whose negative eigenspace V^- is commensurable with E^-. Then*

$$E^+\text{-dim}\, V^- = \max E^+\text{-dim}\, W$$

where the maximum is taken over the family of all closed subspaces W of E which are commensurable with E^- and such that L is strictly negative on W.

PROOF. Let P be the orthogonal projection onto V^-.

Let W be a closed subspace of E commensurable with E^- and such that $L|_W$ is strictly negative. We must prove that

$$E^+\text{-dim}\, W \leq E^+\text{-dim}\, V^-.$$

Since W is commensurable with V^-, the linear operator $I - P$ is compact on W. Moreover P is one-to-one on W. By Proposition 1.1.1, $P(W)$ is commensurable with E^- and

$$E^+\text{-dim}\, W = E^+\text{-dim}\, P(W) \leq E^+\text{-dim}\, V^-.$$

$\square$

Proposition 1.2.2 *Denote by V_L^- the negative eigenspace of the self-adjoint operator L. Let $\mathcal{L}(E^-)$ be the set of invertible self-adjoint operators whose negative eigenspace is commensurable with E^-. Then $\mathcal{L}(E^-)$ is closed as a subset of the set of invertible self-adjoint operators. The function*

$$\mathcal{L}(E^-) \ni L \mapsto E^+\text{-}\dim V_L^- \in \mathbf{Z} \tag{1.3}$$

is continuous.

PROOF. Assume that (L_n) is a sequence of self-adjoint invertible operators which converges to the self-adjoint invertible operator L. We assume that the negative eigenspaces $V_n = V_{L_n}^-$ are commensurable with E^-. Set $V = V_L^-$ and

$$T_n = P_{V_n}|_{E^-} : E^- \to V_n, \quad T = P_V|_{E^-} : E^- \to V.$$

P_{V_n} converges to P_V and from the fact that the set of compact operators is closed, it is easy to deduce that $L \in \mathcal{L}(E^-)$.

The projection

$$S_n = P_V|_{V_n} : V_n \to V$$

is an isomorphism provided n is large enough. Define $R_n : E^- \to V$ as $R_n = S_n \circ T_n$. Then R_n is a Fredholm operator whose index equals the index of T_n, for n large enough.

Since (P_{V_n}) converges to P_V, R_n converges to T. Therefore the index of R_n equals the index of T, for n large enough, and thus (1.1) implies that the function (1.3) is continuous. □

We conclude this section exhibiting an important class of functionals whose critical points have a well-defined E^+-Morse index. Consider a functional of the form

$$f(x) = \frac{1}{2} < Lx, x > + b(x)$$

where L is a bounded self-adjoint operator and b is twice differentiable. Then

$$D^2 f(x) = L + D^2 b(x).$$

If L is a Fredholm operator whose negative eigenspace is commensurable with E^- and $D^2 b(x)$ is a compact operator, then the negative eigenspace of $D^2 f(x)$ is commensurable with E^-, as the next proposition shows.

Proposition 1.2.3 *Assume that L is a self-adjoint Fredholm operator and that K is a self-adjoint compact operator. Then the negative eigenspaces of $L + K$ and of L are commensurable.*

PROOF. We denote by $V^+(T)$ and $V^-(T)$ the positive and the negative eigenspaces of the self-adjoint Fredholm operator T. Moreover, if $T = \int_{-\infty}^{+\infty} \mu \, dP_\mu$, we denote by T^+ and T^- the positive and negative parts of T

$$T^+ = \int_0^{+\infty} \mu \, dP_\mu, \quad T^- = -\int_{-\infty}^0 \mu \, dP_\mu$$

so that $T = T^+ - T^-$ and $|T| = T^+ + T^-$.

Since our problem is symmetric, it is enough to prove the compactness of the quotient projection

$$V^-(L+K) \to E/V^-(L).$$

Equivalently, we have to show that the orthogonal projection P onto $V^+(L) \oplus \operatorname{Ker} L$ is compact on $V^-(L+K)$. Since $(L+K)^-$ is invertible on $V^-(L+K)$, it is enough to prove that $P \circ (L+K)^-$ is compact.

Notice that

$$2P \circ (L+K)^- = P \circ (|L+K| - L - K) = P \circ (|L+K| - |L| - K).$$

Therefore the thesis will follow from the fact that $|L+K| - |L|$ is compact, whenever K is compact. This follows from from a general fact: if L is a bounded self-adjoint operator, K is a compact self-adjoint operator and $h : \mathbf{R} \mapsto \mathbf{R}$ is a continuous function, then the operator $h(L+K) - h(L)$ is compact. In our case $h(s) = |s|$.

To prove the last assertion, notice that $(L+K)^m - L^m$ is compact,for every $m \in \mathbf{N}$. Therefore $p(L+K) - p(L)$ is compact for every polinomyal p. Now choose a sequence of polinomyals (p_n) which converges uniformly to h on a bounded set containing both the spectrum of L and the spectrum of $L+K$. Then $(p_n(L+K))$ converges to $h(L+K)$ and $(p_n(L))$ converges to $h(L)$ in the operator norm. So $h(L+K) - h(L)$ is compact, being the limit of a sequence of compact operators. □

1.3 The E^+-cohomology

A topological pair is a pair of topological spaces (X, A) such that $A \subset X$. Let Top be the category whose objects are topological pairs and whose morphisms $\Phi : (X, A) \mapsto (Y, B)$ are continuous maps $\Phi : X \mapsto Y$ such that $\Phi(A) \subset B$. Let $\mathcal{A}$ be a ring and let $\mathcal{M}(\mathcal{A})$ be the category of moduli over $\mathcal{A}$ and $\mathcal{A}$-homomorphisms.

A generalized cohomology with coefficients in $\mathcal{A}$ on a sub-category $\mathcal{C}$ of Top is a controvariant functor from $\mathcal{C}$ to $\mathcal{M}(\mathcal{A})$, which satisfies all the Eilenberg-Steenrod axioms, except from the dimension axiom.

We want to introduce a generalized cohomology theory on a category consisting of a class of $\mathcal{T}_{E^+}$-closed subsets of E and of a class of $\mathcal{T}_{E^+}$-continuous maps between them.

Let $\pi : E \to E/E^+$ be the quotient projection. The sets we will consider are the E^+-locally-compact subsets of E:

Definition 1.3.1 *A subset X of E is E^+-locally compact if $X \cap \pi^{-1}(\alpha)$ is $\mathcal{T}_{E^+}$-locally compact, for every finite dimensional subspace α of E/E^+.*

Every bounded $\mathcal{T}_{E^+}$-closed subset X of E is E^+-locally compact. In fact $X \cap \pi^{-1}(\alpha) = X \cap (E^+ \oplus Y)$ for a suitable finite dimensional subspace Y of E^- and $\mathcal{T}_{E^+}$ induces the weak topology on $E^+ \oplus Y$: therefore $X \cap \pi^{-1}(\alpha)$ is $\mathcal{T}_{E^+}$-compact.

If E^+ is finite dimensional, $\mathcal{T}_{E^+}$ is the strong topology and every closed subset of E is E^+-locally compact.

If E^+ is infinite dimensional, an unbounded set may not be E^+-locally compact: for example E is not E^+-locally compact, because an infinite dimensional Hilbert space is never locally compact with respect to its weak topology.

The objects of our generalized cohomology theory will be the E^+-pairs.

Definition 1.3.2 *An E^+-pair (X, A) is a topological pair of subsets of E such that X and A are $\mathcal{T}_{E^+}$-closed and E^+-locally compact.*

The morphisms will be the E^+-morphisms.

Definition 1.3.3 *A $\mathcal{T}_{E^+}$-continuous map $\Phi : (X, A) \mapsto (Y, B)$ is an E^+-morphism if:*

1. *it has the form*
$$\Phi(x) = Tx + K(x)$$
where K maps bounded sets into $\mathcal{T}_{E^+}$-pre-compact sets and T is a linear automorphism of E such that $TE^+ = E^+$, $TE^- = E^-$;

2. *$\Phi^{-1}(U)$ is bounded for every bounded U.*

We need also a notion of homotopy.

Definition 1.3.4 *A $\mathcal{T}_{E^+}$-continuous map $\Psi : (X \times [0,1], A \times [0,1]) \mapsto (Y, B)$ is an E^+-homotopy if:*

1. *it has the form*
$$\Psi(x,t) = T_t x + K(x,t)$$
where K maps bounded sets into $\mathcal{T}_{E^+}$-pre-compact sets and T_t is a continuous path of linear automorphisms of E such that $T_t E^+ = E^+$, $T_t E^- = E^-$;

2. *$\Psi^{-1}(U)$ is bounded for every bounded U.*

Two E^+-morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are called E^+-homotopic if there exists an E^+-homotopy Ψ such that $\Psi(\cdot, 0) = \Phi_0$ and $\Psi(\cdot, 1) = \Phi_1$.

Here is the existence theorem for our generalized cohomology theory, which will be called E^+-cohomology.

Theorem 1.3.1 *There exists a generalized cohomology $H^q_{E^+}$, $q \in \mathbf{Z}$, with coefficients in $\mathbf{Z}_2$, which acts on the category of E^+-pairs and E^+-morphisms. More precisely:*

1. *(Controvariant functoriality) If $I : (X, A) \mapsto (X, A)$ is the identity map, $H^q_{E^+}(I)$ is the identity homomorphism on $H^q_{E^+}(X, A)$. If $\Phi : (X, A) \mapsto (Y, B)$ and $\Phi' : (Y, B) \mapsto (Z, C)$ are E^+-morphisms, then $H^q_{E^+}(\Phi' \circ \Phi) = H^q_{E^+}(\Phi) \circ H^q_{E^+}(\Phi')$.*

2. *(Homotopy invariance) If two E^+-morphisms are E^+-homotopic, then $H^q_{E^+}(\Phi) = H^q_{E^+}(\Phi')$.*

3. *(Strong excision) If X and Y are $\mathcal{T}_{E^+}$-closed E^+-locally compact subsets of E and $i : (X, X \cap Y) \hookrightarrow (X \cup Y, Y)$ is the inclusion map, then $H^q_{E^+}(i)$ is an isomorphism.*

4. *(Functoriality of the coboundary) Given three $\mathcal{T}_{E^+}$-closed and E^+-locally compact sets X, Y, Z, such that $X \subset Y \subset Z$, there exists a family of coboundary homomorphisms*

$$\delta^q_{E^+}(Z,Y,X) : H^q_{E^+}(Y,X) \to H^{q+1}_{E^+}(Z,Y).$$

If $\Phi : (Z,Y) \mapsto (Z',Y')$ is an E^+-morphism such that $\Phi(X) \subset X' \subset Y'$, then the following diagram commutes:

$$\begin{array}{ccc} H^q_{E^+}(Y',X') & \xrightarrow{\delta^q_{E^+}(Z',Y',X')} & H^{q+1}_{E^+}(Z',Y') \\ {\scriptstyle H^q_{E^+}(\Phi|_{(Y,X)})}\downarrow & & \downarrow{\scriptstyle H^{q+1}_{E^+}(\Phi)} \\ H^q_{E^+}(Y,X) & \xrightarrow{\delta^q_{E^+}(Z,Y,X)} & H^{q+1}_{E^+}(Z,Y). \end{array}$$

5. *(Long exact sequence) Given three $\mathcal{T}_{E^+}$-closed and E^+-locally compact sets X, Y, Z, such that $X \subset Y \subset Z$, denote by $i : (Y,X) \hookrightarrow (Z,X)$ and $j : (Z,X) \hookrightarrow (Z,Y)$ the inclusion maps. Then the following sequence of homomorphisms is exact*

$$\cdots \to H^q_{E^+}(Z,X) \overset{H^q_{E^+}(i)}{\to} H^q_{E^+}(Y,X) \overset{\delta^q_{E^+}}{\to} H^{q+1}_{E^+}(Z,Y) \overset{H^{q+1}_{E^+}(j)}{\to} H^{q+1}_{E^+}(Z,X) \to \dots$$

6. *(Dimension property) Let V be a closed subspace of E, commensurable with E^-. Let $\overline{B_V}$ be a closed ball in V and ∂B_V be its relative boundary in V. Then*

$$H^q_{E^+}(\overline{B_V}, \partial B_V) = \begin{cases} \mathbf{Z}_2 & \text{if } q = E^+\text{-dim}\, V, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the grading of this cohomology is made on $\mathbf{Z}$, and not on $\mathbf{N}$ as for usual cohomologies: this fact is crucial, because the E^+-dimension of a subspace may also be negative.

The fact that we are unable to use an arbitrary coefficient ring should not surprise. This reminds a well known phenomenon: the same problem arises, for example, when one wants to generalize the Leray-Schauder degree theory from the class of maps of the form Identity + Compact to the class of Fredholm maps. One actually defines only a $\mathbf{Z}_2$-valued degree (see [BZS77]). In our case everything depends on the fact that the group $GL(E^+) \oplus GL(E^-)$ is contained in the class of E^+-morphisms and it is connected, when both E^+ and E^- are infinite dimensional (see [Mit70]).

The construction of the E^+-cohomology and the proof of the above theorem requires a lot of technicalities: Chapter 2 is entirely devoted to this problem. But, as often happens with homology and cohomology, it will be possible to use the theory without knowing the details of the construction, from an axiomatic point of view.

For example, the following assertions are standard properties of every cohomology theory, and they derive directly from the axioms.

Proposition 1.3.2 *Assume that $A \subset Y \subset X$ are $\mathcal{T}_{E^+}$-closed E^+-locally compact subsets of E.*

1. *If there exists an E^+-homotopy*

$$\Psi : (X \times [0,1], A \times [0,1]) \mapsto (X, A)$$

such that $\Psi_0 = id$, $\Psi_1(X) \subset Y$ *and* $\Psi_t(Y) \subset Y$ *for every* t, *then*

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(Y, A).$$

2. *If there exists an E^+-homotopy*

$$\Psi : (X \times [0,1], Y \times [0,1]) \mapsto (X, Y)$$

such that $\Psi_0 = id$, $\Psi_1(Y) \subset A$ *and* $\Psi_t(A) \subset A$ *for every* t, *then*

$$H^*_{E^+}(X, Y) \cong H^*_{E^+}(X, A).$$

PROOF. We prove the first assertion, the argument for the second one being similar. Consider the composition of maps

$$(Y, A) \overset{i}{\hookrightarrow} (X, A) \overset{\Psi_1}{\to} (Y, A)$$

where i is the inclusion map (which is an E^+-morphism). By the assumptions on Ψ, the composition $\Psi_1 \circ i$ is E^+-homotopic to the identity on (Y, A). Therefore $H^*_{E^+}(\Psi_1 \circ i)$ is the identity homomorphism on $H^*_{E^+}(Y, A)$ and by the functorial property, we get that $H^*_{E^+}(i)$ is onto.

Now consider the composition of maps

$$(X, A) \overset{\Psi_1}{\to} (Y, A) \overset{i}{\hookrightarrow} (X, A).$$

Again $i \circ \Psi_1$ is E^+-homotopic to the identity on (X, A) and so $H^*_{E^+}(i \circ \Psi_1)$ is the identity homomorphism on $H^*_{E^+}(X, A)$. Therefore $H^*_{E^+}(i)$ is one-to-one.

We conclude that $H^*_{E^+}(i)$ is an isomorphism between $H^*_{E^+}(X, A)$ and $H^*_{E^+}(Y, A)$. □

Another useful property which can be derived in a standard way from the axioms is the existence of the Mayer-Vietoris exact sequence. We need it in the following form.

Proposition 1.3.3 *Assume that* (X_1, A_1) *and* (X_2, A_2) *are* E^+*-pairs such that*

$$A_i = (A_1 \cup A_2) \cap X_i, \quad i = 1, 2. \tag{1.4}$$

Then there exists a homomorphism

$$T^q : H^q_{E^+}(X_1 \cap X_2, A_1 \cap A_2) \longrightarrow H^{q+1}_{E^+}(X_1 \cup X_2, A_1 \cup A_2)$$

such that the following sequence is exact

$$\begin{aligned} H^q_{E^+}(X_1 \cup X_2, A_1 \cup A_2) &\overset{R^q}{\to} H^q_{E^+}(X_1, A_1) \oplus H^q_{E^+}(X_2, A_2) \overset{S^q}{\to} H^q_{E^+}(X_1 \cap X_2, A_1 \cap A_2) \\ &\overset{T^q}{\to} H^{q+1}_{E^+}(X_1 \cup X_2, A_1 \cup A_2) \overset{R^{q+1}}{\to} H^{q+1}_{E^+}(X_1, A_1) \oplus H^{q+1}_{E^+}(X_2, A_2) \\ &\overset{S^{q+1}}{\to} H^{q+1}_{E^+}(X_1 \cap X_2, A_1 \cap A_2). \end{aligned}$$

Here R^q is induced by the inclusions and S^q is the difference of the two homomorphisms induced by the inclusions. Moreover, the Mayer-Vietoris homomorphism T^q is functorial: if

$$\Phi : (X_1 \cup X_2, A_1 \cup A_2) \mapsto (\tilde{X}_1 \cup \tilde{X}_2, \tilde{A}_1 \cup \tilde{A}_2)$$

is an E^+-morphism which takes X_i into $\tilde{X}_i$ and A_i into $\tilde{A}_i$, the following diagram commutes

$$\begin{array}{ccc} H^q_{E^+}(X_1 \cap X_2, A_1 \cap A_2) & \xrightarrow{T^q} & H^{q+1}_{E^+}(X_1 \cup X_2, A_1 \cup A_2) \\ {\scriptstyle H^q_{E^+}(\Phi)}\downarrow & & \downarrow{\scriptstyle H^{q+1}_{E^+}(\Phi|_{(X_1\cup X_2, A_1 \cup A_2)})} \\ H^q_{E^+}(\tilde{X}_1 \cap \tilde{X}_2, \tilde{A}_1 \cap \tilde{A}_2) & \xrightarrow{\tilde{T}^q} & H^{q+1}_{E^+}(\tilde{X}_1 \cup \tilde{X}_2, \tilde{A}_1 \cup \tilde{A}_2), \end{array}$$

$\tilde{T}^q$ being the Mayer-Vietoris homomorphism associated to the pairs $(\tilde{X}_i, \tilde{A}_i)$.

PROOF. Let

$$\begin{aligned} i &: (X_1 \cap X_2, A_1 \cap A_2) \hookrightarrow ((X_1 \cap X_2) \cup A_2, A_2), \\ j &: (X_2, (X_1 \cap X_2) \cup A_2) \hookrightarrow (X_1 \cup X_2, X_1 \cup A_2), \\ k &: (X_1 \cup X_2, A_1 \cup A_2) \hookrightarrow (X_1 \cup X_2, X_1 \cup A_2), \end{aligned}$$

be the inclusion maps. By (1.4),

$$A_1 \cap A_2 = X_1 \cap A_2, \quad (X_1 \cap X_2) \cup A_2 = X_2 \cap (X_1 \cup A_2),$$

so both i and j are strong excisions. By property 3 of Theorem 1.3.1, $H^*_{E^+}(i)$ and $H^*_{E^+}(j)$ are isomorphisms.

From the inclusions $A_2 \subset (X_1 \cap X_2) \cup A_2 \subset X_2$, we get the coboundary homomorphism

$$\delta^q_{E^+}(X_2, (X_1 \cap X_2) \cup A_2, A_2) : H^q_{E^+}((X_1 \cap X_2) \cup A_2, A_2) \longrightarrow H^{q+1}_{E^+}(X_2, (X_1 \cap X_2) \cup A_2).$$

Set

$$T^q = H^{q+1}_{E^+}(k) \circ H^{q+1}_{E^+}(j)^{-1} \circ \delta^q_{E^+}(X_2, (X_1 \cap X_2) \cup A_2, A_2) \circ H^q_{E^+}(i)^{-1}.$$

The exactness of the above sequence follows easily from the exactness of the long sequence stated by Theorem 1.3.1, assertion 5. The functoriality of T^q follows from the functoriality of $\delta^q_{E^+}$ (assertion 4 in Theorem 1.3.1) and the functoriality of $H^q_{E^+}$ (assertion 1 in Theorem 1.3.1). □

1.4 The E^+-index pairs

Let f be a continuously differentiable real valued function on E. Let K be the critical set of f, i.e. the set $K = \{x \in E \mid df(x) = 0\}$. Denote by f^a the set $\{x \in E \mid f(x) \leq a\}$.

Recall that a Palais-Smale sequence is a sequence (x_n) of elements of E such that $f(x_n)$ is bounded and $\|\nabla f(x_n)\|$ converges to zero.

We make the following assumptions on f:

A1 Each sublevel f^a is $\mathcal{T}_{E^+}$-closed and $\mathcal{T}_{E^+}$-locally compact.

A2 Every bounded Palais-Smale sequence has a converging subsequence.

A3 ∇f is globally Lipschitz.

A4 The flow $\eta : E \times \mathbf{R} \mapsto E$ defined by

$$\begin{cases} \frac{\partial}{\partial t}\eta(x,t) = -\nabla f(\eta(x,t)), \\ \eta(x,0) = x \end{cases}$$

is an E^+-homotopy.

Notice that the flow η exists globally by condition (A3). If $t \in \mathbf{R}$ and $x \in E$, we will use both notations $\eta(x,t) = \eta_t(x)$.

Assumption (A3) implies that both f and ∇f are bounded on bounded sets.

An isolated critical set is an isolated subset of K.

Definition 1.4.1 *Let K_0 be a compact isolated critical set. An E^+-index pair for K_0 is a bounded E^+-pair (X, A) such that:*

1. $(X \setminus A) \cap K = K_0 \subset \mathrm{Int}\,(X)$ *and* $(K \setminus K_0) \cap X \subset \mathrm{Int}_X(A)$. *Here* $\mathrm{Int}\,(X)$ *is the interior part of X with respect to the strong topology of E and* $\mathrm{Int}_X(A)$ *is the interior part of A with respect to the topology induced on X by the strong topology of E;*

2. *A is strongly positively invariant with respect to X: if $x \in A$ and $\eta(x,t) \in X$ for a certain $t > 0$, then $\eta(x,t) \in A$;*

3. *A is an exit set for X: if $x \in X$ and $\eta(\{x\} \times [0,t])$ is not contained in X, then there exists $t^* \in [0,t]$ such that $\eta(x,t^*) \in A$.*

Moreover, an E^+-index pair (X, A) for K_0 is called strict if $K \cap X = K_0$. Notice that condition (2) is stronger than the positive invariance required in Conley's definition (see [Con78]): here the flow lines are not allowed to exit from A and get back again into $X \setminus A$.

Definition 1.4.2 *An elementary critical set is a compact isolated critical set K_0 such that f is constant on K_0.*

We will prove that an elementary critical set K_0 always has an E^+-index pair.

Let δ be a positive number such that

$$\overline{N_\delta}(K_0) = \{x \in E \mid \mathrm{dist}\,(x, K_0) \leq \delta\} \subset U$$

and $K \cap \overline{N_\delta}(K_0) = K_0$.

Since the distance from a compact set is a weakly lower semi-continuous function, $\overline{N_\delta}(K_0)$ is $\mathcal{T}_{E^+}$-closed.

Lemma 1.4.1 *Let K_0 be an elementary critical set. Set*

$$\Gamma_\delta = \left\{\eta(x,t) \mid x \in \overline{N_\delta}(K_0),\ t \geq 0,\ \exists s \geq 0 \text{ such that } \eta(x,t+s) \in \overline{N_\delta}(K_0)\right\} \setminus \overline{N_{\frac{\delta}{2}}}(K_0).$$

If δ is small enough

$$\inf_{z\in\Gamma_\delta} \|\nabla f(z)\| > 0.$$

PROOF. Assume that $f = c$ on K_0. Let $\overline{\delta}$ be a positive number such that $K \cap \overline{N_{\overline{\delta}}}(K_0) = K_0$.

By assumption (A3), there exists k such that

$$\|\nabla f\| \leq k \quad \text{on } \overline{N_{\overline{\delta}}}(K_0).$$

By assumption (A2), there exists $a > 0$ such that

$$\|\nabla f\| > a \quad \text{on } \overline{N_{\overline{\delta}}}(K_0) \setminus N_{\frac{\overline{\delta}}{2}}(K_0).$$

Assume, by contradiction, that there exists an infinitesimal sequence of positive numbers δ_n such that

$$\inf_{z\in\Gamma_{\delta_n}} \|\nabla f(z)\| = 0.$$

We may assume that $\delta_n \leq \frac{\overline{\delta}}{2}$. By assumption (A2)

$$d_n = \inf_{z\in\overline{N_{\overline{\delta}}}(K_0)\setminus N_{\frac{\delta_n}{2}}(K_0)} \|\nabla f(z)\| > 0$$

and therefore we can find $z_n \in \Gamma_{\delta_n}$ such that

$$\|\nabla f(z_n)\| < d_n. \tag{1.5}$$

By the definition of Γ_{δ_n}, $z_n = \eta(x_n,t_n)$, with $x_n \in \overline{N_{\delta_n}}(K_0)$, $t_n \geq 0$ and there exists $s_n \geq 0$ such that $\eta(x_n, t_n + s_n) \in \overline{N_{\delta_n}}(K_0)$.

By (1.5), z_n is not in $\overline{N_{\overline{\delta}}}$. Therefore there exists an interval $[\alpha_n,\beta_n]$, $0 < \alpha_n < \beta_n < t_n$, such that

$$\eta(x_n,s) \in \overline{N_{\overline{\delta}}}(K_0) \setminus N_{\frac{\overline{\delta}}{2}}(K_0) \quad \text{for } s \in [\alpha_n,\beta_n],$$
$$\text{dist}\,(\eta(x_n,\alpha_n),K_0) = \frac{\overline{\delta}}{2},$$
$$\text{dist}\,(\eta(x_n,\beta_n),K_0) = \overline{\delta}.$$

Then

$$\frac{\overline{\delta}}{2} \leq \|\eta(x_n,\beta_n) - \eta(x_n,\alpha_n)\| \leq \int_{\alpha_n}^{\beta_n} \|\frac{\partial}{\partial s}\eta(x_n,s)\|\,ds \leq k(\beta_n - \alpha_n)$$

which implies $\beta_n - \alpha_n \geq \frac{\overline{\delta}}{2k}$. Therefore

$$f(\eta(x_n, t_n + s_n) = f(x_n) - \int_0^{t_n+s_n} \|\nabla f(\eta(x_n, s))\|^2 \, ds \leq$$
$$\leq f(x_n) - \int_{\alpha_n}^{\beta_n} \|\nabla f(\eta(x_n, s))\|^2 \, ds \leq f(x_n) - \frac{\overline{\delta}}{2k} a^2. \tag{1.6}$$

But, since both x_n and $\eta(x_n, t_n + s_n)$ are in $\overline{N_{\delta_n}}(K_0)$, we get

$$\lim_{n\to+\infty} f(x_n) = \lim_{n\to+\infty} f(\eta(x_n, t_n + s_n)) = c$$

and looking at (1.6) we have a contradiction. □

Now we can prove the existence of an E^+-index pair.

Proposition 1.4.2 *Let K_0 be an elementary critical set and let U be a neighborhood of K_0 which is $\mathcal{T}_{E^+}$-closed. Then there exists a strict E^+-index pair (X, A) for K_0 such that $X \subset U$.*

PROOF. Choose $\delta > 0$ so small that the thesis of Lemma 1.4.1 holds and

$$\overline{N_\delta}(K_0) \subset U.$$

Since f and ∇f are bounded on bounded sets, there exists k such that

$$|f| \leq k, \quad \|\nabla f\| \leq k \quad \text{on } \overline{N_\delta}(K_0).$$

By assumption (A2) there exists $a > 0$ such that

$$\|\nabla f\| \geq a \quad \text{on } \overline{N_\delta}(K_0) \setminus N_{\frac{\delta}{2}}(K_0).$$

Choose a number ϵ such that

$$0 < \epsilon < \frac{\delta a^2}{4k}.$$

Assume that $f = c$ on K_0. Choose $0 < \delta_1 \leq \frac{\delta}{2}$ such that

$$\overline{N_{\delta_1}}(K_0) \subset f^{c+\epsilon}.$$

Set

$$\tilde{X} = \left\{\eta(x,t) \mid x \in \overline{N_{\delta_1}}(K_0),\ t \geq 0,\ f(\eta(x,t)) \geq c - \epsilon\right\}.$$

We claim that $\tilde{X} \subset \overline{N_\delta}(K_0)$: otherwise there would exist $x \in \overline{N_{\delta_1}}(K_0)$ and $0 \leq t_1 < t_2$ such that

$$\eta(x,s) \in \overline{N_\delta}(K_0) \setminus N_{\frac{\delta}{2}}(K_0) \quad \forall s \in [t_1, t_2],$$
$$\text{dist}\,(\eta(x,t_1), K_0) = \frac{\delta}{2}, \quad \text{dist}\,(\eta(x,t_2), K_0) = \delta$$

and

$$f(\eta(t_2, x)) \geq c - \epsilon. \tag{1.7}$$

But
$$\frac{\delta}{2} \le \|\eta(x,t_2) - \eta(x,t_1)\| \le \int_{t_1}^{t_2} \|\frac{\partial \eta}{\partial s}(x,s)\| \, ds \le k(t_2 - t_1)$$
which implies that $t_2 - t_1 \ge \frac{\delta}{2k}$. Then
$$f(\eta(x,t_2)) = f(\eta(x,t_1)) - \int_{t_1}^{t_2} \|\nabla f(\eta(x,s))\|^2 \, ds \le$$
$$\le f(x) - a^2(t_2 - t_1) \le c + \epsilon - \frac{a^2\delta}{2k} < c - \epsilon$$

which contradicts (1.7).

In general $\tilde{X}$ needs not to be $\mathcal{T}_{E^+}$-closed. So let X be the $\mathcal{T}_{E^+}$-closure of $\tilde{X}$. Since $\overline{N_\delta}(K_0)$ is $\mathcal{T}_{E^+}$-closed, X is a subset of $\overline{N_\delta}(K_0)$.

Set $A = X \cap f^{c-\epsilon}$. Assumption (A1) implies that A is $\mathcal{T}_{E^+}$-closed. We claim that (X, A) is a strict E^+-index pair for K_0.

Property (1) in the strict form follows from the inclusions
$$K_0 \subset K \cap X \subset K \cap \overline{N_\delta}(K_0) = K_0,$$
$$K_0 \subset \text{Int}\left(\left\{x \in \overline{N_{\delta_1}}(K_0) \mid f(x) \ge c - \epsilon\right\}\right) \subset \text{Int}\,(X).$$

Set $\tilde{A} = \tilde{X} \cap f^{c-\epsilon}$. Then $\tilde{A}$ is strongly positively invariant with respect to $\tilde{x}$, by construction. Equivalently
$$\eta_t(\tilde{A}) \cap \tilde{X} \subset \tilde{A} \quad \forall t > 0.$$
Since the E^+-morphism η_t maps bounded $\mathcal{T}_{E^+}$-closed sets into $\mathcal{T}_{E^+}$-closed sets, from the above relation we obtain
$$\eta_t(A) \cap X \subset A \quad \forall t > 0$$
which proves property (2).

By Lemma 1.4.1 and by assumption (A2) the number
$$b = \inf\left\{\|\nabla f(z)\| \mid z \in [\Gamma_\delta \cup \overline{N_{\frac{\delta}{2}}}(K_0)] \setminus N_{\delta_1}(K_0)\right\}$$
is positive. Choose $T = \frac{1}{b^2}(c + \epsilon + k)$. We claim that
$$\eta(\overline{N_{\delta_1}}(K_0) \times [0,T]) \cap \overline{N_\delta}(K_0) = \eta(\overline{N_{\delta_1}}(K_0) \times [0,+\infty[) \cap \overline{N_\delta}(K_0). \tag{1.8}$$
In fact, let $\eta(x,t) \in \overline{N_\delta}(K_0)$, with $x \in \overline{N_{\delta_1}}(K_0)$ and $t \ge 0$. If $\eta(x,t) \in \overline{N_{\delta_1}}(K_0)$ there is nothing to prove. Otherwise set
$$\bar{t} = \sup\left\{s < t \mid \eta(x,s) \in \overline{N_{\delta_1}}(K_0)\right\}.$$
Obviously $0 \le \bar{t} < t$, $\eta(x,\bar{t}) \in \overline{N_{\delta_1}}(K_0)$ and $\eta(x,s) \notin \overline{N_{\delta_1}}(K_0)$ for every $s \in]\bar{t}, t]$.

Since the E^+-homotopy η maps bounded $\mathcal{T}_{E^+}$-closed sets into $\mathcal{T}_{E^+}$-closed sets, (1.8) implies that the set
$$\eta(\overline{N_{\delta_1}} \times [0,+\infty[) \cap \overline{N_\delta}(K_0)$$

is $\mathcal{T}_{E^+}$-closed. Since the above set contains $\tilde{X}$, we get

$$X \subset \eta(\overline{N_{\delta_1}}(K_0) \times [0, +\infty[).$$

From this fact we easily deduce that

$$X \setminus \tilde{X} \subset A. \tag{1.9}$$

Then property (3) readily follows from the fact that $\tilde{A}$ is an exit set for $\tilde{X}$ and from (1.9). □

We conclude this section giving a useful tool to build an E^+-index pair for the whole critical set of f.

Proposition 1.4.3 *Let $E = V^+ \oplus V^-$ be an orthogonal splitting such that V^+ is commensurable with E^+ (and therefore V^- is commensurable with E^-). Let P^+ and P^- be the orthogonal projections onto V^+ and V^-, respectively. Assume that there exists $\overline{R}$ such that for every $R \geq \overline{R}$ the following inequalities hold*

$$\frac{d}{dt}\|P^-\eta(t,x)\|^2\Big|_{t=0} > 0 \quad \forall x \in \overline{B_{V^+}}(R) \times \partial B_{V^-}(R), \tag{1.10}$$

$$\frac{d}{dt}\|P^+\eta(t,x)\|^2\Big|_{t=0} < 0 \quad \forall x \in \partial\overline{B_{V^+}}(R) \times \overline{B_{V^-}}(R). \tag{1.11}$$

Then the critical set K of f is compact and

$$(X, A) = (\overline{B_{V^+}}(\overline{R}) \times \overline{B_{V^-}}(\overline{R}), \overline{B_{V^+}}(\overline{R}) \times \partial B_{V^-}(\overline{R}))$$

is an E^+-index pair for K.

PROOF. Since $\mathcal{T}_{E^+} = \mathcal{T}_{V^+}$, we must check that X and A are $\mathcal{T}_{V^+}$-closed. Being convex and strongly closed, X is weakly closed and therefore also $\mathcal{T}_{V^+}$-closed. For the same reason $B_{V^+}(\overline{R})$ is $\mathcal{T}_{V^+}$-closed. $\partial B_{V^-}(\overline{R})$ is $\mathcal{T}_{V^+}$-closed because $\mathcal{T}_{V^+}$ induces the strong topology on V^-. Therefore A is $\mathcal{T}_{V^+}$-closed.

Now we must verify the three properties of the E^+-index pairs.

1. The assumptions (1.10) and (1.11) imply that

$$K \subset \text{Int}\,(X) \subset X \setminus A$$

and the first property holds.

2. We want to check that A is strongly positively invariant with respect to X. Assumption (1.10) implies that points in A immediately exit from X. We must show that they can not get back into X at a later time.

By contradiction, assume that there exists $x \in A$ and $\bar{t} > 0$ such that $\eta(x, \bar{t}) \in X$. Let $s \in]0, \bar{t}[$ be such that

$$R_0 = \|P^-\eta(x,s)\| = \max_{t \in [0,\bar{t}]} \|P^-\eta(x,t)\|.$$

Since $x \in A$, $R_0 \geq \overline{R}$. It can not happen that

$$\|P^+\eta(x,s)\| \leq R_0.$$

In fact in this case (1.10) implies that

$$\frac{d}{dt}\|P^-\eta(x,t)\|^2\Big|_{t=s} = \frac{d}{dt}\|P^-\eta(\eta(x,s),t)\|^2\Big|_{t=0} > 0$$

in contradiction with the definition of s.

Therefore

$$\|P^+\eta(x,s)\| > R_0.$$

Since $\|P^+x\| \leq \overline{R} \leq R_0$, there exists $s' \in [0, s[$ such that

$$\|P^+\eta(x,s')\| = R_0 \quad \text{and} \quad \frac{d}{dt}\|P^+\eta(x,t)\|^2\Big|_{t=s'} \geq 0. \tag{1.12}$$

But $\|P^-\eta(x,s')\| \leq R_0$ and (1.11) implies that

$$\frac{d}{dt}\|P^+\eta(x,t)\|^2\Big|_{t=s'} = \frac{d}{dt}\|P^-\eta(\eta(x,s'),t)\|^2\Big|_{t=0} < 0$$

which contradicts (1.12).

3. We want to check that A is an exit set for X. By contradiction assume that there exists x in X and $\overline{t} > 0$ such that $\eta(x,\overline{t}) \notin X$ but $\eta(x,t) \notin A$ for every t in $[0, \overline{t}]$.

Since $\eta(x,t)$ has to pass through the boundary of X, there exists $s \in [0, \overline{t}[$ such that

$$\eta(x,s) \in \partial B_{V^+}(\overline{R}) \times \overline{B_{V^-}}(\overline{R}) \quad \text{and} \quad \frac{d}{dt}\|P^+\eta(x,t)\|^2\Big|_{t=s} \geq 0.$$

However (1.11) implies that

$$\frac{d}{dt}\|P^+\eta(x,t)\|^2\Big|_{t=s} = \frac{d}{dt}\|P^-\eta(\eta(x,s),t)\|^2\Big|_{t=0} < 0$$

which is a contradiction. □

1.5 E^+-index pairs for functionals of class C^2

In this section, besides conditions (A1), (A2), (A3) and (A4), we assume that f is twice continuously differentiable on E.

Definition 1.5.1 *Assume that $f \in C^2(E)$. An elementary critical set K_0 is called E^+-non-degenerate critical manifold if it is a finite dimensional compact C^1-manifold embedded in E and for every $x \in K_0$:*

1. *x is a weakly non-degenerate critical point and the kernel of $D^2f(x)$ coincides with the tangent space of K_0 in x;*

2. the negative eigenspace of $D^2f(x)$ is commensurable with E^-.

For example, every non-degenerate critical point x constitutes an E^+-non-degenerate critical manifold if the negative eigenspace of $D^2f(x)$ is commensurable with E^-.

Proposition 1.2.2 implies that, if the E^+-non-degenerate critical manifold K_0 is connected, the E^+-Morse index $E^+\text{-m}\,(x)$ is constant on K_0. In this case we set

$$E^+\text{-m}\,(K_0) = E^+\text{-m}\,(K_0; f) = E^+\text{-m}\,(x; f) \quad \text{for } x \in K_0.$$

We denote by $T_z^\perp K_0$ the normal space to K_0 in $z \in K_0$, considered as a linear subspaces of E (passing through 0). Then

$$T_z^\perp K_0 = E_z^+ \oplus E_z^-$$

where E_z^+ and E_z^- are the positive and negative eigenspaces of $D^2f(z)$.

Let R_0 be such that

$$\overline{N_{R_0}}(K_0) = \{x \in E \mid \text{dist}\,(x, K_0) \le R_0\}$$

is a tubular neighborhood of K_0: this means that the map

$$\Pi : \overline{N_{R_0}}(K_0) \mapsto K_0$$

which associates to x its closest point on K_0 is well defined and continuous. Moreover the fibers of Π are

$$\Pi^{-1}(z) = \overline{N_{R_0}}(K_0) \cap (z + T_z^\perp K_0) \quad \forall z \in K_0.$$

A first remark is that the weak topology and the strong topology coincide on K_0: in fact the identity $I : \{K_0, \text{strong}\} \mapsto \{K_0, \text{weak}\}$ continuously maps a compact space onto a Hausdorff space, and therefore its inverse is continuous. A second remark is that the projection Π is also weakly continuous.

We may assume that R_0 is so small that $\overline{N_{R_0}}(K_0)$ contains no critical points outside K_0.

For $R \le R_1 = \frac{R_0}{\sqrt{2}}$ set

$$\begin{aligned} Q_R &= \left\{ x \in \overline{N_{R_0}}(K_0) \mid \|P_{E^{\pm}_{\Pi(x)}}(x - \Pi(x))\| \le R \right\}, \\ Z_R^- &= \left\{ x \in Q_R \mid \|P_{E^{-}_{\Pi(x)}}(x - \Pi(x))\| = R \right\}, \\ Z_R^+ &= \left\{ x \in Q_R \mid \|P_{E^{+}_{\Pi(x)}}(x - \Pi(x))\| = R \right\}. \end{aligned}$$

Proposition 1.5.1 *If K_0 is a compact E^+-non-degenerate critical manifold, (Q_R, Z_R^-) is an E^+-index pair for K_0, provided R is small enough.*

PROOF. The maps $x \mapsto P_{E^{\pm}_{\Pi(x)}}(x - \Pi(x))$ are weakly continuous: in fact for every $y \in E$

$$x \mapsto < P_{E^{\pm}_{\Pi(x)}}(x - \Pi(x)), y > = < x - \Pi(x), P_{E^{\pm}_{\Pi(x)}} y >$$

is weakly continuous, being the scalar product between a weakly continuous map and a map which is continuous from the weak topology to the strong one. Since the norm is weakly-lower-semi-continuous, Q_R is weakly closed, and thus also $\mathcal{T}_{E^+}$-closed.

The map $x \mapsto P_{E^-_{\Pi(x)}}(x - \Pi(x))$ is also continuous from the $\mathcal{T}_{E^+}$-topology to the strong one. In fact it is given by the composition

$$x \mapsto x - \Pi(x) = y \mapsto P_{E^-_{\Pi(y)}}(y)$$

The first map is $\mathcal{T}_{E^+}$-continuous, because such is Π, being weakly continuous. The second map is continuous from the $\mathcal{T}_{E^+}$-topology to the strong one. Therefore Z_R^- is $\mathcal{T}_{E^+}$-closed and (Q_R, Z_R^-) is an E^+-pair.

Now we must verify the three properties of Definition 1.4.1.

1. If $R \leq R_1$ the only critical set in Q_R is K_0 and

$$K_0 = K \cap Q_R \subset \text{Int}\,(Q_R) \subset Q_R \setminus Z_R$$

implies the second property.

2. We want to check that Z_R^- is positively invariant with respect to Q_R, provided R is small enough. Let $\alpha > 0$ be such that

$$D^2 f(z) \geq \alpha I \ \text{ on } E_z^+, \quad D^2 f(z) \leq -\alpha I \ \text{ on } E_z^-, \quad \forall z \in K_0.$$

If $x \in Q_{R_1}$

$$\nabla f(x) = D^2 f(\Pi(x))(x - \Pi(x)) + G_{\Pi(x)}(x - \Pi(x))$$

where $G_z(y) = o(\|y\|)$ for $y \to 0$, uniformly with respect to $z \in K_0$. Choose $0 < R_2 \leq R_1$ so small that

$$\|G_z(y)\| \leq \frac{\alpha}{4}\|y\| \quad \forall z \in K_0, \ \|y\| \leq R_2.$$

Then, if $x \in Z_R^-$ with $R \leq R_2$,

$$\begin{aligned}
&\tfrac{d}{dt}\|P_{E^-_{\Pi(x)}}(\eta(x,t) - \Pi(x))\|^2|_{t=0} = -2 < \nabla f(x), P_{E^-_{\Pi(x)}}(x - \Pi(x)) > \\
&= -2 < D^2 f(\Pi(x)) P_{E^-_{\Pi(x)}}(x - \Pi(x)), P_{E^-_{\Pi(x)}}(x - \Pi(x)) > + \\
&-2 < G_{\Pi(x)}(x - \Pi(x)), P_{E^-_{\Pi(x)}}(x - \Pi(x)) > \\
&\geq 2\alpha R^2 - \tfrac{\alpha}{2}\|x - \Pi(x)\| R \geq 2\alpha R^2 - \alpha R^2 = \alpha R^2.
\end{aligned} \tag{1.13}$$

Analogously, if $x \in Z_R^+$ with $R \leq R_2$,

$$\frac{d}{dt}\|P_{E^+_{\Pi(x)}}(\eta(x,t) - \Pi(x))\|^2\Big|_{t=0} \leq -\alpha R^2. \tag{1.14}$$

If $x \in Q_{R_2} \setminus Q_{\frac{R_2}{2}}$

$$\begin{aligned}
&\|\nabla f(x)\| \geq \|D^2 f(\Pi(x))(x - \Pi(x))\| - \|G_{\Pi(x)}(x - \Pi(x))\| \\
&\geq \alpha\|x - \Pi(x)\| - \tfrac{\alpha}{4}\|x - \Pi(x)\| \geq \tfrac{3}{4}\alpha R_2.
\end{aligned} \tag{1.15}$$

Let $M > 0$ be such that

$$\|\nabla f(x)\| \leq M \quad \forall x \in Q_{R_2}. \tag{1.16}$$

Choose a positive $R_3 \leq \frac{R_2}{2}$ such that

$$\sup_{Q_{R_3}} f - \inf_{Q_{R_3}} f < \frac{9\alpha^2}{16M} R_2^3. \tag{1.17}$$

We claim that Z_R^- is positively invariant with respect to Q_R, provided $R \leq R_3$.

In fact let $R \leq R_3$ and assume that $x \in Z_R^-$. Inequality (1.13) implies that $\eta(x,t) \notin Q_R$ for $t > 0$ small. Assume, by contradiction, that $\eta(t_0, x) \in Q_R$ for a certain $t_0 > 0$. Then (1.13) and (1.14) easily imply that there exists a positive number $t_1 < t_0$ such that $\eta(x, t_1) \notin Q_{R_2}$. Therefore by (1.16)

$$\lambda(\left\{t \in [0, t_0] \mid \eta(x,t) \in Q_{R_2} \setminus Q_{\frac{R_2}{2}}\right\}) \geq \frac{R_2}{M}. \tag{1.18}$$

Here $\lambda(\Gamma)$ is the Lebesgue measure of the measurable set $\Gamma \subset \mathbf{R}$. By (1.15) and (1.18)

$$f(\eta(t_0, x)) - f(x) = -\int_0^{t_0} \|\nabla f(\eta(t,x))\|^2 \, dt \leq -\frac{9\alpha^2}{16M} R_2^3$$

which contradicts (1.17).

3. If $R \leq R_2$, (1.14) implies that Z_R^- is an exit set for Q_R. □

The above proposition has the following consequence: let x be a non-degenerate critical point. We may assume that $x = 0$. Let V^+ and V^- be the positive and negative eigenspace of $D^2 f(0)$. If V^- is commensurable with E^-, an E^+-index pair for the isolated critical set $\{0\}$ is given by

$$(\overline{B_{V^+}}(R) \times \overline{B_{V^-}}(R), \overline{B_{V^+}}(R) \times \partial B_{V^-}(R))$$

provided R is small enough.

1.6 Cohomological invariance of the E^+-index pairs

We know from the last section that an elementary critical set K_0 always has an E^+-index pair. Now we want to show that every E^+-index pair has the same E^+-cohomology. We begin with a lemma.

Lemma 1.6.1 *Let K_0 be a compact isolated critical set. Let (X, A) be an E^+-index pair for K_0 and (X_0, A_0) a strict E^+-index pair for K_0 such that $X_0 \subset X$. Then*

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(X_0, A_0).$$

PROOF. Let W_0 be the $\mathcal{T}_{E^+}$-closure of $\eta(A_0 \times [0, +\infty[)$ and set

$$A' = (W_0 \cap X) \cup A.$$

Let W be the $\mathcal{T}_{E^+}$-closure of $\eta(A' \times [0,1])$. Notice that

$$W \subset f^m \quad \text{if} \quad m = \sup_{A'} f,$$

so condition (A1) implies that W is E^+-locally compact.

Look at the inclusion

$$(X_0 \cup W, W) \hookrightarrow (X \cup W, W).$$

The sets W, $X \cup W$ and $X_0 \cup W$ are positively invariant for the flow η. The set $Z = X \setminus (X_0 \cup A')$ has positive distance from K. By (A2) $\|\nabla f\| \geq a > 0$ on Z. Let b be the supremum of f on Z and let c be less than the infimum of f on Z.

Choose $T_1 > \frac{b-c}{a^2}$. We claim that

$$\eta_{T_1}(Z) \subset X_0 \cup W.$$

In fact points of Z can leave Z only by getting into the positively invariant set $X_0 \cup W$. But points of Z are forced to leave Z before time T_1: if $x \in Z$ and $\eta(t,x)$ remains in Z for every $t \in [0, T_1]$ then

$$f(\eta(T_1, x)) \leq f(x) - \int_0^{T_1} \|\nabla f(\eta(s,x)\|^2 \, ds \leq b - a^2 T_1 < c$$

which is impossible because $f > c$ on Z.

Therefore

$$\eta_{T_1}(X \cup W) \subset X_0 \cup W$$

and Lemma 1.3.2 (1) implies that

$$H^*_{E^+}(X_0 \cup W, W) \cong H^*_{E^+}(X \cup W, W).$$

Condition (2) in the definition of E^+-index pairs implies that $W \cap X_0 = A_0 \cup (A \cap X_0)$ and $W \cap X = A'$. Then the strong excision property of our generalized cohomology implies

$$\begin{aligned} H^*_{E^+}(X_0 \cup W, W) &\cong H^*_{E^+}(X_0, A_0 \cup (A \cap X_0)), \\ H^*_{E^+}(X \cup W, W) &\cong H^*_{E^+}(X, A'). \end{aligned}$$

Notice that $(X_0, A_0 \cup (A \cap X_0)$ is a strict E^+-index pair for K_0 and that (X, A') is an E^+-index pair for K_0. In order to conclude the proof, we must prove the following fact: if (Y, B) and (Y, B') are E^+-index pairs for K_0 such that $B \subset B'$ and $K \cap B' = K \cap B \subset \mathrm{Int}\,(B)$ then

$$H^*_{E^+}(Y, B) \cong H^*_{E^+}(Y, B').$$

Let V be the $\mathcal{T}_{E^+}$-closure of $\eta([0, +\infty[\times B)$. We look at the inclusion

$$(Y \cup V, V) \hookrightarrow (Y \cup V, B' \cup V).$$

The sets V, $Y \cup V$ and $B' \cup V$ are positively invariant for the flow η. The set $B' \setminus B$ has positive distance from K. Arguing as before we can find $T_2 > 0$ such that

$$\eta_{T_2}(B' \cup V) \subset V.$$

and Lemma 1.3.2 (2) implies

$$H^*_{E^+}(Y \cup V, V) \cong H^*_{E^+}(Y \cup V, B' \cup V).$$

It is easy to see that $Y \cap V = B$ and $Y \cap (V \cup B') = B'$. Then the strong excision property of our generalized cohomology implies

$$\begin{aligned} H^*_{E^+}(Y \cup V, V) &\cong H^*_{E^+}(Y, B), \\ H^*_{E^+}(Y \cup V, B' \cup V) &\cong H^*_{E^+}(Y, B'), \end{aligned}$$

which concludes the proof. □

Now we can prove the E^+-cohomological invariance of the E^+-index pairs.

Theorem 1.6.2 *Let K_0 be an elementary critical set. If (X, A) and (Y, B) are E^+-index pairs for K_0 then*

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(Y, B)$$

PROOF. By Proposition 1.4.2 we can find a strict E^+-index pair (Z, C) for K_0 such that $Z \subset X \cup Y$. By Lemma 1.6.1

$$H^*_{E^+}(X, A) \cong H^*_{E^+}(Z, C) \cong H^*_{E^+}(Y, B).$$

□

1.7 The E^+-Morse polynomial

As for usual cohomology, the E^+-Poincaré polynomial of an E^+-pair (X, A) is defined as

$$P_{E^+}(X, A) = \sum_{q \in \mathbf{Z}} \dim H^q_{E^+}(X, A) \lambda^q.$$

In general $P_{E^+}(X, A)$ is a formal Laurent series, whose coefficients are non-negative integers or $+\infty$.

Proposition 1.4.2 and Theorem 1.6.2 allow to give the following definition.

Definition 1.7.1 *Let K_0 be an elementary critical set of f. The E^+-critical groups of K_0 are the $\mathbf{Z}_2$-vector spaces*

$$c^q_{E^+}(K_0) = H^q_{E^+}(X, A)$$

where (X, A) is an E^+-index pair for K_0. The E^+-Morse polynomial of K_0 is the E^+-Poincaré polynomial of (X, A):

$$M_{E^+}(K_0) = \sum_{q \in \mathbf{Z}} \dim c^q_{E^+}(K_0) \lambda^q.$$

It is easy to relate the E^+-Morse polynomial a non-degenerate critical point with its E^+-Morse index.

Theorem 1.7.1 *Assume that x_0 is a non-degenerate critical point of a functional of class C^2. Assume that the negative eigenspace of $D^2f(x)$ is commensurable with E^-. Then*

$$M_{E^+}(\{x_0\}) = \lambda^{E^+\text{-m}(x_0;f)}.$$

PROOF. Let V^+ and V^- be the positive and negative eigenspaces of $D^2f(x)$. By Proposition 1.5.1, an E^+-index pair for $\{x_0\}$ is given by

$$(Q, Z^-) = (x_0 + \overline{B_{V^+}}(R) \times \overline{B_{V^-}}(R), x_0 + \overline{B_{V^+}}(R) \times \partial B_{V^-}(R))$$

provided R is small enough. The set

$$Q^- = x_0 + \overline{B_{V^-}}(R)$$

is $\mathcal{T}_{E^+}$-closed and the strong excision property of the E^+-cohomology implies that

$$M_{E^+}(Q, Z^-) = M_{E^+}(Q^-, Z^- \cap Q^-).$$

But $Z^- \cap Q^- = \partial B_{V^-}(R)$ and by the dimension axiom of the E^+-cohomology

$$M_{E^+}(Q^-, Z^- \cap Q^-) = \lambda^{E^+\text{-dim}\, V^-} = \lambda^{E^+\text{-m}\,(x_0)}.$$

□

The computation of the E^+-Morse polynomial of an E^+-non-degenerate critical manifold is more difficult.

Theorem 1.7.2 *If K_0 is a compact E^+-non-degenerate critical manifold of a functional of class C^2 then*

$$M_{E^+}(K_0) = P(K_0)\lambda^{m(K_0)}$$

where $P(K_0)$ is the Poincaré polynomial of K_0 with respect to a standard Z_2-valued cohomology.

PROOF. Using the notations of section 1.5, let $\overline{N_{R_0}}(K_0)$ be a tubular neighborhood of K_0 so small that the projection

$$\Pi : \overline{N_{R_0}}(K_0) \mapsto K_0$$

which associates to x its closest point on K_0 is well defined and continuous. For each $z \in K_0$, the normal space to K_0 in z splits into the positive and the negative eigenspaces of $D^2f(z)$:

$$T_z^\perp K_0 = E_z^+ \oplus E_z^-.$$

Choose a positive $R \le \frac{R_0}{\sqrt{2}}$ and set

$$Q = \left\{x \in \overline{N_{R_0}}(K_0) \mid \|P_{E^\pm_{\Pi(x)}}[x - \Pi(x)]\| \le R\right\}$$
$$Z^- = \left\{x \in Q \mid \|P_{E^-_{\Pi(x)}}[x - \Pi(x)]\| = R\right\}.$$

We know from Proposition 1.5.1 that (Q, Z^-) is an E^+-index pair for K_0, provided R is small enough. We must compute its E^+-cohomology. Such a computation will require several steps.

We can assume that K_0 is connected: otherwise we can choose R so small that Q has as many connected components as K_0 and then work component by component. So let m be the E^+-dimension of E_z^-, for $z \in K_0$.

Set

$$Q^- = \left\{x \in Q \mid x - \Pi(x) \in E^-_{\Pi(x)}\right\}$$
$$W^- = Q^- \cap Z^- = \left\{x \in Q^- \mid \|x - \Pi(x)\| = R\right\}.$$

Since the map

$$x \overset{F}{\mapsto} P_{E^+_{\Pi(x)}}[x - \Pi(x)]$$

is weakly continuous, the set $Q^- = F^{-1}(\{0\})$ is weakly closed and so also $\mathcal{T}_{E^+}$-closed. By the excision property we get

$$H^*_{E^+}(Q, Z^-) \cong H^*_{E^+}(Q^-, W^-). \tag{1.19}$$

For every closed $U \subset K_0$ set

$$Q^-(U) = \left\{x \in Q^- \mid \Pi(x) \in U\right\},$$
$$W^-(U) = \left\{x \in W^- \mid \Pi(x) \in U\right\}.$$

$(Q^-(U), W^-(U))$ is still an E^+-pair.

Since $f \in C^2(E)$, the map which associates to every $z \in K_0$ the linear operator $P_{E_z^-}$ is continuous, when the space of bounded linear operators is equipped with its standard norm. Therefore we can find a positive number d such that

$$\|P_{E_z^-} - P_{E_{z'}^-}\| \leq \frac{1}{2} \quad \text{if } z, z' \in K_0,\ \|z - z'\| \leq d. \tag{1.20}$$

Lemma 1.7.3 *Assume that $U \subset K_0$ is closed, contractible and that its diameter does not exceed d. Then*

$$H^q_{E^+}(Q^-(U), W^-(U)) = \begin{cases} \mathbf{Z}_2 & \text{if } q = m, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Let $\varphi : U \times [0,1] \mapsto U$ be a continuous homotopy to a point $z_0 \in U$: $\varphi_0(z) = z$, $\varphi(z) = z_0$ for every $z \in U$. Consider the map

$$\Psi(x,t) = \varphi(\Pi(x), t) + P_{E^-_{\varphi(\Pi(x),t)}}[x - \Pi(x)].$$

For $z \in K_0$ set $V_z^+ = E_z^+ \oplus T_z K_0$. Notice that

$$\Psi(x,t) = x - \Pi(x) - P_{V^+_{\varphi(\Pi(x),t)}}[x - \Pi(x)] + \varphi(\Pi(x), t).$$

Since V_z^+ is commensurable with E^+, the projection $P_{V_z^+}$ maps bounded sets into $\mathcal{T}_{E^+}$-precompact sets. From this fact it is easy to deduce that Ψ is an E^+-homotopy. Moreover

$$\Psi(x,0) = x \quad \forall x \in Q^-(U), \tag{1.21}$$
$$\Pi(\Psi(x,t)) = \varphi(\Pi(x),t) \quad \forall x \in Q^-(U), \forall t \in [0,1], \tag{1.22}$$
$$\Psi(x,1) \in Q^-(\{z_0\}) \quad \forall x \in Q^-(U). \tag{1.23}$$

Set

$$\tilde{W}^-(U) = \left\{ x \in Q^-(U) \mid \|P_{E^-_{\Pi(x)}}[x - \Pi(x)]\| \geq \frac{R}{2} \right\}.$$

Since the map $x \mapsto P_{E^-_{\Pi(x)}}[x - \Pi(x)]$ is continuous from the $\mathcal{T}_{E^+}$-topology to the strong one, the set $\tilde{W}^-(U)$ is $\mathcal{T}_{E^+}$-closed.

If $x \in Q^-(U)$, (1.20) and (1.22) imply that

$$\frac{1}{2}\|x - \Pi(x)\| \leq \|\Psi(x,t) - \Pi(\Psi(x,t))\| \leq \|x - \Pi(x)\|. \tag{1.24}$$

Therefore Ψ maps $Q^-(U)$ into $Q^-(U)$ and $W^-(U)$ into $\tilde{W}^-(U)$.

Consider the inclusions

$$i : (Q^-(U), W^-(U)) \hookrightarrow (Q^-(U), \tilde{W}^-(U)),$$
$$j : (Q^-(\{z_0\}), W^-(\{z_0\})) \hookrightarrow (Q^-(\{z_0\}), \tilde{W}^-(\{z_0\})),$$
$$l : (Q^-(\{z_0\}), \tilde{W}^-(\{z_0\})) \hookrightarrow (Q^-(U), \tilde{W}^-(U)),$$
$$k : (Q^-(\{z_0\}), W^-(\{z_0\})) \hookrightarrow (Q^-(U), W^-(U)).$$

By the excision property, both $H^*_{E^+}(i)$ and $H^*_{E^+}(j)$ are isomorphisms.

Consider Ψ_1 as an E^+-morphism from $(Q^-(U), W^-(U))$ to $(Q^-(\{z_0\}), \tilde{W}^-(\{z_0\}))$. $l \circ \Psi_1$ is E^+-homotopic to i and so $H^*_{E^+}(l)$ is one-to-one. $\Psi_1 \circ k$ is E^+-homotopic to j and so $H^*_{E^+}(k)$ is onto. Since $i \circ k = l \circ j$,

$$H^*_{E^+}(k) = H^*_{E^+}(j) \circ H^* E^+(l) \circ H^*_{E^+}(i)^{-1}$$

and $H^*_{E^+}(k)$ is an isomorphism. Therefore

$$H^*_{E^+}(Q^-(U), W^-(U)) \cong H^*_{E^+}(Q^-(\{z_0\}), W^-(\{z_0\})).$$

The thesis follows from the fact that $Q^-(\{z_0\})$ is a closed ball in the subspace $E^-_{z_0}$, which has E^+-dimension m, and that $W^-(\{z_0\})$ is its boundary. □

Now we need a version of the Five Lemma in homological algebra.

Lemma 1.7.4 *Assume that in the diagram of homomorphisms between vector spaces*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

the rows are exact, the vertical arrows are isomorphisms and everything commutes. Then there exists a unique isomorphism $\phi : C \to C'$ such that the above diagram still commutes. Moreover the isomorphism ϕ is functorial: if we have another diagram as the one above

$$\begin{array}{ccccccccc} \tilde{A} & \longrightarrow & \tilde{B} & \longrightarrow & \tilde{C} & \longrightarrow & \tilde{D} & \longrightarrow & \tilde{E} \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ \tilde{A}' & \longrightarrow & \tilde{B}' & \longrightarrow & \tilde{C}' & \longrightarrow & \tilde{D}' & \longrightarrow & \tilde{E}' \end{array}$$

and homomorphism between the first diagram and the second one such that everything commutes, also the diagram

$$\begin{array}{ccc} C & \longrightarrow & \tilde{C} \\ {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \tilde{\phi}} \\ C' & \longrightarrow & \tilde{C}' \end{array}$$

commutes, $\tilde{\phi}$ being the isomorphism determined by the second diagram.

This lemma can be proved by a standard and easy diagram chasing technique. In order to prove the theorem, we have got to show that

$$H^{q+m}_{E^+}(Q^-(M), W^-(M)) \cong H^q(M),$$

H^q denoting the singular $\mathbf{Z}_2$-valued cohomology. By Lemma 1.7.3,

$$H^{q+m}_{E^+}(Q^-(U), W^-(U)) \cong H^q(U),$$

when $U \subset M$ is closed, contractible and its diameter does not exceed d. Notice that M can be covered by a finite number of closed, contractible sets, whose diameter does not exceed d. Moreover we can assume that every intersection of a sub-family of such sets is still contractible. More concisely, we will say that M has a good covering consisting of a finite number of sets.

Therefore the thesis will follow from the following assertion:

(P_k). Assume that the closed set $X \subset M$ has a good covering consisting of k sets. Then for every $q \in \mathbf{Z}$ there exists an isomorphism

$$\phi^q_X : H^{q+m}_{E^+}(Q^-(X), W^-(X)) \longrightarrow H^q(X).$$

Moreover if $X' \subset X$ also has a good covering consisting of k sets, the diagram

$$\begin{array}{ccc} H^{q+m}_{E^+}(Q^-(X), W^-(X)) & \longrightarrow & H^{q+m}_{E^+}(Q^-(X'), W^-(X')) \\ {\scriptstyle \phi^q_X}\downarrow & & \downarrow{\scriptstyle \phi^q_{X'}} \\ H^q(X) & \longrightarrow & H^q(X') \end{array}$$

commutes, the horizontal homomorphisms being induced by the inclusions.

We will prove assertion (P_k) by induction on k. If $k = 1$ X itself is contractible and the existence of the isomorphism ϕ_X^q is given by Lemma 1.7.3. The second part of the assertion follows from the fact that there is only one automorphism of $\mathbf{Z}_2$.

Now assume that (P_{k-1}) holds. If X has a good covering

$$X = \bigcup_{j=1}^{k} U_j$$

consisting of k elements, we can set

$$Y = \bigcup_{j=1}^{k-1} U_j.$$

Both Y and $Y \cap U_k$ have good coverings consisting of $k-1$ sets. By Proposition 1.3.3 we have got an exact Mayer-Vietoris sequence for the E^+-cohomology which can be compared to the exact Mayer Vietoris sequence for the standard cohomology:

$$\begin{array}{ccccccccc}
A^{q+m-1} & \longrightarrow & B^{q+m-1} & \longrightarrow & C^{q+m} & \longrightarrow & A^{q+m} & \longrightarrow & B^{q+m} \\
\phi_Y^{q-1} \oplus \phi_{U_k}^{q-1} \big\downarrow & & \big\downarrow \phi_{Y \cap U_k}^{q-1} & & & & \phi_Y^q \oplus \phi_{U_k}^q \big\downarrow & & \big\downarrow \phi_{Y \cap U_k}^q \\
\tilde{A}^{q-1} & \longrightarrow & \tilde{B}^{q-1} & \longrightarrow & \tilde{C}^q & \longrightarrow & \tilde{A}^q & \longrightarrow & \tilde{B}^q,
\end{array}$$

where, to simplify the notations, we have set

$$\begin{aligned}
A^q &= H^q_{E^+}(Q^-(Y), W^-(Y)) \oplus H^q_{E^+}(Q^-(U_k), W^-(U_k)), \\
B^q &= H^q_{E^+}(Q^-(Y \cap U_k), W^-(Y \cap U_k)), \quad C^q = H^q_{E^+}(Q^-(X), W^-(X)), \\
\tilde{A}^q &= H^q(Y) \oplus H^q(U_k), \quad \tilde{B}^q = H^q(Y \cap U_k), \quad \tilde{C}^q = H^q(X).
\end{aligned}$$

The vertical arrows are the isomorphisms given by (P_{k-1}). By the second statement of (P_{k-1}) this diagram commutes. Therefore we can apply the Five Lemma 1.7.4 and we obtain an isomorphism

$$\phi_X^q : H^{q+m}_{E^+}(Q^-(X), W^-(X)) \longrightarrow H^q(X)$$

which keeps the diagram commutative. The second assertion of (P_k) follows from the functoriality of the Mayer-Vietoris homomorphism and from the functoriality of the isomorphism given by the Five Lemma. □

Remark 1.7.1 *The argument of the proof of the above Theorem could also be used to compute the (standard) relative cohomology of the pair (finite dimensional vector bundle, vector bundle minus the zero section). It is an elementary proof, meaning that it does not make use of the Thom isomorphisms, which are the standard tool to study these objects (see [MS74]). Of course everything works because we are taking* $\mathbf{Z}_2$ *coefficients: the second statement in assertion* (P_1) *fails when the coefficient ring has more than one automorphism. As a consequence the diagram connecting the Mayer-Vietoris sequences needs not to commute, unless the bundle is trivial. That is why the Thom isomorphisms give different formulas when the bundle is non-trivial and the coefficient ring is not* $\mathbf{Z}_2$.

1.8 The E^+-Morse-Conley relations

We need a lemma, which is a standard and easy consequence of the exactness of the long sequence in cohomology.

Lemma 1.8.1 *Assume that $X_0 \subset X_1 \subset \cdots \subset X_r$ are bounded $\mathcal{T}_{E^+}$-closed subsets of E. Then there exists a Laurent series Q with positive or infinite coefficients such that*

$$\sum_{j=0}^{r-1} P_{E^+}(X_{j+1}, X_j) = P_{E^+}(X_r, X_0) + (1+\lambda)Q(\lambda). \tag{1.25}$$

PROOF. Arguing by induction on r, it is easy to see that it is enough to prove formula (1.25) when $r = 2$. In this case the long exact sequence for the sets $X_0 \subset X_1 \subset X_2$ is

$$\to H^q_{E^+}(X_2, X_1) \to H^q_{E^+}(X_2, X_0) \to H^q_{E^+}(X_1, X_0) \to H^{q+1}_{E^+}(X_2, X_1) \to .$$

From the exactness of the above sequence we get the splittings

$$H^q_{E^+}(X_1, X_0) \cong A_q \oplus B_q,$$
$$H^q_{E^+}(X_2, X_0) \cong B_q \oplus C_q,$$
$$H^q_{E^+}(X_2, X_1) \cong C_q \oplus A_{q-1}.$$

Set $a_q = \dim A_q$, $b_q = \dim B_q$, $c_q = \dim C_q$. Some of these numbers may be $+\infty$. Then the formula

$$P_{E^+}(X_1, X_0) + P_{E^+}(X_2, X_1) = \sum_q (a_q + b_q)\lambda^q + \sum_q (c_q + b_{q-1})\lambda^q$$
$$= \sum_q (a_q + c_q)\lambda^q + (1+\lambda)\sum_q a_q \lambda^q = P_{E^+}(X_2, X_0) + (1+\lambda)\sum_q a_q\lambda^q$$

concludes the proof. □

Now we are ready to prove the Morse-Conley relations.

Theorem 1.8.2 *Let K be an isolated critical set of f and let (X, A) be an E^+-index pair for K. Assume that $K = K_1 \cup \cdots \cup K_m$, where $K_1, \ldots, K_m$ are pairwise disjoint elementary critical sets. Then there exists a Laurent series Q with positive or infinite coefficients such that*

$$\sum_{i=1}^{m} M_{E^+}(K_i) = P_{E^+}(X, A) + (1+\lambda)Q(\lambda).$$

If K consists of E^+-non-degenerate critical manifolds, the above relation is an equality between true Laurent polynomials with finite coefficients.

PROOF. First assume that all the critical sets K_i are at the same level. In this case we choose an E^+-index pair (X_i, A_i) for each K_i, built as in Proposition 1.4.2. We can assume that the X_i are pairwise disjoint. Then

$$\left(\bigcup_{i=1}^{m} X_i, \bigcup_{i=1}^{m} A_i\right)$$

is an E^+-index pair for K. By Theorem 1.6.2

$$H^q_{E^+}(X, A) \cong H^q_{E^+}\left(\bigcup_{i=1}^{m} X_i, \bigcup_{i=1}^{m} A_i\right) \cong \bigoplus_{i=1}^{m} c^q_{E^+}(K_i).$$

Therefore in this case

$$\sum_{i=1}^{m} M_{E^+}(K_i) = P_{E^+}(X, A).$$

Now we pass to the general case. Let $c_1 < c_2 < \cdots < c_r$ be the critical values of f in X. Choose numbers a_j such that

$$-\infty = a_0 < c_1 < a_1 < c_2 < \cdots < a_{r-1} < c_r < a_r = +\infty.$$

Set $U_j = (X \cap f^{a_j}) \cup A$. It is easy to check that (U_{j+1}, U_j) is an E^+-index pair for the elementary critical set

$$K \cap \{x \in X \mid f(x) = c_j\}$$

From the preliminary discussion there holds

$$\sum_{f|_{K_i} = c_j} M_{E^+}(K_i) = P_{E^+}(U_{j+1}, U_j)$$

and therefore

$$\sum_{i=1}^{m} M_{E^+}(K_i) = \sum_{j=0}^{r-1} P_{E^+}(U_{j+1}, U_j).$$

From Lemma 1.8.1 we get the existence of a Laurent series Q with positive or infinite coefficients such that

$$\sum_{j=0}^{r-1} P_{E^+}(U_{j+1}, U_j) = P_{E^+}(U_r, U_0) + (1 + \lambda)Q(\lambda).$$

The Morse-Conley relations now follow from the fact that $(U_r, U_0) = (X, A)$.

If every K_i is an E^+-non-degenerate critical manifold, then each $M_{E^+}(K_i)$ is a true Laurent polynomial with finite coefficients and so must be $P_{E^+}(X, A)$ and Q. □

In the case of a Morse function, we have the following immediate corollary.

Corollary 1.8.3 *Assume that $x_1, \ldots, x_k$ are non-degenerate critical points of f with finite E^+-Morse index. If (X, A) is an E^+-index pair for the critical set $\{x_1, \ldots, x_k\}$, there exists a Laurent polynomial Q with positive coefficients such that*

$$\sum_{i=1}^{k} \lambda^{E^+\text{-m}(x_i; f)} = P_{E^+}(X, A) + (1 + \lambda)Q(\lambda).$$

1.9 Some bibliography and further remarks

Originally Morse theory was developed for smooth functions with non-degenerate critical points on compact manifolds by Marston Morse (see [Mor25], [Mor34] and [Mil63] for a modern exposition). The theory can be applied also to non-compact finite dimensional manifolds, provided the function is proper. The non-degeneration condition was weakened by Bott, who proved the Morse relations assuming that the critical points are gathered in non-degenerate critical manifolds (see [Bot54]).

The first infinite dimensional generalizations were discovered by Palais (see [Pal63]), who developed Morse theory on Hilbert manifolds, introducing the celebrated Palais-Smale condition, which replaces the properness assumption. Palais' work opened the way to many beautiful applications to variational problems in nonlinear analysis (see, for example, the review paper by Bott, [Bot82], and the beautiful book by Chang, [Cha93]). Perturbation arguments are usually applied in order to deal with degenerate critical points: in fact a well known result by Marino and Prodi asserts that every C^2 function, whose second differential at critical points is a Fredholm operator, can be approximated in a suitable way by functions with only non-degenerate critical points (see [MP75]).

An important generalization, in the direction of degenerations, was made by Charles Conley (see [Con78]). In Conley's theory one works on locally compact metric spaces and replaces the index of a non-degenerate critical point with the homotopy type of an index pair, for an arbitrary isolated critical set. Moreover, also flows which are not obtained by integrating a gradient vector field can be considered: all flows which admit a Lyapunov function can be studied.

Conley's theory has been generalized to metric spaces which may not be locally compact by Benci (see [Ben91]). Notice that the topology $\mathcal{T}_{E^+}$, introduced in this chapter, is not metrizable and therefore Benci's results can not be used directly.

In the framework of strongly indefinite functionals, a Conley-type theory similar to ours has been developed by Szulkin (see [Szu92]). The main difference is in the underlying cohomology. The class of admissible maps in Szulkin cohomology is much more restricted than ours and the gradient flow does not belong to this class. So approximation arguments are needed. More will be said about Szulkin's theory at the end of Chapter 2.

We should also mention that many other approaches, developed for strongly indefinite functionals, could be derived from our E^+-Conley theory: this is the case, for example, of the infinite dimensional linking theorems proved by Benci and Rabinowitz, [BR79], or of the symmetry arguments used by Benci in [Ben82].

The two fundamental ingredients of Conley's theory are the existence of an index pair for every isolated critical set and the homotopical (or homological, or cohomological) invariance of such index pair. Both Conley's and Benci's proofs of these facts make use of suitable deformations, built up using Urysohn functions, partitions of unity or other continuous functions obtained by the metric structure. In our case the topology is too weak and the class of admissible deformations too restricted to use such tools. So it was somewhat a surprise for us realizing that everything can be done using only the deformations given by the gradient flow. Therefore we think that our proofs can be adapted in all those cases in which the topology is too weak, or there are too few admissible deformations.

In view of our applications we made some rather strong assumptions, namely the request that the gradient of f should be globally Lipschitz and that the isolated critical sets consist of points at the same level. These assumptions will not cause any trouble for us, because the functionals we study in Chapter 4 have the first property, and the second one can be obtained by using a Marino-Prodi perturbation argument. However we believe that the condition on the gradient of f can be eliminated, by using suitable pseudo-gradient vector fields (see [Rab78]). It should be possible to eliminate also the second condition, using some techniques introduced by Benci in [Ben91], but things could become really complicated.

The condition on the E^+-local compactness of the sublevels of f, involving a lower bound on f, is quite strange and it has been introduced because we can compute the E^+-cohomology only of sets which have this property (see Chapter 2). However the Morse-Conley relations involve only bounded sets, which have always this property, so one could think that the condition can be eliminated. We do not know the answer. We will see that this property holds for functionals coming from the asymptotically linear problems we study. However, it could fail when the non-linearities are stronger and it would be important to understand if it is really necessary.

Chapter 2

The E^+-cohomology

In this chapter we want to build the E^+-cohomology functor, whose properties have been anticipated in section 1.3. The E^+-cohomology of a set X will be defined as a direct limit of standard cohomology groups of intersections of X with suitable subspaces.

The E^+-finite morphisms will be defined as those E^+-morphisms whose compact part has finite rank in E^-. It will be shown that an E^+-finite morphism induces a homomorphism between the E^+-cohomology groups. The homotopical invariance for the corresponding restricted class of homotopies will be proved.

Then continuity properties for the E^+-cohomology will be proved. These properties allow to define the induced homomorphisms for general E^+-morphisms, by approximating them with E^+-finite morphisms. All the remaining axioms listed in Theorem 1.3.1 will be finally proved.

2.1 Alexander-Spanier cohomology with compact supports

Since the E^+-cohomology will be defined as a limit of moduli obtained with the Alexander-Spanier cohomology with compact supports, it is useful to review some properties of such functor. Chapter 6 of [Spa66] is the standard reference for these topics.

By H^*_{AS} we denote the Alexander-Spanier cohomology, with coefficients in a given ring $\mathcal{A}$ (see section 6.4 in [Spa66]). By H^*_c we denote the Alexander-Spanier cohomology with compact supports, on the same ring (see section 6.6 in [Spa66]). These functors coincide on topological pairs (X, A) such that $X \setminus A$ has compact closure in X (Lemma 6.6.9 in [Spa66]).

H^*_{AS} satisfies the Eilenberg-Steenrod axioms (see [ES52]) and moreover it has a strong excision property (see the more general Theorem 6.6.5 of [Spa66]).

Proposition 2.1.1 *Let X and Y be two closed sets in a paracompact Hausdorff space. Let $i : (Y, X \cap Y) \hookrightarrow (X \cup Y, X)$ be the inclusion map. Then*

$$H^*_{AS}(i) : H^*_{AS}(X \cup Y, X) \longrightarrow H^*_{AS}(Y, X \cap Y)$$

is an isomorphism.

H_c^* satisfies the Eilenberg-Steenrod axioms in the following modified form (see section 6.6 of [Spa66]).

Proposition 2.1.2 *H_c^* acts on arbitrary topological pairs and on proper maps between them. The following properties hold.*

1. *(Controvariant functoriality) If $I : (X, A) \mapsto (X, A)$ is the identity map, then $H_c^*(I)$ is the identity homomorphism on $H_c^*(X, A)$. If $\Phi : (X, A) \mapsto (Y, B)$ and $\Phi' : (Y, B) \mapsto (Z, C)$ are proper maps, then $H_c^*(\Phi' \circ \Phi) = H_c^*(\Phi) \circ H_c^*(\Phi')$.*

2. *(Homotopy invariance) If two proper maps Φ and Φ' are homotopic via a proper homotopy, then $H_c^*(\Phi) = H_c^*(\Phi')$.*

3. *(Excision) Let (X, A) be a topological pair and let U be open in X, such that $\overline{U} \subset \mathrm{Int}\,(A)$. If $i : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ is the inclusion map, $H_c^*(i)$ is an isomorphism.*

4. *(Functoriality of the coboundary) If $X \subset Y \subset Z$ and X,Y are closed in Z, there exists a coboundary homomorphism $\delta_c^q(Z, Y, X) : H_c^q(Y, X) \mapsto H_c^{q+1}(Z, Y)$. If $\Phi : (Z, Y) \mapsto (Z', Y')$ is a proper map such that $\Phi(X) \subset X' \subset Y'$, then the following diagram commutes*

$$\begin{array}{ccc} H_c^q(Y', X') & \xrightarrow{H_c^q(\Phi|_{(Y,X)})} & H_c^q(Y, X) \\ \delta_c^q(Z',Y',X') \big\downarrow & & \big\downarrow \delta_c^q(Z,Y,X) \\ H_c^{q+1}(Z', Y') & \xrightarrow{H_c^{q+1}(\Phi)} & H_c^{q+1}(Z, Y). \end{array}$$

5. *(Long exact sequence) Given three topological spaces X, Y, Z, such that $X \subset Y \subset Z$ and X, Y, are closed in Z, denote by $i : (Y, X) \hookrightarrow (Z, X)$ and $j : (Z, X) \hookrightarrow (Z, Y)$ the inclusion maps. Then the following sequence of homomorphisms is exact*

$$\cdots \to H_c^q(Z, X) \stackrel{H_c^q(i)}{\to} H_c^q(Y, X) \stackrel{\delta_c^q}{\to} H_c^{q+1}(Z, Y) \stackrel{H_c^{q+1}(j)}{\to} H_c^{q+1}(Z, X) \to \dots$$

6. *(Dimension property) $H_c^0(\{point\}) = \mathcal{A}$ and $H_c^q(\{point\}) = 0$ for $q \neq 0$.*

Moreover H_c^* satisfies the following continuity property.

Proposition 2.1.3 *Let $\{(X^m, A^m)\}$, $m \in \mathbf{N}$, be a sequence of compact Hausdorff pairs in some space, directed downward by the inclusion, and let*

$$X = \bigcap_{m \in \mathbf{N}} X^m, \quad A = \bigcap_{m \in \mathbf{N}} A^m.$$

The inclusion maps $i^m : (X, A) \hookrightarrow (X^m, A^m)$ induce an isomorphism

$$\varinjlim_{m \in \mathbf{N}} H_c^*(i^m) : \varinjlim_{m \in \mathbf{N}} H_c^*(X^m, A^m) \longrightarrow H_c^*(X, A).$$

In fact the same property holds for the Alexander-Spanier cohomology theory (Theorem 6.6.6 of [Spa66]), and the two theories coincide on compact pairs.

The following property is peculiar of cohomology theories with compact supports.

Proposition 2.1.4 *Let X be a paracompact, locally compact, Hausdorff space and let $A \subset X$ be closed. Set*

$$\mathcal{S}(X,A) = \left\{U \mid A \subset U \subset X,\ U \text{ is closed, } \overline{X \setminus U} \text{ is compact}\right\}.$$

$\mathcal{S}(X,A)$ is directed downward by the inclusion. Then the inclusion maps induce the isomorphism

$$\varinjlim_{U \in \mathcal{S}(X,A)} H_c^*(X,U) \cong H_c^*(X,A). \tag{2.1}$$

Proof. Since X is paracompact and Hausdorff, for every closed subset U

$$H_{AS}^*(X,U) \cong \varinjlim_{U \subset V \text{ open}} H_{AS}^*(X,V)$$

by Corollary 6.6.3 of [Spa66]. Therefore

$$\varinjlim_{U \in \mathcal{S}(X,A)} H_c^*(X,U) \cong \varinjlim_{U \in \mathcal{S}(X,A)} H_{AS}^*(X,U) \cong \varinjlim_{U \in \mathcal{S}(X,A)} \varinjlim_{U \subset V \text{ open}} H_{AS}^*(X,V). \tag{2.2}$$

Since X is a locally compact Hausdorff space, by Theorem 6.6.16 of [Spa66]

$$\varinjlim_{\substack{A \subset V \text{ open} \\ X \setminus V \text{ compact}}} H_{AS}^*(X,V) \cong H_c^*(X,A). \tag{2.3}$$

(2.2) and (2.3) imply the thesis if we can prove that for every open neighborhood V of A such that $X \setminus V$ is compact, there exists a set $U \in \mathcal{S}(X,A)$ such that $U \subset V$.

X is locally compact and Hausdorff, $X \setminus V$ is compact and $X \setminus A$ is open. By Theorem 5.18 of [Kel55] there exists an open neighborhood Ω of $X \setminus V$ such that $\Omega \subset X \setminus A$ and $\overline{\Omega}$ is compact. Therefore $U = X \setminus \Omega$ is in $\mathcal{S}(X,A)$ and $U \subset V$. □

With some assumptions on the topological spaces, also H_c^* has a strong excision property.

Proposition 2.1.5 *Let X and Y be two closed subsets of a paracompact, locally compact Hausdorff space. If $i : (Y, X \cap Y) \hookrightarrow (X \cup Y, X)$ is the inclusion map, $H_c^*(i)$ is an isomorphism.*

Proof. Take $U \in \mathcal{S}(X \cup Y, X)$. Then $U \cap Y$ is in $\mathcal{S}(Y, X \cap Y)$. Let

$$i_U : (Y, U \cap Y) \hookrightarrow (X \cup Y, U)$$

be the inclusion map. Since $X \cup Y = U \cup Y$, $H_{AS}^*(i_U)$ is an isomorphism, by Proposition 2.1.1. But $H_c^*(i_U) = H_c^*(i_U)$, because the cohomologies H_{AS}^* and H_c^* coincide on pairs (X,A) where $\overline{X \setminus A}$ is compact.

Notice that the family $\{U \cap Y \mid U \in \mathcal{S}(X \cup Y, X)\}$ is cofinal in $\mathcal{S}(Y, X \cap Y)$. Therefore, by Proposition 2.1.4,

$$H_c^*(i) = \varinjlim_{U \in \mathcal{S}(X \cup Y, X)} H_c^*(i_U)$$

and it must be an isomorphism, being the direct limit of isomorphisms. □

The last part of this section is devoted to the Mayer-Vietoris homomorphism. In order to simplify our statements, we introduce some notations.

Capital calligraphic letters (such as $\mathcal{X}$) will denote topological pairs. $T = (\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2)$ is called a triad if $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$. By $-T$ we denote the triad $(\mathcal{X}; \mathcal{X}_2, \mathcal{X}_1)$. $\Phi : (\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2) \mapsto (\mathcal{X}'; \mathcal{X}_1', \mathcal{X}_2')$ is a map between triads if it maps $\mathcal{X}$ into $\mathcal{X}'$, taking $\mathcal{X}_i$ into $\mathcal{X}_i'$, for $i = 1, 2$.

A triad $((X, A); (X_1, A_1), (X_2, A_2))$ is called proper if $A_i = A \cap X_i$, $i = 1, 2$.

Proposition 2.1.6 *Let $T = (\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2) = ((X, A); (X_1, A_1), (X_2, A_2))$ be a proper triad consisting of closed pairs in a paracompact, locally compact, Hausdorff space. Then there exists a homomorphism*

$$\Delta^q(T) : H_c^q(X_1 \cap X_2, A_1 \cap A_2) \to H_c^{q+1}(X, A)$$

called Mayer-Vietoris homomorphism of the triad T. Δ^q is functorial with respect to proper maps: if $\Phi : T \mapsto T'$ is a proper map between proper triads, then the following diagram commutes

$$\begin{array}{ccc} H_c^q(\mathcal{X}_1' \cap \mathcal{X}_2') & \xrightarrow{H_c^q(\Phi|_{\mathcal{X}_1 \cap \mathcal{X}_2})} & H_c^q(\mathcal{X}_1 \cap \mathcal{X}_2) \\ {\scriptstyle \Delta^q(T')}\downarrow & & \downarrow{\scriptstyle \Delta^q(T)} \\ H_c^{q+1}(\mathcal{X}') & \xrightarrow{H_c^q(\Phi)} & H_c^{q+1}(\mathcal{X}). \end{array}$$

Finally, $\Delta^q(-T) = -\Delta^q(T)$.

Proof. Set $(X_0, A_0) = (X_1 \cap X_2, A_1 \cap A_2)$. Let

$$\begin{aligned} i : (X_0, X_0 \cap A_2) &\hookrightarrow (X_0 \cup A_2, A_2), \\ j : (X_2, X_2 \cap (X_1 \cup A)) &\hookrightarrow (X_2 \cup X_1, X_1 \cup A), \\ k : (X, A) &\hookrightarrow (X, X_1 \cup A), \end{aligned}$$

be inclusion maps. Since the triad T is proper, $X_0 \cap A_2 = A_0$ and $X_2 \cap (X_1 \cup A) = X_0 \cup A_2$. By strong excision (Proposition 2.1.5),

$$\begin{aligned} H_c^q(i) : H_c^q(X_0 \cup A_2) &\longrightarrow H_c^q(X_0, A_0), \\ H_c^{q+1}(j) : H_c^{q+1}(X, X_1 \cup A) &\longrightarrow H_c^{q+1}(X_2, X_0 \cup A_2) \end{aligned}$$

are isomorphisms. From the inclusions $A_2 \subset X_0 \cup A_2 \subset X_2$ we get the relative coboundary homomorphism

$$\delta_c^q(X_2, X_0 \cup A_2, A_2) : H_c^q(X_0 \cup A_2, A_2) \longrightarrow H_c^q(X_0, A_0).$$

Set

$$\Delta^q(T) = H_c^{q+1}(k) \circ H_c^{q+1}(j)^{-1} \circ \delta_c^q(X_2, X_0 \cup A_2, A_2) \circ H_c^q(i)^{-1}.$$

The functoriality of Δ^q follows immediately from the functoriality of the coboundary homomorphism and from the functoriality of H_c^* with respect to proper inclusions. □

A pair of triads $(T, T') = ((\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2), \mathcal{X}'; \mathcal{X}_1', \mathcal{X}_2'))$ is called a consecutive pair of triads if

$$\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 = \mathcal{X}_1' \cap \mathcal{X}_2'.$$

In symbols, we write $T \Rightarrow T'$. In this case we say that the consecutive pair of triads starts at $\mathcal{X}_1 \cap \mathcal{X}_2$ and ends at $\mathcal{X}' = \mathcal{X}_1' \cup \mathcal{X}_2'$.

Assume that $T_0 \Rightarrow T$ and $T_0' \Rightarrow T'$ are consecutive pairs of triads, consisting of closed subsets of a paracompact, locally compact, Hausdorff space. Assume that T_0' is contained in T_0 and T' is contained in T. In symbols

$$\begin{array}{ccc} T_0 & \Longrightarrow & T \\ \cup & & \cup \\ T_0' & \Longrightarrow & T'. \end{array}$$

If both consecutive pairs share the same starting pair and the same end pair, the functoriality of the Mayer-Vietoris homomorphism immediately implies that

$$\Delta^{q+1}(T) \circ \Delta^q(T_0) = \Delta^{q+1}(T') \circ \Delta^q(T_0').$$

Here is a technical lemma which will be useful later on.

Lemma 2.1.7 *Assume that*

$$((\mathcal{Y}; \mathcal{Y}_1, \mathcal{Y}_2),\ (\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2))$$

and

$$((\mathcal{Z}; \mathcal{Z}_1, \mathcal{Z}_2),\ (\mathcal{X}; \mathcal{W}_1, \mathcal{W}_2))$$

are two consecutive pairs of proper triads, in a paracompact, locally compact Hausdorff space, both starting at

$$\mathcal{Y} \cap \mathcal{Z} = \mathcal{Y}_1 \cap \mathcal{Y}_2 = \mathcal{Z}_1 \cap \mathcal{Z}_2$$

and both ending at $\mathcal{X}$. Assume further that

$$\mathcal{Z}_i = \mathcal{Z} \cap \mathcal{X}_i \quad \text{and} \quad \mathcal{Y}_i = \mathcal{Y} \cap \mathcal{W}_i, \quad \text{for } i = 1, 2.$$

Then we have

$$\Delta^{q+1}(\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2) \circ \Delta^q(\mathcal{Y}; \mathcal{Y}_1, \mathcal{Y}_2) = -\Delta^{q+1}(\mathcal{X}; \mathcal{W}_1, \mathcal{W}_2) \circ \Delta^q(\mathcal{Z}; \mathcal{Z}_1, \mathcal{Z}_2).$$

PROOF. Consider the following triads

$$\begin{array}{ll} T_1 = (\mathcal{Y}; \mathcal{Y}_1, \mathcal{Y}_2), & T_2 = (\mathcal{X}; \mathcal{X}_1, \mathcal{X}_2), \\ T_3 = ((\mathcal{W}_1 \cap \mathcal{X}_2) \cup \mathcal{Y}_2; \mathcal{W}_1 \cap \mathcal{X}_2, \mathcal{Y}_2), & T_4 = (\mathcal{X}; \mathcal{X}_1 \cup \mathcal{W}_1, \mathcal{X}_2), \\ T_5 = (\mathcal{Y}_2 \cup \mathcal{Z}_2; \mathcal{Z}_2, \mathcal{Y}_2), & T_6 = (\mathcal{X}; \mathcal{X}_1 \cup \mathcal{W}_1, \mathcal{X}_2 \cap \mathcal{W}_2), \\ T_7 = ((\mathcal{X}_1 \cap \mathcal{W}_2) \cup Z_2; \mathcal{X}_1 \cap \mathcal{W}_2, \mathcal{Z}_2), & T_8 = (\mathcal{X}; \mathcal{X}_1 \cup \mathcal{W}_1, \mathcal{W}_2), \\ T_9 = (\mathcal{Z}; \mathcal{Z}_1, \mathcal{Z}_2), & T_{10} = (\mathcal{X}; \mathcal{W}_1, \mathcal{W}_2). \end{array}$$

We claim that every pair (T_{2i-1}, T_{2i}), $i = 1, \dots, 5$, is a consecutive pair of triads starting at $\mathcal{Y} \cap \mathcal{Z}$ and ending at $\mathcal{X}$.

This is true by assumption for i=1,5. We have the inclusions

$$\mathcal{Y}_2 = \mathcal{Y} \cap \mathcal{W}_2 \subset \mathcal{X}_2, \quad \mathcal{Y}_1 \subset \mathcal{W}_1$$

and so

$$\begin{aligned}(\mathcal{W}_1 \cap \mathcal{X}_2) \cap \mathcal{Y}_2 &= \mathcal{W}_1 \cap \mathcal{Y}_2 = \mathcal{W}_1 \cap \mathcal{W}_2 \cap \mathcal{Y} = \mathcal{Z} \cap \mathcal{Y}, \\ (\mathcal{X}_1 \cup \mathcal{W}_1) \cap \mathcal{X}_2 &= (\mathcal{X}_1 \cap \mathcal{X}_2) \cup (\mathcal{W}_1 \cap \mathcal{W}_2) = \mathcal{Y} \cup (\mathcal{W}_1 \cap \mathcal{W}_2) = \mathcal{Y}_2 \cup (\mathcal{W}_1 \cap \mathcal{X}_2).\end{aligned}$$

Therefore our claim is true for $i = 2$. The same argument works for $i = 4$. Finally

$$\mathcal{Z}_2 \cap \mathcal{Y}_2 = (\mathcal{Z} \cap \mathcal{X}_2) \cap (\mathcal{Y} \cap \mathcal{W}_2) = (\mathcal{Z} \cap \mathcal{W}_2) \cap (\mathcal{Y} \cap \mathcal{X}_2) = \mathcal{Z} \cap \mathcal{Y}$$

and

$$\begin{aligned}(\mathcal{X}_1 \cup \mathcal{W}_1) \cap (\mathcal{X}_2 \cap \mathcal{W}_2) &= (\mathcal{X}_1 \cap \mathcal{X}_2 \cap \mathcal{W}_2) \cup (\mathcal{W}_1 \cap \mathcal{X}_2 \cap \mathcal{W}_2) \\ &= (\mathcal{Y} \cap \mathcal{W}_2) \cup (\mathcal{Z} \cap \mathcal{X}_2) = \mathcal{Y}_2 \cup \mathcal{Z}_2,\end{aligned}$$

which implies our claim for $i = 3$.

The relations between these triads can be displayed as follows:

$$\begin{array}{ccccccc} T_5 & \Longrightarrow & T_6 & \qquad & -T_5 & \Longrightarrow & T_6 \\ \cap & & \cap & & \cap & & \cap \\ T_3 & \Longrightarrow & T_4 & & T_7 & \Longrightarrow & T_8 \\ \cup & & \cup & & \cup & & \cup \\ T_1 & \Longrightarrow & T_2 & & T_9 & \Longrightarrow & T_{10}. \end{array}$$

Therefore

$$\begin{aligned}\Delta^{q+1}(T_2) \circ \Delta^q(T_1) &= \Delta^{q+1}(T_4) \circ \Delta^q(T_3) = \Delta^{q+1}(T_6) \circ \Delta^q(T_5) \\ = -\Delta^{q+1}(T_6) \circ \Delta^q(-T_5) &= -\Delta^{q+1}(T_8) \circ \Delta^q(T_7) = -\Delta^{q+1}(T_{10}) \circ \Delta^q(T_9),\end{aligned}$$

as we wished to prove. □

Proposition 2.1.8 *Consider the triads*

$$T = ((X, A); (X_1, A_1), (X_2, A_2)) \quad \textit{and} \quad T' = (A; A_1, A_2),$$

where (X_1, A_1), (X_2, A_2) are closed pairs in a paracompact locally compact Hausdorff space. Assume that T is proper and that A is compact. Then the following diagram is anti-commutative

$$\begin{array}{ccc} H_c^{q-1}(A_1 \cap A_2) & \xrightarrow{\delta_c^q} & H_c^q(X_1 \cap X_2, A_1 \cap A_2) \\ {\scriptstyle \Delta^{q-1}(T')}\downarrow & & \downarrow{\scriptstyle \Delta^q(T)} \\ H_c^q(A_1 \cup A_2) & \xrightarrow{\delta_c^{q+1}} & H_c^{q+1}(X_1 \cup X_2, A_1 \cup A_2). \end{array}$$

PROOF. If X is a topological space, let $CX = X \times [0,1]/X \times \{1\}$ be the cone over X. In CX we identify X with $X \times \{0\}$. Given a closed pair (X, A) in a paracompact, locally compact, Hausdorff space, we want to define an injective homomorphism

$$\eta^q(X,A) : H^q_x(X,A) \longrightarrow H^q_c(X \cup CA).$$

Let

$$\begin{aligned} e : (X,A) &\hookrightarrow (X \cup CA, CA), \\ j : (X \cup CA) &\hookrightarrow (X \cup CA, CA), \\ k : CA &\hookrightarrow X \cup CA, \end{aligned}$$

be the inclusion maps. Since e is an excision,

$$H^q_c(e) : H^q_c(X \cup CA, CA) \longrightarrow H^q_c(X,A)$$

is an isomorphism. Set

$$\eta^q(X,A) = H^q_c(j) \circ H^q_c(e)^{-1}.$$

Notice that CA is homotopically equivalent to a point, via a proper homotopy. From the exact sequence of the pair $(X \cup CA, CA)$ and from the fact that

$$H^q_c(CA) = 0 \quad \forall q \neq 0$$

we get that $H^q_c(j)$, and thus $\eta^q(X,A)$, is injective.

Set $(X_0, A_0) = (X_1 \cap X_2, A_1 \cap A_2)$. We introduce the triads

$$\begin{aligned} T'' &= (X_0 \cup CA_0; CA_0, X_0), \\ T''' &= (X \cup CA; CA, X), \\ T'''' &= (X \cup CA; X_1 \cup CA_1, X_2 \cup CA_2). \end{aligned}$$

Since T is proper, we easily get that the consecutive pairs

$$(T'', T'''') \quad \text{and} \quad (T', T''')$$

satisfy the assumptions of Lemma 2.1.7. Therefore

$$\Delta^q(T'''') \circ \Delta^{q-1}(T'') = -\Delta^q(T''') \circ \Delta^{q-1}(T'). \tag{2.4}$$

Consider the diagram

$$\begin{array}{ccccc} H^{q-1}_c(A_0) & \xrightarrow{\delta^{q-1}_c(X_0,A_0)} & H^q_c(X_0,A_0) & \xrightarrow{\eta^q(X_0,A_0)} & H^q_c(X_0 \cup CA_0) \\ {\scriptstyle \Delta^{q-1}(T')}\downarrow & & \downarrow{\scriptstyle \Delta^q(T)} & & \downarrow{\scriptstyle \Delta^q(T'''')} \\ H^q_c(A) & \xrightarrow{\delta^q_c(X,A)} & H^{q+1}_c(X,A) & \xrightarrow{\eta^{q+1}(X,A)} & H^{q+1}_c(X \cup CA). \end{array}$$

It is easy to see that the composition of the top row homomorphisms coincides with $\Delta^{q-1}(T'')$, while the composition of the bottom row homomorphisms coincides with $\Delta^q(T''')$. So (2.4) implies that the outer diagram anti-commutes.

The functoriality of Δ^q implies that the right-hand square commutes. Therefore, using the fact that $\eta^{q+1}(X,A)$ is injective, we conclude that the left-hand square anti-commutes, as we wished to prove. □

2.2 The E^+-cohomology of an E^+-pair

Let E be a real Hilbert space, given with a fixed splitting into closed orthogonal subspaces: $E = E^+ \oplus E^-$. On E we will consider the topology $\mathcal{T}_{E^+}$ introduced in Chapter 1: $\mathcal{T}_{E^+}$ is the product topology between the weak topology of E^+ and the strong topology of E^-.

Let $\pi : E \to E/E^+$ be the quotient projection. Notice that $\mathcal{T}_{E^+}$ induces the weak topology on $\pi^{-1}(\alpha)$, for every finite dimensional subspace α of $E/E^+ \cong E^-$.

We recall that a set $X \subset E$ is said E^+-locally compact if $X \cap \pi^{-1}(\alpha)$ endowed with the topology $\mathcal{T}_{E^+}$ is locally compact, for every finite dimensional subspace α of $E/E^+ \cong E^-$. An E^+-pair is a pair of subsets of E which are $\mathcal{T}_{E^+}$-closed and E^+-locally compact. The E^+-locally compact sets have the following property.

Proposition 2.2.1 *If the $\mathcal{T}_{E^+}$-closed set $X \subset H$ is E^+-locally compact, then $X \cap \pi^{-1}(\alpha)$ is weakly paracompact for every finite dimensional linear subspace α of E/E^+.*

In fact $X \cap \pi^{-1}(\alpha)$ is a countable union of weakly compact sets and, being weakly locally compact, it must be weakly paracompact.

So all the properties of the Alexander-Spanier cohomology theory with compact supports seen in the previous section hold for $(X \cap \pi^{-1}(\alpha), A \cap \pi^{-1}(\alpha))$, whenever (X, A) is an E^+-pair.

Let $\mathcal{V}$ be the set of all finite dimensional linear subspaces of E/E^+. $\mathcal{V}$ is a partially ordered set with the inclusion ordering $\subset$. Since for each α and β in $\mathcal{V}$ there exists $\gamma \in \mathcal{V}$ such that $\alpha \subset \gamma$ and $\beta \subset \gamma$, $(\mathcal{V}, \subset)$ is a directed set.

The dimension of $\alpha \in \mathcal{V}$ will be denoted by $d(\alpha)$. Choose arbitrarily an orientation for each $\alpha \in \mathcal{V}$. These orientations will be considered fixed once for all. If $\alpha \subset \beta$ and $d(\beta) = d(\alpha) + 1$, α divides β in two half spaces. The orientations of α and β allow to denote uniquely them as β_α^+ and β_α^-. They satisfy

$$\beta_\alpha^+ \cup \beta_\alpha^- = \beta, \quad \beta_\alpha^+ \cap \beta_\alpha^- = \alpha.$$

If $X \subset E$, α and β as above, set

$$X_\alpha = X \cap \pi^{-1}(\alpha), \quad X_{\beta_\alpha^+} = X \cap \pi^{-1}(\beta_\alpha^+), \quad X_{\beta_\alpha^-} = X \cap \pi^{-1}(\beta_\alpha^-).$$

As before

$$X_{\beta_\alpha^+} \cup X_{\beta_\alpha^-} = X_\beta, \quad X_{\beta_\alpha^+} \cap X_{\beta_\alpha^-} = X_\alpha.$$

Let (X, A) be an E^+-pair. Consider the following proper triad of paracompact, locally compact, Hausdorff pairs

$$T = ((X_\beta, A_\beta); (X_{\beta_\alpha^+}, A_{\beta_\alpha^+}), (X_{\beta_\alpha^-}, A_{\beta_\alpha^-})).$$

By Proposition 2.1.6, for each $q \in \mathbf{Z}$ we have the relative Mayer-Vietoris homomorphism

$$\Delta^q_{\alpha\beta}(X, A) = \Delta^{q+d(\alpha)}(T) : H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \longrightarrow H_c^{q+d(\beta)}(X_\beta, A_\beta).$$

Definition 2.2.1 *If $\alpha = \alpha_0 \subset \alpha_1 \subset \cdots \subset \alpha_k = \beta$ are in $\mathcal{V}$ and $d(\alpha_{i+1}) = d(\alpha_i) + 1$, for $i = 0, \ldots, k-1$, set*

$$\Delta^q_{\alpha\beta}(X, A) = \Delta^q_{\alpha_{k-1}\alpha_k}(X, A) \circ \cdots \circ \Delta^q_{\alpha_0\alpha_1}(X, A).$$

The next result justifies the above definition.

Proposition 2.2.2 *The definition of $\Delta^q_{\alpha\beta}(X,A)$ does not depend on the choice of α_i, $i = 1,\dots,k-1$.*

PROOF. Arguing by induction, it is enough to consider the case $k=2$. So let

$$\alpha \subset \gamma \subset \beta, \quad \alpha \subset \tilde{\gamma} \subset \beta$$

be linear subspaces of E/E^+ such that

$$d(\beta) = d(\gamma)+1 = d(\tilde{\gamma})+1 = d(\alpha)+2.$$

We want to show that

$$\Delta^q_{\gamma\beta}(X,A) \circ \Delta^q_{\alpha\gamma}(X,A) = \Delta^q_{\tilde{\gamma}\beta}(X,A) \circ \Delta^q_{\alpha\tilde{\gamma}}(X,A).$$

If $\gamma = \tilde{\gamma}$ there is nothing to prove. Otherwise dimensional reasons imply that $\gamma \cap \tilde{\gamma} = \alpha$.

$\tilde{\gamma} \cap \beta^+_\gamma$ is a half-space in $\tilde{\gamma}$, bounded by $\gamma \cap \tilde{\gamma} = \alpha$, so it must coincide with either $\tilde{\gamma}^+_\alpha$ or $\tilde{\gamma}^-_\alpha$. It is easy to see that

$$\tilde{\gamma} \cap \beta^+_\gamma = \tilde{\gamma}^+_\alpha \quad \text{implies} \quad \gamma \cap \beta^+_{\tilde{\gamma}} = \gamma^-_\alpha, \tag{2.5}$$
$$\tilde{\gamma} \cap \beta^+_\gamma = \tilde{\gamma}^-_\alpha \quad \text{implies} \quad \gamma \cap \beta^+_{\tilde{\gamma}} = \gamma^+_\alpha. \tag{2.6}$$

Assume that case (2.5) holds. Consider the proper triads

$$\begin{aligned}
T_1 &= ((X_\gamma, A_\gamma);\ (X_{\gamma^+_\alpha}, A_{\gamma^+_\alpha}), (X_{\gamma^-_\alpha}, A_{\gamma^-_\alpha})),\\
T_2 &= ((X_\beta, A_\beta);\ (X_{\beta^+_\gamma}, A_{\beta^+_\gamma}), (X_{\beta^-_\gamma}, A_{\beta^-_\gamma})),\\
T_3 &= ((X_{\tilde{\gamma}}, A_{\tilde{\gamma}});\ (X_{\tilde{\gamma}^+_\alpha}, A_{\tilde{\gamma}^+_\alpha}), (X_{\tilde{\gamma}^-_\alpha}, A_{\tilde{\gamma}^-_\alpha})),\\
T_4 &= ((X_\beta, A_\beta);\ (X_{\beta^-_{\tilde{\gamma}}}, A_{\beta^-_{\tilde{\gamma}}}), (X_{\beta^+_{\tilde{\gamma}}}, A_{\beta^+_{\tilde{\gamma}}})).
\end{aligned}$$

A straightforward analysis shows that (T_1,T_2) and (T_3,T_4) are consecutive pairs of triads satisfying the assumptions of Lemma 2.1.7. Therefore

$$\begin{aligned}
\Delta^q_{\gamma\beta}(X,A) \circ \Delta^q_{\alpha\gamma}(X,A) &= \Delta^{q+d(\alpha)+1}(T_2) \circ \Delta^{q+d(\alpha}(T_1) = -\Delta^{q+d(\alpha)+1}(T_4) \circ \Delta^{q+d(\alpha)}(T_3)\\
&= \Delta^{q+d(\alpha)+1}(-T_4) \circ \Delta^{q+d(\alpha)}(T_3) = \Delta^q_{\tilde{\gamma}\beta}(X,A) \circ \Delta^q_{\alpha\tilde{\gamma}}(X,A)
\end{aligned}$$

as claimed. An analogous argument works for case (2.6). □

The above proposition also shows that $\{H^{q+d(\alpha)}_c(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X,A)\}$ is a direct system of $\mathcal{A}$-modules over the directed set $\mathcal{V}$.

Definition 2.2.2 *Let $q \in \mathbf{Z}$; the E^+-cohomology module of the E^+-pair (X,A) of index q is the direct limit*

$$H^q_{E^+}(X,A) = \varinjlim_{\alpha\in\mathcal{V}} \{H^{q+d(\alpha)}_c(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X,A)\}.$$

As usual, $H^*_{E^+}(X) = H^*_{E^+}(X, \emptyset)$.

Now we want to compute the E^+-cohomology of an important family of E^+-pairs. Let V^+ be a linear subspace of E^+ of finite dimension r and let V^- be a closed subspace of E^- of finite codimension s. Set $V = V^+ \oplus V^-$ and

$$\overline{B_V}(R) = \{x \in V \mid \|x\| \leq R\}\,, \quad \partial B_V(R) = \{x \in V \mid \|x\| = R\}\,.$$

The topology $\mathcal{T}_{E^+}$ induces the strong topology on V: therefore $(\overline{B_V}(R), \partial B_V(R))$ is an E^+-pair.

Proposition 2.2.3 *The E^+-cohomology of $(\overline{B_V}(R), \partial B_V(R))$ is*

$$H^q_{E^+}(\overline{B_V}(R), \partial B_V(R)) = \begin{cases} \mathcal{A} & \text{if } q = r - s = E^+\text{-}\dim V, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Set $(X, A) = (\overline{B_V}(R), \partial B_V(R))$. Let S be an s-dimensional complement of V^- in E^-. Then π is one-to-one on S and $\sigma = \pi(S)$ has dimension s. Set

$$\mathcal{V}_\sigma = \{\alpha \in \mathcal{V} \mid \sigma \subset \alpha\}\,.$$

$\mathcal{V}_\sigma$ is a cofinal subset of $\mathcal{V}$, meaning that it is a directed subset of $\mathcal{V}$ such that for each $\alpha \in \mathcal{V}$ there exists $\beta \in \mathcal{V}_\sigma$ such that $\alpha \subset \beta$.

Since the limit of a direct system over a directed set is naturally isomorphic to the limit of the same direct system restricted to any cofinal subset

$$H^q_{E^+}(X, A) = \varinjlim_{\alpha \in \mathcal{V}_\sigma} \{H_c^{q+d(\alpha)}(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X, A)\}.$$

Let $\alpha \in \mathcal{V}_\sigma$. The restriction of π from $V \cap \pi^{-1}(\alpha)$ to α has kernel V^+ and image complementary to σ in α. Thus

$$\dim V \cap \pi^{-1}(\alpha) = d(\alpha) - s + r.$$

Therefore X_α is a closed ball in an Euclidean space of dimension $d(\alpha) - s + r$ and A_α is its boundary. So

$$H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) = \begin{cases} \mathcal{A} & \text{if } q = r - s, \\ 0 & \text{otherwise.} \end{cases}$$

Taking the direct limit, $H^q_{E^+}(X, A) = 0$ if $q \neq r - s$. It is easy to show that, if $\alpha \subset \beta$ are in $\mathcal{V}_\sigma$ and $d(\beta) = d(\alpha) + 1$, the Mayer-Vietoris homomorphism $\Delta^{r-s}_{\alpha\beta}(X, A)$ is an isomorphism. Therefore

$$H^{r-s}_{E^+}(X, A) = \mathcal{A}.$$

□

Remark 2.2.1 *Notice that, if we change our choice of the orientations of the elements of $\mathcal{V}$, some of the homomorphisms $\Delta^q_{\alpha\beta}(X, A)$ may change sign. If the coefficient ring $\mathcal{A}$ is $\mathbf{Z}_2$, we do not see any difference and the E^+-cohomology of a pair does not depend on the orientations.*

2.3 E^+-finite morphisms

Definition 2.3.1 *A $\mathcal{T}_{E^+}$-continuous map $\Phi : (X, A) \mapsto (Y, B)$ is an E^+-finite morphism if*

1. *it has the form*
$$\Phi(x) = Tx + R(x), \tag{2.7}$$
where T is a linear automorphism of E such that $TE^+ = E^+$, $TE^- = E^-$, R maps bounded sets into bounded sets and $\pi(R(X)) \subset \alpha_0$, a finite dimensional subspace of E/E^+;

2. *$\Phi^{-1}(U)$ is bounded for every bounded U.*

Keeping notations as in the above definition, we will say that Φ is an E^+-finite morphism with respect to α_0.

We will see that the composition of two E^+-finite morphisms is again an E^+-finite morphism. Therefore we have a category, whose objects are all the E^+-pairs in E and whose morphisms are all the E^+-finite morphisms. The purpose of this section is to extend the definition of $H^*_{E^+}$ to the E^+-finite morphisms, so to construct a controvariant functor on this category. We will do this only for $\mathbf{Z}_2$ coefficients. So from now on, $\mathcal{A} = \mathbf{Z}_2$.

Let $\Phi : (X, A) \mapsto (Y, B)$ be an E^+-finite morphism with respect to α_0 between E^+-pairs. Since $TE^+ = E^+$, T induces a linear automorphism $\tilde{T}$ on E/E^+. Set $\beta_0 = \tilde{T}^{-1}\alpha_0$ and

$$\mathcal{V}_{\beta_0} = \{\beta \in \mathcal{V} \mid \beta_0 \subset \beta\}.$$

If $\beta \in \mathcal{V}_{\beta_0}$, Φ maps $X \cap \pi^{-1}(\beta)$ into $Y \cap \pi^{-1}(\tilde{T}\beta)$. Therefore we can define a map

$$\Phi_\beta = \Phi|_{X_\beta} : (X_\beta, Y_\beta) \mapsto (Y_{\tilde{T}\beta}, B_{\tilde{T}\beta}).$$

Let $q \in \mathbf{Z}$. Since $\mathcal{T}_{E^+}$ induces the weak topology on $\pi^{-1}(\beta)$ and (X_β, A_β) is a $\mathcal{T}_{E^+}$-closed pair, property (2) of the E^+-finite morphism(see Definition 2.3.1) implies that Φ_β is a $\mathcal{T}_{E^+}$-proper map. Therefore it induces homomorphisms

$$H_c^{q+d(\beta)}(\Phi_\beta) : H_c^{q+d(\beta)}(Y_{\tilde{T}\beta}, B_{\tilde{T}\beta}) \longrightarrow H_c^{q+d(\beta)}(X_\beta, A_\beta).$$

Now let $\alpha \subset \beta$ be in $\mathcal{V}_{\beta_0}$, with $d(\beta) = d(\alpha) + 1$. Notice that $\tilde{T}$ maps β^+_α isomorphically onto either

$$(\tilde{T}\beta)^+_{\tilde{T}\alpha} \quad \text{or} \quad (\tilde{T}\beta)^-_{\tilde{T}\alpha}.$$

Therefore Φ_β maps $(X_{\beta^+_\alpha}, A_{\beta^+_\alpha})$ into either

$$(Y_{(\tilde{T}\beta)^+_{\tilde{T}\alpha}}, B_{(\tilde{T}\beta)^+_{\tilde{T}\alpha}}) \quad \text{or} \quad (Y_{(\tilde{T}\beta)^-_{\tilde{T}\alpha}}, B_{(\tilde{T}\beta)^-_{\tilde{T}\alpha}}).$$

In the first case, by the functoriality of the Mayer-Vietoris homomorphism (see Proposition 2.1.6), the following diagram commutes

$$\begin{array}{ccc} H_c^{q+d(\tilde{T}\alpha)}(Y_{\tilde{T}\alpha}, B_{\tilde{T}\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(\Phi_\alpha)} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\tilde{T}\alpha\tilde{T}\beta}(Y,B)} \downarrow & & \downarrow {\scriptstyle \Delta^q_{\alpha\beta}(X,A)} \\ H_c^{q+d(\tilde{T}\beta)}(Y_{\tilde{T}\beta}, B_{\tilde{T}\beta}) & \xrightarrow{H_c^{q+d(\beta)}(\Phi_\beta)} & H_c^{q+d(\beta)}(X_\beta, A_\beta). \end{array} \tag{2.8}$$

In the second case, since exchanging the role of the two sets changes the sign of the Mayer-Vietoris homomorphism, the same diagram anticommutes. Since we are using $\mathbf{Z}_2$ coefficients, diagram (2.8) commutes in every case.

As an immediate consequence we get:

Proposition 2.3.1 $\{H_c^{q+d(\alpha)}(\Phi_\alpha)\}$ *is a direct system of homomorphisms from the direct system of* $\mathbf{Z}_2$*-vector spaces*

$$\{H_c^{q+d(\alpha)}(Y_{\tilde{T}\alpha}, B_{\tilde{T}\alpha}); \Delta^q_{\tilde{T}\alpha\tilde{T}\beta}(Y, B)\}$$

to the direct system of $\mathbf{Z}_2$*-vector spaces*

$$\{H_c^{q+d(\alpha)}(X_\alpha, A_\alpha); \Delta^q_{\alpha\beta}(X, A)\}$$

over the directed set $\mathcal{V}_{\beta_0}$.

Taking the direct limit of the above system we get a homomorphism

$$\lim_{\substack{\longrightarrow \\ \alpha\in\mathcal{V}_{\beta_0}}} H_c^{q+d(\alpha)}(\Phi_\alpha) : \lim_{\substack{\longrightarrow \\ \alpha\in\mathcal{V}}}\{H_c^{q+d(\alpha)}(Y_{\tilde{T}\alpha}, B_{\tilde{T}\alpha}); \Delta^q_{\tilde{T}\alpha\tilde{T}\beta}(Y, B)\} \longrightarrow H^q_{E^+}(X, A).$$

Here we have identified limits over $\mathcal{V}_{\alpha_0}$ and $\mathcal{V}_{\beta_0}$ with limits over $\mathcal{V}$ and this is correct because $\mathcal{V}_{\alpha_0}$ and $\mathcal{V}_{\beta_0}$ are cofinal in $\mathcal{V}$.

$\tilde{T}$ acts on $\mathcal{V}$ as an order preserving bijection. Therefore there exists an isomorphism

$$\hat{T}^q(Y, B) : H^q_{E^+}(Y, B) \longrightarrow \lim_{\substack{\longrightarrow \\ \alpha\in\mathcal{V}}}\{H_c^{q+d(\alpha)}(Y_{\tilde{T}\alpha}, B_{\tilde{T}\alpha}); \Delta^q_{\tilde{T}\alpha\tilde{T}\beta}(Y, B)\}.$$

We are ready to define the cohomology homomorphism induced by the E^+-finite morphism Φ:

Definition 2.3.2 *The* $\mathbf{Z}_2$*-homomorphism*

$$H^q_{E^+}(\Phi) : H^q_{E^+}(Y, B) \to H^q_{E^+}(X, A)$$

is defined as

$$H^q_{E^+}(\Phi) = \lim_{\substack{\longrightarrow \\ \alpha\in\mathcal{V}_{\beta_0}}} H_c^{q+d(\alpha)}(\Phi_\alpha) \circ \hat{T}^q(Y, B).$$

Neither the space α_0 nor the decomposition (2.7) are uniquely determined in Definition 2.3.1. Therefore, in order to prove that Definition 2.3.2 is well posed, we must check that it does not depend on α_0 and on such a decomposition.

If Φ is an E^+-finite morphism with respect to α_0 and α_0', then it is an E^+-finite morphism with respect to $\alpha_0 + \alpha_0'$. Since $\mathcal{V}_{\alpha_0+\alpha_0'}$ is cofinal in both $\mathcal{V}_{\alpha_0}$ and $\mathcal{V}_{\alpha_0'}$, it is easy to see that Definition 2.3.2 does not depend on the choice of α_0.

Now assume that

$$\Phi(x) = T_1 x + R_1(x) = T_2 x + R_2(x).$$

We may choose $\alpha_0 \in \mathcal{V}$ large enough so that

$$\pi(R_1(X)) \subset \alpha_0 \quad \text{and} \quad \pi(R_2(X)) \subset \alpha_0.$$

Then

$$\pi(T_1x - T_2x) = \pi(R_1(x) - R_2(x)) \in \alpha_0 \quad \forall x \in X.$$

Since the definition of $H^q_{E^+}(\Phi)$ depends only on the restriction of T to the minimal closed subspace which contains X, we may assume that

$$\pi((T_1 - T_2)E) \subset \alpha_0.$$

Therefore

$$(\tilde{T}_1 - \tilde{T}_2)E/E^+ \subset \alpha_0. \tag{2.9}$$

By (2.9) and by the fact that $\tilde{T}_1$ and $\tilde{T}_2$ are isomorphisms, we easily deduce that

$$\beta_0 = \tilde{T}_1^{-1}\alpha_0 = \tilde{T}_2^{-1}\alpha_0, \tag{2.10}$$

$$\tilde{T}_1\alpha = \tilde{T}_2\alpha \quad \forall \alpha \in \mathcal{V}_{\beta_0}. \tag{2.11}$$

Therefore the two decompositions generate the same direct system of homomorphisms. Moreover (2.11) implies that $\hat{T}_1^q(Y,B) = \hat{T}_2^q(Y,B)$, and the two definitions of $H^q_{E^+}(\Phi)$ coincide.

We study now the functorial properties of H^*_E.

Proposition 2.3.2 *Assume that $\Phi^1 : (X,A) \mapsto (Y,B)$ and $\Phi^2 : (Y,B) \mapsto (Z,C)$ are E^+-finite morphisms between E^+-pairs. Then:*

1. *if $I : (X,A) \mapsto (X,A)$ is the identity map, $H^q_{E^+}(I)$ is the identity homomorphism on $H^q_{E^+}(X,A)$, for each $q \in \mathbf{Z}$;*
2. *$\Phi^2 \circ \Phi^1$ is an E^+-finite morphism and $H^*_{E^+}(\Phi^2 \circ \Phi^1) = H^*_{E^+}(\Phi^1) \circ H^*_{E^+}(\Phi^2)$.*

PROOF. Assertion (1) is trivial. We prove assertion (2).

Assume that

$$\Phi^1(x) = T_1x + R_1(x), \quad \Phi^2(x) = T_2x + R_2(x)$$

and $\pi(R_1(X)) \subset \alpha_1$, $\pi(R_2(Y)) \subset \alpha_2$. Then

$$\Phi^2 \circ \Phi^1(x) = T_2 \circ T_1x + R(x)$$

where

$$R(x) = T_2R_1(x) + R_2(T_1x + R_1(x)).$$

$\pi(R(X))$ is contained in the finite dimensional space

$$\beta_2 = \tilde{T}_2\alpha_1 + \alpha_2$$

and so $\Phi^2 \circ \Phi^1$ is an E^+-finite morphism with respect to β_2. Set $\beta_1 = \tilde{T}_2^{-1}\beta_2$ and $\beta_0 = \tilde{T}_1^{-1}\beta_1$.

Then $\alpha_i \subset \beta_i$, and so Φ^i is an E^+-finite morphism with respect to β_i, for $i = 1, 2$. Assume that $\alpha \in \mathcal{V}_{\beta_0}$. Since $(\Phi^2 \circ \Phi^1)_\alpha = \Phi^2_{\tilde{T}_1 \alpha} \circ \Phi^1_\alpha$, by the functoriality of H^*_c

$$\varinjlim_{\alpha \in \mathcal{V}_{\beta_0}} H_c^{q+d(\alpha)}((\Phi^2 \circ \Phi^1)_\alpha) = \varinjlim_{\alpha \in \mathcal{V}_{\beta_0}} H_c^{q+d(\alpha)}(\Phi^1_\alpha) \circ \varinjlim_{\alpha \in \mathcal{V}_{\beta_0}} H_c^{q+d(\alpha)}(\Phi^2_{\tilde{T}_1 \alpha}). \tag{2.12}$$

Applying the order preserving bijection $\tilde{T}_1$ on $\mathcal{V}$ we get

$$\varinjlim_{\alpha \in \mathcal{V}_{\beta_0}} H_c^{q+d(\alpha)}(\Phi^2_{\tilde{T}_1 \alpha}) = \hat{T}_1{}^q(Y, B) \circ \varinjlim_{\alpha \in \mathcal{V}_{\beta_1}} H_c^{q+d(\alpha)}(\Phi^2_\alpha) \circ \hat{T}_2{}^q(Z, C) \circ \widehat{T_2 \circ T_1}(Z, C)^{-1}. \tag{2.13}$$

The thesis follows from (2.12) and (2.13). □

Since inclusions are E^+-finite morphisms, we can already prove the strong excision property for $H^*_{E^+}$, which constitutes assertion (3) of Theorem 1.3.1.

Proposition 2.3.3 *Let X, Y be two closed E^+-locally compact sets. Let $i : (X, X \cap Y) \hookrightarrow (X \cup Y, Y)$ be the inclusion map. Then*

$$H^*_{E^+}(i) : H^*_{E^+}(X \cup Y, Y) \longrightarrow H^*_{E^+}(X, X \cap Y)$$

is an isomorphism.

Proof. By Proposition 2.2.1, $X_\alpha \cup Y_\alpha$ is a paracompact locally compact Hausdorff space. By Proposition 2.1.5

$$H^*_c(i_\alpha) : H^*_c(X_\alpha \cup Y_\alpha, X_\alpha) \longrightarrow H^*_c(Y_\alpha, X_\alpha \cap Y_\alpha) \tag{2.14}$$

is an isomorphism. Therefore $H^*_{E^+}(i)$ is an isomorphism, being the direct limit of isomorphisms. □

Remark 2.3.1 *We were forced to use $\mathbf{Z}_2$ coefficients in order to make diagram (2.8) commute. If we consider only E^+-finite morphisms of the form $\Phi(x) = x + R(x)$, we can deal with an arbitrary coefficient ring. Arguing as in the remaining part of this chapter, one could derive a cohomology with arbitrary coefficients which is functorial with respect to maps of the form $\Phi(x) = x + K(x)$, where $\pi \circ K$ maps bounded sets into pre-compact ones (this was done explicitly in [Abb97]). The fact that it is impossible to use arbitrary coefficients when dealing with the whole class of E^+-morphisms reminds a well known phenomenon: if one tries to extend the Leray-Schauder degree theory from maps of the form $Identity + Compact$ to Fredholm maps, one actually gets only a $\mathbf{Z}_2$-valued degree (see, for example, [BZS77]). It would not be difficult to change Definition 2.3.2, in order to have $H^*_{E^+}(\Phi)$ for an arbitrary ring $\mathcal{A}$. However, if S denotes the unit sphere in E^- and $\Phi : S \mapsto S$ is a mirror symmetry with respect to a hyperplane, then we would have $H^0_{E^+}(\Phi) = -identity$ on $H^0_{E^+}(S) = \mathcal{A}$. Since the linear group of an infinite dimensional Hilbert space is always connected (see [Mit70]), Φ is E^+-finitely homotopic to the identity map (provided E^- is infinite dimensional). Therefore the homotopy property could not hold.*

2.4 Continuity properties of $H^*_{E^+}$

We will define the homomorphism induced by an E^+-morphism by approximating it with E^+-finite morphisms. In order to do this, we must prove some continuity properties for $H^*_{E^+}$.

Definition 2.4.1 *Let (X, A) be a bounded E^+-pair. An approximating sequence for (X, A) is a sequence of bounded E^+-pairs $\{(U^m, V^m)\}$ such that:*

1. $U^m \subset U^n$, $V^m \subset V^n$ *if* $n \leq m$;
2. $\bigcap_{m\in\mathbf{N}} U^m = X$, $\bigcap_{m\in\mathbf{N}} V^m = A$.

Given a bounded E^+-pair, consider the family of bounded E^+-pairs

$$(X^m, A^m) = (\mathrm{Cl}_{\mathcal{T}_{E^+}}(X + B_{E^-}(\tfrac{1}{m})), \mathrm{Cl}_{\mathcal{T}_{E^+}}(A + B_{E^-}(\tfrac{1}{m}))), \quad m \in \mathbf{N}^*,$$

where $\mathrm{Cl}_{\mathcal{T}_{E^+}}(Z)$ denotes the closure of Z with respect to the $\mathcal{T}_{E^+}$-topology. We claim that $\{(X^m, A^m)\}$ is an approximating sequence for (X, A): it is enough to show that

$$\bigcap_m X^m = X, \quad \bigcap_m A^m = A.$$

If $y = x_m + z_m$ with $x_m \in X$ and $z_m \in B_{E^-}(\frac{1}{m})$, x_m converges strongly to y. Since X is strongly closed, y must belong to X.

Given an approximating sequence $\{(U^m, V^m)\}$ for a bounded E^+-pair (X, A), consider the inclusion maps

$$i^m : (X, A) \hookrightarrow (U^m, V^m), \quad j^{m,n} : (U^m, V^m) \hookrightarrow (U^n, V^n) \quad \text{if } n \leq m.$$

Since every inclusion map is an E^+-finite morphism, we can consider the induced homomorphisms

$$\begin{aligned} &H^*_{E^+}(i^m) : H^*_{E^+}(U^m, V^m) \longrightarrow H^*_{E^+}(X, A), \\ &H^*_{E^+}(j^{m,n}) : H^*_{E^+}(U^n, V^n) \longrightarrow H^*_{E^+}(U^m, V^m), \quad \text{if } n \leq m. \end{aligned}$$

By the functoriality of $H^*_{E^+}$ on the category of E^+-finite morphisms, $\{H^*_{E^+}(i^m)\}$ is a direct system of homomorphisms from the direct system of moduli

$$\{H^*_{E^+}(U^m, V^m); H^*_{E^+}(j^{m,n})\}$$

to the module $H^*_{E^+}(X, A)$. The following continuity property holds.

Proposition 2.4.1 *Let (X, A) be a bounded E^+-pair. Then the direct limit of the above system*

$$\varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(i^m) : \varinjlim_{m\in\mathbf{N}^*} \{H^*_{E^+}(U^m, V^m); H^*_{E^+}(j^{m,n})\} \longrightarrow H^*_{E^+}(X, A)$$

is an isomorphism.

PROOF. Let $\alpha \in \mathcal{V}$ and let $i^m_\alpha : (X_\alpha, A_\alpha) \hookrightarrow (U^m_\alpha, V^m_\alpha)$, $j^{m,n}_\alpha : (U^m_\alpha, V^m_\alpha) \hookrightarrow (U^n_\alpha, V^n_\alpha)$, for $n \leq m$, be the inclusion maps.

Since α is finite dimensional, the topology induced on $\pi^{-1}(\alpha)$ by $\mathcal{T}_{E^+}$ coincides with the weak topology. U^m_α, V^m_α, X_α and A_α are weakly closed and bounded and therefore they are $\mathcal{T}_{E^+}$-compact. By the continuity property of H^*_c, stated in Proposition 2.1.3, the following homomorphism is an isomorphism

$$\varinjlim_{m\in\mathbf{N}^*} H_c^{*+d(\alpha)}(i^m_\alpha) : \varinjlim_{m\in\mathbf{N}^*} \{H_c^{*+d(\alpha)}(U^m_\alpha, V^m_\alpha); H_c^{*+d(\alpha)}(j^{m,n}_\alpha)\} \longrightarrow H_c^{*+d(\alpha)}(X_\alpha, A_\alpha).$$

Since direct limits of Abelian groups commute,

$$\varinjlim_{m\in\mathbf{N}^*} H^*_E(i^m) = \varinjlim_{m\in\mathbf{N}^*} \varinjlim_{\alpha\in\mathcal{V}} H_c^{*+d(\alpha)}(i^m_\alpha) = \varinjlim_{\alpha\in\mathcal{V}} \varinjlim_{m\in\mathbf{N}^*} H_c^{*+d(\alpha)}(i^m_\alpha).$$

Therefore $\varinjlim_{m\in\mathcal{M}} H^*_{E^+}(i^m)$ is an isomorphism, being a direct limit of isomorphisms. □

Now assume that (X, A) is any E^+-pair and set

$$\mathcal{T}(X, A) = \{S \mid A \subset S \subset X,\ S \text{ is } \mathcal{T}_{E^+}\text{−closed},\ X \setminus S \text{ is bounded}\}.$$

$\mathcal{T}(X, A)$ is ordered downward by the inclusion and it is a directed set. If $i_S : (X, A) \hookrightarrow (X, S)$ is the inclusion map, $H^*_{E^+}(i_S)$ is a direct system of homomorphisms from the direct system of moduli $\{H^*_{E^+}(X, S)\}$ to $H^*_{E^+}(X, A)$, over the directed set $\mathcal{T}(X, A)$.

Proposition 2.4.2 *Assume that (X, A) is an E^+-pair. Then the direct limit of the above system*

$$\varinjlim_{S\in\mathcal{T}(X,A)} H^*_{E^+}(i_S) : \varinjlim_{S\in\mathcal{T}(X,A)} H^*_{E^+}(X, S) \longrightarrow H^*_{E^+}(X, A)$$

is an isomorphism.

PROOF. It follows immediately from Proposition 2.1.4 and from the possibility of changing the order of two direct limits. □

2.5 E^+-finite homotopies

Definition 2.5.1 *A $\mathcal{T}_{E^+}$-continuous map $\Psi : (X \times [0, 1], A \times [0, 1]) \mapsto (Y, B)$ is an E^+-finite homotopy if:*

1. *it has the form*
$$\Psi(x, t) = T_t x + R(x, t),$$
where T_t is a continuous path of linear automorphisms of E such that $T_t E^+ = E^+$, $T_t E^- = E^-$, R maps bounded sets into bounded sets and $\pi(R(X \times [0, 1])) \subset \alpha_0$, a finite dimensional subspace of E/E^+;

2. *$\Psi^{-1}(U)$ is bounded for every bounded U.*

Two E^+-finite morphisms Φ^0 and Φ^1 from (X, A) to (Y, B) are called E^+-finitely homotopic if there exists an E^+-finite homotopy $\Psi : (X \times [0,1], A \times [0,1]) \mapsto (Y, B)$ such that $\Psi(\cdot, 0) = \Phi^0$ and $\Psi(\cdot, 1) = \Phi^1$.

We would like to prove that, if the E^+-finite morphisms Φ^0 and Φ^1 are E^+-finitely homotopic, then $H^*_{E^+}(\Phi^0) = H^*_{E^+}(\Phi^1)$.

Let Φ^0 and Φ^1 be two E^+-finite morphisms of the form

$$\Phi^i(x) = T_i x + R_i(x), \quad i = 0, 1$$

where $\pi(R_i(X)) \subset \alpha_i$. Set $\beta_i = \tilde{T_i}^{-1}\alpha_i$ and set $\overline{\beta} = \beta_0 + \beta_1$. Then Φ^i is an E^+-finite morphism with respect to $\tilde{T}_i\overline{\beta}$. So for every $\alpha \in \mathcal{V}_{\overline{\beta}}$, Φ^i maps (X_α, A_α) into $(Y_{\tilde{T}_i\alpha}, B_{\tilde{T}_i\alpha})$.

Lemma 2.5.1 *Assume that for each $\alpha \in \mathcal{V}_{\overline{\beta}}$ there exists $\gamma \in \mathcal{V}$, $\alpha \subset \gamma$, $\tilde{T}_0\alpha \subset \tilde{T}_1\gamma$, such that the following diagram commutes*

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(Y_{\tilde{T}_0\alpha}, B_{\tilde{T}_0\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(\Phi^0_\alpha)} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\tilde{T}_0\alpha\tilde{T}_1\gamma}(Y,B)}\downarrow & & \downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\ H_c^{q+d(\gamma)}(Y_{\tilde{T}_1\gamma}, B_{\tilde{T}_1\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(\Phi^1_\gamma)} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma). \end{array} \qquad (2.15)$$

Then $H^q_{E^+}(\Phi^0) = H^q_{E^+}(\Phi^1)$.

Proof. By the definition of direct limit, $H^q_{E^+}(X, A)$ is obtained by taking the quotient of the direct sum of all $H_c^{q+d(\alpha)}(X_\alpha, A_\alpha)$, $\alpha \in \mathcal{V}$, with respect to the following equivalence relation: $\eta \in H_c^{q+d(\alpha)}(X_\alpha, A_\alpha)$ is equivalent to $\zeta \in H_c^{q+d(\beta)}(X_\beta, A_\beta)$ if there exists $\gamma \in \mathcal{V}$, $\alpha \subset \gamma$, $\beta \subset \gamma$, such that

$$\Delta^q_{\alpha\gamma}(X, A)\eta = \Delta^q_{\beta\gamma}(X, A)\zeta.$$

Choose an element Ξ of $H^q_{E^+}(Y, B)$: Ξ is the equivalence class of a certain

$$\xi \in H_c^{q+d(\tau)}(Y_\tau, B_\tau)$$

and we may assume that $\tilde{T}_0\overline{\beta} + \tilde{T}_1\overline{\beta} \subset \tau$. Set $\alpha = \tilde{T_0}^{-1}\tau$ and $\overline{\alpha} = \tilde{T_1}^{-1}\tau$, so that both α and $\overline{\alpha}$ are in $\mathcal{V}_{\overline{\beta}}$.

Then $H^q_{E^+}(\Phi^0)\Xi$ is represented by

$$H_c^{q+d(\alpha)}(\Phi^0_\alpha)\xi \in H_c^{q+d(\alpha)}(X_\alpha, A_\alpha), \qquad (2.16)$$

while $H^q_{E^+}(\Phi^1)\Xi$ is represented by

$$H_c^{q+d(\overline{\alpha})}(\Phi^1_{\overline{\alpha}})\xi \in H_c^{q+d(\overline{\alpha})}(X_{\overline{\alpha}}, A_{\overline{\alpha}}). \qquad (2.17)$$

To prove that (2.16) and (2.17) are equivalent in $H^q_{E^+}(X, A)$, we must find $\gamma \in \mathcal{V}$, $\alpha \subset \gamma$, $\overline{\alpha} \subset \gamma$, such that

$$\Delta^q_{\alpha\gamma}(X, A) \circ H_c^{q+d(\alpha)}(\Phi^0_\alpha)\xi = \Delta^q_{\overline{\alpha}\gamma}(X, A) \circ H_c^{q+d(\overline{\alpha})}(\Phi^1_{\overline{\alpha}})\xi. \qquad (2.18)$$

Since $\{H_c^{q+d(\omega)}(\Phi^1_\omega)\}_{\omega\in\mathcal{V}_{\overline{\beta}}}$ is a direct system of homomorphisms

$$\Delta^q_{\overline{\alpha}\gamma}(X,A)\circ H_c^{q+d(\overline{\alpha})}(\Phi^1_{\overline{\alpha}}) = H_c^{q+d(\gamma)}(\Phi^1_\gamma)\circ\Delta^q_{\tilde{T}_1\overline{\alpha}\tilde{T}_1\gamma}(Y,B) = H_c^{q+d(\gamma)}(\Phi^1_\gamma)\circ\Delta^q_{\tilde{T}_0\alpha\tilde{T}_1\gamma}(Y,B).$$

If we choose γ as in the hypotheses, the above relation and the commutativity of diagram (2.15) imply (2.18). □

Now we can prove the invariance of the E^+-cohomology with respect to E^+-finite homotopies in a special case.

Lemma 2.5.2 *Let (X,A) and (Y,B) be two bounded E^+-pairs. Assume that*

$$\Phi(x,t)=\Phi^t(x)=T_tx+R(x,t)\ :(X,A)\mapsto(Y,B)$$

is an E^+-finite homotopy between Φ^0 and Φ^1. Let $a>0$ be such that

$$\|T_ix\|\geq a\|x\|,\quad \forall x\in X,\ i=0,1$$

and let $M=\sup\|x\| x\in X$. Assume that $m>0$ is an integer and

$$\|T_t-T_0\|<\min\{\frac{\sqrt{2}}{2a},\frac{1}{2mM}\}. \tag{2.19}$$

Let

$$(Y^m,B^m)=\left(\mathrm{Cl}_{\mathcal{T}_{E^+}}\left(Y+B_{E^-}\left(\tfrac{1}{m}\right)\right)\right),\mathrm{Cl}_{\mathcal{T}_{E^+}}\left(B+B_{E^-}\left(\tfrac{1}{m}\right)\right)$$

Then, if $i_m:(Y,B)\hookrightarrow(Y^m,B^m)$ is the inclusion map, we have

$$H^*_{E^+}(i_m\circ\Phi^0)=H^*_{E^+}(i_m\circ\Phi^1).$$

Proof. Assume that

$$\pi\circ R(X\times[0,1])\subset\alpha_0\in\mathcal{V}.$$

Let $\alpha\in\mathcal{V}$. We may assume that

$$\tilde{T}_0^{-1}\alpha_0+\tilde{T}_1^{-1}\alpha_0\subset\alpha. \tag{2.20}$$

Set

$$\tilde{\gamma}=\tilde{T}_0\alpha+\tilde{T}_1\alpha,\quad \gamma=\tilde{T}_0^{-1}\tilde{\gamma}.$$

Then $\alpha\subset\gamma$ and $\tilde{T}_1\alpha\subset\tilde{T}_0\gamma$. Moreover, by (2.20), $\alpha_0\subset\tilde{\gamma}$.

Let P be the orthogonal projection onto $\pi^{-1}(\tilde{\gamma})$. We introduce the family of maps

$$\Psi^t(x)=P\circ T_tx+R(x,t).$$

We claim that Ψ^t maps (X_γ,A_γ) into (Y^m,B^m). In fact

$$\Psi^t(x)-\Phi^t(x)=(P-I)\circ T_tx\in E^-. \tag{2.21}$$

Moreover, if $x \in X_\gamma$, $P \circ T_0 x = T_0 x$ and

$$\|\Psi^t(x) - \Phi^t(x)\| \le \|P \circ T_t - P \circ T_0\| + \|T_0 x - T_t x\| \le 2\|T_t - T_0\|\|x\| < \frac{1}{m} \tag{2.22}$$

by assumption (2.19). (2.21) and (2.22) imply our claim.

Since $P \circ T_1 E^+ = E^+$, $P \circ T_1$ induces a linear operator $\tilde{T}$ from E/E^+ to E/E^+. We claim that $\tilde{T}$ is one-to-one on γ. In fact, assume that

$$v \in \gamma, \quad \tilde{T}v = 0.$$

Then there exists $x \in \pi^{-1}(\gamma) \cap E^-$ such that $P \circ T_1 x = 0$. Therefore $T_1 x \in \pi^{-1}(\tilde{\gamma})^\perp$ and so

$$< T_1 x, T_0 x >= 0.$$

Thus

$$\|T_1 x - T_0 x\|^2 = \|T_1 x\|^2 + \|T_0 x\|^2 \ge 2a^2\|x\|^2,$$

which contradicts assumption (2.19), unless $x = 0$.

Now choose linear spaces β_i such that

$$\alpha = \beta_0 \subset \beta_1 \subset \cdots \subset \beta_k = \gamma,$$

where $d(\beta_{i+1}) = d(\beta_i) + 1$, for $i = 0, \ldots, k-1$. Since $\tilde{T}$ is one-to-one on γ, $d(\tilde{T}\beta_i) = d(\beta_i)$.

Assume that $x \in X_{\beta_i}$. Then

$$\Psi^1(x) \in \pi^{-1}(\tilde{T}\beta_i) + \pi^{-1}(\alpha_0) = \pi^{-1}(\tilde{T}\beta_i),$$

because, by (2.20), $\alpha_0 \subset \tilde{T}_1\alpha = \tilde{T}\alpha \subset \tilde{T}\beta_i$. Therefore Ψ^1 maps $(X_{\beta_i}, A_{\beta_i})$ into $(Y^m_{\beta_i}, B^m_{\beta_i})$. So we have proper maps

$$\Psi^1_{\beta_i} : (X_{\beta_i}, A_{\beta_i}) \mapsto (Y^m_{\beta_i}, B^m_{\beta_i}).$$

Since $\tilde{T}$ maps β_i onto $\tilde{T}\beta_i$, it maps $\beta_{i+1}{}^{+}_{\beta_i}$ onto either $\tilde{T}\beta_{i+1}{}^{+}_{\tilde{T}\beta_i}$ or $\tilde{T}\beta_{i+1}{}^{-}_{\tilde{T}\beta_i}$. Therefore $\Psi^1_{\beta_i}$ maps

$$(X_{\beta_{i+1}{}^{+}_{\beta_i}}, A_{\beta_{i+1}{}^{+}_{\beta_i}})$$

into either

$$(Y^m_{\tilde{T}\beta_{i+1}{}^{+}_{\tilde{T}\beta_i}}, B^m_{\tilde{T}\beta_{i+1}{}^{+}_{\tilde{T}\beta_i}}) \quad \text{or} \quad (Y^m_{\tilde{T}\beta_{i+1}{}^{-}_{\tilde{T}\beta_i}}, B^m_{\tilde{T}\beta_{i+1}{}^{-}_{\tilde{T}\beta_i}}).$$

Then the functoriality of the Mayer-Vietoris homomorphism and the fact that we are using $\mathbf{Z}_2$ coefficients implies that the following diagram commutes

$$\begin{array}{ccc} H_c^{q+d(\beta_i)}(Y^m_{\tilde{T}\beta_i}, B^m_{\tilde{T}\beta_i}) & \xrightarrow{H_c^{q+d(\beta_i)}(\Psi^1_{\beta_i})} & H_c^{q+d(\beta_i)}(X_{\beta_i}, A_{\beta_i}) \\ \Delta^q_{\tilde{T}\beta_i\tilde{T}\beta_{i+1}}(Y^m,B^m)\downarrow & & \downarrow\Delta^q_{\beta_i\beta_{i+1}}(X,A) \\ H_c^{q+d(\beta_{i+1})}(Y^m_{\tilde{T}\beta_{i+1}}, B^m_{\tilde{T}\beta_{i+1}}) & \xrightarrow{H_c^{q+d(\beta_{i+1})}(\Psi^1_{\beta_{i+1}})} & H_c^{q+d(\beta_{i+1})}(X_{\beta_{i+1}}, A_{\beta_{i+1}}). \end{array} \tag{2.23}$$

Using the fact that $P \circ T_1|_{\pi^{-1}(\alpha)} = T^1|_{\pi^{-1}\alpha}$, from the commutativity of diagrams (2.23), we get the commutativity of diagram

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(Y^m_{\tilde{T}_1\alpha}, B^m_{\tilde{T}_1\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(i_m\circ\Phi^1_\alpha)} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\tilde{T}_1\alpha\tilde{T}\gamma}(Y^m,B^m)}\downarrow & & \downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\ H_c^{q+d(\gamma)}(Y^m_{\tilde{T}\gamma}, B^m_{\tilde{T}\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(\Psi^1_\gamma)} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma). \end{array}$$

Now, Ψ^t_γ is a proper homotopy from (X_γ, A_γ) to $(Y^m_{\tilde{T}\gamma}, B^m_{\tilde{T}\gamma})$, which connects Ψ^1_γ to $\Psi^0|_\gamma = i_m \circ \Phi^0_\gamma$. Using the homotopy property of H_c^*, we see that we have proved the following fact: for every $\alpha \in \mathcal{V}$ satisfying (2.20), there exists $\gamma \in \mathcal{V}$ such that $\alpha \subset \gamma$, $\tilde{T}_1\alpha \subset \tilde{T}_0\gamma$ and the following diagram commutes

$$\begin{array}{ccc} H_c^{q+d(\alpha)}(Y^m_{\tilde{T}_1\alpha}, B^m_{\tilde{T}_1\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}(i_m\circ\Phi^1_\alpha)} & H_c^{q+d(\alpha)}(X_\alpha, A_\alpha) \\ {\scriptstyle \Delta^q_{\tilde{T}_1\alpha\tilde{T}_0\gamma}(Y^m,B^m)}\downarrow & & \downarrow{\scriptstyle \Delta^q_{\alpha\gamma}(X,A)} \\ H_c^{q+d(\gamma)}(Y^m_{\tilde{T}_0\gamma}, B^m_{\tilde{T}_0\gamma}) & \xrightarrow{H_c^{q+d(\gamma)}(i_m\circ\Phi^0_\gamma)} & H_c^{q+d(\gamma)}(X_\gamma, A_\gamma). \end{array}$$

By Lemma 2.5.1 this fact implies the thesis. □

Now it is easy to conclude.

Proposition 2.5.3 *Assume that (X, A) and (Y, B) are bounded E^+-pairs. If the E^+-finite morphisms Φ^0 and Φ^1 from (X, A) to (Y, B) are E^+-finitely homotopic, then $H^*_{E^+}(\Phi^0) = H^*_{E^+}(\Phi^1)$.*

PROOF. Let

$$\Phi^t(x) = \Phi(x, t) = T_t x + R(x, t)$$

be the E^+-finite homotopy between Φ^0 and Φ^1.

Let $M = \sup\{\|x\| \mid x \in X\}$ and let $a > 0$ be such that

$$\|T_t x\| \geq a\|x\|, \quad \forall t \in [0, 1], \ \forall x \in E.$$

Let $m > 0$ be an integer. By the compactness of the unit interval we can find numbers $0 = s_0 < s_1 < \cdots < s_r = 1$ such that

$$\|T_t - T_{s_i}\| < \min\{\frac{\sqrt{2}}{2a}, \frac{1}{2mM}\}, \quad \forall t \in [s_i, s_{i+1}], \ i = 0, \ldots, r-1.$$

Applying Lemma 2.5.2 to every E^+-finite homotopy $\Phi|_{(X,A)\times[s_i,s_{i+1}]}$, we get

$$H^*_{E_+}(i_m \circ \Phi^0) = H^*_{E^+}(i_m \circ \Phi^1).$$

By the continuity property stated in Proposition 2.4.1, we conclude that

$$H^*_{E^+}(\Phi^0) = H^*_{E^+}(\Phi^1).$$

□

2.6 Approximating systems

A pair (X, A) of subsets of E will be called cobounding if $X \setminus A$ is bounded.

Definition 2.6.1 *Let (X, A), (Y, B) be two cobounding E^+-pairs. Let $\Phi : (X, A) \mapsto (Y, B)$ be an E^+-morphism. An approximating system for Φ is given by:*

(a) *two bounded $\mathcal{T}_{E^+}$-closed sets U, V such that:*

- **(a1)** $X \setminus A \subset U \subset X$, $Y \setminus B \subset V \subset Y$;
- **(a2)** $\Phi(U) \subset V$;

(b) *an approximating sequence $\{(Y^m, B^m)\}$ for $(V, B \cap V)$; denote by $i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$, $j^{m,n} : (Y^m, B^m) \hookrightarrow (Y^n, B^n)$, for $n \leq m$, the inclusion maps;*

(c) *a sequence of E^+-finite morphisms*

$$\Phi^m : (U, A \cap U) \mapsto (Y^m, B^m), \quad m \in \mathbf{N},$$

such that, if $\tilde{\Phi} = \Phi|_{(U,A\cap U)} : (U, A \cap U) \mapsto (V, B \cap V)$, there holds:

- **(c0)** $P_{E^+} \circ \Phi^m = P_{E^+} \circ \tilde{\Phi}$;
- **(c1)** $\pi \circ (\Phi^m - \tilde{\Phi})$ *has pre-compact image;*
- **(c2)** Φ^m *is E^+-homotopic to $i^m \circ \tilde{\Phi}$;*
- **(c3)** Φ^n *is E^+-finitely homotopic to $j^{m,n} \circ \Phi^m$.*

In order to prove the existence of approximating systems, we need a lemma.

Lemma 2.6.1 *Assume that Φ_0 and Φ_1 are E^+-morphisms [E^+-finite morphisms] from the bounded E^+-pair (X, A) to the E^+-pair (Y, B), such that:*

1. $P_{E^+} \circ \Phi_0 = P_{E^+} \circ \Phi_1$;
2. $\pi \circ (\Phi_1 - \Phi_0)$ *has pre-compact image;*
3. *there holds*

$$\|\Phi_0(x) - \Phi_1(x)\| \leq \text{dist}\,(\Phi_0(x), (E \setminus Y) \cap (\Phi_0(x) + E^-)) \quad \forall x \in X, \tag{2.24}$$

$$\|\Phi_0(x) - \Phi_1(x)\| \leq \text{dist}\,(\Phi_(x), (E \setminus B) \cap (\Phi_0(x) + E^-)) \quad \forall x \in A. \tag{2.25}$$

Then Φ_0 and Φ_1 are E^+-homotopic [E^+-finitely homotopic] by means of an E^+-homotopy [E^+-finite homotopy] Ψ such that

$$P_{E^+} \circ \Psi(x, t) = P_{E^+} \circ \Phi_0(x) = P_{E^+} \circ \Phi_1(x) \quad \forall x \in X,\ \forall t \in [0, 1],$$

and such that $\pi \circ (\Psi(\cdot, t) - \Phi_i)$ has pre-compact image, for $t \in [0, 1]$, $i = 0, 1$.

PROOF. It is easy to show that Φ_0 and Φ_1 can be written in the form

$$\Phi_0(x) = T^+x + T_0^-x + K^+(x) + K_0^-(x),$$
$$\Phi_1(x) = T^+x + T_1^-x + K^+(x) + K_1^-(x),$$

where T^+ is a linear automorphism of E^+, T_0^- and T_1^- are linear automorphisms of E^-, $K^+ : X \mapsto E^+$ is continuous and compact from the $\mathcal{T}_{E^+}$-topology to the weak one, $K_0^-, K_1^- : X \mapsto E^-$ are continuous and compact [with finite dimensional image] from the $\mathcal{T}_{E^+}$-topology to the strong one. Moreover, by (2), $T_1^- - T_0^-$ is a compact operator.

Therefore

$$T_t^- = tT_1^- + (1-t)T_0^-$$

is a continuous path of Fredholm endomorphisms of E^-, with index zero, which connects T_0^- to T_1^-. Set

$$\Psi(x,t) = T^+x + T_t^-x + K^+(x) + tK_1^-(x) + (1-t)K_0^-(x).$$

Since T_t^- is a finite dimensional perturbation of a linear automorphism of E^-, Ψ is an E^+-homotopy [E^+-finite homotopy] between Φ_0 and Φ_1. By construction $P_{E^+} \circ \Psi(\cdot, t) = P_{E^+} \circ \Phi_0 = P_{E^+} \circ \Phi_1$ for every $t \in [0,1]$.

We claim that Ψ maps $(X \times [0,1], A \times [0,1])$ into (Y, B). In fact, if $x \in X$, by (2.24) we get

$$\|\Psi(x,t) - \Phi_0(x)\| = t\|\Phi_1(x) - \Phi_0(x)\| \le \operatorname{dist}(\Phi_0(x), (E \setminus Y) \cap (\Phi_0(x) + E^-)).$$

Since $\Psi(x,t) \in \Phi_0(x) + E^-$, the above inequality implies that $\Psi(x,t) \in Y$. An analogous argument making use of (2.25) proves that $\Psi(x,t) \in B$ when $x \in A$. □

Proposition 2.6.2 *Let (X, A), (Y, B) be two cobounding $\mathcal{T}_{E^+}$-closed pairs. Let*

$$\Psi : (X \times [0,1], A \times [0,1]) \longrightarrow (Y, B)$$

be an E^+-homotopy. Let U, V be two bounded $\mathcal{T}_{E^+}$-closed sets such that:

1. *$X \setminus A \subset U \subset X$, $Y \setminus B \subset V \subset Y$;*
2. *$\Psi(U \times [0,1]) \subset V$.*

Then for each $m \in \mathbf{N}^*$ there exists an E^+-finite homotopy

$$\Psi^m : (U \times [0,1], (A \cap U) \times [0,1]) \longrightarrow (Y^m, B^m)$$
$$Y^m = \mathrm{Cl}_{\mathcal{T}_{E^+}}(V + B_{E^\perp}(\tfrac{1}{m})), \quad B^m = \mathrm{Cl}_{\mathcal{T}_{E^+}}((B \cap V) + B_{E^\perp}(\tfrac{1}{m})),$$

such that $\{\Psi^m(\cdot, t)\}$ is an approximating system for $\Psi(\cdot, t)$, for every $t \in [0,1]$. Moreover

$$\|\Psi^m(x,t) - \Psi(x,t)\| < \tfrac{1}{3m} \quad \forall (x,t) \in U \times [0,1]. \tag{2.26}$$

PROOF. Assume that
$$\Psi(x,t) = T_t x + K(x,t).$$
Set $K^+ = P_{E^+} \circ K$ and $K^- = P_{E^-} \circ K$, so that
$$K(x,t) = K^+(x,t) + K^-(x,t).$$

We are going to show that there exists a sequence of continuous maps $K_m^- : U \times [0,1] \mapsto W_m$, where W_m is a finite dimensional linear subspace of E^-, such that
$$\|K_m^-(x,t) - K^-(x,t)\| < \frac{1}{3m} \qquad \forall (x,t) \in U \times [0,1]. \tag{2.27}$$

Since $K^-(U \times [0,1])$ is pre-compact in the complete metric space E^-, it is totally bounded and there exist $x_1, \dots, x_N$ in E^- such that
$$K^-(U \times [0,1]) \subset \bigcup_{i=1}^{N} B_{E^-}(x_i, \tfrac{1}{3m}). \tag{2.28}$$

Let W_m be the linear space spanned by $\{x_1, \dots, x_N\}$. Let P_m be the orthogonal projection onto W_m. Set
$$K_m^-(x,t) = P_m \circ K^-(x,t), \quad K^m(x,t) = K^+(x,t) + K_m^-(x,t).$$

Let $(x,t) \in U \times [0,1]$. By (2.28) there exists $j \le N$ such that $K^-(x,t) \in B_{E^-}(x_j, \frac{1}{3m})$. Therefore
$$\|P_m \circ K^-(x,t) - K^-(x,t)\| \le \|K^-(x,t) - x_j\| < \frac{1}{3m}$$
so K_m^- satisfies (2.27). Now set
$$\Psi^m(x,t) = T_t x + K^m(x,t).$$

By (2.27), Ψ^m maps $(U \times [0,1], (A \cap U) \times [0,1])$ into (Y^m, B^m). Moreover $\pi \circ K^m(U \times [0,1]) \subset \pi(W_m)$ and Ψ^m is an E^+-finite morphism. By (2.27)
$$\begin{aligned}\|\Psi^m(x,t) - \Psi(x,t)\| &\le \tfrac{1}{3m} \le \operatorname{dist}(\Psi(x,t), (E \setminus Y^m) \cap (\Psi(x,t) + E^-)),\\ &\qquad \forall (x,t) \in U \times [0,1]\\ \|\Psi^m(x,t) - \Psi(x,t)\| &\le \tfrac{1}{3m} \le \operatorname{dist}(\Psi(x,t), (E \setminus B^m) \cap (\Psi(x,t) + E^-)),\\ &\qquad \forall (x,t) \in (A \cap U) \times [0,1].\end{aligned}$$

Set
$$\tilde{\Psi} = \Psi|_{(U \times [0,1], A \cap U \times [0,1])} : (U \times [0,1], (A \cap U) \times [0,1]) \longrightarrow (V, B \cap V).$$
By Lemma 2.6.1, $\Psi^m(\cdot,t)$ and $i^m \circ \tilde{\Psi}(\cdot,t)$ are E^+-homotopic.

Again, by (2.27), if $n < m$
$$\begin{aligned}\|\Psi^m(x,t) - \Psi^n(x,t)\| &< \tfrac{1}{3m} + \tfrac{1}{3n} < \tfrac{1}{n} - \tfrac{1}{3m} \le\\ &\le \operatorname{dist}(\Psi^m(x,t), (E \setminus Y^n) \cap (\Psi^m(x,t) + E^-)),\end{aligned}$$

for all $(x,t) \in U \times [0,1]$ and

$$\|\Psi^m(x,t) - \Psi^n(x,t)\| < \tfrac{1}{3m} + \tfrac{1}{3n} < \tfrac{1}{n} - \tfrac{1}{3m} \leq \\ \leq \operatorname{dist}(\Psi^m(x,t), (E \setminus B^n) \cap (\Psi^m(x,t) + E^-)),$$

for all $(x,t) \in (A \cap U) \times [0,1]$. By Lemma 2.6.1, Ψ^n and $j^{m,n} \circ \Psi^m$ are E^+-finitely homotopic. Therefore $\{\Psi^m(\cdot,t)\}$ is an approximating system for $\Psi(\cdot,t)$. □

Corollary 2.6.3 *Let (X,A), (Y,B) be cobounding E^+-pairs. Let $\Phi : (X,A) \mapsto (Y,B)$ be an E^+-morphism. Then there exists an approximating system for Φ.*

PROOF. Let $U = \mathrm{Cl}_{\mathcal{T}_{E^+}}(X \setminus A)$. Since Φ maps bounded $\mathcal{T}_{E^+}$-closed sets into bounded $\mathcal{T}_{E^+}$-closed sets, $V = \Phi(U) \cup \mathrm{Cl}_{\mathcal{T}_{E^+}}(Y \setminus B)$ is bounded and $\mathcal{T}_{E^+}$-closed. Apply Proposition 2.6.2 with these U, V and with $\Psi(\cdot,t) = \Phi(\cdot)$ for each $t \in [0,1]$. □

2.7 E^+-morphisms

Let (X,A), (Y,B) be two cobounding E^+-pairs. Let $\Phi : (X,A) \mapsto (Y,B)$ be an E^+-morphism.

Consider an approximating system $\{\Phi^m : (U, A \cap U) \mapsto (Y^m, B^m)\}$ for Φ, (notations as in Definition 2.6.1). Since Φ^m is an E^+-finite morphism, we can consider the induced homomorphism

$$H^*_{E^+}(\Phi^m) : H^*_{E^+}(Y^m, B^m) \to H^*_{E^+}(U, A \cap U).$$

By property (c3) of Definition 2.6.1, $\{H^*_{E^+}(\Phi^m)\}$ is a direct system of homomorphisms from the direct system of $\mathbf{Z}_2$-modules

$$\{H^*_{E^+}(Y^m, B^m); H^*_{E^+}(j^{m,n})\}$$

to the $\mathbf{Z}_2$-module $H^*_{E^+}(U, A \cap U)$. Consider the direct limit of this system

$$\varinjlim_{m \in \mathbf{N}} H^*_{E^+}(\Phi^m) : \varinjlim_{m \in \mathbf{N}} \{H^*_{E^+}(Y^m, B^m); H^*_{E^+}(j^{m,n})\} \longrightarrow H^*_{E^+}(U, A \cap U).$$

Since $(V, B \cap V)$ is a bounded E^+-pair, the domain of this homomorphism is isomorphic to $H^*_{E^+}(V, B \cap V)$, via the isomorphism

$$\varinjlim_{m \in \mathbf{N}} H^*_{E^+}(j_m) : \varinjlim_{m \in \mathbf{N}} H^*_{E^+}(Y^m, B^m) \longrightarrow H^*_{E^+}(V, B \cap V),$$

where $j_m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ are the inclusion maps. Since X and Y are E^+-locally compact, the strong excision property stated in Proposition 2.3.3 holds and the inclusion maps induce isomorphisms

$$H^*_{E^+}(X,A) \cong H^*_{E^+}(U, A \cap U), \quad H^*_{E^+}(Y,B) \cong H^*_{E^+}(V, B \cap V).$$

Therefore we can give the following definition.

Definition 2.7.1 *Assume that Φ is an E^+-morphism between two cobounding E^+-pairs (X,A) and (Y,B). Φ induces a homomorphism*

$$H^*_{E^+}(\Phi) : H^*_{E^+}(Y,B) \longrightarrow H^*_{E^+}(X,A)$$

defined as

$$H^*_{E^+}(\Phi) = H^*_{E^+}(i)^{-1} \circ \varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(\Phi^m) \circ [\varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(j_m)]^{-1} \circ H^*_E(j),$$

where $i : (U, A\cap U) \hookrightarrow (X,A)$ and $j : (V, B\cap V) \hookrightarrow (Y,B)$ are the inclusion maps.

In order to prove that this is a good definition, we must check that it does not depend on the choice of the approximating system for Φ.

Proposition 2.7.1 *The definition of $H^q_{E^+}(\Phi)$ does not depend on the choice of the approximating system for Φ.*

Proof. Consider two approximating systems for Φ.

1. Here we assume that the approximating system share the same U and V. Moreover we assume that they share the same approximating sequence $\{(Y^m, B^m)\}$ of $(V, B\cap V)$. With these assumptions, let $\{\Phi^m\}$ and $\{\Phi'^m\}$ be two approximating systems for Φ

$$\Phi^m : (U, A\cap U) \longrightarrow (Y^m, B^m),$$
$$\Phi'^m : (U, A\cap U) \longrightarrow (Y^m, B^m).$$

By property (c2) of Definition 2.6.1, we can find an E^+-homotopy $\Psi^m : (U\times[0,1], (A\cap U)\times[0,1]) \mapsto (Y^m, B^m)$ between Φ^m and Φ'^m. For $r\in\mathbf{N}^*$ let $Z^m_r = \mathrm{Cl}_{\mathcal{T}_{E^+}}(Y^m + B_{E^-}(\frac{1}{r}))$, $W^m_r = \mathrm{Cl}_{\mathcal{T}_{E^+}}(B^m + B_{E^-}(\frac{1}{r}))$. Apply Proposition 2.6.2 and find an E^+-finite homotopy

$$\Psi^m_r : (U\times[0,1], (A\cap U)\times[0,1]) \longrightarrow (Z^m_r, W^m_r)$$

such that $\{\Psi^m_r(\cdot,t)\}_{r\in\mathbf{N}^*}$ is an approximating system for $\Psi^m(\cdot,t)$, for each $t\in[0,1]$, and such that

$$\|\Psi^m_r(x,t) - \Psi^m(x,t)\| < \frac{1}{3r} \quad \forall (x,t)\in U\times[0,1]. \tag{2.29}$$

Set $\tilde{Z}^m_r = \mathrm{Cl}_{\mathcal{T}_{E^+}}(Z^m_r + B_{E^-}(\frac{1}{r}))$, $\tilde{W}^m_r = \mathrm{Cl}_{\mathcal{T}_{E^+}}(W^m_r + B_{E^-}(\frac{1}{r}))$ and let

$$h^m_r : (Z^m_r, W^m_r) \hookrightarrow (\tilde{Z}^m_r, \tilde{W}^m_r), \quad f^m_r : (Y^m, B^m) \hookrightarrow (\tilde{Z}^m_r, \tilde{W}^m_r)$$

be the inclusion maps.

By (2.29) we can apply Lemma 2.6.1: the E^+-finite morphisms $f^m_r \circ \Phi^m$ and $h^m_r \circ \Psi^m_r(\cdot,0)$ from $(U, A\cap U)$ to $(\tilde{Z}^m_r, \tilde{W}^m_r)$ are E^+-finitely homotopic. For the same reason $f^m_r \circ \Phi'^m$ and $h^m_r \circ \Psi^m_r(\cdot,1)$ are E^+-finitely homotopic. Therefore

$$H^*_{E^+}(\Phi^m) \circ H^*_{E^+}(f^m_r) = H^*_{E^+}(f^m_r \circ \Phi^m) = H^*_{E^+}(h^m_r \circ \Psi^m_r(\cdot,0)) =$$
$$= H^*_{E^+}(h^m_r \circ \Psi^m_r(\cdot,1)) = H^*_{E^+}(f^m_r \circ \Phi'^m) = H^*_{E^+}(\Phi'^m) \circ H^*_{E^+}(f^m_r).$$

By Proposition 2.4.1, $\varinjlim_{r\in\mathbf{N}^*} H^*_{E^+}(f^m_r)$ is an isomorphism and therefore

$$H^*_{E^+}(\Phi^m) = H^*_{E^+}(\Phi'^m), \quad \forall m \in \mathbf{N}^*.$$

Thus we have showed that we can choose different approximating systems for Φ, provided the use the same U, V and the same approximating sequence for $(V, V \cap B)$.

2. Now we only assume that U and V are the same for the two approximating systems. Therefore we have two approximating sequences $\{(Y^m, B^m)\}$ and $\{(Y'^m, B'^m)\}$ for $(V, B\cap V)$. Moreover we have two sequences of E^+-finite morphisms

$$\Phi^m : (U, A \cap U) \longrightarrow (Y^m, B^m),$$
$$\Phi'^m : (U, A \cap U) \longrightarrow (Y'^m, B'^m).$$

Denote by:

$$i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m), \quad i'^m : (V, B \cap V) \hookrightarrow (Y'^m, B'^m).$$

the inclusion maps. The sequence

$$(\tilde{Y}^m, \tilde{B}^m) = (Y^m \cup Y'^m, B^m \cup B'^m), \quad m \in \mathbf{N}^*,$$

is an approximating sequence for $(V, B \cap V)$. Let

$$u_m : (Y^m, B^m) \hookrightarrow (\tilde{Y}^m, \tilde{B}^m), \quad u'_m : (Y'^m, B'^m) \hookrightarrow (\tilde{Y}^m, \tilde{B}^m),$$
$$z_m : (V, B \cap V) \hookrightarrow (\tilde{Y}^m, \tilde{B}^m)$$

denote the inclusion maps.

By our previous argument

$$H^*_{E^+}(\Phi^m) \circ H^*_{E^+}(u_m) = H^*_{E^+}(u_m \circ \Phi^m) = H^*_{E^+}(u'_m \circ \Phi'^m) = H^*_{E^+}(\Phi'^m) \circ H^*_{E^+}(u'_m).$$

Moreover

$$H^*_{E^+}(i^m) \circ H^*_{E^+}(u_m) = H^*_{E^+}(z_m) = H^*_{E^+}(i'^m) \circ H^*_{E^+}(u'_m).$$

By Proposition 2.4.1

$$\varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(u_m) = \varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(u'_m) = Id_{H^*_{E^+}(V, B\cap V)}.$$

Therefore

$$\varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(\Phi^m) = \varinjlim_{m\in\mathbf{N}^*} H^*_{E^+}(\Phi'^m).$$

Thus we have showed that we can choose different approximating systems for Φ, provided they use the same U, V.

3. Finally consider the general case. The smallest U and V which can be chosen are

$$U = \mathrm{Cl}_{\mathcal{T}_{E^+}}(X \setminus A), \quad V = \Phi(U) \cup \mathrm{Cl}_{\mathcal{T}_{E^+}}(Y \setminus B).$$

Let U' and V' be another choice. Apply twice Proposition 2.6.2 and find an approximating sequence $\{(Y^m, B^m)\}$ for $(V, B \cap V)$, an approximating sequence $\{(Y'^m, B'^m)\}$ for $(V', B \cap V')$ and two sequences of E^+-finite morphisms

$$\Phi^m : (U, A \cap U) \longrightarrow (Y^m, B^m),$$
$$\Phi'^m : (U', B \cap V') \longrightarrow (Y'^m, B'^m),$$

such that the following diagram commutes

$$\begin{array}{ccc} (U, A \cap U) & \xrightarrow{\Phi^m} & (Y^m, B^m) \\ i \big\downarrow & & \big\downarrow j^m \\ (U', A \cap U') & \xrightarrow{\Phi'^m} & (Y'^m, B'^m) \end{array}$$

where i and j^m are inclusion mappings.

Taking the limit over $\mathbf{N}^*$ and then using the strong excision property of Proposition 2.3.3, we conclude the proof. □

Therefore we have defined $H^*_{E^+}(\Phi)$ for every E^+-morphism Φ between cobounding E^+-pairs. $H^*_{E^+}$ is invariant under E^+-homotopies, as the next proposition shows.

Proposition 2.7.2 *Assume that (X, A) and (Y, B) are cobounding E^+-pairs. If the E^+-morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are E^+-homotopic, then $H^*_{E^+}(\Phi_0) = H^*_{E^+}(\Phi_1)$.*

PROOF. Let $\Psi : (X \times [0,1], A \times [0,1]) \mapsto (Y, B)$ an E^+-homotopy between Φ_0 and Φ_1. Choose $U = \mathrm{Cl}_{\mathcal{T}_{E^+}}(X \setminus A)$, $V = \Psi(U \times [0,1]) \cup \mathrm{Cl}_{\mathcal{T}_{E^+}}(Y \setminus B)$.

By Proposition 2.6.2 we can find a sequence of E^+-finite homotopies

$$\Psi^m : (U \times [0,1], (A \cap U) \times [0,1]) \longrightarrow (Y^m, B^m)$$

such that $\Psi^m(\cdot, 0)$ is an approximating system for Φ_0 and $\Psi^m(\cdot, 1)$ is an approximating system for Φ_1. By Proposition 2.5.3, $H^*_{E^+}(\Psi^m(\cdot,0)) = H^*_{E^+}(\Psi^m(\cdot,1))$. Taking the direct limit over $\mathbf{N}^*$ we find $H^*_{E^+}(\Phi_0) = H^*_{E^+}(\Phi_1)$. □

Proving the functoriality of $H^*_{E^+}$ is a bit more difficult.

Proposition 2.7.3 *Assume that (X, A), (Y, B) and (Z, C) are cobounding E^+-pairs. Assume that $\Phi : (X, A) \mapsto (Y, B)$ and $\Psi : (Y, B) \mapsto (Z, C)$ are E^+-morphisms. Then:*

1. *if $I : (X, A) \mapsto (X, A)$ is the identity map, $H^*_{E^+}(I)$ is the identity homomorphism on $H^*_{E^+}(X, A)$;*

2. *$\Psi \circ \Phi$ is an E^+-morphism and $H^*_{E^+}(\Psi \circ \Phi) = H^*_{E^+}(\Phi) \circ H^*_{E^+}(\Psi)$.*

PROOF. (1) is trivial. We prove (2). Set $\mathrm{Cl}_{\mathcal{T}_{E^+}}(X \setminus A)$, $V = \Phi(U) \cup \mathrm{Cl}_{\mathcal{T}_{E^+}}(Y \setminus B)$, $W = \Psi(V) \cup \mathrm{Cl}_{\mathcal{T}_{E^+}}(Z \setminus C)$. Set

$$\tilde{\Phi} = \Phi|_{(U, A\cap U)} : (U, A \cap U) \longrightarrow (V, B \cap V),$$
$$\tilde{\Psi} = \Psi|_{(V, B\cap V)} : (V, B \cap V) \longrightarrow (W, C \cap W).$$

By our excision argument, we must show that

$$H^*_{E^+}(\tilde{\Psi} \circ \tilde{\Phi}) = H^*_{E^+}(\tilde{\Phi}) \circ H^*_{E^+}(\tilde{\Psi}).$$

1. First we assume that $\tilde{\Psi}$ is an E^+-finite morphism:

$$\tilde{\Psi}(x) = Tx + \tilde{R}(x)$$

where $\tilde{R} : V \mapsto \pi^{-1}(\alpha_0)$ is continuous and has pre-compact image, T is a linear automorphism of E which preserves the splitting $E^+ \oplus E^-$.

$\mathcal{T}_{E^+}$ induces the weak topology on $\pi^{-1}(\alpha_0)$, which is therefore a locally convex topologically vector space. By Dugundji's generalization of Tietze's Theorem [Dug51], we can find a $\mathcal{T}_{E^+}$-continuous extension of $\tilde{R}$

$$\overline{R} : E \longrightarrow \pi^{-1}(\alpha_0)$$

such that $\overline{R}(E) \subset \overline{conv}(\tilde{R}(V))$, and thus $\overline{R}(E)$ is $\mathcal{T}_{E^+}$-pre-compact. Therefore $\overline{\Psi}(x) = Tx + \overline{R}(x)$ is an E^+-finite morphism on E.

Let $\Phi^m : (U, A \cap U) \mapsto (Y^m, B^m)$ be an approximating system for Φ. Set

$$Z^m = \overline{\Psi}(Y^m) \cup W, \quad C^m = \overline{\Psi}(B^m) \cup (C \cap W).$$

$\{(Z^m, C^m)\}$ is an approximating sequence for $(W, C \cap W)$: if $z \in \bigcap_m Z^m \setminus W$, $z = \overline{\Psi}(y_m)$ for some sequence $y_m \in Y^m$. Since $\overline{\Psi}$ is $\mathcal{T}_{E^+}$-proper, there exists a subsequence y_{m_k} $\mathcal{T}_{E^+}$-converging to a certain $y \in \bigcap_k Y^{m_k} = V$. Therefore $\overline{\Psi}(y) = z$ and $\bigcap_m Z^m = W$. With the same argument we find that $\bigcap_m C^m = C$.

Let $i^m : (V, B \cap V) \hookrightarrow (Y^m, B^m)$ and $j^m : (W, C \cap W) \hookrightarrow (Z^m, C^m)$ be the inclusion maps. Set

$$\Psi^m = j^m \circ \tilde{\Psi} : (V, B \cap V) \longrightarrow (Z^m, C^m),$$
$$\overline{\Psi}^m = \overline{\Psi}|_{Y^m, B^m} : (Y^m, B^m) \longrightarrow (Z^m, C^m).$$

Since $\Psi^m \circ \tilde{\Phi}$ is E^+-homotopic to $\overline{\Psi}^m \circ \Phi^m$, by Proposition 2.7.2:

$$H^*_{E^+}(\Psi^m \circ \tilde{\Phi}) = H^*_{E^+}(\overline{\Psi}^m \circ \Phi^m) = H^*_{E^+}(\Phi^m) \circ H^*_{E^+}(\overline{\Psi}^m).$$

Therefore, by Proposition 2.4.1,

$$\varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Psi^m \circ \tilde{\Phi}) = \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Phi^m) \circ \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\overline{\Psi}^m) =$$
$$= \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Phi^m) \circ [\varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(i^m)]^{-1} \circ \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Psi^m).$$

Since $\{\Psi^m \circ \tilde{\Phi}\}$ is an approximating system for $\tilde{\Psi} \circ \tilde{\Phi}$,

$$H^*_{E^+}(\tilde{\Psi} \circ \tilde{\Phi}) = \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Psi^m \circ \tilde{\Phi}) \circ [\varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(j^m)]^{-1} =$$
$$= H^*_{E^+}(\tilde{\Phi}) \circ \varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(\Psi^m) \circ [\varinjlim_{m \in \mathbf{N}^*} H^*_{E^+}(j^m)\}]^{-1} = H^*_{E^+}(\tilde{\Phi}) \circ H^*_{E^+}(\tilde{\Psi}).$$

2. We pass now to the general case. Let

$$\Psi^m : (V, B \cap V) \mapsto (Z^m, C^m), \quad \Theta^m : (U, A \cap U) \mapsto (Z^m, C^m)$$

be two approximating systems for $\tilde{\Psi}$ and $\tilde{\Psi} \circ \tilde{\Phi}$, constructed as in Proposition 2.6.2. By (2.26) we can apply Lemma 2.6.1: $\Psi^m \circ \tilde{\Phi}$ and Θ^m are E^+-homotopic. By Proposition 2.7.2, $H^*_{E^+}(\Psi^m \circ \tilde{\Phi}) = H^*_{E^+}(\Theta^m)$. By the first part of this proof, $H^*_{E^+}(\Psi^m \circ \tilde{\Phi}) = H^*_{E^+}(\tilde{\Phi}) \circ H^*_{E^+}(\Psi^m)$. Taking the direct limit over $m \in \mathbf{N}^*$ we conclude the proof. □

Now let (X, A), (Y, B) be arbitrary E^+-pairs. Let $\Phi : (X, A) \mapsto (Y, B)$ be an E^+-morphism. Recall that

$$\mathcal{T}(X, A) = \{S \mid A \subset S \subset X,\ S \text{ is } \mathcal{T}_{E^+}\text{−closed},\ X \setminus S \text{ is bounded}\}.$$

For each $T \in \mathcal{T}(Y, B)$, $\Phi^{-1}(T)$ is in $\mathcal{T}(X, A)$: in fact $A \subset \Phi^{-1}(B) \subset \Phi^{-1}(T)$ and $X \setminus \Phi^{-1}(T) = \Phi^{-1}(Y \setminus T)$ is bounded by property (2) of the definition of E^+-morphisms. Let $i_T : (X, A) \hookrightarrow (X, \Phi^{-1}(T))$ be the inclusion map. Let Φ_T be Φ seen as a map from $(X, \Phi^{-1}(T))$ to (Y, T). Then we have the homomorphisms

$$H^*_{E^+}(Y, T) \overset{H^*_{E^+}(\Phi_T)}{\longrightarrow} H^*_{E^+}(X, \Phi^{-1}(T)) \overset{H^*_{E^+}(i_T)}{\longrightarrow} H^*_{E^+}(X, A).$$

Definition 2.7.2 *The E^+-morphism Φ between the E^+-pairs (X, A) and $Y, B)$ induces the homomorphism*

$$H^*_{E^+}(\Phi) = \varinjlim_{T \in \mathcal{T}(Y,B)} H^*_{E^+}(i_T) \circ H^*_{E^+}(\Phi_T) : H^*_{E^+}(Y, B) \longrightarrow H^*_{E^+}(X, A).$$

The following Proposition proves assertion (2) of Theorem 1.3.1.

Proposition 2.7.4 *Assume that (X, A) and (Y, B) are E^+-pairs. If the E^+-morphisms Φ_0 and Φ_1 from (X, A) to (Y, B) are E^+-homotopic, $H^*_{E^+}(\Phi_0) = H^*_{E^+}(\Phi_1)$.*

PROOF. Let Φ_t, $t \in [0, 1]$, be an E^+-homotopy between Φ_0 and Φ_1. For $T \in \mathcal{T}(Y, B)$, set

$$S = \bigcap_{t \in [0,1]} \Phi_t^{-1}(T)$$

Since $A \subset \Phi_t^{-1}(B) \subset \Phi_t^{-1}(T)$ for every t, A is a subset of S. Moreover

$$X \setminus S = \bigcup_{t \in [0,1]} \Phi_t^{-1}(X \setminus T)$$

is bounded by condition (2) in the definition of E^+-homotopy (see Definition 1.3.4). Therefore $S \in \mathcal{T}(X, A)$.

Let $\tilde{\Phi}_{tT}$ be Φ_t seen as a map from (X, S) to (Y, T) and let

$$i^t_T : (X, A) \hookrightarrow (X, \Phi_t^{-1}(T)), \quad j^t_T : (X, A) \hookrightarrow (X, \Phi_t^{-1}(T)), \quad k_T : (X, A) \hookrightarrow (X, S)$$

be the inclusion maps.

By Proposition 2.7.3, $H^*_{E^+}(\tilde{\Phi}_{tT}) = H^*_{E^+}(j^t_T) \circ H^*_{E^+}(\Phi_{tT})$. By Proposition 2.3.2, $H^*_{E^+}(i^t_T) = H^*_{E^+}(k_T) \circ H^*_{E^+}(j^t_T)$. Therefore for every $t \in [0,1]$, the following diagram commutes

$$\begin{array}{ccc} H^*_{E^+}(Y,T) & \xrightarrow{H^*_{E^+}(\tilde{\Phi}_{tT})} & H^*_{E^+}(X,S) \\ {\scriptstyle H^*_{E^+}(\Phi_{tT})}\downarrow & & \downarrow{\scriptstyle H^*_{E^+}(k_T)} \\ H^*_{E^+}(X, \Phi_t^{-1}(T)) & \xrightarrow{H^*_{E^+}(i^t_T)} & H^*_{E^+}(X,A). \end{array}$$

By Proposition 2.7.2, the thesis is true for E^+-homotopies between cobounding pairs and therefore $H^*_{E^+}(\tilde{\Phi}_{0T}) = H^*_{E^+}(\tilde{\Phi}_{1T})$. Thus

$$H^*_{E^+}(i^0_T) \circ H^*_{E^+}(\Phi_{0T}) = H^*_{E^+}(i^1_T) \circ H^*_{E^+}(\Phi_{1T})$$

and the thesis follows. □

Now it is easy to prove assertion (1) of Theorem 1.3.1.

Proposition 2.7.5 *Assume that (X,A), (Y,B) and (Z,C) are E-pairs. Assume that $\Phi : (X,A) \mapsto (Y,B)$ and $\Psi : (Y,B) \mapsto (Z,C)$ are E^+-morphisms. Then:*

1. *if $I : (X,A) \mapsto (X,A)$ is the identity map, $H^*_{E^+}(I)$ is the identity homomorphism on $H^*_{E^+}(X,A)$;*
2. *$H^*_{E^+}(\Psi \circ \Phi) = H^*_{E^+}(\Phi) \circ H^*_{E^+}(\Psi)$.*

PROOF. (1) is trivial. We prove (2).

If $X \in \mathcal{T}(Z,C)$, $\Psi^{-1}(W)$ is in $\mathcal{T}(Y,B)$ and $\Phi^{-1}(\Psi^{-1}(W))$ is in $\mathcal{T}(X,A)$. By the definition of direct limit the following diagram commutes

$$\begin{array}{ccccc} H^*_{E^+}(Z,W) & \xrightarrow{H^*_{E^+}(\Psi_W)} & H^*_{E^+}(Y,\Psi^{-1}(W)) & \xrightarrow{H^*_{E^+}(\Phi_{\Psi^{-1}(W)})} & H^*_{E^+}(\Phi^{-1}(\Psi^{-1}(W))) \\ \downarrow & & \downarrow & & \downarrow \\ H^*_{E^+}(Z,C) & \xrightarrow{H^*_{E^+}(\Psi)} & H^*_{E^+}(Y,B) & \xrightarrow{H^*_{E^+}(\Phi)} & H^*_{E^+}(X,A) \end{array}$$

where the vertical arrows are induced by the inclusion maps. By Proposition 2.7.3, the functoriality holds for E^+-morphisms between cobounding pairs and therefore the composition of the upper homomorphisms in the diagram equals $H^*_{E^+}((\Psi \circ \Phi)_W)$. By the unicity property of direct limits, $H^*_{E^+}(\Psi \circ \Phi) = H^*_{E^+}(\Phi) \circ H^*_{E^+}(\Psi)$. □

Now we can prove the dimension axiom, which constitutes assertion (6) of Theorem 1.3.1.

Proposition 2.7.6 *Let W be a closed subspace of E, commensurable with E^-. Let $\overline{B_W}$ be a closed ball in W and let ∂B_W be its relative boundary in W. Then*

$$H^q_{E^+}(\overline{B_W}, \partial B_W) = \begin{cases} \mathbf{Z}_2 & \text{if } q = E^+\text{-}\dim W, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Since translations and homoteties are invertible E^+-morphisms, we can assume that $\overline{B_W}$ is the unit closed ball of W, centered in 0. It is easy to prove that there exists a linear automorphism T of E, of the form Identity + Compact, such that

$$TW = V = V^+ \oplus V^-$$

where V^+ is a finite codimensional subspace of E^+ and V^- is a finite dimensional subspace of E^-. By Proposition 1.1.1,

$$E^+\text{-}\dim W = E^+\text{-}\dim V. \tag{2.30}$$

Since both T and its inverse are positive E^+-morphisms, by the functoriality property,

$$H^q_{E^+}(\overline{B_W}, \partial B_W) = H^q_{E^+}(T(\overline{B_W}), \partial T(B_W)).$$

Now let S be an invertible operator on V such that $S(T(\overline{B_W})) = \overline{B_V}$, the unit closed ball of V, centered in 0. Also S and its inverse are E^+-morphisms. Therefore

$$H^q_{E^+}(\overline{B_W}, \partial B_W) = H^q_{E^+}(\overline{B_V}, \partial B_V).$$

The thesis follows from (2.30) and Proposition 2.2.3. □

2.8 The E^+-coboundary homomorphism

Let (X, A) be an E^+-pair. For each $\alpha \in \mathcal{V}$ we have the coboundary homomorphism

$$\delta_c^{q+d(\alpha)}(X_\alpha, A_\alpha) : H_c^{q+d(\alpha)}(A_\alpha) \longrightarrow H_c^{q+1+d(\alpha)}(X_\alpha, A_\alpha).$$

Set $\partial^q_\alpha(X, A) = (-1)^{d(\alpha)}\delta_c^{q+d(\alpha)}(X_\alpha, A_\alpha)$. By Proposition 2.1.8, $\{\partial^q_\alpha(X, A)\}$ is a direct system of homomorphisms from the direct system

$$\{H_c^{q+d(\alpha)}(A_\alpha); \Delta^q_{\alpha\beta}(A)\}$$

to the direct system

$$\{H_c^{q+1+d(\alpha)}(X_\alpha, A_\alpha); \Delta^{q+1}_{\alpha\beta}(X, A)\}$$

over the directed set $\mathcal{V}$.

Definition 2.8.1 *We define the E^+-coboundary homomorphism $\delta^q_{E^+}(X, A)$ as the direct limit of this system:*

$$\delta^q_{E^+}(X, A) = \varinjlim_{\alpha\in\mathcal{V}}\{\partial^q_\alpha(X, A)\} : H^q_{E^+}(A) \longrightarrow H^q_{E^+}(X, A).$$

If $X \subset Y \subset Z$ are $\mathcal{T}_{E^+}$-closed E^+-locally compact subsets of E, we can define a relative coboundary homomorphism in the usual way:

$$\delta^q_{E^+}(Z, Y, X) = \delta^q_{E^+}(Z, Y) \circ H^q_{E^+}(k) : H^q_{E^+}(Y, X) \longrightarrow H^{q+1}_{E^+}(Z, Y),$$

where $k : Y \hookrightarrow (Y, X)$ is the inclusion map.

Proposition 2.8.1 *Given three $\mathcal{T}_{E^+}$-closed and E^+-locally compact sets X, Y, Z, such that $X \subset Y \subset Z$, denote by $i : (Y,X) \hookrightarrow (Z,X)$ and $j : (Z,X) \hookrightarrow (Z,Y)$ the inclusion maps. Then the following sequence of homomorphisms is exact*

$$\cdots \to H^q_{E^+}(Z,X) \xrightarrow{H^q_{E^+}(i)} H^q_{E^+}(Y,X) \xrightarrow{\delta^q_{E^+}} H^{q+1}_{E^+}(Z,Y) \xrightarrow{H^{q+1}_{E^+}(j)} H^{q+1}_{E^+}(Z,X) \to \dots$$

PROOF. Assume that $\alpha \subset \beta$ are in $\mathcal{V}$ and that $d(\beta) = d(\alpha)+1$. By Proposition 2.1.8, the following diagram anti-commutes

$$\begin{array}{ccccc}
H_c^{q+d(\alpha)}(Y_\alpha, X_\alpha) & \xrightarrow{H_c^{q+d(\alpha)}(k_\alpha)} & H_c^{q+d(\alpha)}(Y_\alpha) & \xrightarrow{\delta_c^{q+d(\alpha)}(Z_\alpha,Y_\alpha)} & H_c^{q+1+d(\alpha)}(Z_\alpha, Y_\alpha) \\
\Delta^q_{\alpha\beta}(Y,X)\downarrow & & \Delta^q_{\alpha\beta}(Y)\downarrow & & \downarrow \Delta^{q+1}_{\alpha\beta}(Z,Y) \\
H_c^{q+d(\beta)}(Y_\beta, X_\beta) & \xrightarrow{H_c^{q+d(\beta)}(k_\beta)} & H_c^{q+d(\beta)}(Y_\beta) & \xrightarrow{\delta_c^{q+d(\beta)}(Z_\beta,Y_\beta)} & H_c^{q+1+d(\beta)}(Z_\beta, Y_\beta).
\end{array}$$

Set

$$\partial^q_\alpha(Z,Y,X) = (-1)^{d(\alpha)}\delta_c^{q+d(\alpha)}(Z_\alpha, Y_\alpha) \circ H_c^{q+d(\alpha)}(k_\alpha) = (-1)^{d(\alpha)}\delta_c^{q+d(\alpha)}(Z_\alpha, Y_\alpha, X_\alpha).$$

$\{\partial^q_\alpha(Z,Y,X)\}$ is a direct system of homomorphisms, and its direct limit coincides with $\delta^q_{E^+}(Z,Y,X)$, as previously defined.

By the exactness of the Alexander-Spanier cohomology with compact supports, the following sequence is exact

$$\cdots \to H_c^{q+d(\alpha)}(Z_\alpha, X_\alpha) \xrightarrow{H_c^{q+d(\alpha)}(i_\alpha)} H_c^{q+d(\alpha)}(Y_\alpha, X_\alpha) \xrightarrow{\partial^q_\alpha(Z,Y,X)} H_c^{q+1+d(\alpha)}(Z_\alpha, Y_\alpha) \to$$
$$\xrightarrow{H_c^{q+1+d(\alpha)}(j_\alpha)} H_c^{q+1+d(\alpha)}(Z_\alpha, X_\alpha) \to \dots$$

Since the direct limit takes exact sequences into exact sequences, the thesis follows. □

The above Proposition proves assertion (5) of Theorem 1.3.1.

Proposition 2.8.2 *Let X, Y and Z be $\mathcal{T}_{E^+}$-closed and E^+-locally such that $X \subset Y \subset Z$. If $\Phi : (Z,Y) \mapsto (Z',Y')$ is an E^+-finite morphism such that $\Phi(X) \subset X'$, the following diagram commutes*

$$\begin{array}{ccc}
H^q_{E^+}(Y',X') & \xrightarrow{H^q_{E^+}(\Phi|_{(Y,X)})} & H^q_{E^+}(Y,X) \\
\delta^q_{E^+}(Z',Y')\downarrow & & \downarrow \delta^q_{E^+}(Z,Y) \\
H^{q+1}_{E^+}(Z',Y') & \xrightarrow{H^{q+1}_{E^+}(\Phi)} & H^{q+1}_{E^+}(Z,Y).
\end{array}$$

PROOF. If $\Phi(x) = Tx + R(x)$, with $\pi \circ R(Z) \subset \alpha_0$, and $\alpha \in \mathcal{V}_{\tilde{T}\alpha_0}$, by the functoriality of the coboundary for H^*_c, the following diagram commutes

$$\begin{array}{ccc}
H_c^{q+d(\alpha)}(Y'_{\tilde{T}\alpha}, X'_{\tilde{T}\alpha}) & \xrightarrow{H_c^{q+d(\alpha)}((\Phi|_{(Y,X)})_\alpha)} & H_c^{q+d(\alpha)}(Y_\alpha, X_\alpha) \\
\partial^q_\alpha(Z',Y')\downarrow & & \downarrow \partial^q_\alpha(Z,Y) \\
H_c^{q+d(\alpha)+1}(Z'_{\tilde{T}\alpha}, Y'_{\tilde{T}\alpha}) & \xrightarrow{H_c^{q+d(\alpha)+1}(\Phi_\alpha)} & H_c^{q+d(\alpha)+1}(Z_\alpha, Y_\alpha).
\end{array}$$

Taking the direct limit over $\alpha \in \mathcal{V}_{\tilde{T}\alpha_0}$, the thesis follows. □

Proposition 2.8.3 *Let X, Y and Z be $\mathcal{T}_{E^+}$-closed and E^+-locally such that $X \subset Y \subset Z$. If $\Phi : (Z, Y) \mapsto (Z', Y')$ is an E^+-morphism such that $\Phi(X) \subset X'$, then the following diagram commutes*

$$\begin{array}{ccc} H^q_{E^+}(Y', X') & \xrightarrow{H^q_{E^+}(\Phi|_{(Y,X)})} & H^q_{E^+}(Y, X) \\ {\scriptstyle \delta^q_{E^+}(Z',Y')}\downarrow & & \downarrow{\scriptstyle \delta^q_{E^+}(Z,Y)} \\ H^{q+1}_{E^+}(Z', Y') & \xrightarrow{H^{q+1}_{E^+}(\Phi)} & H^{q+1}_{E^+}(Z, Y). \end{array}$$

PROOF. Set $U = \mathrm{Cl}_{\mathcal{T}_{E^+}}(Z \setminus Y)$, $V = \mathrm{Cl}_{\mathcal{T}_{E^+}}(Z' \setminus Y') \cup \Phi(U)$. We must prove the thesis for the restriction $\tilde{\Phi} : (U, Y \cap U) \mapsto (V, Y' \cap V)$.

Let $\Phi^m : (U, Y \cap U) \mapsto (Z'^m, Y'^m)$ be an approximating system for $\tilde{\Phi}$. By Proposition 2.8.2, the following diagram commutes

$$\begin{array}{ccc} H^q_{E^+}(Y'^m, X'^m) & \xrightarrow{H^q_{E^+}(\Phi^m|_{(Y\cap U, X\cap U)})} & H^q_{E^+}(Y \cap U, X \cap U) \\ {\scriptstyle \delta^q_{E^+}(Z'^m,Y'^m)}\downarrow & & \downarrow{\scriptstyle \delta^q_{E^+}(U,Y\cap U)} \\ H^{q+1}_{E^+}(Z'^m, Y'^m) & \xrightarrow{H^{q+1}_{E^+}(\Phi^m)} & H^{q+1}_{E^+}(U, Y \cap U). \end{array}$$

By Proposition 2.8.2 applied to the maps $i_m : (V, Y' \cap V) \hookrightarrow (Z'^m, Y'^m)$ and by Proposition 2.4.1, the direct limit of the left vertical arrow is $\delta^q_{E^+}(V, Y' \cap V)$. Therefore we get the commutativity of

$$\begin{array}{ccc} H^q_{E^+}(Y' \cap V) & \xrightarrow{H^q_{E^+}(\tilde{\Phi}|_{(Y\cap U, X\cap U)})} & H^q_{E^+}(Y \cap U, X \cap U) \\ {\scriptstyle \delta^q_{E^+}(V,Y'\cap V)}\downarrow & & \downarrow{\scriptstyle \delta^q_{E^+}(U,Y\cap U)} \\ H^{q+1}_{E^+}(V, B \cap V) & \xrightarrow{H^{q+1}_{E^+}(\tilde{\Phi})} & H^{q+1}_{E^+}(U, Y \cap U). \end{array}$$

The thesis now follows from the excision property of Proposition 2.3.3 and from Proposition 2.8.2 applied to the inclusions $(V, Y' \cap V) \hookrightarrow (Z', Y')$ and $(U, Y \cap U) \hookrightarrow (Z, Y)$.

□

The above Proposition proves assertion (4) of Theorem 1.3.1, which is now completely proved.

2.9 Some bibliography and further remarks

The idea of considering cohomology theories with different dimension axioms goes back to G. W. Whitehead, [Whi62].

Our construction of the E^+-cohomology is basically a generalization of a theory developed by Gęba and Granas, [GG73]. Actually, a large part of the techniques used in this chapter, were introduced in [GG73]. Their cohomology has the bounded closed subset of a Banach space as objects and the Leray-Schauder maps as morphism (Φ is a Leray-Schauder map if it has the form $\Phi(x) = x + K(x)$, where K is a compact map). The aim of their work was to establish an Alexander type duality in infinite dimensional spaces and to generalize degree theory.

Apart the fact that we are dealing with Hilbert spaces, the E^+-cohomology coincides with Gęba and Granas' theory when $E^+ = E$. Actually it generalizes it, because both the objects and the morphisms belong to larger classes. In fact we allow also unbounded sets: this has been done by using a cohomology with compact support as starting point, and then by approximating unbounded sets with bounded ones from the inside. Besides that, we allow also maps whose linear part is not the identity: as we explained in Remark 2.3.1, this fact forces to use $\mathbf{Z}_2$ as coefficient ring. Moreover, it makes the homotopy property more difficult to prove, as it was shown in section 2.5. When $E^+ = \{0\}$ the E^+-cohomology coincides with the Alexander cohomology theory on the weak topology of E.

Another generalized cohomology, suitable for the analysis of strongly indefinite functionals, was developed by Szulkin, [Szu92]. His construction is considerably simpler: one fixes a flag of finite dimensional linear spaces of E^- and takes a direct limit of a standard cohomology over these spaces. Consequently the class of morphism is very restricted: only the maps which definitively preserve the given flag. For this reason is not even clear if Szulkin's cohomology is translation-invariant.

The definition that Szulkin gives of the relative Morse index involves this flag. Since the gradient flows do not belong to the class of admissible homotopies, these flows must be suitably approximated, in order to prove the Morse relations.

Probably Szulkin's theory and the E^+-theory share the same range of applications. However we believe that our theory makes some definitions (the dimension of a subspace, the Morse index) easier and more clear. The technicalities required for its construction do not seem to be a problem: the final axioms are rich enough to allow the use of the theory without worrying about the details of the construction, as we have shown in the first chapter.

Finally, it should be possible to generalize the E^+-cohomology to manifolds modeled on a Hilbert space with an orthogonal splitting, provided there exists an atlas consisting of invertible E^+-morphisms. Such a construction would provide a Morse theory for strongly indefinite functionals on Hilbert manifolds.

Chapter 3

The Maslov index

The aim of this chapter is to associate an integer, the Maslov index, to every non-resonant linear periodic Hamiltonian system. This integer labels the connected components of the space of non-resonant linear periodic Hamiltonian systems. Roughly speaking, the Maslov index of such a system is the number of half windings made by its resolvent in the symplectic group.

We deal first with the two-dimensional case. In this case it is easy to describe the topology of the symplectic group. Moreover in dimension 2, the Maslov index has some special features, which will be used in the next chapter.

Then we deal with the general case and we compute the Maslov index of an autonomous system. Finally, the Maslov index of a periodic solution of a non-linear system is defined as the Maslov index of the linearized system.

3.1 The symplectic group

Definition 3.1.1 *A symplectic vector space* (V, ω) *is a finite dimensional real vector space* V *endowed with a non-degenerate antisymmetric bilinear form* ω.

The basic example is $\mathbf{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, equipped with the 2-form

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Equivalently, if $u \cdot v$ denotes the usual scalar product in $\mathbf{R}^{2n}$,

$$\omega_0(u, v) = Ju \cdot v, \quad \text{where} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

$(\mathbf{R}^{2n}, \omega_0)$ is called standard symplectic vector space.

Definition 3.1.2 *A linear subspace* W *of a symplectic vector space* (V, ω) *is called symplectic if the restriction of* ω *to* W *is non-degenerate.*

Proposition 3.1.1 *Let (V, ω) be a symplectic vector space. Then V is even dimensional and it has a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ such that*

$$\begin{aligned} \omega(e_i, e_j) = \omega(f_i, f_j) = 0 \quad \forall i, j, \\ \omega(e_i, f_j) = 0 \quad if\ i \neq j, \quad \omega(e_i, f_i) = 1 \quad \forall i. \end{aligned} \tag{3.1}$$

PROOF. Choose a vector e_1 in V. Since ω is non-degenerate, we can find a vector $v \in V$ such that $\omega(e_1, v) \neq 0$. We can normalize v to obtain a vector f_1 such that $\omega(e_1, f_1) = 1$. Since ω is anti-symmetric, f_1 is not a multiple of e_1; therefore e_1 and f_1 span a two-dimensional subspace V_1 of V.

Notice that V_1 is a symplectic subspace of V. If V has dimension two the proof is finished, otherwise we can consider the subspace

$$W = \{w \in V \mid \omega(w, u) = 0 \; \forall u \in V_1\}.$$

Also W is a symplectic subspace of V and it is a complement of V_1: $V = V_1 \oplus W$. Therefore we can apply the same construction to W. □

A basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ which satisfies (3.1) is called symplectic basis of (V, ω).

Definition 3.1.3 *Let (V_1, ω_1) and (V_2, ω_2) be symplectic vector spaces. A linear map $\varphi : V_1 \to V_2$ is called a symplectic isomorphism if it is bijective and it preserves the symplectic forms:*

$$\varphi^* \omega_2 = \omega_1, \; \textit{that is } \omega_2(\varphi(u), \varphi(v)) = \omega_1(u, v) \; \forall u, v \in V_1.$$

In the case $(V_1, \omega_1) = (V_2, \omega_2)$, φ is called symplectic automorphism.

Remark 3.1.1 *Proposition 3.1.1 essentially shows that every symplectic vector space of dimension $2n$ is symplectically isomorphic to $(\mathbf{R}^{2n}, \omega_0)$.*

The matrices which correspond to symplectic automorphisms of $(\mathbf{R}^{2n}, \omega_0)$ are called symplectic and they are characterized by the equation

$$A^T J A = J \tag{3.2}$$

where A^T is the transpose of A.

Let $\mathcal{E} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a symplectic basis of (V, ω). A linear automorphism φ of V is symplectic if and only if the matrix associated to φ with respect to the basis $\mathcal{E}$ is symplectic.

The set of symplectic automorphisms of (V, ω) forms a group, denoted by $Sp(V, \omega)$. The set of the symplectic matrices is a group, denoted by $Sp(n)$. Remark 3.1.1 shows that $Sp(V, \omega)$ is isomorphic to $Sp(n)$.

Since the symplectic automorphisms preserve ω, they also preserve the $2n$-form $\sigma = \omega \wedge \cdots \wedge \omega$, n times the wedge product of ω by itself. In the case of the standard symplectic space, σ coincides with the standard volume form of $\mathbf{R}^{2n}$, possibly with the opposite sign, and therefore every symplectic matrix has determinant 1.

In order to study the geometry of $Sp(n)$, notice that

$$Sp(n) = \{A \in L(2n) \mid \Phi(A) = J\}$$

where $L(2n)$ is the vector space of all real matrices of order $2n$ and

$$\Phi(A) = A^T JA.$$

Φ is a smooth map from $L(2n)$ to $Skew(2n)$, the vector space of all skew-symmetric matrices of order $2n$. The differential of Φ in $A \in L(2n)$ is

$$D\Phi(A)H = H^T JA + A^T JH.$$

If A is invertible, then $D\Phi(A)$ is onto. In fact, if $B \in Skew(2n)$,

$$D\Phi(A)\left[\frac{1}{2}J(A^{-1})^T B^T\right] = \frac{1}{2}B - \frac{1}{2}B^T = B.$$

In particular $D\Phi(A)$ is onto whenever $A \in Sp(n)$. This proves that $Sp(n)$ is a smooth submanifold of $L(2n)$. Being also a group, $Sp(n)$ is Lie subgroup of the Lie group $GL(2n)$.

The tangent space to $Sp(n)$ in the identity matrix I is the vector space

$$sp(n) = \operatorname{Ker} D\Phi(I) = \left\{H \in L(2n) \mid H^T J + JH = 0\right\}.$$

The matrices in $sp(n)$ are called infinitesimally symplectic. From the theory of Lie subgroups of $GL(2n)$ we deduce that

$$e^H = \sum_{j=0}^{\infty} \frac{1}{j!} H^j$$

belongs to $Sp(n)$ if $H \in sp(n)$. Moreover every symplectic matrix is the exponential of an infinitesimally symplectic one. The first assertion can also be proved directly as follows.

The smooth path $\gamma(t) = e^{tH}$ is the solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}\gamma(t) = H\gamma(t) \\ \gamma(0) = I \end{cases}.$$

If $H \in sp(n)$ then

$$\frac{d}{dt}[\gamma(t)^T J\gamma(t)] = \gamma(t)^T H^T J\gamma(t) + \gamma(t)^T JH\gamma(t) = 0.$$

Since $\gamma(0)$ is symplectic, $\gamma(t)$ must be symplectic for every $t \in \mathbf{R}$.

3.2 The topology of $Sp(1)$

In this section the symplectic group of $\mathbf{R}^2$ is studied in detail. On $\mathbf{R}^2$ the standard symplectic form ω_0 is nothing else but the usual area form $dx \wedge dy$. Therefore an automorphism is symplectic if and only if it preserves the oriented areas: the symplectic group $Sp(1)$ coincides with $SL(2)$, the group of real matrices two by two having determinant 1.

Every invertible matrix A can be decomposed in polar form

$$A = PO, \quad P = (AA^T)^{\frac{1}{2}}, \quad O = P^{-1}A$$

where P is symmetric and positive, O is orthogonal. Such a decomposition is unique.

If $A \in Sp(1)$, P must have determinant 1 and therefore $P \in Sp(1)$. Thus also $O \in Sp(1)$. Therefore O belongs to the group of rotations $SO(2)$, the orthogonal matrices with determinant 1.

$SO(2)$ is isomorphic to $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. The map $\mathcal{U} : Sp(1) \mapsto S^1$ associates to every symplectic matrix the rotation angle of its orthogonal part:

$$\mathcal{U}(PO) = e^{i\theta} \quad \text{if} \quad O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Since P and O in the polar decomposition are unique and depend continuously on A, $Sp(1)$ is homeomorphic to the topological product of S^1 and the set of symmetric positive matrices two by two with determinant 1. Call $\mathcal{P}(1)$ the latter set.

$\mathcal{P}(1)$ is homeomorphic to the plane because every matrix in $\mathcal{P}(1)$ is the exponential of one and only one matrix in the two-dimensional vector space consisting of symmetric infinitesimally symplectic matrices. However, we will build a different homeomorphism, which will make some later calculations easier.

The trace of P is the sum of its eigenvalues, which are positive and whose product is 1. Therefore $\operatorname{tr} P \geq 2$ and we can set $\operatorname{tr} P = 2\cosh\tau$, with $\tau \geq 0$. Then P can be written as

$$P = \begin{pmatrix} \cosh\tau + a & b \\ b & \cosh\tau - a \end{pmatrix}.$$

There is a relationship between a and b, determined by the determinant condition

$$1 = \det P = \cosh^2\tau - a^2 - b^2.$$

Then b can be expressed in terms of a as

$$b^2 = \cosh^2\tau - a^2 - 1 = \sinh^2\tau - a^2.$$

The above equation makes sense if and only if $|a| \leq |\sinh\tau|$. Therefore we can set $a = \cos\sigma\sinh\tau$, with $\sigma \in \mathbf{R}$. Then $b^2 = \sin^2\sigma\sinh^2\tau$ and we obtain a one-to-one parameterization of $\mathcal{P}(1)$ if we set $b = \sin\sigma\sinh\tau$ and we let σ vary in $[0, 2\pi[$. If we consider $(\tau = |z|, \sigma = \arg z)$ as polar coordinates on the complex plane, an homeomorphism between $\mathbf{C}$ and $\mathcal{P}(1)$ is given by

$$P = \begin{pmatrix} \cosh\tau + \cos\sigma\sinh\tau & \sin\sigma\sinh\tau \\ \sin\sigma\sinh\tau & \cosh\tau - \cos\sigma\sinh\tau \end{pmatrix}.$$

In order to draw better pictures, it is more convenient to parameterize $\mathcal{P}(1)$ with the open unit disk $B = \{z \in \mathbf{C} \mid |z| < 1\}$. To do this it is enough to consider polar coordinates $(r = |z|, \sigma = \arg z)$ on B and to set $r = \tanh^2\tau$.

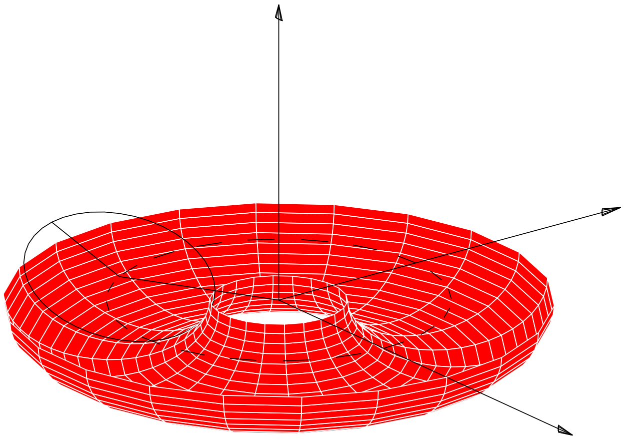

Figure 3.1: The parameterization of $Sp(1)$.

Therefore $Sp(1)$ is homeomorphic to the product space between the circle and the open disk, i.e. to the interior of the full torus, as it is shown in Fig. 1. The above homeomorphism is actually a diffeomorphism, if $Sp(1)$ is equipped with its standard structure of differentiable manifold.

3.3 The rotation function on $Sp(1)$

We will always consider coordinates (θ, r, σ) on $Sp(1)$, as in the previous section. Notice that the group of rotations $SO(2)$ corresponds to the circle

$$\{(\theta, r, \sigma) \in Sp(1) \mid r = 0\}.$$

Thus $SO(2)$ is a deformation retract of $Sp(1)$. Moreover $\mathcal{U}(\theta, 0, \sigma) = e^{i\theta}$, so $\mathcal{U}$ is the standard isomorphism from $SO(2)$ onto S^1.

The problem with the map $\mathcal{U}$ is that it is no longer a homomorphism on $Sp(1)$: for example, if

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad O = \frac{1}{2}\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}, \quad A = PO$$

then $\mathcal{U}(A) = e^{i\frac{\pi}{4}}$. If $\mathcal{U}$ is a homomorphism, $\mathcal{U}(A^2)$ is equal to $e^{i\frac{\pi}{2}} = i$. This means that the polar decomposition of A^2 takes the form

$$A^2 = \begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If we compute A^2 and we solve the above equation for a, b and c, we find the system

$$\begin{cases} c = -1 \\ 2a = 3 \\ b = 3 \\ 2c = -7 \end{cases}$$

which is clearly unsolvable: so $\mathcal{U}(A^2)$ must be different from $\mathcal{U}(A)^2$.

In order to avoid this problem, we are going to define a map

$$\rho : Sp(1) \mapsto S^1$$

homotopic to $\mathcal{U}$ which is still not a homomorphism but has the property that $\rho(A^k) = \rho(A)^k$, for every $A \in Sp(1)$. This map will be called the rotation function.

The eigenvalues of $A \in Sp(1)$ must be of the form λ, $\frac{1}{\lambda}$, where $\lambda \in \mathbf{R} \cup S^1$. An eigenvalue $\lambda \neq \pm 1$ must be therefore simple. The eigenvalues 1 and -1 are always double.

Now we need to define the Krein sign of the eigenvalues which lie on S^1: we do it only in the two-dimensional case, the general case being considered in section 5. We introduce the Hermitian matrix $G = iJ$ (Hermitian means that $G^* = G$, where G^* is the adjoint of G with respect to the standard Hermitian product $(\cdot, \cdot)$ of $\mathbf{C}^2$: $G^* = \overline{G}^T$). Notice that a real matrix A is symplectic if and only if $A^* G A = G$.

Assume that $A \in Sp(1)$ has eigenvalues $\lambda \neq \pm 1$ and $\overline{\lambda}$ on the unit circle and that ξ and $\overline{\xi}$ are the corresponding eigenvectors. Then

$$(G\xi, \overline{\xi}) = (A^* G A \xi, \overline{\xi}) = (GA\xi, A\overline{\xi}) = \lambda^2 (G\xi, \overline{\xi}).$$

Since $\lambda \neq \pm 1$, $(G\xi, \overline{\xi})$ must be null. So $\{\xi, \overline{\xi}\}$ is a G-orthogonal basis of $\mathbf{C}^2$. Remembering that G is Hermitian and invertible, we conclude that $(G\xi, \xi)$ and $(G\overline{\xi}, \overline{\xi})$ are real and not zero. This fact justifies the following definition.

Definition 3.3.1 *If $\lambda \in S^1 \backslash \{-1, 1\}$ is an eigenvalue of $A \in Sp(1)$ and ξ is the corresponding eigenvector, the Krein sign of λ is the sign of $(G\xi, \xi)$.*

Since

$$(G\overline{\xi}, \overline{\xi}) = (\overline{G}\xi, \xi) = -(G\xi, \xi)$$

if the eigenvalue $\lambda \in S^1 \setminus \{-1, 1\}$ is Krein-positive, the eigenvalue $\overline{\lambda}$ is Krein-negative. Sometimes it is useful to consider the double eigenvalue $\lambda = \pm 1$ as a pair of eigenvalues, one of which is Krein-positive, the other Krein-negative.

Now we are ready to define the rotation function $\rho : Sp(1) \mapsto S^1$:

$$\rho(A) = \begin{cases} \lambda & \text{if } \lambda \in S^1 \setminus \{-1, 1\} \text{ is the Krein} - \text{positive eigenvalue of } A, \\ 1 & \text{if the eigenvalues of } A \text{ are real and positive,} \\ -1 & \text{if the eigenvalues of } A \text{ are real and negative.} \end{cases}$$

In order to see that ρ is continuous, notice that

$$\rho(A) = \frac{\lambda}{|\lambda|}$$

where λ is any eigenvalue of A, in the case $\lambda \in \mathbf{R}$, and it is the Krein-positive eigenvalue, in the case $\lambda \in S^1 \setminus \{-1, 1\}$. The rotation

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

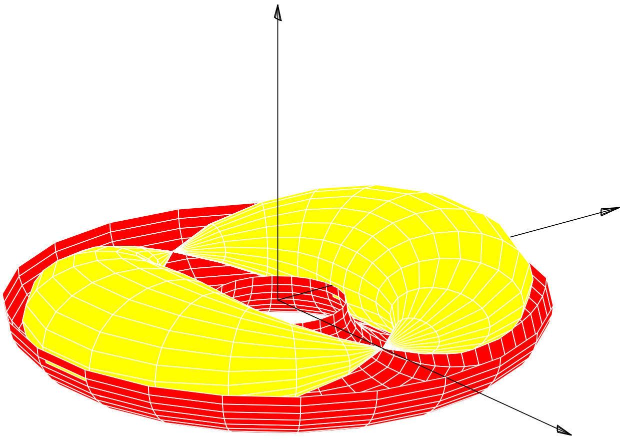

Figure 3.2: The rotation function ρ.

has eigenvalues $e^{i\theta}$, $e^{-i\theta}$ and

$$\xi = \begin{pmatrix} \cos\theta + i\sin\theta \\ \sin\theta - i\cos\theta \end{pmatrix}$$

is an eigenvector corresponding to $e^{i\theta}$. A direct calculation shows that

$$(G\xi, \xi) = 2$$

so $e^{i\theta}$ is Krein-positive and $\rho(R(\theta)) = e^{i\theta}$. Therefore ρ coincides with the map $\mathcal{U}$ on the subgroup $SO(2)$.

If λ is an eigenvalue of A, λ^k is an eigenvalue of A^k, the eigenvectors being the same. Therefore

$$\rho(A^k) = \rho(A)^k.$$

However, since the eigenvalues of the product of two matrices need not to be the products of the eigenvalues, ρ is not a homomorphism on the whole $Sp(1)$.

In order to study the function ρ we must find the eigenvalues of A, so we must solve

$$\det(\lambda I - A) = \lambda^2 - (\operatorname{tr} A)\lambda + 1 = 0$$

for λ. The discriminant of this polynomial is

$$\Delta = (\operatorname{tr} A)^2 - 4 = 4\cosh^2\tau\cos^2\theta - 4.$$

A has a double eigenvalue ± 1 if and only if $\Delta = 0$, which turns out to be equivalent to

$$r = \sin^2\theta.$$

So the set of symplectic matrices with double eigenvalue is the light surface of Fig. 2. It consists of two connected components (remember that the full torus has no boundary). One component contains I and thus it consists of matrices with eigenvalue 1. The other component contains $-I$ and thus it consists of matrices with eigenvalue -1.

The inequality $\Delta > 0$ is equivalent to

$$r > \sin^2\theta.$$

Therefore $\rho = \pm 1$ outside the light surface. By continuity, $\rho = 1$ in the right and $\rho = -1$ on the left.

If A is in the region inside the light surface, Δ is less than zero and A must have two eigenvalues λ, $\overline{\lambda} \in S^1 \setminus \{-1, 1\}$. The region inside the light surface consists of two components, one which contains the rotations of angle θ with $\sin\theta > 0$, the other which contains the rotations of angle θ with $\sin\theta < 0$. Call these regions Ω^+ and Ω^-, respectively.

The map ρ is continuous, it never takes the values ± 1 in the region inside the checked surface and $\rho(U) = e^{i\theta}$ whenever U is a θ-rotation: therefore ρ must take values on the upper half circle $S^1_+ = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ in Ω^+, and on the lower half circle $S^1_- = \{z \in \mathbf{C} \mid \operatorname{Im} z < 0\}$ in Ω^-.

Notice that $\rho(A) = \mathcal{U}(A) = 1$ if $A \in D^+ = \{(\theta, r, \sigma) \in Sp(1) \mid \theta = 0\}$ and that $\rho(A) = \mathcal{U}(A) = -1$ if $A \in D^- = \{(\theta, r, \sigma) \in Sp(1) \mid \theta = \pi\}$. Moreover both ρ and $\mathcal{U}$ take value on the upper half circle $\overline{S^1_+}$ in the tube $T^+ = \{(\theta, r, \sigma) \mid \sin\theta \geq 0\}$. Consider the class of all maps from T^+ to the closed interval S^1_+ which take the values 1 on D^+ and -1 on D^-. Both ρ and $\mathcal{U}$ restricted to T^+ belong to this class. Since all the maps in this class are homotopic via a homotopy which stays in the same class, so are $\rho|_{T^+}$ and $\mathcal{U}|_{T^+}$. The same happens on $T^- = \{(\theta, r, \sigma) \in Sp(1) \mid \sin\theta \leq 0\}$ and gluing together the two homotopies we deduce that ρ and $\mathcal{U}$ are homotopic.

We summarize the above discussion in the following proposition:

Proposition 3.3.1 *There exists a continuous map*

$$\rho : Sp(1) \mapsto S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$$

such that:

1. *ρ is homotopic to $\mathcal{U}$;*

2. *$\rho(R(\theta)) = \mathcal{U}(R(\theta)) = e^{i\theta}$ if $R(\theta)$ is the θ-rotation;*

3. *$\rho(A^k) = \rho(A)^k$ for every $A \in Sp(1)$ and for every integer k.*

3.4 The Maslov index of paths in $Sp(1)$

The set $Sp(1)$ can be divided into three subsets:

$$\begin{aligned}
\Gamma^+ &= \{A \in Sp(1) \mid \det(I - A) > 0\} \\
\Gamma^- &= \{A \in Sp(1) \mid \det(I - A) < 0\} \\
\Gamma^0 &= \{A \in Sp(1) \mid \det(I - A) = 0\}\,.
\end{aligned}$$

Recall that the light surface of Fig. 2 consisted of the matrices with double eigenvalue 1 or -1. Therefore Γ^0 is the component of such surface containing I, that is the lower-right part.

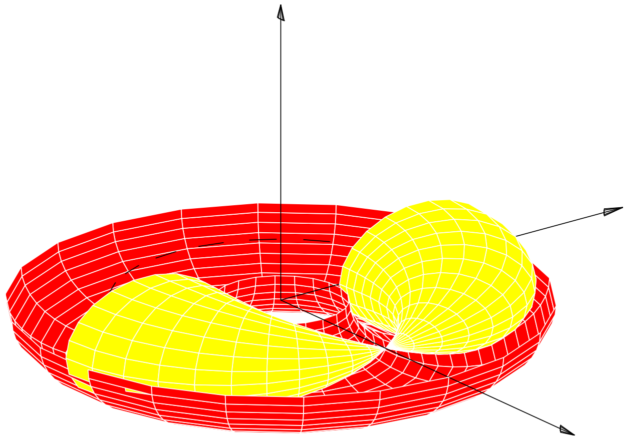

Figure 3.3: The sets Γ^0, Γ^+ and Γ^-.

Fig. 3 represents Γ^0, Γ^+ and Γ^-: Γ^0 is a surface with a 2-codimensional singularity and it divides $Sp(1)$ into two connected components, Γ^+ and Γ^-. Notice that both these components are contractible in $Sp(1)$ (also if Γ^- is not contractible in itself).

We want to associate to every continuous path

$$\begin{cases} \gamma : [0,T] \mapsto Sp(1) \\ \gamma(0) = I, \quad \gamma(T) \notin \Gamma^0 \end{cases}$$

an integer, which will be called the Maslov index of the path γ. Loosely speaking, the Maslov index of γ is the number of half windings made by γ in $Sp(1)$, $Sp(1)$ being divided into two components by Γ^0.

The matrix $-I$ lies in Γ^+; we fix a matrix W in Γ^-, for instance

$$W = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

The path $\gamma : [0,T] \mapsto Sp(1)$ can be extended to a path $\tilde{\gamma} : [0, T+1] \mapsto Sp(1)$ in such a way that

$$\begin{cases} \tilde{\gamma}(t) = \gamma(t) & \text{if } t \in [0,T] \\ \tilde{\gamma}(t) \notin \Gamma^0 & \text{if } t \in [T, T+1] \\ \tilde{\gamma}(T+1) \in \{-I, W\}. \end{cases}$$

There exists a unique continuous function $\tilde{\delta} : [0, T+1] \mapsto \mathbf{R}$ such that

$$\begin{cases} \rho(\tilde{\gamma}(t)) = e^{i\tilde{\delta}(t)} \\ \tilde{\delta}(0) = 0. \end{cases}$$

Notice that, since $\tilde{\gamma}(t) \notin \Gamma^0$ for every $t \in [T, T+1]$,

$$|\tilde{\delta}(T+1) - \tilde{\delta}(T)| < \pi. \tag{3.3}$$

Since $\rho(W) = 1$ and $\rho(-I) = -1$, the number $\tilde{\delta}(T+1)$ is an integer multiple of π.

Definition 3.4.1 *The Maslov index of the path γ at time T is the integer*

$$\mu_T(\gamma) = \frac{1}{\pi}\tilde{\delta}(T+1).$$

The above definition does not depend on the extension of γ chosen: let $\tilde{\gamma}_0$, $\tilde{\gamma}_1$ be two such extensions and $\tilde{\delta}_0$, $\tilde{\delta}_1$ be the liftings of $\rho \circ \tilde{\gamma}_0$ and $\rho \circ \tilde{\gamma}_1$. Since the closed loops in Γ^+ and Γ^- are contractible in $Sp(1)$, $\tilde{\gamma}_0|_{[T,T+1]}$ and $\tilde{\gamma}_1|_{[T,T+1]}$ are homotopic via a homotopy which fixes the values at T and $T+1$. Therefore

$$\tilde{\delta}_1(T+1) - \tilde{\delta}_1(T) = \tilde{\delta}_0(T+1) - \tilde{\delta}_0(T)$$

and so $\tilde{\delta}_0(T+1) = \tilde{\delta}_1(T+1)$.

Some more properties of the rotation function and of the Maslov index will be proved in section 8 and 9, where these quantity are defined for every dimension.

3.5 The Krein signature on $Sp(n)$

In order to define the Krein signature of the eigenvalues of a symplectic matrix A, we must consider A as acting on $\mathbf{C}^{2n}$ in the usual way

$$A(\xi + i\eta) = A\xi + iA\eta, \quad \forall \xi, \eta \in \mathbf{R}^{2n}.$$

It is useful to introduce the Hermitian form

$$g(v, w) = (iJv, w)$$

The complex symplectic group $Sp(n; \mathbf{C})$ consists of the complex linear automorphisms of $\mathbf{C}^{2n}$ which preserve the form g. Equivalently $A \in Sp(n; \mathbf{C})$ if and only if

$$A^* G A = G \tag{3.4}$$

where $G = iJ$ and A^* is the transpose conjugate of A. A matrix belongs to $Sp(n)$ if and only if it is in $Sp(n; \mathbf{C})$ and it is real.

We want to study the spectrum of complex and real symplectic matrices. The discussion will be divided into several steps.

1. Equation (3.4) can be written in an equivalent form as

$$A = G^{-1}(A^*)^{-1}G.$$

Therefore A and $(A^*)^{-1}$ are similar and they must have the same eigenvalues and the same Jordan forms. To each eigenvalue λ of A there corresponds an eigenvalue $\bar{\lambda}$ of A^* and an eigenvalue $\bar{\lambda}^{-1}$ of $(A^*)^{-1}$.

Then the similarity between A and $(A^*)^{-1}$ implies the following fact: if $A \in Sp(n; \mathbf{C})$ has an eigenvalue λ, then it has also the eigenvalue $\bar{\lambda}^{-1}$. Moreover A has the same Jordan form on the generalized eigenspaces E_λ and $E_{\bar{\lambda}^{-1}}$. Here and in the following

$$E_\lambda = \bigcup_{j=1}^{2n} \operatorname{Ker}(A - \lambda I)^j.$$

2. Assume that λ and μ are two eigenvalues of $A \in Sp(n;\mathbf{C})$ such that

$$\lambda\bar{\mu} \neq 1. \tag{3.5}$$

If v and w are corresponding eigenvectors, then

$$g(v,w) = g(Av, Aw) = g(\lambda v, \mu w) = \lambda\bar{\mu}\, g(v,w).$$

Since (3.5) holds, we conclude that v and w are g-orthogonal.

More generally we can show that the generalized eigenspaces E_λ and E_μ are g-orthogonal, provided (3.5) holds. In fact if $v \in E_\lambda$ and $w \in E_\mu$, and if $k \in \mathbf{N}$ is large enough, $A^k v$ and $A^k w$ are eigenvectors of A, corresponding to the eigenvalues λ and μ, respectively. Then

$$g(v,w) = g(A^k v, A^k w) = 0.$$

3. By what we proved in step 1, we can arrange the spectral decomposition of $A \in Sp(n;\mathbf{C})$ in the following way:

$$\mathbf{C}^{2n} = \bigoplus_{\substack{\lambda\in\sigma(A)\\ |\lambda|\geq 1}} F_\lambda$$

where $F_\lambda = E_\lambda$ if $|\lambda| = 1$ and $F_\lambda = E_\lambda \oplus E_{\bar{\lambda}^{-1}}$ if $|\lambda| > 1$.

By step 2, the above decomposition is g-orthogonal. Therefore g must be non-degenerate on each of the spaces F_λ.

4. Let λ be an eigenvalue of $A \in Sp(n;\mathbf{C})$ outside the unit circle, with algebraic multiplicity d. Again by step 2, $g(v,v) = 0$ for every $v \in E_\lambda$. Then g restricted to the $2d$-dimensional space F_λ has a d-dimensional isotropic subspace. We conclude that g must have signature (d,d) on F_λ.

5. On the contrary, g may have any signature on $F_\lambda = E_\lambda$ in the case $|\lambda| = 1$. This fact justifies the following definition.

Definition 3.5.1 *Let λ be an eigenvalue on the unit circle of a complex symplectic matrix. The Krein signature of λ is the signature of the restriction of the hermitian form g to the generalized eigenspace E_λ.*

6. Now let A be a real symplectic matrix. Since A is real, it has pairs of eigenvalues λ and $\bar{\lambda}$, with the same Jordan form on the corresponding generalized eigenspaces. If we put this fact together with what we proved at step 1, we get the following result: if $A \in Sp(n)$ has an eigenvalue λ, then it has also the eigenvalues $\bar{\lambda}$, $\bar{\lambda}^{-1}$ and λ^{-1}. Moreover A has the same Jordan form on each generalized eigenspace. Therefore the eigenvalues of a real symplectic matrix come out in groups, according to the following list:

- the real eigenvalues different from ± 1 are always in pairs λ, $\frac{1}{\lambda}$;
- the eigenvalues on the unit circle, different from ± 1, are always in pairs λ, $\bar{\lambda}$;

- the eigenvalues away from the real line and from the unit circle come out four by four: λ, $\bar{\lambda}^{-1}$ and λ^{-1};

- the eigenvalues 1 and -1 always have even algebraic multiplicity.

The last assertion follows from the fact that $\det A = 1$.

7. Assume that A is real symplectic. If the eigenvalue $\lambda \in S^1$ has Krein signature (p, q) then the eigenvalue $\bar{\lambda}$ has Krein signature (q, p). In fact $E_{\bar{\lambda}} = \overline{E_\lambda}$ and

$$g(\bar{v}, \bar{v}) = (iJ\bar{v}, \bar{v}) = \overline{(-iJv, v)} = -g(v, v).$$

In particular the Krein signature of the eigenvalue 1 or -1 has always the form (p, p).

If the real symplectic matrix A has an eigenvalue λ on the unit circle of Krein signature (p, q), it is often convenient to say that A has $p + q$ eigenvalues λ, and that p of them are Krein-positive, while q of them are Krein-negative. With this notations, if $2k$ is the total algebraic multiplicity of the eigenvalues of A on the unit circle, we can say that there are

$$\lambda_1, \dots, \lambda_k$$

Krein-positive eigenvalues and

$$\overline{\lambda_1}, \dots, \overline{\lambda_k}$$

Krein-negative eigenvalues.

Recall from section 1 that the space of infinitesimally symplectic matrices $sp(n)$ consists of the matrices H such that

$$H^T J + JH = 0.$$

Moreover e^H is symplectic for every $H \in sp(n)$ and every symplectic matrix can be written in this way.

Arguing as before, it is easy to describe the spectrum of an infinitesimally symplectic matrix H. In fact H belongs to $sp(n)$ if and only if

$$J^{-1} H^T J = -H.$$

Therefore, if λ is an eigenvalue of H, so are $\overline{\lambda}$, $-\lambda$ and $-\overline{\lambda}$, all with the same multiplicities. Hence three situations are possible:

- the non zero real or purely imaginary eigenvalues occur in pairs $\{\lambda, -\lambda\}$, with λ in $\mathbf{R}$ or in $i\mathbf{R}$;

- all the other non zero eigenvalues occur in groups of four $\{\lambda, -\lambda, \overline{\lambda}, -\overline{\lambda}\}$;

- the eigenvalue zero has always even algebraic multiplicity.

To each eigenvalue λ of H there corresponds an eigenvalue e^λ of $A = e^H$. The reader can easily check that the above description of the spectrum of H agrees with the description of the spectrum of the symplectic matrix A.

The Krein signature of the eigenvalues of H is defined as before.

Definition 3.5.2 *Let λ be a purely imaginary eigenvalue of an infinitesimally symplectic matrix. The Krein signature of λ is the signature of the restriction of the Hermitian form g to the generalized eigenspace E_λ*

With the convention of repeating the eigenvalues according to their multiplicity, if $2k$ is the total number of purely imaginary eigenvalues of H, we can say that there are

$$\lambda_1, \dots, \lambda_k$$

Krein-positive eigenvalues and

$$-\lambda_1, \dots, -\lambda_k$$

Krein-negative eigenvalues.

3.6 Normal forms of semi-simple symplectic matrices

Let H be an infinitesimally symplectic matrix. Notice that H is semi-simple if and only if $A = e^H$ is semi-simple (a matrix is called semi-simple if the algebraic and geometric multiplicities of its eigenvalues coincide). We want to derive normal forms for H and A in the semi-simple case.

There exists a decomposition of $\mathbf{R}^{2n}$ into symplectic subspaces

$$\mathbf{R}^{2n} = V^{(1)} \oplus V^{(2)} \oplus V^{(3)}$$

where:

1. $V^{(1)}$ is the sum of the eigenspaces of H corresponding to the purely imaginary eigenvalues. Therefore $V^{(1)}$ has the symplectic decomposition
$$V^{(1)} = E_0 \oplus \bigoplus_{\lambda \in i\mathbf{R}^+} (E_\lambda \oplus E_{-\lambda}).$$

2. $V^{(2)}$ is the sum of the eigenspaces of H corresponding to the non zero real eigenvalues. Therefore $V^{(2)}$ has the symplectic decomposition
$$V^{(2)} = \bigoplus_{\lambda \in \mathbf{R}^+} (E_\lambda \oplus E_{-\lambda}).$$

3. $V^{(3)}$ is the sum of the eigenspaces of H corresponding to all the other eigenvalues. Therefore $V^{(3)}$ has the symplectic decomposition
$$V^{(3)} = \bigoplus_{\lambda \in \mathbf{C} \setminus (\mathbf{R} \cup i\mathbf{R})} (E_\lambda \oplus E_{-\lambda} \oplus E_{\overline{\lambda}} \oplus E_{-\overline{\lambda}}).$$

Since H is semi-simple, $V^{(1)}$, $V^{(2)}$ and $V^{(3)}$ actually have the symplectic decomposition

$$V^{(j)} = \oplus_{s=1}^{k_j} V_s^{(j)}, \quad j = 1, 2, 3.$$

where the spaces $V_s^{(j)}$ are two-dimensional for $j = 1, 2$ and four dimensional for $j = 3$.

Now we want to examine the form of H and $A = e^H$ on each of the spaces $V_s^{(j)}$. Call $H_s^{(j)}$ and $A_s^{(j)}$ the above restrictions.

1. The space $V_s^{(1)}$ corresponds to a pair of purely imaginary eigenvalues $\{\alpha i, -\alpha i\}$, with $\alpha \in \mathbf{R}$. Then there exists a symplectic basis of $V_s^{(1)}$ such that

$$H_s^{(1)} = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \quad A_s^{(1)} = e^{H_s^{(1)}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

2. The space $V_s^{(2)}$ corresponds to a pair of non zero real eigenvalues $\{\beta, -\beta\}$, with $\beta \in \mathbf{R} \setminus \{0\}$. Then there exists a symplectic basis of $V_s^{(2)}$ such that

$$H_s^{(2)} = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \quad A_s^{(2)} = e^{H_s^{(2)}} = \begin{pmatrix} e^{\beta} & 0 \\ 0 & e^{-\beta} \end{pmatrix}.$$

3. The space $V_s^{(3)}$ corresponds to a group of eigenvalues $\{\beta+\alpha i, -\beta-\alpha i, \beta-\alpha i, -\beta+\alpha i\}$, with $\alpha, \beta \in \mathbf{R} \setminus \{0\}$. Then there exists a symplectic basis of $V_s^{(3)}$ such that

$$H_s^{(3)} = \begin{pmatrix} \beta & -\alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & -\beta & -\alpha \\ 0 & 0 & \alpha & -\beta \end{pmatrix},$$

$$A_s^{(3)} = e^{H_s^{(3)}} = \begin{pmatrix} e^{\beta}\cos\alpha & -e^{\beta}\sin\alpha & 0 & 0 \\ e^{\beta}\sin\alpha & e^{\beta}\cos\alpha & 0 & 0 \\ 0 & 0 & e^{-\beta}\cos\alpha & -e^{-\beta}\sin\alpha \\ 0 & 0 & e^{-\beta}\sin\alpha & e^{-\beta}\cos\alpha \end{pmatrix}.$$

It is not difficult to derive normal forms for infinitesimally symplectic and symplectic matrices which are not semi-simple. However we will be able not to use these normal forms by approximating the given matrix with semi-simple ones. The whole list of normal forms can be found in the book of Shui-Nee Chow, Chengzhi Li and Duo Wang [CLW94].

3.7 The topology of $Sp(n)$

Every invertible matrix A can be put in polar form

$$A = PO, \quad P = (AA^T)^{\frac{1}{2}}, \quad O = P^{-1}A,$$

where P is symmetric and positive definite, O is orthogonal. Such a decomposition is unique and it depends continuously on the matrix A.

If A is symplectic, both P and O turn out to be symplectic. To see this notice that the condition for a matrix M to be symplectic can be written as

$$M = J^{-1}(M^T)^{-1}J.$$

Applying the above formula with $M = A = PO$ we get

$$PO = J^{-1}(P^T)^{-1}(O^T)^{-1}J = J^{-1}(P^T)^{-1}J \cdot J^{-1}(O^T)^{-1}J.$$

Since J is orthogonal, the matrix $J^{-1}(P^T)^{-1}J$ is still symmetric and positive definite, while $J^{-1}(O^T)^{-1}J$ is orthogonal. By the uniqueness of the polar decomposition, there must hold

$$J^{-1}(P^T)^{-1}J = P, \quad J^{-1}(O^T)^{-1}J = O$$

which means that both P and O are symplectic.

Therefore $Sp(n)$ is homeomorphic to the topological product of $\mathcal{P}(n)$, the set of symplectic positive definite matrices, and $O(2n) \cap Sp(n)$, the group of orthogonal and symplectic matrices.

The latter group is isomorphic to the unitary group $U(n)$, the group of complex matrices which preserve the standard Hermitian product of $\mathbf{C}^n$. In fact $\mathbf{R}^{2n}$ can be identified to $\mathbf{C}^n$ by the map

$$\mathbf{R}^{2n} \ni \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow x + iy \in \mathbf{C}^n.$$

Applying J to $\mathbf{R}^{2n}$ corresponds to the multiplication by i in $\mathbf{C}^n$. If A is both orthogonal and symplectic, $JA = AJ$ which means that A is $\mathbf{C}$-linear. Notice that

$$(x + iy, x' + iy') = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} + i\omega_0(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x' \\ y' \end{pmatrix}).$$

Therefore the orthogonal and symplectic automorphisms of $\mathbf{R}^{2n}$ correspond to the automorphisms of $\mathbf{C}^n$ which preserve the Hermitian product, that is the unitary matrices.

Moreover it can be shown that A is orthogonal and symplectic if and only if

$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

where $X^TY - Y^TX = 0$ and $X^TX + Y^TY = I$. Then, if U is the unitary matrix which corresponds to A,

$$U(x + iy) = Xx - Yy + i(Yx + Xy) = (X + iY)(x + iy) \tag{3.6}$$

and so

$$U = X + iY. \tag{3.7}$$

The space $\mathcal{P}(n)$ is homeomorphic to a $n(n + 1)$-dimensional Euclidean space. In fact every symmetric positive definite matrix P can be uniquely represented in the form

$$P = e^S$$

where S is symmetric (S is the logarithm of P). Then P is symplectic if and only if S is infinitesimally symplectic. Therefore $\mathcal{P}(n)$ is homeomorphic, via the logarithm function, to the vector space

$$Sym(2n) \cap Sp(n) = \{S \in Sym(2n) \mid SJ + JS = 0\}.$$

It is easy to see that S belongs to $Sym(2n) \cap Sp(n)$ if and only if

$$S = \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix}$$

where both X and Y are n by n symmetric matrices. Therefore $\mathcal{P}(n)$ has dimension $n(n+1)$.

The unitary group $U(n)$ is homeomorphic to the topological product of S^1 and the special unitary group $SU(n)$, the group of unitary matrices with determinant one. In fact, if $U \in U(n)$ and $\det U = e^{i\theta}$, then

$$U = R_1(\theta) \cdot (R_1(-\theta)U), \quad \text{where } R_1(\theta) = \begin{pmatrix} e^{i\theta} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix}$$

and $R_1(-\theta)U \in SU(n)$.

Therefore $Sp(n)$ is homeomorphic to $\mathbf{R}^{n(n+1)} \times S^1 \times SU(n)$.

Finally, we show that $SU(n)$ is simply connected: let v be a fixed vector in $\mathbf{C}^n$. Then it is easy to see that the map

$$SU(n) \ni U \longrightarrow Uv \in S^{2n-1} \subset \mathbf{C}^n$$

is a fibration with fiber $SU(n-1)$. By the exact homotopy sequence we get the exact sequence

$$\pi_2(S^{2n-1}) \to \pi_1(SU(n-1)) \to \pi_1(SU(n)) \to \pi_1(S^{2n-1}).$$

If $n \geq 2$, $\pi_2(S^{2n-1}) \cong \pi_1(S^{2n-1}) = 0$ and then

$$\pi_1(SU(n)) \cong \pi_1(SU(n-1)) \cong \cdots \cong \pi_1(SU(1)) = \pi_1(\{1\}) = 0.$$

We conclude that the fundamental group of $Sp(n)$ is infinite cyclic: $\pi_1(Sp(n)) = \mathbf{Z}$.

3.8 The rotation function on $Sp(n)$

Let (V, ω) be a real symplectic space and let $\mathcal{E} = \{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ be a symplectic basis of (V, ω). If φ is a symplectic automorphism of (V, ω) and $A \in Sp(n)$ is the matrix which corresponds to φ with respect to the basis $\mathcal{E}$, we can define the rotation function

$$\rho : Sp(V, \omega) \mapsto S^1$$

by setting

$$\rho(\varphi) = \rho(A) = (-1)^m \prod_{j=1}^{k} \lambda_j$$

where $2m$ is the total algebraic multiplicity of the real negative eigenvalues of A and $\lambda_1, \ldots, \lambda_k$ are the Krein-positive eigenvalues of A on the unit circle. Here we are repeating the eigenvalues according to their multiplicity. By step 7 of section 3.5, we also have the formula

$$\rho(\varphi) = \rho(A) = (-1)^m \prod_{\lambda \in \sigma(A) \cap S^1_+} \lambda^{p(\lambda)-q(\lambda)} \tag{3.8}$$

where $S^1_+ = \{z \in \mathbf{C} \mid |z| = 1 \text{ and } \operatorname{Im} z > 0\}$ and $(p(\lambda), q(\lambda))$ is the Krein signature of the eigenvalue λ.

In order to see that ρ is well defined, we must check that its value does not depend on the choice of the symplectic basis $\mathcal{E}$. If $\mathcal{E}'$ is another symplectic basis of (V, ω) then the matrix associated to φ with respect to $\mathcal{E}'$ has the form

$$A' = BAB^{-1}.$$

The matrix B corresponds to the symplectic automorphism of (V, ω) which takes $\mathcal{E}$ into $\mathcal{E}'$, with respect to the basis $\mathcal{E}$. Therefore B is symplectic. Let λ be an eigenvalue of A, with generalized eigenspace E_λ. Then BE_λ is the generalized eigenspace of A' corresponding the eigenvalue λ. Since B is symplectic, g has the same signature on E_λ and on BE_λ. Therefore

$$\rho(A) = \rho(A').$$

Theorem 3.8.1 *The rotation function ρ satisfies the following properties.*

1. *(Continuity) The function ρ is continuous.*

2. *(Naturality) If $\psi : (V_1, \omega_1) \mapsto (V_2, \omega_2)$ is a symplectic isomorphism then*

   $$\rho(\psi \circ \varphi \circ \psi^{-1}) = \rho(\varphi)$$

   for every $\varphi \in Sp(V_1, \omega_1)$.

3. *(Decomposition) If $(V, \omega) = (V_1 \oplus V_2, \omega_1 \oplus \omega_2)$, then*

   $$\rho(\varphi) = \rho(\varphi_1)\rho(\varphi_2)$$

   if $\varphi \in Sp(V, \omega)$ has the form $\varphi(z_1 \oplus z_2) = \varphi_1(z_1) \oplus \varphi_2(z_2)$.

4. *(Value on the unitary group) If $A \in Sp(n) \cap O(2n)$ and U is the corresponding matrix in $U(n)$, then*

   $$\rho(A) = \det U.$$

5. *(Normalization) If $\varphi \in Sp(V, \omega)$ has no eigenvalue on the unit circle then*

   $$\rho(\varphi) = \pm 1.$$

The above properties characterize the family of functions ρ. Moreover:

6. *(Homotopy) The map ρ induces an isomorphism of the fundamental groups*

$$\rho_* : \pi_1(Sp(V,\omega)) \to \pi_1(S^1).$$

7. *(Powers) If $\varphi \in Sp(V,\omega)$ and $k \in \mathbf{Z}$, then*

$$\rho(\varphi^k) = \rho(\varphi)^k.$$

PROOF. 1. It is enough to prove the continuity of ρ on the standard symplectic group $Sp(n)$ and it is useful to use formula (3.8). Let A be a symplectic matrix. We want to show that if we perturb A slightly, the value of ρ will not change much.

The eigenvalues of A which do not lie on the closure of S^1_+ or on the negative real axis will not contribute to the value of ρ on the perturbed matrix.

Let $\lambda \in S^1_+ \cup \{1,-1\}$ be an eigenvalue of A with Krein signature (p,q). If we perturb A slightly, the eigenvalue λ will split into eigenvalues $\mu_1,\dots,\mu_h$ on S^1_+ and eigenvalues $\nu_1, \overline{\nu_1}^{-1}, \dots, \nu_k, \overline{\nu_k}^{-1}$ outside the unit circle. The sum of the algebraic multiplicities of all these eigenvalues is $p+q$, whereas the direct sum of the corresponding generalized eigenspaces - call it Y - lies near E_λ. Therefore we can assume that g has signature (p,q) on Y.

Let (p_j,q_j) be the Krein signature of μ_j, for $j=1,\dots,h$, and let d_j be the algebraic multiplicity of ν_j, for $j=1,\dots,k$. By step 4 section, 3.5 g has signature (d_j,d_j) on $E_{\nu_j} \oplus E_{\bar{\nu_j}^{-1}}$. Therefore

$$p-q = \sum_{j=1}^{h} p_j + \sum_{j=1}^{k} d_j - \sum_{j=1}^{h} q_j - \sum_{j=1}^{k} d_j = \sum_{j=1}^{h} (p_j - q_j). \tag{3.9}$$

Thus $\mu_1^{p_1-q_1} \dots \mu_h^{p_h-q_h}$ is near λ^{p-q}. In the case $\lambda = \pm 1$, $\mu_1^{p_1-q_1} \dots \mu_h^{p_h-q_h}$ is near 1.

Let λ be a real negative eigenvalue of A different from -1 and let s be its algebraic multiplicity. If we perturb A the pair of eigenvalues $\lambda, \frac{1}{\lambda}$ can generate groups of four eigenvalues outside the real axis. Therefore the parity of the number of pairs of eigenvalues which remain on the real axis is the parity of s.

In the case $\lambda = -1$ one may think that the number r of pairs of eigenvalues which leave the real axis, remaining on the unit circle, can be odd. But equation (3.9) gives

$$0 = \sum_{j=1}^{h} (p_j - q_j)$$

and therefore

$$r = \sum_{j=1}^{h} (p_j + q_j)$$

is even. Therefore ρ is continuous.

2. Let $\mathcal{E}$ be a symplectic basis of (V_1, ω_1). Then $\mathcal{F} = \{\psi(\mathcal{E})\}$ is a symplectic basis of (V_2, ω_2). If A is the matrix associated to φ with respect to the basis $\mathcal{E}$, A is also the matrix associated to $\psi \circ \varphi \circ \psi^{-1}$ with respect to the basis $\mathcal{F}$. So assertion 2 is trivial.

3. Assertion 3 follows immediately from the definition of ρ.

4. IfA is symplectic and orthogonal, all its eigenvalues have modulus one and

$$\rho(A) = \prod_{\lambda \in \sigma(A)} \lambda^{p(\lambda)}$$

where $(p(\lambda), q(\lambda))$ is the Krein signature of λ.

We notice that J is semi-simple and has eigenvalues i and $-i$. Then $\mathbf{C}^{2n} = F_+ \oplus F_-$ where

$$\begin{aligned} F_+ &= \operatorname{Ker}(J + iI) = \{(x, -ix) \mid x \in \mathbf{C}^n\}, \\ F_- &= \operatorname{Ker}(J - iI) = \{(x, ix) \mid x \in \mathbf{C}^n\}. \end{aligned}$$

If v is in F_-

$$g(v, v) = (iJv, v) = -(v, v).$$

Therefore g is negative definite on F_- and positive definite on F_+.

Let λ be an eigenvalue of A and let E_λ be the corresponding generalized eigenspace. Let $v \in E_\lambda$. If the integer k is large enough $w = A^k v$ is an eigenvector of A, with eigenvalue λ. The fact that $A \in Sp(n) \cap O(2n)$ implies that $AJ = JA$. Then

$$AJw = JAw = \lambda Jw$$

and therefore also Jw is an eigenvector of A. Thus $Jv = JA^{-k}w = A^{-k}Jw$ belongs to E_λ. This proves that the subspace E_λ is J-invariant. Hence

$$E_\lambda = E_\lambda^+ \oplus E_\lambda^- \quad \text{where } E_\lambda^+ = E_\lambda \cap F_+,\ E_\lambda^- = E_\lambda \cap F_-.$$

Then the eigenvalue λ contributes to the value of $\rho(A)$ by

$$\lambda^{\dim E_\lambda^+}.$$

From formula (3.6) we know that A has the form

$$A = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}.$$

From the above equation it is easy to deduce that A has the form

$$A = \begin{pmatrix} X + iY & 0 \\ 0 & X - iY \end{pmatrix}$$

with respect to the splitting $\mathbf{C}^{2n} = F_+ \oplus F_-$. Then to each eigenvalue λ of A there corresponds an eigenvalue λ of $U = X + iY$ with algebraic multiplicity $\dim E_\lambda^+$. Therefore λ contributes to the value of $\det U$ by

$$\lambda^{\dim E_\lambda^+}$$

and so $\rho(A) = \det U$.

5. Assertion 5 follows immediately from the definition of ρ.

To see that the properties (1), (2), (3), (4) and (5) characterize the family of functions ρ we assume that $\tilde{\rho}$ is another family of functions satisfying the same assumptions. We will show that $\tilde{\rho} = \rho$.

Since the set of semi-simple symplectic automorphisms of (V, ω) is dense in $Sp(V, \omega)$, the continuity of $\tilde{\rho}$ allows us to verify the equality $\tilde{\rho} = \rho$ only on such automorphisms.

By the naturality and decomposition properties, it is enough to check the value of $\tilde{\rho}$ on the three symplectic matrices in normal form (see section 3.6)

$$A^{(1)}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbf{R}$$

$$A^{(2)}_\mu = \begin{pmatrix} \mu & 0 \\ 0 & \frac{1}{\mu} \end{pmatrix}, \qquad \mu \in \mathbf{R} \setminus \{0\}$$

$$A^{(3)}_{\nu,\sigma} = \begin{pmatrix} \nu\cos\sigma & -\nu\sin\sigma & 0 & 0 \\ \nu\sin\sigma & \nu\cos\sigma & 0 & 0 \\ 0 & 0 & \frac{1}{\nu}\cos\sigma & -\frac{1}{\nu}\sin\sigma \\ 0 & 0 & \frac{1}{\nu}\sin\sigma & \frac{1}{\nu}\cos\sigma \end{pmatrix}, \qquad \nu \in \mathbf{R}^+, \ \sigma \in \mathbf{R}.$$

Assumption (4) guarantees that

$$\tilde{\rho}(A^{(1)}_\theta) = e^{i\theta} = \rho(A^{(1)}_\theta).$$

The set of matrices of the second kind consists of two connected components, one for $\mu > 0$ and the other for $\mu < 0$. By assumption (5), $\tilde{\rho}$ must take the value 1 or -1 on these matrices. Since $\tilde{\rho}$ is continuous and

$$\tilde{\rho}(I) = \tilde{\rho}(A^{(1)}_0) = 1, \quad \tilde{\rho}(-I) = \tilde{\rho}(A^{(1)}_\pi) = -1,$$

we deduce that

$$\begin{aligned} \tilde{\rho}(A^{(2)}_\mu) &= 1 = \rho(A^{(2)}_\mu), \quad \text{if } \mu > 0, \\ \tilde{\rho}(A^{(2)}_\mu) &= -1 = \rho(A^{(2)}_\mu), \quad \text{if } \mu < 0. \end{aligned}$$

Finally, the set of matrices of the third kind is connected and it contains the identity matrix. Again, by assumption (5) and by the continuity of $\tilde{\rho}$,

$$\tilde{\rho}(A^{(3)}_{\nu,\sigma}) = 1 = \rho(A^{(3)}_{\nu,\sigma}).$$

6. It is enough to verify assertion 6 on the standard symplectic group. Identifying $U(n)$ with $Sp(n) \cap O(n)$, we have the inclusion map

$$U(n) \overset{i}{\hookrightarrow} Sp(n).$$

Since $Sp(n) \cong U(n) \times \mathcal{P}(n)$ and $\mathcal{P}(n)$ is simply connected,

$$i_* : \pi_1(U(n)) \to \pi_1(Sp(n))$$

is an isomorphism. The determinant function $\det : U(n) \mapsto S^1$ induces an isomorphism of the fundamental groups. Since $\rho \circ i = \det$, also ρ_* is an isomorphism.

7. Assertion 7 easily follows from the fact that λ^k is an eigenvalue of φ^k if λ is an eigenvalue of φ. □

3.9 The Maslov index of paths in $Sp(n)$

As in the two-dimensional case, it is useful to divide $Sp(n)$ into three subsets:

$$\begin{aligned} \Gamma^+ &= \{A \in Sp(n) \mid \det(I - A) > 0\}, \\ \Gamma^- &= \{A \in Sp(n) \mid \det(I - A) < 0\}, \\ \Gamma^0 &= \{A \in Sp(n) \mid \det(I - A) = 0\}. \end{aligned}$$

The matrix $-I$ lies in Γ^+; we fix a matrix W in Γ^-, for instance

$$W = \begin{pmatrix} 2 & & & & & & & 0 \\ & 1 & & & & & & \\ & & \cdots & & & & & \\ & & & 1 & & & & \\ & & & & \frac{1}{2} & & & \\ & & & & & 1 & & \\ & & & & & & \cdots & \\ 0 & & & & & & & 1 \end{pmatrix}.$$

Lemma 3.9.1 *There exist continuous functions*

$$\alpha_j : \Gamma^+ \cup \Gamma^- \mapsto [0, 2\pi], \quad j = 1, \dots, n,$$

such that

$$\rho(A) = e^{i \sum_{j=1}^n \alpha_j(A)}$$

for every $A \in \Gamma^+ \cup \Gamma^-$. Moreover $\alpha_j(-I) = \pi$ for every $j = 1, \dots, n$. α_1 is identically zero on Γ^- and $\alpha_2(W) = \cdots = \alpha_n(W) = \pi$.

Proof. Assume that $A \in \Gamma^+$. We can arrange the eigenvalues of A, repeating them according to their algebraic multiplicity, in the following way

$$\lambda_1, \dots, \lambda_n, \lambda_1^{-1}, \dots, \lambda_n^{-1}$$

where $|\lambda_j| \geq 1$ and λ_j is Krein-positive, when $|\lambda_j| = 1$.

Then we can assume that the λ_j's are ordered in such a way that

$$0 \leq \alpha_1(A) \leq \cdots \leq \alpha_n(A) \leq 2\pi$$

where we have set

$$\frac{\lambda_j}{|\lambda_j|} = e^{\alpha_j(A)}.$$

Such an ordering and the numbers $\alpha_j(A)$ are well defined when there are no real positive eigenvalues. Since $A \in \Gamma^+$, there is an even number of real eigenvalues greater than 1. We choose the ordering and the $\alpha_j(A)$ in such a way that there is the same number of indices j with $\alpha_j(A) = 0$ and with $\alpha_j(A) = 2\pi$. It is easy to see that this choice makes the functions α_j continuous. If $A = -I$ the above construction gives

$$\alpha_1(-I) = \cdots = \alpha_n(-I) = \pi.$$

The same construction works also for Γ^-. In this case there is an odd number $2k+1$ of real eigenvalues greater than 1 and we choose the $\alpha_j(A)$ in such a way that there are $k+1$ indices j with $\alpha_j(A) = 0$ and k indices j with $\alpha_j(A) = 2\pi$. With such a choice α_1 vanishes on Γ^- because matrices in Γ^- have at least one pair of positive real eigenvalues. If $A = W$ the above construction gives

$$\alpha_1(W) = 0,\ \alpha_2(W) = \cdots = \alpha_n(W) = \pi.$$

□

Proposition 3.9.2 *Both Γ^+ and Γ^- are arc-connected. Every closed loop in Γ^+ or in Γ^- is contractible in $Sp(n)$.*

PROOF. Assume that A is in $\Gamma^+ \cup \Gamma^-$. We want to build a continuous path which joins A to $-I$ or to W, remaining in Γ^+ or in Γ^-.

Since Γ^+ and Γ^- are open subsets of a manifold, they are locally arc-connected. Since the set of semi-simple symplectic matrices is dense in $Sp(n)$, we can find a path in Γ^+ or in Γ^- which joins A to a semi-simple symplectic matrix $\tilde{A}$.

By the results of section 6, we can find a splitting of $\mathbf{R}^{2n}$ into symplectic subspaces such that $\tilde{A}$ has normal form on each of them. Since 1 is not an eigenvalue of $\tilde{A}$, the possible blocks are:

$$\tilde{A}^{(1)}_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbf{R} \setminus 2\pi\mathbf{R},$$

$$\tilde{A}^{(2)}_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}, \qquad \lambda \in \mathbf{R}^+ \setminus \{1\},$$

$$\tilde{A}^{(3)}_{\mu,\sigma} = \begin{pmatrix} \mu\cos\sigma & -\mu\sin\sigma & 0 & 0 \\ \mu\sin\sigma & \mu\cos\sigma & 0 & 0 \\ 0 & 0 & \frac{1}{\mu}\cos\sigma & -\frac{1}{\mu}\sin\sigma \\ 0 & 0 & \frac{1}{\mu}\sin\sigma & \frac{1}{\mu}\cos\sigma \end{pmatrix}, \quad \mu \in \mathbf{R}^+,\ \sigma \in \mathbf{R} \setminus 2\pi\mathbf{R}.$$

A direct computation gives

$$\det(I - \tilde{A}^{(1)}_\theta) = 2(1 - \cos\theta) > 0,$$

$$\det(I - \tilde{A}^{(2)}_\lambda) = (1 - \lambda)(1 - \frac{1}{\lambda}) < 0,$$

$$\det(I - \tilde{A}^{(3)}_{\mu,\sigma}) = (\mu + \frac{1}{\mu} - 2\cos\theta)^2 > 0.$$

Then $\tilde{A}$ lies in Γ^+ if and only if there is an even number of blocks of the first kind.

1. Assume that $\tilde{A} \in \Gamma^+$. Then we must show that each block can be joined to $-I$, without introducing the eigenvalue 1.

The blocks of the first kind can be joined to $-I$ by letting θ vary monotonically. A block $\tilde{A}^{(3)}_{\mu,\sigma}$ can be joined to a block $\tilde{A}^{(3)}_{\mu,\pi}$. Then $\tilde{A}^{(3)}_{\mu,\pi}$ can be joined to $\tilde{A}^{(3)}_{1,\pi} = -I$. Each block of the second kind with negative eigenvalue can be joined to $-I$. Since there is an even number of blocks of the second kind, they can be grouped two by two. Every double block

$$\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \frac{1}{\lambda_1} & \\ 0 & & & \frac{1}{\lambda_2} \end{pmatrix}$$

can be joined to the matrix

$$\begin{pmatrix} 2 & & & 0 \\ & 2 & & \\ & & \frac{1}{2} & \\ 0 & & & \frac{1}{2} \end{pmatrix}.$$

The above matrix can be joined to

$$\begin{pmatrix} -2 & & & 0 \\ & -2 & & \\ & & -\frac{1}{2} & \\ 0 & & & -\frac{1}{2} \end{pmatrix}$$

using blocks of the third kind, and then to $-I$.

2. Assume now that $\tilde{A} \in \Gamma^-$. There must be at least one block of the second kind. This block can be joined to

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Once this block has been removed, we can proceed as in the first part.

Let $\gamma : S^1 \mapsto \Gamma^+$ be a continuous loop. Since the rotation function $\rho : Sp(n) \mapsto S^1$ induces an isomorphism of the fundamental groups, γ is contractible if and only if $\rho \circ \gamma : S^1 \mapsto S^1$ is contractible. By Lemma 3.9.1

$$\rho(\gamma(t)) = e^{i \sum_{j=1}^{n} \alpha_j(\gamma(t))}$$

is contractible, because the interval $[0, 2\pi]$ is contractible. □

Set

$$\mathcal{S}_T(n) = \left\{ \gamma \in C^0([0,T]; Sp(n)) \mid \gamma(0) = I,\ 1 \text{ is not an eigenvalue of } \gamma(T) \right\}$$

$\mathcal{S}_T(n)$ is a topological space, with the uniform convergence topology. Two paths γ_0 and γ_1 are in the same connected component of $\mathcal{S}_T(n)$ if and only if there exists a continuous map

$$G : [0,1] \times [0,T] \mapsto Sp(n)$$

such that $G(0,t) = \gamma_0(t)$, $G(1,t) = \gamma_1(t)$ and 1 is not an eigenvalue of $G(s,T)$ for every $s \in [0,1]$.

We want to show that there is a bijective correspondence between the set of connected components of $\mathcal{S}_T(n)$ and $\mathbf{Z}$.

Let $\gamma \in \mathcal{S}_T(n)$. $\gamma(T)$ belongs to Γ^+ or to Γ^-. Therefore we can extend γ to a path $\tilde{\gamma} : [0, T+1] \mapsto Sp(n)$ such that

$$\begin{aligned}
&\tilde{\gamma}(t) = \gamma(t) \quad \text{if } t \in [0,T], \\
&1 \text{ is not an eigenvalue of } \tilde{\gamma}(t) \text{ if } t \in [T, T+1], \\
&\tilde{\gamma}(T+1) = -I \quad \text{or} \quad \tilde{\gamma}(T+1) = W.
\end{aligned}$$

There exists a unique continuous function $\tilde{\delta} : [0, T+1] \mapsto \mathbf{R}$ such that

$$\rho(\tilde{\gamma}(t)) = e^{i\tilde{\delta}(t)} \quad \text{and} \quad \tilde{\delta}(0) = 0.$$

Since $\rho(-I) = (-1)^n$ and $\rho(W) = (-1)^{n+1}$, $\tilde{\delta}(T+1)$ is an integer multiple of π.

Definition 3.9.1 *The Maslov index of the path γ at time T (or T-Maslov index) is the integer*

$$\mu_T(\gamma) = \frac{1}{\pi}\tilde{\delta}(T+1).$$

It is easy to see that the above definition does not depend on the extension of γ chosen. This follows from the fact that the closed loops in Γ^+ and in Γ^- are contractible, as proved in Proposition 3.9.2.

Here is an useful estimate of $\mu_T(\gamma)$ in terms of $\tilde{\delta}(T)$:

$$|\pi\mu_T(\gamma) - \tilde{\delta}(T)| < n\pi. \tag{3.10}$$

To see that (3.10) holds, notice that Lemma 3.9.1 implies that there exists an integer k such that

$$\tilde{\delta}(t) = 2k\pi + \sum_{j=1}^{n} \alpha_j(\tilde{\gamma}(t)) \quad \forall t \in [T, T+1].$$

Therefore

$$|\tilde{\delta}(T+1) - \tilde{\delta}(T)| \leq \sum_{j=1}^{n} |\alpha_j(\tilde{\gamma}(T+1)) - \alpha_j(\tilde{\gamma}(T))|. \tag{3.11}$$

If $\tilde{\gamma}(T+1) = -I$ then $\alpha_j(\tilde{\gamma}(T+1)) = \pi$ for every $j = 1, \ldots, n$ and (3.11) implies (3.10). If $\tilde{\gamma}(T+1) = W$ then $\alpha_1(\tilde{\gamma}(T+1)) = \alpha_1(\tilde{\gamma}(T) = 0$ while $\alpha_j(\tilde{\gamma}(T+1)) = \pi$ for $j = 2, \ldots, n$. Again (3.10) follows from (3.11).

Theorem 3.9.3 *Two paths γ_0 and γ_1 are in the same connected component of $\mathcal{S}_T(n)$ if and only if they have the same T-Maslov index.*

PROOF. We can extend a path $\gamma \in \mathcal{S}_T(n)$ to a continuous paths $\tilde{\gamma} : [0, T+1] \mapsto Sp(n)$ which agree with γ on $[0,T]$, satisfies $\tilde{\gamma}(t) \notin \Gamma^0$ for $T \leq t \leq T+1$ and $\tilde{\gamma}(T+1) \in \{-I, W\}$.

By Proposition 3.9.2, γ_0 and γ_1 are in the same connected component of $\mathcal{S}_T(n)$ if and only if $\tilde{\gamma}_0$ and $\tilde{\gamma}_1$ are homotopic with fixed end points. Since the rotation function ρ induces an isomorphism of fundamental groups, this is equivalent to $\mu_T(\gamma_0) = \mu_T(\gamma_1)$. □

3.10 The Maslov index of a linear Hamiltonian system and the iteration formula

A linear T-periodic Hamiltonian system in $\mathbf{R}^{2n}$ has the form

$$\dot{z}(t) = JB(t)z(t) \tag{3.12}$$

where $B(t)$ is a T-periodic path of symmetric matrices.

Let $\gamma(t)$ be the fundamental solution of (3.12), i.e. the solution of the matrix differential problem

$$\begin{cases} \dot{\gamma}(t) = JB(t)\gamma(t), \\ \gamma(0) = I. \end{cases}$$

Then $\gamma(t)$ is symplectic for every t. This follows from the fact that $JB(t)$ is infinitesimally symplectic. Otherwise it can be verified directly noticing that

$$\frac{d}{dt}[\gamma(t)^T J\gamma(t)] = \gamma(t)^T B(t)^T J^T J\gamma(t) + \gamma(t)^T JJB(t)\gamma(t) = 0$$

and that $\gamma(0)$ is symplectic.

Definition 3.10.1 *The T-nullity of a T-periodic linear Hamiltonian system* (3.12) *is the algebraic multiplicity of* 1 *as an eigenvalue of* $\gamma(T)$. *It is denoted by* null_T.

Definition 3.10.2 *The T-periodic linear Hamiltonian system* (3.12) *is said T-resonant if* 1 *is an eigenvalue of* $\gamma(T)$. *In the opposite case, it is said T-non-resonant.*

Therefore the T-non-resonant systems are exactly those systems whose fundamental solution belongs to $\mathcal{S}_T(n)$.

Definition 3.10.3 *The eigenvalues of* $\gamma(T)$ *are called T-Floquet multipliers of the linear Hamiltonian system* (3.12).

Definition 3.10.4 *Assume that the T-periodic linear Hamiltonian system* (3.12) *if T-non-resonant. The T-Maslov index of system* (3.12) *is the T-Maslov index of its fundamental solution.*

Notice that the fundamental solution γ of a T-periodic linear system satisfies

$$\gamma(kT + t) = \gamma(t)\gamma(T)^k \quad \forall k \in \mathbf{Z},\ \forall t \in \mathbf{R}.$$

In particular $\gamma(kT) = \gamma(T)^k$ and the eigenvalues of $\gamma(kT)$ are k-th powers of the eigenvalues of $\gamma(T)$. Therefore, if system (3.12) is T-non-resonant, it is also kT-non-resonant, at least for every large prime number k. Thus it makes sense to study the kT-Maslov index of a T-periodic system, as k grows up.

As before, let $\delta : \mathbf{R} \mapsto \mathbf{R}$ be the only continuous function such that

$$\rho(\gamma(t)) = e^{i\delta(t)} \quad \text{and} \quad \delta(0) = 0.$$

Proposition 3.10.1 *$\delta(kT) = k\delta(T)$ for every integer k.*

PROOF. We have seen in section 3.7 that $Sp(n)$ is the topological product of $O(2n) \cap Sp(n)$ and $\mathcal{P}(n)$. Therefore there exists a homotopy

$$\gamma_\lambda : [0, T] \mapsto Sp(n), \quad \lambda \in [0, 1],$$

such that $\gamma_0 = \gamma$, $\gamma_1(t) \in O(2n) \cap Sp(n)$ for every $t \in [0, T]$ and $\gamma_\lambda(0) = I$. We can extend the paths γ_λ over all $\mathbf{R}$ by setting

$$\gamma_\lambda(t) = \gamma_\lambda(t - kT)\gamma_\lambda(T)^k \quad \text{for } \; t \in [kT, (k+1)T[.$$

Then $\gamma.(\cdot)$ is continuous on $(\lambda, t) \in [0, 1] \times \mathbf{R}$. By construction

$$\gamma_\lambda(kT + t) = \gamma_\lambda(t)\gamma_\lambda(T)^k \quad \forall k \in \mathbf{Z}, \; \forall t \in \mathbf{R}. \tag{3.13}$$

and $\gamma_1(t) \in O(2n) \cap Sp(n)$ for every $t \in \mathbf{R}$.

Let $\delta_\lambda : \mathbf{R} \mapsto \mathbf{R}$ be the unique continuous function such that

$$\rho(\gamma_\lambda(t)) = e^{i\delta_\lambda(t)} \quad \text{and } \delta_\lambda(0) = 0.$$

Again $\Delta.(\cdot)$ is continuous on $(\lambda, t) \in [0, 1] \times \mathbf{R}$. By (3.13) and assertion 7 of Proposition 3.8.1

$$\rho(\gamma_\lambda(kT)) = \rho(\gamma_\lambda(T)^k) = \rho(\gamma_\lambda(T))^k$$

and so

$$\delta_\lambda(kT) - k\delta(T) \tag{3.14}$$

must be an integer multiple of 2π. Moreover the quantity (3.14) depends continuously on λ and therefore

$$\delta(kT) - k\delta(T) = \delta_0(kT) - k\delta_0(T) = \delta_1(kT) - k\delta_1(T). \tag{3.15}$$

So it is enough to show that the right-hand member of (3.15) vanishes for every integer k.

We start with the case $k > 0$ and we argue by induction on k. (3.15) vanishes for $k = 1$. Assume that it vanishes for $k = \bar{k}$. Since ρ is a homomorphism on $O(2n) \cap Sp(n)$

$$\rho(\gamma_1(T + t)) = \rho(\gamma_1(t)) \cdot \rho(\gamma_1(T)).$$

Therefore the quantity

$$\delta_1(T + t) - \delta_1(t) - \delta_1(T)$$

must be an integer multiple of 2π. It depends continuously on t and it vanishes for $t = 0$, so

$$\delta_1(T + t) - \delta_1(t) - \delta_1(T) = 0. \tag{3.16}$$

For $t = \bar{k}T$, (3.16) gives

$$\delta_1((\bar{k} + 1)T) - \delta_1(\bar{k}T) - \delta_1(T) = 0.$$

Since (3.14) vanishes for $k = \bar{k}$, we have

$$\delta_1((\bar{k}+1)T) - (\bar{k}+1)\delta_1(T) = 0.$$

This proves the induction step and the thesis for $k > 0$. To conclude the proof notice that formula (3.16) with $t = -T$ gives

$$\delta_1(-T) = -\delta_1(T).$$

Then the same induction argument for negative k's proves the proposition. □

The above proposition, together with estimate (3.10), implies that the kT-Maslov index grows linearly with k. The rate of this growth is called mean winding number.

Definition 3.10.5 *The mean winding number of the linear Hamiltonian system* (3.12) *is*

$$\tau = \lim_{k\to+\infty} \frac{\mu_{kT}}{kT} = \lim_{k\to+\infty} \frac{\Delta(kT)}{\pi kT} = \frac{1}{\pi T}\Delta(T)$$

The first limit in the above formula is taken over all k's such that system (3.12) is kT-non-resonant. Notice that estimate (3.10) implies

$$|\mu_{kT} - kT\tau| < n. \tag{3.17}$$

If we know the mean winding number of a linear system, inequality (3.10) can be used to give estimates of its Maslov index. In general the mean winding number does not determine the Maslov index exactly. However this happens in dimension 2.

Proposition 3.10.2 *Let τ be the mean winding number of a T-non-resonant linear T-periodic Hamiltonian system in dimension* 2 *(i.e. $n = 1$). Let $\nu = \frac{1}{T}$ be the frequency of the system.*

1. *If $\tau = 2k\nu$, for an integer k, the T-Maslov index of the system equals $2k$.*
2. *If $\tau = (2k+\sigma)\nu$, for an integer k and $\sigma \in]0,2[$, the T-Maslov index of the systems equals $2k+1$.*

PROOF. Everything follows from the fact that ρ equals 1 in Γ^- and $\rho \neq 1$ in Γ^+. □

3.11 The Maslov index of a linear autonomous Hamiltonian system

A linear autonomous Hamiltonian system in $\mathbf{R}^{2n}$ takes the form

$$\dot{z}(t) = JBz(t) \tag{3.18}$$

where B is a symmetric matrix. This system can be considered T-periodic for every positive number T. The aim of this section is to compute its T-Maslov index, which will be denoted as $\mu_T(B)$.

The fundamental solution of system (3.18) is

$$\gamma(t) = e^{tJB}.$$

First assume that JB is semi-simple. Let S be an infinitesimally symplectic matrix in normal form symplectically similar to JB, that is

$$JB = M^{-1}SM$$

where M is symplectic. Then

$$\gamma(t) = e^{tJB} = M^{-1}e^{tS}M.$$

Since the Maslov index of a path is invariant under a symplectic change of coordinates, $\mu_T(\gamma)$ equals the T-Maslov index of the path e^{tS}. S has the form:

$$S = S_1 \oplus \dots S_h$$

where each S_j is a block in normal form as in the list of section 3.6. By assertion 3 of Proposition 3.8.1

$$\rho(e^{tS}) = \prod_{j=1}^{h} \rho(e^{tS_j}).$$

1. If

$$S_j = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$$

where $\alpha \in \mathbf{R}$, we have

$$e^{tS_j} = \begin{pmatrix} \cos\alpha t & -\sin\alpha t \\ \sin\alpha t & \cos\alpha t \end{pmatrix}$$

and $\rho(e^{tS_j}) = e^{i\alpha t}$. Moreover

$$\det(I - e^{tS_j}) = 2(1 - \cos\alpha t) \geq 0.$$

2. If

$$S_j = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}$$

where $\beta \in \mathbf{R} \setminus \{0\}$, we have

$$e^{tS_j} = \begin{pmatrix} e^{t\beta} & 0 \\ 0 & e^{-t\beta} \end{pmatrix}$$

and $\rho(e^{tS_j}) = 1$ for every t. Moreover

$$\det(I - e^{tS_j}) = 2(1 - \cosh\beta t) \leq 0.$$

3. If

$$S_j = \begin{pmatrix} \beta & -\alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & -\beta & -\alpha \\ 0 & 0 & \alpha & -\beta \end{pmatrix} \tag{3.19}$$

where $\alpha \in \mathbf{R}$, $\beta \in \mathbf{R}^+$, we have

$$e^{tS_j} = \begin{pmatrix} e^{\beta t}\cos\alpha t & -e^{\beta t}\sin\alpha t & 0 & 0 \\ e^{\beta t}\sin\alpha t & e^{\beta t}\cos\alpha t & 0 & 0 \\ 0 & 0 & e^{-\beta t}\cos\alpha t & -e^{-\beta t}\sin\alpha t \\ 0 & 0 & e^{-\beta t}\sin\alpha t & e^{-\beta t}\cos\alpha t \end{pmatrix}$$

and $\rho(e^{tS_j}) = 1$ for every t. Moreover

$$\det(I - e^{tS_j}) = 4(\cosh\beta t - \cos\alpha t)^2 \geq 0.$$

Therefore only blocks of the first kind give contribution to $\rho(e^{tS})$. We can assume that $S_1, \ldots, S_k$, $0 \leq k \leq h$, are all the blocks of the first kind:

$$S_j = \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}$$

where $\alpha_j \in \mathbf{R}$, $j = 1, \ldots, k$. Then

$$\rho(e^{tS}) = \prod_{j=1}^{k} e^{i\alpha_j t} = e^{i\sum_{j=1}^{k}\alpha_j t}.$$

Moreover e^{tS} belongs always to $\Gamma^+ \cup \Gamma^0$ or to $\Gamma^- \cup \Gamma^0$ depending on the number of blocks of the second kind, even in the first case, odd in the second case.

Assume that e^{TS} is in Γ^+. Then to compute the T-Maslov index we must join e^{TS} to $-I$. Again, it is enough to join each single block to $-I$, without introducing the eigenvalue 1. We repeat the construction introduced when we proved that Γ^+ and Γ^- are arc-connected (Proposition 3.9.2), taking in account the value of ρ along the path.

1. Consider a block

$$S_j = \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}, \quad e^{TS_j} = \begin{pmatrix} \cos\alpha_j T & -\sin\alpha_j T \\ \sin\alpha_j T & \cos\alpha_j T \end{pmatrix}.$$

Since $e^{TS_j} \notin \Gamma^0$ we must have

$$\alpha_j T = 2m_j\pi + \sigma_j$$

where $m_j \in \mathbf{Z}$ and $\sigma_j \in]0, 2\pi[$. e^{TS_j} can be joined to $-I$ by the path of rotations $R(s)$, where s goes monotonically from $\alpha_j T$ to $(2m_j + 1)\pi$. This shows that each block of the first kind S_j gives as contribution

$$\pi - \sigma_j.$$

2. Since there is an even number of blocks of the second kind, we can group them two by two, generating blocks of the form

$$\tilde{S} = \begin{pmatrix} \beta_1 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 \\ 0 & 0 & -\beta_1 & 0 \\ 0 & 0 & 0 & -\beta_2 \end{pmatrix}$$

Then

$$e^{T\tilde{S}} = \begin{pmatrix} e^{T\beta_1} & 0 & 0 & 0 \\ 0 & e^{T\beta_2} & 0 & 0 \\ 0 & 0 & e^{-T\beta_1} & 0 \\ 0 & 0 & 0 & e^{-T\beta_2} \end{pmatrix}.$$

We can join $e^{T\tilde{S}}$ to the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

by a path of matrices formed by two blocks of the second kind. The function ρ has value 1 over all this path.

Then we can join the above matrix with the matrix

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

by a path of matrices of the third kind. Again the function ρ has value 1 over all this path.

Finally we can join the above matrix with $-I$ by a path of matrices formed by two blocks of the second kind. The function ρ has still value 1 over all this path.

Therefore all the blocks of the second kind can be joined with $-I$ without affecting the value of ρ.

3. If S_j is a block of the third kind as in (3.19), we can join e^{TS_j} to the matrix

$$\begin{pmatrix} -e^{\alpha T} & 0 & 0 & 0 \\ 0 & -e^{\alpha T} & 0 & 0 \\ 0 & 0 & -e^{-\alpha T} & 0 \\ 0 & 0 & 0 & -e^{-\alpha T} \end{pmatrix}$$

by a path of matrices of the third kind. The function ρ takes value 1 over all this path.

The above matrix can be joined to $-I$ by a path of matrices formed by two blocks of the second kind. Again the function ρ has value 1 over all this path.

Therefore only blocks of the first kind give a contribution which amounts to

$$k\pi - \sum_{j=1}^{k} \sigma_j.$$

Therefore the T-Maslov index of the path e^{tS} is

$$\mu_T = \frac{1}{\pi}\sum_{j=1}^{k}(\alpha_j T + \pi - \sigma_j) = \sum_{j=1}^{k}(2m_j + 1) = \sum_{j=1}^{k}[[\frac{\alpha_j T}{\pi}]]$$

where $[[\cdot]] : \mathbf{R} \setminus 2\mathbf{Z} \mapsto 2\mathbf{Z} + 1$ is the function which approximates to the closest odd integer

$$[[x]] = 1 + 2[\frac{x}{2}].$$

Using the above relation, we have also the formula

$$\mu_T = k + 2\sum_{j=1}^{k}[\frac{\alpha_j T}{2\pi}]$$

where $[\cdot] : \mathbf{R} \mapsto \mathbf{Z}$ is the integer part.

If e^{TS} belongs to Γ^-, we have to join it with

$$W = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}.$$

Since there is an odd number of blocks of the second kind, there is at least one of them. We can join this block with

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

by a path of matrices of the second kind. The function ρ takes value 1 over all this path.

The remaining part of the matrix e^{TS} has now an even number of blocks of the first kind, and it can be joined with the $(2n-2) \times (2n-2)$ matrix $-I$ as in the previous discussion.

Therefore the same formula for μ_T holds.

By a simple perturbation argument, it is easy to see that the same formula must hold also if JB is not semi-simple.

Recall that the blocks of the first kind

$$S_j = \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}$$

have a pair of purely imaginary eigenvalues $\alpha_j i$ and $-\alpha_j i$. Moreover $\alpha_j i$ is the Krein-positive one. Since S and JB are symplectically similar, they have the same eigenvalues with the same Krein signs. From these facts we deduce the following theorem.

Theorem 3.11.1 *Let B be a real symmetric matrix. Assume that $\alpha_1 i, \ldots, \alpha_k i$ are the Krein-positive purely imaginary eigenvalues of JB, counted with algebraic multiplicity. Then the linear autonomous Hamiltonian system*

$$\dot{z}(t) = JBz(t)$$

has T-Maslov index

$$\mu_T(B) = \sum_{j=1}^{k}[[\frac{\alpha_j T}{\pi}]] = k + 2\sum_{j=1}^{k}[\frac{\alpha_j T}{2\pi}].$$

3.12 The Maslov index of a periodic solution of a non-linear system

A nonlinear T-periodic Hamiltonian system in $\mathbf{R}^{2n}$ has the form

$$\dot{z}(t) = J\nabla H(z(t), t) \tag{3.20}$$

where $H \in C^2(\mathbf{R}^{2n}; \mathbf{R})$ is T-periodic in the last variable and ∇H denotes the gradient of H with respect to the first $2n$ variables.

Assume that $z \in C^1(\mathbf{R}; \mathbf{R}^{2n})$ is a T-periodic solution of system (3.20). We can linearize system (3.20) near the solution z, obtaining the T-periodic linear Hamiltonian system

$$\dot{w}(t) = JH''(z(t), t)w(t) \tag{3.21}$$

where H'' denotes the Hessian matrix of H with respect to the first $2n$ variables.

Definition 3.12.1 *The T-periodic solution z of system* (3.20) *is said T-resonant if the linear system* (3.21) *is T-resonant.*

Definition 3.12.2 *The T-nullity of a T-periodic solution z of system* (3.20) *is the T-nullity of the linear system* (3.21). *It is denoted by* $\mathrm{null}_T(z)$.

Definition 3.12.3 *The T-Floquet multipliers of a T-periodic solution z of system* (3.20) *are the T-Floquet multipliers of the linear system* (3.21).

Definition 3.12.4 *Assume that the T-periodic solution z of the Hamiltonian system* (3.20) *is T-non-resonant. The T-Maslov index of z is the T-Maslov index of the linear system* (3.21). *It is denoted by* $\mu_T(z)$.

The mean winding number of z is the mean winding number of the linear system (3.21). *It is denoted by* $\tau(z)$.

3.13 Some bibliography and further remarks

The Maslov index of a linear periodic Hamiltonian system was introduced by Gel'fand and Lidskiǐ [GL58], who call it index of rotation. Their aim was to enumerate the connected components in the space of strongly stable linear periodic Hamiltonian systems. The approach of this pioneering paper does not differ much from our presentation.

Under the present name, the Maslov index was defined by Maslov in 1965 as a slightly different object: it is an integer associated to closed loops of Lagrangian subspaces of a symplectic vector space (see the translation from Russian of the book of Maslov's [Mas72]). The study of loops of Lagrangian subspaces turns out to be equivalent to the study of loops of symplectic automorphisms. Although it is defined as an intersection number, this Maslov index is basically the class of the given loop in $\pi_1(Sp(V, \omega)) \cong \mathbf{Z}$. The equivalence between these different descriptions was discovered by Arnold. See [Arn67] and Appendix A in the book of Maslov's [Mas72]. Connections between the Maslov index, the number of negative eigenvalues of Sturm-Lioville problems, the Morse index and the asymptotic expansions of elliptic operators can be found, for example, in [CD77], [Dui76], [RS95].

More recently the Maslov index was used by Conley and Zehnder in a famous paper [CZ84], where its variational meaning was clarified. The definition of Conley and Zehnder is somewhat different. First they define the index of an autonomous system using the formula of Theorem 3.11.1. Then they prove that in every connected component of $\mathcal{S}_T(n)$ there are paths which are fundamental solutions of autonomous systems, and that autonomous systems giving paths in the same component of $\mathcal{S}_T(n)$ have the same index. The former assertion is true only for $n \geq 2$: for this reason in the paper by Conley and Zehnder the case $n = 1$ is excluded.

Then the Maslov index was used in many papers in which Hamiltonian systems were studied with variational methods (for example [CZ86], [Lon90], [LZ90], [BF94] and [CLL97]). In particular, Long [Lon90] defines the Maslov index also for resonant systems. In some of these papers there is a mistake in the proof of the iteration formula: the rotation function is replaced by the determinant of the unitary part and we have seen that the latter function does not have nice iteration properties.

The rotation function was already in the paper by Gel'fand and Lidskiǐ and it was reintroduced by Salamon and Zehnder [SZ92]. Our presentation follows closely this latter paper.

Chapter 4

Periodic orbits of Hamiltonian systems

In this chapter we want to apply the general theory developed in Chapter 1 to the problem of finding periodic solutions of asymptotically linear periodic Hamiltonian systems.

First of all we will prove that the Maslov index of a periodic solution coincides with its E^+-Morse index, when such a solution is seen as a critical point of the action functional.

Then the E^+-Conley theory will be used to prove a well known theorem of Conley and Zehnder, which provides the Morse relations for the periodic solutions of systems which are non-resonant at infinity.

In particular, this theorem implies that a system with a periodic solution, whose Maslov index is different from the Maslov index at infinity, has at least another periodic solution. We will prove that a similar assertion holds also in a case of resonance at infinity.

Then we will study two-dimensional systems and we will prove that the existence of two or more periodic solutions forces the system to have infinitely many sub-harmonics. Such a result is strictly related with a recent result by Franks about area-preserving homeomorphisms of the disc.

Finally we will study autonomous systems and we will give a lower bound for the number $n(T)$ of T-periodic solutions, namely

$$n(T) \geq \frac{1}{2}\Theta T - M.$$

The constants Θ and M can be computed explicitly. Θ is called global twist of the system and it depends only on the mean winding number of the equilibrium solutions.

4.1 The variational formulation

We will consider the Hamiltonian system

$$\dot{z}(t) = J\nabla H(z(t), t) \tag{4.1}$$

where $H : \mathbf{R}^{2N} \times \mathbf{R} \mapsto \mathbf{R}$ (the Hamiltonian) is T-periodic in the last variable (the time variable). We are interested in the T-periodic solutions of (4.1). In order to give a variational formulation of this problem, some preliminaries are needed.

Let $L^2(\mathbf{R}/T\mathbf{Z};\mathbf{R}^{2N})$ be the space of T-periodic curves in $\mathbf{R}^{2N}$, which are square integrable. If $u \in L^2(\mathbf{R}/T\mathbf{Z};\mathbf{R}^{2N})$, it has Fourier expansion

$$u(t) = a_0 + \sum_{k=1}^{\infty}(a_k \cos\frac{2\pi kt}{T} + b_k \sin\frac{2\pi kt}{T}),$$

where

$$a_k, b_k \in \mathbf{R}^{2N} \text{ and } \sum_{k=1}^{\infty}(|a_k|^2 + |b_k|^2) < \infty.$$

Set

$$\|u\|^2 = 2\pi|a_0|^2 + \pi\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2).$$

We can define the Sobolev space

$$E = H^{\frac{1}{2}}(\mathbf{R}/T\mathbf{Z};\mathbf{R}^{2N}) = \left\{u \in L^2(\mathbf{R}/T\mathbf{Z};\mathbf{R}^{2N}) \mid \|u\| < \infty\right\}.$$

Then E is a separable Hilbert space and its norm $\|\cdot\|$ is induced by the scalar product

$$< u, u' >= 2\pi a_0 \cdot a_0' + \pi\sum_{k=1}^{\infty} k(a_k \cdot a_k' + b_k \cdot b_k').$$

See, for example, [Fri69]. We will often use the following consequence of the Sobolev embedding theorems.

Proposition 4.1.1 *For every $p \in [1, +\infty[$, E is compactly embedded into the Banach space $L^p(\mathbf{R}/T\mathbf{Z};\mathbf{R}^{2N})$.*

Now consider the symmetric bilinear form

$$\varphi(u, u') = \int_0^T J\dot{u}(t) \cdot u'(t)\, dt,$$

where u and u' are smooth T-periodic curves in $\mathbf{R}^{2N}$. It is easy to check that

$$|\varphi(u, u')| = \left|\pi\sum_{k=1}^{\infty} k(Jb_k \cdot a_k' - Ja_k \cdot b_k')\right| \leq$$

$$\leq \pi\left(\sum_{k=1}^{\infty} k|b_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} k|a_k'|^2\right)^{\frac{1}{2}} + \pi\left(\sum_{k=1}^{\infty} k|a_k|^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{\infty} k|b_k'|^2\right)^{\frac{1}{2}} \leq \|u\|\|u'\|.$$

Since the smooth curves are dense in E, φ extends uniquely to a continuous bilinear form on E. Let L be the bounded self-adjoint operator on E such that

$$\varphi(u, u') =< Lu, u' > .$$

L can be computed explicitly in terms of the Fourier expansion of u:

$$Lu = \sum_{k=1}^{\infty} (Jb_k \cos \frac{2\pi kt}{T} - Ja_k \sin \frac{2\pi kt}{T}).$$

Define the following family of subspaces of E

$$E^+(0) = \text{Span}\,\{e_1, \ldots, e_N\}, \; E^-(0) = \text{Span}\,\{e_{N+1}, \ldots, e_{2N}\},$$
$$E^\pm(k) = \left\{ (\cos \frac{2\pi}{T} kt) a \mp (\sin \frac{2\pi}{T} kt) Ja \mid a \in \mathbf{R}^{2N} \right\} \quad \forall k \geq 1,$$

where $\{e_1, \ldots, e_{2N}\}$ is the standard basis of $\mathbf{R}^{2N}$. Then $Lu = 0$ on $E^+(0) \oplus E^-(0)$, $Lu = u$ on $E^+(k)$ and $Lu = -u$ on $E^-(k)$.

Our fixed splitting $E = E^+ \oplus E^-$ will be given by

$$E^+ = \bigoplus_{k=0}^{+\infty} E^+(k), \quad E^- = \bigoplus_{k=0}^{+\infty} E^-(k).$$

Notice that L is a Fredholm operator.

We will always make the following assumption on the Hamiltonian H.

H0 $H \in C^2(\mathbf{R}^{2N} \times \mathbf{R}/T\mathbf{Z})$ has polynomial growth together with its spatial derivatives.

If $u \in E$, by assumption (H0) and Proposition 4.1.1, the function $H(u(t), t)$ is integrable in $[0, T]$ and we can define

$$b(u) = \int_0^T H(u(t), t)\, dt.$$

We recall the following definition.

Definition 4.1.1 *A map $\Phi : E \mapsto E$ is called completely continuous if it is continuous from the weak to the strong topology of E.*

Proposition 4.1.2 *Assume that (H0) holds. Then $b \in C^2(E)$ and*

$$db(u)[v] = < \nabla b(u), v > = \int_0^T \nabla H(u(t), t) \cdot v(t)\, dt, \tag{4.2}$$

$$d^2 b(u)[v, w] = < D^2 b(u) v, w > = \int_0^T D^2 H(u(t), t) v(t) \cdot w(t)\, dt. \tag{4.3}$$

Moreover the map $\nabla b : E \mapsto E$ is completely continuous.

For the proof see, for example, Proposition B.37 in [Rab86].

The above proposition implies that the functional

$$f(u) = \frac{1}{2} < Lu, u > + b(u) = \frac{1}{2} \int_0^T J\dot{u}(t) \cdot u(t)\, dt + \int_0^T H(u(t), t)\, dt \tag{4.4}$$

is twice continuously differentiable on E.

Proposition 4.1.3 *Assume that (H0) holds. Every critical point of f is twice continuously differentiable. $u \in E$ is a critical point of f if and only if it is a T-periodic solution of* (4.1). *Moreover, if u is a critical point of f, the linear operator $D^2 b(u)$ is compact.*

PROOF. If u is a critical point of f, an integration by parts gives

$$\int_0^T (-Ju(t) - \phi(t)) \cdot \dot{v}(t)\, dt = 0 \tag{4.5}$$

for every smooth T-periodic v, where

$$\phi(t) = \int_0^t \nabla H(u(s), s)\, ds$$

is continuous. The function $Ju(t) + \phi(t)$ is in $L^1([0,T]; \mathbf{R}^{2N})$ and (4.5) implies

$$-Ju(t) - \phi(t) = \text{const}.$$

Therefore u is continuous. Thus ϕ must be continuously differentiable and so must be u, which has to solve (4.1). Since $H \in C^2$, also $u \in C^2$.

Since the critical point u is continuous, the bilinear form

$$d^2 b(u)[v,w] = \int_0^T D^2 H(u(t), t) v(t) \cdot w(t)\, dt$$

is continuous in $L^2 = L^2(\mathbf{R}/T\mathbf{Z}; \mathbf{R}^{2N})$. So there exists a bounded operator B on L^2 such that

$$d^2 b(u)[v,w] = < Bv, w >_{L^2}, \quad \forall v, w \in L^2.$$

Now let $\mathcal{I} : E \to L^2$ be the immersion operator. We know from Proposition 4.1.1 that $\mathcal{I}$ is compact and so also its adjoint $\mathcal{I}^* : L^2 \to E$ must be compact. If $v, w \in E$,

$$< D^2 b(u) v, w > = < Bv, w >_{L^2} = < \mathcal{I}^* \circ Bv, w > .$$

Therefore $D^2 b(u) = \mathcal{I}^* \circ B$ is compact. □

4.2 The relation between the Maslov index and the E^+-Morse index

Assume now that the Hamiltonian system (4.1) is linear: it can be written as

$$\dot{z}(t) = JA(t) z(t) \tag{4.6}$$

where A is a T-periodic loop of symmetric matrices, of class C^1. The corresponding functional is quadratic:

$$f(u) = \frac{1}{2} < Lu, u > + \frac{1}{2} < Bu, u > \tag{4.7}$$

where B is the self-adjoint operator such that

$$< Bu, v >= \int_0^T A(t)u(t) \cdot v(t)\, dt.$$

The same argument used in the second part of the proof of Proposition 4.1.3 shows that B is compact. Since f is quadratic, 0 is a critical point of f and its E^+-Morse index is the E^+-dimension of the negative eigenspace of $L + B$.

Theorem 4.2.1 *The T-nullity of the linear system* (4.6) *equals the dimension of the kernel of the quadratic form f given by* (4.7). *In particular f is non-degenerate if and only if the linear system* (4.6) *is T-non-resonant. If this case the E^+-Morse index of the origin equals the T-Maslov index of the system* (4.6).

PROOF. Since $\nabla f(u) = Lu + Bu$, $(L + B)u = 0$ if and only if u is a critical point of f. By Proposition 4.1.3, $(L + B)u = 0$ if and only if u is a T-periodic solution of the linear system (4.6). This proves the first assertion.

Since B is compact, Proposition 1.2.3 implies that the negative eigenspace of $L + B$ is commensurable with E^-. We must determine its E^+-dimension.

We start by assuming that the system (4.6) is autonomous, that is $A(t) = A$ for every t. In this case it is easy to give an explicit expression for the operator B:

$$Bu = \frac{T}{2\pi}Aa_0 + \frac{T}{2\pi}\sum_{k=1}^{\infty}\frac{1}{k}(Aa_k \cos\frac{2\pi}{T}kt + Ab_k \sin\frac{2\pi}{T}kt).$$

Moreover we will assume that JA is in normal form with respect to the standard basis $\{e_1, \dots, e_{2N}\}$ of $\mathbf{R}^{2N}$, and that it has only blocks of the first and the second kind (see section 3.6). Hence $\mathbf{R}^{2N}$ splits into a direct sum of A-invariant symplectic sub-spaces of dimension 2:

$$\mathbf{R}^{2N} = \bigoplus_{j=1}^{N} W_j, \quad W_j = \operatorname{span}\{e_j, e_{N+j}\}.$$

By Theorem 3.8.1, the T-Maslov index of the linear system 4.6 equals the sum of the T-Maslov indices of the restrictions of the system to the spaces W_j.

The above spitting of $\mathbf{R}^{2N}$ induces a splitting of the space E:

$$E = \bigoplus_{j=1}^{N} E_j$$

where $E_j = H^{\frac{1}{2}}(\mathbf{R}/T\mathbf{Z}; W_j)$. Such a splitting is L-invariant and B-invariant. Set

$$E_j^{\pm}(k) = E^{\pm}(k) \cap E_j, \quad E_j^{\pm} = E^{\pm} \cap E_j.$$

Then the negative eigenspace of $L + B$ has the form

$$V^- = \bigoplus_{j=1}^{N} V_j^-$$

where V_j^- is the negative eigenspace of $L+B$ restricted to E_j. It is easy to see that

$$E^+\text{-}\dim V^- = \sum_{j=1}^{N} E_j^+\text{-}\dim V_j^-.$$

Therefore it is enough to prove that the E_j^+-dimension of V_j^- equals the T-Maslov index of the system restricted to W_j.

Notice that the subspaces $E_j^+(k) \oplus E_j^-(k)$, for $k \geq 0$, are $(L+B)$-invariant. Therefore the negative eigenspace of $L+B$ on E_j is

$$V_j^- = \bigoplus_{k=0}^{\infty} V_j^-(k),$$

where $V_j^-(k)$ is the negative eigenspace of $(L+B)|_{E_j^+(k)\oplus E_j^-(k)}$. We will use analogue notations for the positive eigenspace V_j^+ of $L+B$ on E_j. With these notations

$$\begin{aligned} E_j^+\text{-}\dim V_j^- &= \dim V_j^- \cap E_j^+ - \dim V_j^+ \cap E_j^- \\ = \dim \bigoplus_{k=0}^{\infty} V_j^-(k) \cap \bigoplus_{k=0}^{\infty} E_j^+(k) &- \dim \bigoplus_{k=0}^{\infty} V_j^+(k) \cap \bigoplus_{k=0}^{\infty} E_j^-(k) \\ = \sum_{k=0}^{\infty} \dim V_j^-(k) \cap E_j^+(k) &- \sum_{k=0}^{\infty} \dim V_j^+(k) \cap E_j^-(k). \end{aligned}$$

Since V_j^- and E_j^- are commensurable, the terms in the above sums are definitively null and

$$E_j^+\text{-}\dim V_j^- = \sum_{k=0}^{\infty} [\dim V_j^-(k) \cap E_j^+(k) - \dim V_j^+(k) \cap E_j^-(k)].$$

Let

$$T_k = P_{V_j^-(k)}|_{E_j^-(k)} : E_j^-(k) \to V_j^-(k).$$

Then

$$\begin{aligned} \dim V_j^-(k) \cap E_j^+(k) - \dim V_j^+(k) \cap E_j^-(k) &= \operatorname{codim}\ T_k(E_j^-(k)) - \dim \operatorname{Ker} T_k \\ &= \dim V_j^-(k) - \dim E_j^-(k). \end{aligned}$$

Therefore we get the following useful formula

$$E_j^+\text{-}\dim V_j^- = \sum_{k=0}^{\infty} (\dim V_j^-(k) - \dim E_j^-(k)). \tag{4.8}$$

On the subspace of constant curves $E_j^+(0) \oplus E_j^-(0)$

$$L + B = \frac{T}{2\pi} A|_{W_j},$$

therefore $V_j^-(0)$ is the negative eigenspace of $A|_{W_j}$.

1. Assume that

$$JA|_{W_j} = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \quad T\alpha \notin 2\pi\mathbf{Z}.$$

We can assume that $\alpha > 0$, the case $\alpha < 0$ being similar. Then

$$A|_{W_j} = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

and $L + B$ is strictly negative on $E_j^+(0) \oplus E_j^-(0)$.

On the subspace $E_j^+(k) \oplus E_j^-(k)$, $k \geq 1$, $L + B$ takes the form

$$\begin{pmatrix} (1 - \frac{T\alpha}{2k\pi})I & 0 \\ 0 & -(1 + \frac{T\alpha}{2k\pi})I \end{pmatrix}.$$

So $L + B$ is strictly negative on $E_j^-(k)$. On $E_j^+(k)$ $L + B$ is strictly positive if $\frac{T\alpha}{2k\pi} < 1$ and strictly negative if $\frac{T\alpha}{2k\pi} > 1$. Therefore

$$\dim V_j^-(k) = \begin{cases} 2 & \text{if } k > \frac{T\alpha}{2\pi} \\ 4 & \text{if } k < \frac{T\alpha}{2\pi} \end{cases}$$

and from formula (4.8) we get

$$E_j^+\text{-}\dim V_j^- = 1 + 2[\frac{T\alpha}{2\pi}],$$

where $[x]$ denotes the integer part of x.

2. Assume that

$$JA|_{W_j} = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \quad \beta \in \mathbf{R} \setminus \{0\}.$$

Then

$$A|_{W_j} = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}.$$

$A|_{W_j}$ has signature $(1, 1)$ and thus $\dim V_j^-(0) = \dim E_j^-(0) = 1$.

Let $k \geq 1$. In the basis of $E_j^+(k) \oplus E_j^-(k)$

$$\xi_1(t) = (\cos\frac{2\pi}{T}kt)e_1 - (\sin\frac{2\pi}{T}kt)Je_1, \quad \xi_2(t) = (\cos\frac{2\pi}{T}kt)e_2 - (\sin\frac{2\pi}{T}kt)Je_2$$
$$\xi_3(t) = (\cos\frac{2\pi}{T}kt)e_1 + (\sin\frac{2\pi}{T}kt)Je_1, \quad \xi_4(t) = (\cos\frac{2\pi}{T}kt)e_2 + (\sin\frac{2\pi}{T}kt)Je_2$$

$L + B$ has the form

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{k}\frac{T}{2\pi}\beta \\ 0 & 1 & \frac{1}{k}\frac{T}{2\pi}\beta & 0 \\ 0 & \frac{1}{k}\frac{T}{2\pi}\beta & -1 & 0 \\ \frac{1}{k}\frac{T}{2\pi}\beta & 0 & 0 & -1 \end{pmatrix}.$$

Therefore $(L+B)|_{E_j^+(k)\oplus E_j^-(k)}$ has signature $(2,2)$. Thus the E_j^+-dimension of V_j^- is

$$E_j^+\text{-}\dim V_j^- = \sum_{k=0}^{\infty}(\dim V_j^-(k) - \dim E_j^-(k)) = 0.$$

By our assumption on JA, there are no blocks of the third kind, so we do not have to worry about that case. From the above discussion we deduce that only blocks of the first kind give a contribution to the E_j^+-dimension of V_j^-. The amount of this contribution equals

$$1+2[\frac{T\alpha}{2\pi}]$$

which is exactly the contribution given to the T-Maslov index by the same block, as shown by Theorem 3.11.1. The thesis in the particular case chosen easily follows.

Now we examine the case of a general T-periodic linear system:

$$\dot{z}(t) = JA(t)z(t) \tag{4.9}$$

but we assume that $N \geq 2$. Notice that in this case, by Theorem 3.11.1, there exists a symmetric matrix A_1 which satisfies our special assumptions and such that the system

$$\dot{z}(t) = JA_1 z(t) \tag{4.10}$$

is T-non-resonant and has the same T-Maslov index of (4.9). By Theorem 3.9.3 the systems (4.9) and (4.10) can be joined by a continuous path of T-non-resonant systems.

Looking at the corresponding operators on E, we get the following result: the operator $L+B$, corresponding to (4.9), can be joined to the operator $L+B_1$, corresponding to (4.10), by a continuous path of invertible self-adjoint operators. Then Proposition 1.2.2 implies that the E^+-dimensions of the negative eigenspaces of $L+B$ and $L+B_1$ coincide. The thesis follows.

The proof in the case $N=1$ is not complete because there are no autonomous linear systems with non-zero and even Maslov index. So let

$$\dot{z}(t) = JA(t)z(t) \tag{4.11}$$

be a two dimensional system with T-Maslov index $2m$. Then, by the additivity property of the Maslov index, the four-dimensional system

$$\begin{cases} \dot{z}_1(t) = JA(t)z_1(t) \\ \dot{z}_2(t) = JA(t)z_2(t) \end{cases} \tag{4.12}$$

has T-Maslov index $4m$. By what we proved before, the negative eigenspace of the self-adjoint operator T corresponding to system (4.12) has E^+-dimension $4m$. However, it is clear that the E^+-dimension of the negative eigenspace of T is the double of the E^+-dimension of the negative eigenspace of the operator corresponding to system (4.11). □

The above theorem has the following fundamental corollary.

Corollary 4.2.2 *If z is a T-periodic solution of* (4.13) *then*

$$\mathrm{null}_T(z) = \dim \mathrm{Ker}\, D^2 f(z).$$

In particular a T-periodic solution of (4.13) *is T-non-resonant if and only if it is a non-degenerate critical point of the functional f. In this case the T-Maslov index of z equals its Morse index:*

$$\mu_T(z) = E^+\text{-m}\,(z).$$

The above theorems allow to enlarge the definition of Maslov index so to include the resonant cases.

Definition 4.2.1 *The T-Maslov index of the possibly T-resonant linear system* (4.6) *is the E^+-Morse index of the origin for the quadratic functional* (4.7)*. The T-Maslov index of the possibly T-resonant T-periodic solution z is the E^+-Morse index of z as a critical point of the functional* (4.4)*.*

It is also possible to give a geometric definition of the Maslov index for resonant systems (see [Lon90]).

4.3 Analytical properties of the functional f

In this section we want to show that, under suitable assumptions on H, the functional f satisfies the conditions (A1), (A2), (A3) and (A4) of section 1.4, so that it is possible to apply the E^+-Conley theory to f.

The following lemma provides an useful families of functions with E^+-locally compact sublevels.

Lemma 4.3.1 *Let V be a finite codimensional subspace of E^+. If $\sigma, \theta, \lambda^+, \lambda^-$ are positive constants, the function*

$$g(u) = \lambda^+ \|P_V u\|^\sigma - \lambda^- \|P_{V^\perp} u\|^\theta$$

has E^+-locally compact sublevels.

PROOF. It is enough to show that g has V-locally compact sublevels. Let W be a finite dimensional subspace of $V^\perp$. We must show that for every $a \in \mathbf{R}$, the set

$$g^a \cap (V \oplus W) = \{u \in V \oplus W \mid g(x) \leq a\}$$

is weakly locally compact. Since W has finite dimension, the function

$$h : V \oplus W \mapsto \mathbf{R}, \quad h(u) = \|P_W u\|$$

is weakly continuous. Therefore the set

$$U_R = \{u \in V \oplus W \mid \|P_W u\| < R\}$$

is weakly open in $V \oplus W$ for every R. If $u \in g^a \cap U_R$,

$$\|P_V u\|^\sigma \leq \frac{\lambda^-}{\lambda^+}\|P_W u\|^\theta + a < \frac{\lambda^-}{\lambda^+}R^\theta + a$$

and thus $g^a \cap U_R$ is bounded. Therefore the weak closure of $g^a \cap U_R$ in $V \oplus W$ is weakly compact and $g^a \cap (V \oplus W)$ is weakly locally compact. □

Proposition 4.3.2 *Assume that H satisfies (H0). Then the sublevels of f are $\mathcal{T}_{E^+}$-closed. If H has quadratic growth, the sublevels of f are also E^+-locally compact, and so (A1) holds.*

PROOF. First we want to verify that

$$f(u) = \frac{1}{2} < Lu, u > + b(u)$$

is $\mathcal{T}_{E^+}$-lower-semicontinuous. Its quadratic part is convex on E^+, and thus weakly lower-semicontinuous on E^+. Since it is strongly continuous on E^-, it is $\mathcal{T}_{E^+}$-lower-semicontinuous. The assumption (H0) implies that b is continuous on L^p, if p is large enough. By Proposition 4.1.1, b is weakly continuous on E, and therefore also $\mathcal{T}_{E^+}$-continuous. We conclude that f is $\mathcal{T}_{E^+}$-lower-semicontinuous.

To prove that f has E^+-locally compact sublevels, it is enough to prove that $f \geq h$, where h is a function with E^+-locally compact sublevels.

Let V^0, V^+ and V^- be the kernel, the positive and the negative eigenspaces of L. Since H has quadratic growth

$$b(u) \geq -C_1\|u\|_{L^2}^2.$$

Since E embeds compactly into L^2, for every $\epsilon > 0$ we can find a finite codimensional subspace V of V^+ such that

$$\|u\|_{L^2}^2 \leq \epsilon\|u\|^2 \quad \forall u \in V.$$

Then

$$\begin{aligned} f(u) = \frac{1}{2}\|P_{V^+}u\|^2 - \frac{1}{2}\|P_{V^-}u\|^2 + b(u) &\geq \frac{1}{2}\|P_{V^+}u\|^2 - \frac{1}{2}\|P_{V^-}u\|^2 - C_1\|u\|_{L^2}^2 \\ \geq \frac{1}{2}\|P_V u\|^2 - \frac{1}{2}\|P_{V^-}u\|^2 &- 2\epsilon C_1\|P_V u\|^2 - 2C_1\|P_{V^\perp}u\|_{L^2} \\ &\geq (\frac{1}{2} - 2\epsilon C_1)\|P_V u\|^2 - C_2\|P_{V^\perp}u\|^2 \end{aligned}$$

which has E^+-locally compact sublevels, when ϵ is small enough, by Lemma 4.3.1. Therefore also f has E^+-locally compact sublevels and (A1) is proved. □

Testing the Palais-Smale condition is a standard task.

Proposition 4.3.3 *Assume that (H0) holds. Then all the bounded Palais-Smale sequences are pre-compact. So (A2) holds.*

PROOF. Assume that $(u_n) \subset E$ is a bounded Palais-Smale sequence. Since it is bounded, it has a subsequence (u_{n_k}) which converges weakly to a certain u.

Let E^0 be the kernel of L. We recall that E^0 is finite dimensional. Since $\nabla f(u_{n_k}) \to 0$, we get

$$(L + P_{E^0})u_{n_k} - P_{E^0}u_{n_k} + \nabla b(u_{n_k}) \to 0.$$

By Proposition 4.1.2, ∇b is completely continuous and thus the sequence

$$\nabla b(u_{n_k}) - P_{E^0}u_{n_k}$$

is strongly converging. So $(L + P_{E^0})u_{n_k}$ converges and, since $L + P_{E^0}$ is invertible, also u_{n_k} must converge. □

Proposition 4.3.4 *Assume that H is Lipschitz in the spatial variables, uniformly with respect to the time variable. Then ∇f is Lipschitz and (A3) holds.*

PROOF. It is enough to show that ∇b is Lipschitz. If H is K-Lipschitz in the spatial variables,

$$\|\nabla b(u) - \nabla b(v)\| = \sup_{\|w\|=1} |db(u)[w] - db(v)[w]| \leq$$

$$\leq \sup_{\|w\|=1} \int_0^T |\nabla H(u(t),t) - \nabla H(v(t),t)||w(t)|\,dt \leq K\|u-v\|_{L^2} \sup_{\|w\|=1} \|w\|_{L^2}.$$

The thesis follows from the fact that E is continuously embedded into L^2. □

Proposition 4.3.5 *Assume that H is Lipschitz in the spatial variables, uniformly with respect to the time variable. Then the flow of $-\nabla f$ is an E^+-homotopy, and (A4) holds.*

PROOF. By induction, we can define a sequence of flows

$$\begin{cases} \eta_0(u,t) = u, \\ \eta_n(u,t) = u - \int_0^t \nabla f(\eta_{n-1}(u,s))\,ds, \quad n \geq 1. \end{cases}$$

It is a standard fact in the theory of ordinary differential equations in Banach spaces that, since, by Proposition 4.3.4, ∇f is Lipschitz, η_n converges uniformly on the bounded subsets of $E \times \mathbf{R}$ to the solution $\eta : E \times \mathbf{R} \mapsto E$ of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}\eta(u,t) = -\nabla f(\eta(u,t)), \\ \eta(u,0) = u. \end{cases}$$

Since ∇f maps bounded sets into bounded sets, so does η_n. Therefore also η maps bounded sets into bounded sets.

Since ∇f is $\mathcal{T}_{E^+}$-continuous, also η_n is $\mathcal{T}_{E^+}$-continuous. To show that also η is $\mathcal{T}_{E^+}$-continuous, it is enough to show that both $P_{E^+} \circ \eta$ and $g_v(u,t) =< \eta(u,t), v >$ are $\mathcal{T}_{E^+}$-continuous, for every $v \in E^-$.

Both $P_{E^+} \circ \eta_n$ and $g_v^n(u,t) =< \eta_n(u,t), v >$, $v \in E^-$, are $\mathcal{T}_{E^+}$-continuous. Moreover $(P_{E^+} \circ \eta_n)$ and (g_v^n) converge uniformly on bounded sets to $P_{E^+} \circ \eta$ and g_v, respectively. Therefore $P_{E^+} \circ \eta$ and g_v are $\mathcal{T}_{E^+}$-continuous, for every $v \in E^-$.

Since η solves the following non-homogeneous equation

$$\frac{\partial}{\partial t}\eta(u,t) + L\eta(u,t) = -\nabla b(\eta(u,t)),$$

it can be represented as

$$\eta(u,t) = e^{-tL}u - \int_0^t e^{(s-t)L}\nabla b(\eta(u,s))\, ds.$$

e^{-tL} is a continuous path of invertible operators which preserve the splitting $E = E^+ \oplus E^-$. Set

$$K(u,t) = -\int_0^t e^{(s-t)L}\nabla b(\eta(u,s))\, ds.$$

If $X \subset E$ is bounded and $T > 0$, $\eta(X \times [-T,T])$ is bounded, as we have showed before. Therefore $\nabla b(\eta(X \times [-T,T]))$ is $\mathcal{T}_{E^+}$-pre-compact. Since $e^{(s-t)L}$ is $\mathcal{T}_{E^+}$-continuous, we conclude that $K(X \times [-T,T])$ is $\mathcal{T}_{E^+}$-pre-compact.

Finally, since $\eta(\cdot,t)$ is a diffeomorphism whose inverse is $\eta(\cdot,-t)$,

$$(\eta|_{E\times[-T,T]})^{-1}(X) = \eta(X \times [-T,T])$$

must be bounded for every bounded X. □

4.4 Asymptotically linear systems

A T-periodic Hamiltonian system

$$\dot{z}(t) = J\nabla H(z(t),t) \tag{4.13}$$

is said asymptotically linear if

H1 there exists a T-periodic loop of symmetric matrices $A_\infty(t)$ of class C^2 and a function $G \in C^2(\mathbf{R}^{2N} \times \mathbf{R})$ such that

$$\nabla H(\xi,t) = A_\infty(t)\xi + \nabla G(\xi,t)$$

where $\nabla G(\xi,t) = o(|\xi|)$ for $|\xi| \to \infty$, uniformly with respect to t.

Moreover we will assume

H2 D^2G is bounded.

Notice that (H1) and (H2) imply (H0).

The T-nullity, the T-Maslov index and the mean winding number of the linear system

$$\dot{w}(t) = JA_\infty(t)w(t) \tag{4.14}$$

are said T-nullity at infinity, T-Maslov index at infinity and mean winding number at infinity and they are denoted by $\mathrm{null}_T(\infty)$, $\mu_T(\infty)$ and $\tau(\infty)$.

The system (4.13) is said T-resonant at infinity if (4.14) is T-resonant, T-non-resonant at infinity in the opposite case.

Let B_∞ be the self-adjoint operator on E such that

$$< B_\infty u, v >= \int_0^T A_\infty(t)u(t) \cdot v(t)\, dt$$

and let

$$g(u) = \int_0^T G(u(t), t)\, dt.$$

Therefore

$$f(u) = \frac{1}{2} < (L + B_\infty)u, u > +g(u).$$

Since B_∞ is compact, $L + B_\infty$ is a Fredholm operator. Let V_∞^0, V_∞^+ and V_∞^- be the kernel, the positive and the negative eigenspace of $L + B_\infty$. Let P_∞^0, P_∞^+ and P_∞^- be the orthogonal projections onto V_∞^0, V_∞^+ and V_∞^-.

If the system (4.13) is T-non-resonant at infinity, the set of critical points of f is compact and it is easy to exhibit an E^+-index pair for it, as the following proposition shows.

Proposition 4.4.1 *Assume that (H1) holds. If the system* (4.13) *is T-non-resonant at infinity, the critical set K of f is compact and*

$$(\overline{B_{V_\infty^+}}(R) \times \overline{B_{V_\infty^-}}(R), \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^-}(R))$$

is an E^+-index pair for K, provided R is large enough.

PROOF. Since the system is T-non-resonant at infinity, the operator $L+B_\infty$ is invertible and $V_\infty^0 = \{0\}$. Let $\alpha > 0$ be such that $L + B_\infty \geq \alpha I$ on V_∞^+ and $L + B_\infty \leq -\alpha I$ on V_∞^-.

Moreover, since B_∞ is compact, Proposition 1.2.3 implies that V_∞^+ and V_∞^- are commensurable with E^+ and E^-, respectively.

By Proposition 4.3.3, all the bounded Palais-Smale sequences are pre-compact: in order to prove that K is compact, it is enough to show that it is bounded.

This fact and the remaining part of the thesis will follow from Proposition 1.4.3 if we can prove (1.10) and (1.11). So let $x \in \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^-}(R)$. Then

$$\begin{aligned} \frac{d}{dt}\|P_\infty^-\eta(t,x)\|^2|_{t=0} &= -2 < \nabla f(x), P_\infty^- x > \\ = -2 < (L + B_\infty)P_\infty^- x, P_\infty^- x > &-2 < \nabla g(x), P_\infty^- x > \\ &\geq 2\alpha\|P_\infty^- x\|^2 - 2 < \nabla g(x), P_\infty^- x > . \end{aligned}$$

For every $\epsilon > 0$, by (H1) we can find r such that

$$|\nabla G(\xi, t)| \leq \epsilon|\xi| \quad \text{if} \quad |\xi| \geq r.$$

Let C_1 be such that

$$|\nabla G(\xi, t)| \leq C_1 \quad \text{if} \quad |\xi| \leq r.$$

Then

$$| < \nabla g(x), v > | \leq \int_{\{t||x(t)|\geq r\}} |\nabla G(x(t), t)||v(t)|\, dt +$$
$$+ \int_{\{t||x(t)|\leq r\}} |\nabla G(x(t), t)||v(t)|\, dt \leq \epsilon\|x\|_{L^2}\|v\|_{L^2} + C_1\|v\|_{L^1}.$$

Since E embeds continuously into L^1 and into L^2, we have shown that for every positive ϵ there exists C such that

$$| < \nabla g(x), v > | \leq \epsilon\|x\|\|v\| + C\|v\|.$$

Therefore

$$\frac{d}{dt}\|P_\infty^- \eta(t,x)\|^2|_{t=0} \geq 2\alpha\|P_\infty^- x\|^2 - 2\epsilon\|x\|\|P_\infty^- x\| - 2C\|P_\infty^- x\|$$
$$\geq 2(\alpha - \epsilon)\|P_\infty^- x\|^2 - 2\epsilon\|P_\infty^+ x\|\|P_\infty^- x\| - 2C\|P_\infty^- x\| \geq 2(\alpha - 2\epsilon)R^2 - 2CR$$

which is positive when ϵ is small and R is large enough. (1.11) is proved in the same way.

□

Here is a proof in our framework of a famous result of Conley and Zehnder (see [CZ84]).

Theorem 4.4.2 *Assume that H satisfies (H1) and (H2). Assume that the system* (4.13) *is T-non-resonant at infinity. If all the T-periodic solutions of* (4.13) *are T-non-resonant then there is an odd number of them and the following relation holds*

$$\sum \lambda^{\mu_T(z)} = \lambda^{\mu_T(\infty)} + (1 + \lambda)Q(\lambda) \tag{4.15}$$

where the sum is taken over all the T-periodic solutions and Q is a Laurent polynomial with positive coefficients.

PROOF. Notice that (H1) implies that H has quadratic growth. (H2) implies that ∇H is Lipschitz in the spatial variables. Therefore Propositions 4.3.2, 4.3.3, 4.3.4 and 4.3.5 show that we can apply the E^+-Conley theory to the functional f.

By Proposition 4.4.1, if R is large enough, the pair

$$(\overline{B_{V_\infty^+}}(R) \times \overline{B_{V_\infty^-}}(R), \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^-}(R))$$

is an E^+-index pair for the critical set of f, which is compact.

By Theorem 1.7.1, the E^+-Poincaré polynomial of the above E^+-pair is the monomial $\lambda^{E^+\text{-dim}\, V_\infty^-}$. Then the Morse relations of Corollary 1.8.3 are

$$\sum \lambda^{E^+\text{-m}\,(z)} = \lambda^{E^+\text{-dim}\, V_\infty^-} + (1+\lambda)Q(\lambda)$$

where the sum is taken over all the T-periodic solutions and Q is a Laurent polynomial with positive coefficients.

By Theorem 4.2.1 and by Corollary 4.2.2,

$$E^+\text{-dim}\, V_\infty^- = \mu_T(\infty), \quad E^+\text{-m}\,(z) = \mu_T(z)$$

and the thesis follows. □

4.5 Systems with resonance at infinity

Conley and Zehnder's Theorem 4.4.2 has the following consequence: assume that an asymptotically linear system, non-resonant at infinity, has a non-resonant periodic solution z_0 such that

$$\mu_T(z_0) \neq \mu_T(\infty).$$

Then the system must have another T-periodic solution. In fact the Laurent polynomial Q cannot be zero in the relation (4.15).

In this section we want to show that a similar result holds also when the system is resonant at infinity. We shall need stronger assumptions on the Hamiltonian H.

H3 (H1) holds and $|D^2G(\xi,t)|$ tends to zero for $|\xi| \to \infty$, uniformly with respect to t.

Notice that (H3) implies (H2). Moreover we will assume

H4 ∇G is bounded.

Here is our result.

Theorem 4.5.1 *Assume that (H3) and (H4) hold. Assume that z_0 is a T-non-resonant T-periodic solution for the system* (4.13). *If $\mu_T(z_0)$ does not belong to the interval*

$$[\mu_T(\infty) - 1, \mu_T(\infty) + \text{null}_T(\infty) + 1]$$

the system has at least another T-periodic solution.

PROOF. Let $\varphi \in C^\infty(\mathbf{R})$ be a non-decreasing function such that $\varphi(s) = 0$ for $s \leq 0$ and $\varphi(s) = s$ for $s \geq 1$.

For $R > \|z_0\|$, we define two new functionals on E, f_R^+ and f_R^-, of class C^2 as

$$f_R^\pm(u) = f(u) \pm \varphi(\|P_\infty^0 u\|^2 - R^2).$$

The gradients of these functionals are

$$\nabla f_R^{\pm}(u) = (L + B_\infty)u + \nabla g(u) \pm 2\varphi'(\|P_\infty^0 u\|^2 - R^2)P_\infty^0 u. \tag{4.16}$$

Critical points u of $f_R^{\pm}$ such that $\|P_\infty^0 u\| \leq R$ are also critical points of f, and thus T-periodic solutions. As we will see, these perturbed functionals satisfy the Palais-Smale condition.

Arguing by contradiction, we assume that these functionals have no critical points u such that $\|P_\infty^0 u\| \leq R$, apart from z_0. Then we would like to have estimates on the E^+-Morse index of critical points v such that $\|P_\infty^0 v\| > R$, provided R is large enough. The E^+-Morse relations would finally imply the existence of a critical point $u \neq 0$ which does not fit these estimates, and thus $\|P_\infty^0 u\| \leq R$, which is a contradiction.

However, since the critical points of $f_R^{\pm}$ need not to be non-degenerate, we will have to make a second perturbation, in the spirit of some results by Marino and Prodi in [MP75].

By (H4), there exists a constant C such that

$$\|\nabla g(u)\| \leq C \quad \forall u \in E. \tag{4.17}$$

Let $\alpha > 0$ be such that

$$\|((L + B_\infty)|_{V_\infty^+ \oplus V_\infty^-})^{-1}\| \leq \frac{1}{\alpha}. \tag{4.18}$$

Lemma 4.5.2 *Assume that (H3) and (H4) hold. For every $\epsilon > 0$ and for every $M > 0$ there exists $Q \in \mathbf{R}$, such that, if $\|P_\infty^{\pm} u\| \leq M$ and $\|P_\infty^0 u\| \geq Q$,*

$$\|D^2 g(u)\| < \epsilon.$$

PROOF. Since $D^2 g(u)$ is self-adjoint, we must show that

$$|D^2 g(u)[v, v]| \leq \epsilon \|v\|^2 \quad \forall v \in E.$$

Since E embeds continuously into L^p, there exists a constant c_p such that

$$\|v\|_{L^p} \leq c_p \|v\|.$$

Let k be the supremum of $|D^2 G(\xi, t)|$ on $\mathbf{R}^{2N} \times \mathbf{R}$ and let r be such that

$$|D^2 G(\xi, t)| \leq \frac{1}{2c_2^2}\epsilon \quad \text{if } |\xi| \geq r.$$

Therefore

$$\begin{aligned} |D^2 g(u)[v, v]| &\leq \int_0^T |D^2 G(u(t), t)||v(t)|^2\, dt \\ \leq \frac{1}{2c_2^2}\epsilon \|v\|_{L^2}^2 + k \int_{\Gamma_r} |v(t)|^2\, dt &\leq \frac{1}{2}\epsilon \|v\|^2 + k \int_{\Gamma_r} |v(t)|^2\, dt, \end{aligned} \tag{4.19}$$

where $\Gamma^r = \{t \in [0, T] \mid |u(t)| \leq r\}$.

We must estimate the last term. By Hölder inequality,

$$\int_{\Gamma_r} |v(t)|^2\, dt \le \lambda(\Gamma_r)^{\frac{1}{2}} \|v\|_{L^4}^2 \le c_4^2 \lambda(\Gamma_r)^{\frac{1}{2}} \|v\|^2, \tag{4.20}$$

where λ denotes the Lebesgue measure.

Set $\tilde{u} = P_\infty^+ u + P_\infty^- u$ and $u^0 = P_\infty^0 u$, so that

$$u(t) = \tilde{u}(t) + u^0(t).$$

By our assumptions $\|\tilde{u}\| \le \sqrt{2}M$. Let s be such that for every $w \in V_\infty^+ \oplus V_\infty^-$ with $\|w\| \le \sqrt{2}M$ there holds

$$\lambda(\{t \in [0,T] \mid |w(t)| \ge s\}) \le \frac{\epsilon^2}{4k^2 c_4^4}. \tag{4.21}$$

Then

$$\lambda(\Gamma^r) \le \lambda(\{t \in [0,T] \mid |\tilde{u}(t)| \ge s\}) + \lambda(\big\{t \in [0,T] \mid |u^0(t)| \le r + s\big\}). \tag{4.22}$$

The subspace V_∞^0 consists of the T-periodic solutions of the linear system (4.14). Therefore V_∞^0 is a finite dimensional subspace of $C^0([0,T]; \mathbf{R}^{2N})$ such that every non-zero curve never vanishes. Then it is easy to see that there exists $q > 0$ such that

$$|v(t)| > r + s \quad \forall v \in V_\infty^0 \text{ such that } \|v\|_\infty \ge q. \tag{4.23}$$

Since V_∞^0 is finite dimensional, the norms $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on V_∞^0: there exists $e > 1$ such that

$$\frac{1}{e}\|y\|_\infty \le \|y\| \le e\|y\|_\infty \quad \forall y \in V_\infty^0.$$

Set $Q = eq$. If $\|P_\infty^0 u\| = \|u^0\| \ge Q$ then $\|u^0\|_\infty \ge q$ and by (4.21), (4.22), (4.23)

$$\lambda(\Gamma^r)^{\frac{1}{2}} \le \frac{\epsilon}{2kc_4^2}. \tag{4.24}$$

The thesis follows from (4.19), (4.20) and (4.24). □

Lemma 4.5.3 *Assume that (H3) and (H4) hold. For every $M > 0$, there exists $Q \in \mathbf{R}$, not depending on R, such that the following property holds: if $\|P_\infty^+ u\| \le M$, $\|P_\infty^- u\| \le M$ and $\|P_\infty^0 u\| \ge Q$, we have*

$$\begin{aligned} E^+\text{-}\dim V_u^- &\ge E^+\text{-}\dim V_\infty^-, \\ E^+\text{-}\dim V_u^- \oplus V_u^0 &\le E^+\text{-}\dim V_\infty^- \oplus V_\infty^0. \end{aligned}$$

where V_u^0 and V_u^- are the kernel and the negative eigenspace of $D^2 f_R^\pm(u)$.

PROOF. Notice that $D^2 f_R^{\pm}(u) = D^2 f(u)$ on $V_\infty^+ \oplus V_\infty^-$. By Lemma 4.5.2 we can find Q such that

$$\|D^2 g(u)\| < \alpha,$$

where α is the constant introduced in (4.18). Therefore $D^2 f_R^{\pm}(u)$ is strictly negative on V_∞^- and strictly positive on V_∞^+. The thesis follows from Proposition 1.2.1. □

Now let $M = \frac{2C}{\alpha}$, with C and α as in (4.17) and (4.18), and choose Q as in Lemma 4.5.3. We may assume that $\|P_\infty^0 z_0\| < Q$.

We assume, by contradiction, that the functionals f_Q^+ and f_Q^- have no critical points u such that $\|P_\infty^0 u\| \leq Q$, apart from z_0.

Lemma 4.5.4 *The maps ∇f_Q^+ and ∇f_Q^- are proper Fredholm maps of index 0.*

PROOF. Notice that $\nabla f_Q^{\pm}$ can be written as

$$\nabla f_Q^{\pm}(u) = (L + B_\infty \pm 2P_\infty^0)u + \nabla g(u) \pm 2[\varphi'(\|P_\infty^0 u\|^2 - Q^2) - 1]P_\infty^0 u.$$

The first term is an invertible linear operator. The last two terms are bounded completely continuous maps. Maps of this kind are proper. In fact, assume that $\Phi(x) = Tx + \psi(x)$ is such a map. Assume that $(\Phi(x_n))$ converges. We must show that (x_n) has a converging subsequence.

Since ψ is bounded, (Tx_n) is bounded and so also (x_n) is bounded. Thus there exists a subsequence (x_{n_k}) which converges weakly. Since ψ is completely continuous, $(\psi(x_{n_k}))$ converges strongly, and so also (Tx_{n_k}) must converge strongly. Therefore (x_{n_k}) is the required converging subsequence.

Moreover the differentials of the last two terms are compact self-adjoint operators. Therefore $D^2 f_Q^{\pm}(u)$ is a Fredholm operator of index 0, for every $u \in E$. □

The above lemma implies that the critical set of $f_Q^{\pm}$ is compact. Since, by our assumption, $f_Q^{\pm}$ has no critical points u such that $\|P_\infty^0 u\| = Q$, there exists $\epsilon > 0$ such that there are no critical points u such that

$$Q \leq \|P_\infty^0 u\| \leq Q + \epsilon.$$

Let $\tilde{\omega} \in C^\infty(\mathbf{R})$ be a non decreasing function such that $\tilde{\omega}(s) = 0$ for $s \leq Q^2$ and $\tilde{\omega}(s) = 1$ for $s \geq (Q+\epsilon)^2$. Set

$$\omega(u) = \tilde{\omega}(\|P_\infty^0 u\|^2) \quad \forall u \in E.$$

ω is a smooth functional on E. $\nabla\omega(u)$ always belongs to V_∞^0 and there exists a constant D such that

$$\|\nabla\omega(u)\| \leq D \quad \forall u \in E. \tag{4.25}$$

For $y \in E$, we define two new functionals h_y^+ and h_y^- on E as

$$h_y^{\pm}(u) = f_Q^{\pm}(u) - \omega(u) < u, y > .$$

The gradients of these functionals are

$$\nabla h_y^{\pm}(u) = \nabla f_Q^{\pm}(u) - \nabla\omega(u) < u, y > -\omega(u)y. \tag{4.26}$$

Lemma 4.5.5 *If $\|y\|$ is small enough, for every critical point u of $h_y^{\pm}$ there holds*

$$\|P_\infty^+ u\| \le M = \frac{2C}{\alpha}, \quad \|P_\infty^- u\| \le M.$$

PROOF. From (4.16) and (4.26) we get

$$0 = dh_y^{\pm}(u)[P_\infty^+ u] =< (L + B_\infty)P_\infty^+ u, P_\infty^+ > + dg(u)[P_\infty^+ u] +$$
$$-\omega(u) < y, P_\infty^+ u > \ge \alpha\|P_\infty^+\|^2 - C\|P_\infty^+ u\| - \|y\|\|P_\infty^+ u\|.$$

Therefore

$$\|P_\infty^+ u\| \le \frac{C + \|y\|}{\alpha}$$

and it is enough to assume $\|y\| \le C$. A similar argument works for the estimate of $\|P_\infty^- u\|$. □

Lemma 4.5.6 *If $\|y\|$ is small enough, there are no critical points u of h_y^+ or h_y^- such that*

$$Q \le \|P_\infty^0 u\| \le Q + \epsilon.$$

PROOF. By Lemma 4.5.5, we can assume that $\|y\|$ is so small that

$$\|P_\infty^+ u\| \le M, \quad \|P_\infty^- u\| \le M.$$

for every critical point u of $h_y^{\pm}$.

Since $f_Q^{\pm}$ has no critical point v such that

$$Q \le \|P_\infty^0 v\| \le Q + \epsilon.$$

by Lemma 4.5.4 we can find a positive constant m such that

$$\|\nabla f_Q^{\pm}(v)\| \ge m \quad \text{if } Q \le \|P_\infty^0 v\| \le Q + \epsilon.$$

Therefore, if u is a critical point of $h_y^{\pm}$ such $Q \le \|P_\infty^0 u\| \le Q + \epsilon$, by (4.25) and (4.26) we get

$$0 = \|\nabla h_y^{\pm}(u)\| = \|\nabla f_Q^{\pm}(u) - \nabla\omega(u) < u, y > -\omega(u)y\|$$
$$\ge \|\nabla f_Q^{\pm}(u)\| - \|y\|(\|\nabla\omega(u)\|\|u\| + \omega(u)) \ge m - \|y\|(D\sqrt{2M^2 + (Q+\epsilon)^2}) + 1).$$

We arrive to a contradiction, provided

$$\|y\| < \frac{m}{D\sqrt{2M^2 + (Q+\epsilon)^2} + 1}.$$

□

Lemma 4.5.7 *The functional h_y^+ and h_y^- satisfy the assumptions (A1), (A2), (A3) and (A4) of section 3.*

PROOF. We have

$$h_y^{\pm}(u) = f(u) \pm \varphi(\|P_\infty^0 u\|^2 - Q^2) - \omega(u) < u, y > .$$

f is $\mathcal{T}_{E^+}$-lower-semi-continuous by Proposition 4.3.2. The last two terms are weakly continuous, and thus also $\mathcal{T}_{E^+}$-continuous. So the sublevels of $h_y^{\pm}$ are $\mathcal{T}_{E^+}$-closed. Arguing as in the proof of Proposition 4.3.2, it is easy to see that the sublevels of $h_y^{\pm}$ are E^+-locally compact. This proves (A1).

Assume that u_n is a bounded Palais-Smale sequence for $h_y^{\pm}$. We can assume that it converges weakly. Since ω is weakly continuous and $\nabla\omega$ is completely continuous, by (4.26) we get that $\nabla f_Q^{\pm}(u_n)$ converges strongly. By Lemma 4.5.4, u_n must converge strongly and (A2) is proved.

A simple computation proves (A3).

Since $h_y^{\pm}$ can be seen as the sum of the quadratic form $\frac{1}{2} < Lu, u >$ and of a completely continuous function, the proof of (A4) goes as the proof of Proposition 4.3.5. □

Lemma 4.5.8 *The critical set K^+ of h_y^+ is compact and*

$$(\overline{B_{V_\infty^+ \oplus V_\infty^0}}(R) \times \overline{B_{V_\infty^-}}(R), \overline{B_{V_\infty^+ \oplus V_\infty^0}}(R) \times \partial B_{V_\infty^-}(R)) \tag{4.27}$$

is an E^+-index pair for K^+, with respect to the functional h_y^+, provided R is large enough and $\|y\|$ is small enough.

The critical set K^- of h_y^- is compact and

$$(\overline{B_{V_\infty^+}}(R) \times \overline{B_{V_\infty^- \oplus V_\infty^0}}(R), \overline{B_{V_\infty^+}}(R) \times \partial B_{V_\infty^- \oplus V_\infty^0}(R))$$

is an E^+-index pair for K^-, with respect to the functional h_y^-, provided R is large enough and $\|y\|$ is small enough.

PROOF. Let η^+ be the flow of the vector field $-\nabla h_y^+$, and let η^- be the flow of the vector field $-\nabla h_y^-$.

The thesis for h_y^+ will follow from Proposition 1.4.3, with $V^+ = V_\infty^+ \oplus V_\infty^0$ and $V^- = V_\infty^-$, if we can prove (1.10) and (1.11).

So let $x \in B_{V_\infty^+ \oplus V_\infty^0}(R) \times \partial B_{V_\infty^-}(R)$. Then by (4.17), (4.18) and (4.26)

$$\begin{aligned} \frac{d}{dt}\|P_\infty^- \eta^+(t,x)\|^2|_{t=0} &= -2 < \nabla h_y^+(x), P_\infty^- x > \\ = -2 < (L + B_\infty)P_\infty^- x, P_\infty^- x > &-2 < \nabla g(x), P_\infty^- x > -\omega(x) < y, P_\infty^- x > \\ &\geq 2\alpha\|P_\infty^- x\|^2 - (2C + \|y\|)\|P_\infty^- x\| = R(2\alpha R - 2C - \|y\|). \end{aligned}$$

Therefore (1.10) holds when $R > \frac{2C + \|y\|}{2\alpha}$.

Now let $x \in \partial B_{V_\infty^+ \oplus V_\infty^0}(R) \times B_{V_\infty^-}(R)$. Again, using also (4.25),

$$\begin{aligned} \frac{d}{dt}\|(P_\infty^+ + P_\infty^0)\eta^+(t,x)\|^2|_{t=0} &= -2 < \nabla h_y^+(x), (P_\infty^+ + P_\infty^0)x > \\ = -2 < (L + B_\infty)P_\infty^+ x, P_\infty^+ x > &-2 < \nabla g(x), (P_\infty^+ + P_\infty^0)x > + \\ -4\varphi'(\|P_\infty^0 x\|^2 - Q^2)\|P_\infty^0\|^2 &- < x, y >< \nabla\omega(x), P_\infty^0 x > + \\ -\omega(x) < y, (P_\infty^+ + P_\infty^0)x > \leq -2\alpha\|P_\infty^+ x\|^2 &- 4\varphi'(\|P_\infty^0 x\|^2 - Q^2)\|P_\infty^0 x\|^2 + \\ &+(2C + \|y\|)\|(P_\infty^+ + P_\infty^0)x\| + D\|y\|\|x\|\|P_\infty^0 x\|. \end{aligned}$$

Notice that $\|x\| \leq \sqrt{2}R$. Since $\|(P^+_\infty + P^0_\infty)x\| = R$, either $\|P^+_\infty x\|^2 \geq \frac{R^2}{2}$ or $\|P^0_\infty x\|^2 \geq \frac{R^2}{2}$. In the first case

$$\frac{d}{dt}\|(P^+_\infty + P^0_\infty)\eta^+(t,x)\|^2|_{t=0} \leq -\alpha R^2 + 2CR = -R(\alpha R - 2C),$$

which is negative for $R > \frac{2C}{\alpha}$. In the second case assume that $\frac{R^2}{2} - Q^2 \geq 1$, so that $\varphi'(\|P^0_\infty x\|^2 - Q^2) = 1$. Then

$$\begin{aligned}\frac{d}{dt}\|(P^+_\infty + P^0_\infty)\eta^+(t,x)\|^2|_{t=0} &\leq -2R^2 + (2C + \|y\|)R + \sqrt{2}D\|y\|R^2 \\ &= -R[(\alpha - \sqrt{2}D\|y\|)R - 2C - \|y\|],\end{aligned}$$

which is negative if

$$\|y\| < \frac{\alpha}{\sqrt{2}D} \quad \text{and} \quad R \geq \frac{2C + \|y\|}{\alpha - \sqrt{2}D\|y\|}.$$

We conclude that both (1.10) and (1.11) hold when $\|y\|$ is small enough and R is large enough, and in this case (4.27) is an E^+-index pair for the critical set of f^+.

An analogous argument works for h^-_y. □

We claim that we can choose y^+ and y^-, such that $\|y^+\|$ and $\|y^-\|$ are so small that the theses of Lemmas 4.5.5, 4.5.6 and 4.5.8 hold and such that all the critical points of $h^+_{y^+}$ and $h^-_{y^-}$ are non-degenerate.

Since it corresponds to a T-non-resonant periodic solution, the critical point z_0 is non-degenerate. By our assumptions and by Lemma 4.5.6 all the other critical points u are such that

$$\|P^0_\infty u\| > Q + \epsilon.$$

If v is in a neighborhood of such a critical point u, by (4.26)

$$\nabla h^\pm_y(v) = \nabla f^\pm_Q(v) - y. \tag{4.28}$$

Therefore

$$D^2 h^\pm_y(u) = D^2 f^\pm_Q(u). \tag{4.29}$$

By Lemma 4.5.4, $\nabla f^\pm_Q$ is a continuously differentiable Fredholm map of index 0. By Sard-Smale theorem (see [Sma65]) the set of its critical values has first category. Therefore we can choose y^+ and y^-, such that $\|y^+\|$ and $\|y^-\|$ are so small that the theses of Lemmas 4.5.5, 4.5.6 and 4.5.8 hold and such that y^+ and y^- are regular values for ∇f^+_Q and ∇f^-_Q, respectively.

If u is a critical point of $h^+_{y^+}$ different from z_0, by (4.28)

$$\nabla f^+_Q(u) = y^+$$

and by (4.29) the linear map $D^2 h^+_{y^+}(u) = D^2 f^+_Q(u)$ is invertible, because y^+ is a regular value for ∇f^+_Q. The same argument works for $h^-_{y^-}$.

Having fixed such y^+ and y^-, set $h^+ = h^+_{y^+}$ and $h^- = h^-_{y^-}$.

Now we are ready to prove the theorem. First we deal with the case

$$\mu_T(z_0) < \mu_T(\infty) - 1. \tag{4.30}$$

In this case we use the functional h^+.

By Lemma 4.5.7 we can apply the E^+-Conley theory to h^+. By Lemma 4.5.8, Proposition 1.7.2 and Corollary 1.8.3 we get the following Morse relations

$$\lambda^{E^+\text{-m}\,(z_0;h^+)} + \sum_{u\in\mathcal{U}^+} \lambda^{E^+\text{-m}\,(u;h^+)} = \lambda^{E^+\text{-dim}\,V_\infty^-} + (1+\lambda)Q(\lambda),$$

where $\mathcal{U}^+$ is the (finite) set of all critical points u of h^+ such that $\|P_\infty^0 u\| > Q$. Recall that $E^+\text{-m}\,(z_0;h^+) = E^+\text{-m}\,(z_0;f) = \mu_T(z_0)$ and $E^+\text{-dim}\,V_\infty^- = \mu_T(\infty)$.

By (4.30), $E^+\text{-m}\,(0) \neq E^+\text{-dim}\,V_\infty^-$ and from the above equation we deduce the existence of a critical point $u \in \mathcal{U}^+$ such that

$$|E^+\text{-m}\,(u;h^+) - \mu_T(z_0)| = 1.$$

Therefore by (4.30)

$$E^+\text{-m}\,(u;h^+) \le \mu_T(z_0) + 1 < \mu_T(\infty) = E^+\text{-dim}\,V_\infty^-.$$

But, by Lemmas 4.5.5 and 4.5.3, every critical point $u \in \mathcal{U}^+$ has E^+-Morse index

$$E^+\text{-m}\,(u;h^+) \ge E^+\text{-dim}\,V_\infty^-,$$

which is a contradiction.

Now assume that

$$\mu_T(z_0) > \mu_T(\infty) + \text{null}_T(\infty) + 1. \tag{4.31}$$

In this case we use the functional h^-.

We get the following Morse relations

$$\lambda^{E^+\text{-m}\,(z_0;h^-)} + \sum_{u\in\mathcal{U}^-} \lambda^{E^+\text{-m}\,(u;h^-)} = \lambda^{E^+\text{-dim}\,V_\infty^-\oplus V_\infty^0} + (1+\lambda)Q(\lambda),$$

where $\mathcal{U}^-$ is the (finite) set of all critical points u of h^- such that $\|P_\infty^0 u\| > Q$.

From the above equation we deduce the existence of a critical point $u \in \mathcal{U}^-$ such that

$$|E^+\text{-m}\,(u;h^-) - \mu_T(z_0)| = 1.$$

Therefore by (4.31)

$$E^+\text{-m}\,(u;h^-) \ge \mu_T(z_0) - 1 > \mu_T(\infty) + \text{null}_T(\infty) = E^+\text{-dim}\,V_\infty^- \oplus V_\infty^0.$$

But, by Lemmas 4.5.5 and 4.5.3, every critical point $u \in \mathcal{U}^-$ has E^+-Morse index

$$E^+\text{-m}\,(u;h^-) \le E^+\text{-dim}\,V_\infty^- \oplus V_\infty^0,$$

which is a contradiction. □

4.6 Sub-harmonics in two-dimensional systems

A kT-periodic solution of a T-periodic Hamiltonian system, for $k \in \mathbf{N}$, is called a sub-harmonic. The minimum of the numbers hT such that z is hT-periodic is the minimal period of z.

We want to prove an existence results for sub-harmonics in two-dimensional asymptotically Hamiltonian systems.

Theorem 4.6.1 *Assume that the T-periodic system* (4.13) *is two-dimensional, satisfies (H1), (H2) and is T-non resonant at infinity. Assume moreover that all its T-periodic solutions are T-non resonant. If there are two or more T-periodic solutions, then for every large prime p there is at least a sub-harmonic with minimal period pT. If such a sub-harmonic is pT-non resonant, then there must be a second sub-harmonic with minimal period pT.*

PROOF. First of all, we would like to remark that we need two or more T-periodic solutions: a non-resonant linear Hamiltonian system satisfies all the assumptions of this theorem and may have no sub-harmonics, apart from the periodic solution 0.

By Conley and Zehnder's Theorem 4.4.2, there is a finite number of T-periodic solutions. We can group them into subsets Z_i, $i = 1, \dots, k$, according on the value of their mean winding number:

$$\tau(z) = \tau_i \quad \forall z \in Z_i \quad \text{and} \quad \tau_i \neq \tau_j \quad \text{if } i \neq j.$$

Let $\{\lambda_\infty, \lambda_\infty^{-1}\}$ be the T-Floquet multipliers of the linearization at infinity (4.14) and let $\{\lambda_z, \lambda_z^{-1}\}$ be the T-Floquet multipliers of z, for every T-periodic solution z. By the non-resonance assumptions

$$\lambda_\infty \neq 1 \quad \text{and} \quad \lambda_z \neq 1 \ \forall z.$$

Assume that $\{\alpha_1, \dots, \alpha_r\} \subset \{\lambda_\infty\} \cup \{\lambda_z \,|\, z \text{ is a T-periodic solution}\}$ are those Floquet multipliers which are roots of 1:

$$\alpha_i^{m_i} = 1, \quad \alpha_i^n \neq 1 \text{ if } 1 \leq n < m_i, \quad i = 1, \dots, r.$$

Let $M = \max\{m_1, \dots, m_r\}$. Let p be a prime number greater than M. Since p can not be an integer multiple of m_i, for any $i = 1, \dots, r$, we have

$$\alpha_i^p \neq 1 \quad \forall i = 1, \dots, r.$$

Therefore our system is pT-non resonant at infinity and all the T-periodic solutions are pT-non resonant.

If $z \in Z_i$ is a T-periodic solution, then by the estimate (3.17),

$$|\mu_{pT}(z) - pT\tau_i| < 1.$$

If z is in Z_i and w is in Z_j

$$|\mu_{pT}(z) - \mu_{pT}(w)| \geq pT|\tau_i - \tau_j| - |\mu_{pT}(z) - pT\tau_i + pT\tau_j - \mu_{pT}(w)| > pT|\tau_i - \tau_j| - 2.$$

So we can find a large number $N > M$ such that, for every $p \geq N$

$$|\mu_{pT}(z) - \mu_{pT}(w)| \geq 2 \quad \text{if } z \in Z_i,\ w \in Z_j,\ i \neq j. \tag{4.32}$$

Let $p \geq N$ be a prime number and assume, by contradiction, that there are no sub-harmonics with minimal period pT. Since p is prime, all the pT-periodic solutions must be T-periodic.

So (4.13) is pT-non resonant at infinity and all its pT-periodic solutions are pT-non resonant; therefore the Morse relations of Theorem 4.4.2 hold

$$\sum_z \lambda^{\mu_{pT}(z)} = \lambda^{\mu_{pT}(\infty)} + (1+\lambda)Q(\lambda), \tag{4.33}$$

where the sum is taken over all the T-periodic solutions.

Since there are at least two T-periodic solutions, $Q \neq 0$. Let $n \in \mathbf{Z}$ be such that the coefficient of λ^n in Q is not zero. Then the Morse relations imply that there exist T-periodic solutions z and w such that

$$\mu_{pT}(z) = n \quad \text{and} \quad \mu_{pT}(w) = n+1. \tag{4.34}$$

But this is impossible: by (4.32) these solution must belong to the same Z_i and so

$$\tau(z) = \tau(w).$$

But then $\mu_{pT}(z) = \mu_{pT}(w)$, because Proposition 3.10.2 implies that the pT-Maslov index is uniquely determined by the mean winding number.

The existence of the second sub-harmonic in the pT-non resonant case follows again from relation (4.33). □

4.7 Asymptotically linear autonomous systems

In this section we want to study systems of the form

$$\dot{z}(t) = J\nabla H(z(t)). \tag{4.35}$$

which satisfy (H1) and (H2). In this case $A_\infty(t) = A_\infty$ is constant.

The critical points of H are equilibrium solutions for the system (4.35), and thus they are T-periodic solutions for every $T > 0$. We will also consider ∞ as a virtual critical point, using the following convention

$$D^2H(\infty) = A_\infty.$$

We want to give a lower bound for the number $n(T)$ of non-constant T-periodic solutions of (4.35).

Assume that $z \in \mathbf{R}^{2N} \cup \{\infty\}$ is a critical point of H. Let $\alpha_1 i, \dots, \alpha_h i$ be the purely imaginary eigenvalues of $JD^2H(z)$ such that $e^{\alpha_1 i}, \dots, e^{\alpha_h i}$ are the Krein-positive eigenvalues of $\exp(D^2H(z))$ (see section 3.6). Then Theorem 3.11.1 implies that z is T-non-resonant if and only if $\alpha_j T \notin 2\pi\mathbf{Z}$ for every $j = 1, \ldots, h$. In this case the T-Maslov index of z is

$$\mu_T(z) = h + 2\sum_{j=1}^{h} \left[\frac{\alpha_j T}{2\pi}\right]. \tag{4.36}$$

Notice that if the critical point z is T-non-resonant, $D^2H(z)$ must be invertible. If $D^2H(z)$ is invertible, then z is T-non-resonant for every positive T, apart from a countable set. Since in this section we will study systems which are non-resonant at infinity, we will assume A_∞ invertible.

On the contrary a non constant T-periodic solution u is always T-resonant: in fact $\dot{u}$ is a non-trivial T-periodic solution of the linearized system

$$\dot{w}(t) = JD^2H(u(t))w(t). \tag{4.37}$$

Therefore we give the following definition.

Definition 4.7.1 *A non constant T-periodic solution u of an autonomous Hamiltonian system is T-non-resonant in the autonomous sense if the space of T-periodic solutions of the linear system* (4.37) *is spanned by $\dot{u}$.*

Proposition 4.7.1 *A non-constant T-periodic solution u of* (4.35) *is T-non-resonant in the autonomous sense if and only if the kernel of $D^2f(u)$ is one-dimensional. In this case the set*

$$K_u = \{v \in E \mid \exists r \text{ such that } v(t) = u(t+r)\}$$

is an isolated elementary critical set for the functional f. Its E^+-Morse polynomial is

$$M_{E^+}(K_u) = \sum_{q\in\mathbf{Z}} \dim c^q_{E^+}(K_u)\lambda^q = (1+\lambda)\lambda^{\mu_T(u)}. \tag{4.38}$$

PROOF. The first assertion is easy to prove and well known. If $D^2f(u)$ has a one-dimensional kernel, K_u is a non-degenerate critical manifold homeomorphic to S^1. Then Theorem 1.7.2 implies (4.38). □

If $z \in \mathbf{R}^{2N} \cup \{\infty\}$ is a non-degenerate critical point of H, let $(p(z), q(z))$ be the signature of $D^2H(z)$: $p(z)$ is the number of positive eigenvalues of $D^2H(z)$, while $q(z)$ is the number of negative eigenvalues of $D^2H(z)$. Since we are in $\mathbf{R}^{2N}$, $p(z) - q(z)$ is always even.

Definition 4.7.2 *A non-degenerate true critical point $z \in \mathbf{R}^{2N}$ of H is called τ-positive if the integer*

$$\frac{1}{2}(p(z) - q(z))$$

is even. Otherwise it is called τ-negative. The virtual critical point ∞ is called τ-positive if the integer

$$\frac{1}{2}(p(\infty) - q(\infty))$$

is odd. Otherwise it is called τ-negative.

Proposition 4.7.2 *A true critical point z of H is τ-positive if and only if $\mu_T(z)$ is even for every $T > 0$. The virtual critical point ∞ is τ-positive if and only if $\mu_T(z)$ is odd for every $T > 0$.*

PROOF. It is enough to consider the case of a true critical point z, the case $z = \infty$ being completely symmetric. Let $A = D^2H(z)$. By a simple perturbation argument we can consider the case JA semi-simple. Then there exists a symplectic matrix R such that

$$JA = R^{-1}BR$$

where B is a matrix made of blocks B_j, $j = 1, \dots, r$, each of which is in normal form (see section 3.6). Since

$$A = R^T(-JB)R$$

A and $-JB$ have the same eigenvalues. To compute the eigenvalues of $-JB$ we can work separately with each block.

1. We consider a block of the first kind

$$B_j = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \quad \alpha \in \mathbf{R}.$$

In this case

$$-JB_j = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$$

which has two eigenvalues with the same sign.

2. We consider a block of the second kind

$$B_j = \begin{pmatrix} \beta & 0 \\ 0 & -\beta \end{pmatrix}, \quad \beta \in \mathbf{R} \setminus \{0\}.$$

In this case

$$-JB_j = \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix}$$

which has two eigenvalues with opposite signs.

3. We consider a block of the third kind

$$B_j = \begin{pmatrix} \beta & -\alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \\ 0 & 0 & -\beta & -\alpha \\ 0 & 0 & \alpha & -\beta \end{pmatrix}, \quad \alpha, \beta \in \mathbf{R} \setminus \{0\}.$$

In this case

$$-JB_j = \begin{pmatrix} 0 & 0 & \beta & \alpha \\ 0 & 0 & -\alpha & \beta \\ \beta & -\alpha & 0 & 0 \\ \alpha & \beta & 0 & 0 \end{pmatrix}$$

and an easy computation shows that this matrix has two positive and two negative eigenvalues.

Let n_1, n_2 and n_3 be the numbers of blocks of the first, second and third kind, respectively, in the decomposition of B. Moreover set

$$n_1 = n_1^+ + n_1^-$$

where n_1^+ is the number of blocks B_j of the first kind such that $-JB_j$ has two positive eigenvalues, while n_1^- is the number of blocks B_j of the first kind such that $-JB_j$ has two negative eigenvalues.

By the above discussion, if A has signature (p, q)

$$p - q = (2n_1^+ + n_2 + 2n_3) - (2n_1^- + n_2 + 2n_3) = 2(n_1^+ - n_1^-).$$

Therefore z is τ-positive if and only if $n_1^+ - n_1^-$ is even or, equivalently, iff $n_1 = n_1^+ + n_1^-$ is even. But formula (4.36) shows that $\mu_T(z)$ is even if and only if $-JB$ has an even number of blocks of the first kind. □

Proposition 4.7.3 *Assume that (H1) holds, that all the critical points of H are non-degenerate and that A_∞ is invertible. Then the number of τ-positive critical points equals the number of τ-negative ones (true or virtual). In particular, there are $2k-1$ true critical points, $k-1$ of which are τ-positive, $k-1$ τ-negative and one whose sign is the opposite of the sign of ∞.*

PROOF. By our assumptions H is a Morse function. By (H1) it is easy to see that H has an odd number of true critical points and that the Morse relations for H are

$$\sum_{j=1}^{2k-1} \lambda^{q(z_j)} = \lambda^{q(\infty)} + (1+\lambda)Q(\lambda).$$

Here z_j are the true critical points of H and Q is a polynomial with positive coefficients. Multiplying each member of the above equation by λ^{-N} and evaluating it for $\lambda = -1$ we get

$$\sum_{j=1}^{2k-1} (-1)^{q(z_j)-N} - (-1)^{q(\infty)-N} = 0.$$

Then the thesis follows from the fact that z_j is τ-positive if and only if $q(z_j) - N$ is even, while ∞ is τ-positive if and only if $q(\infty) - N$ is odd. □

Let $x_1, \dots, x_k$ be the τ-positive critical points of H and let $y_1, \dots, y_k$ be the τ-negative ones (∞ is included).

Theorem 4.7.4 *Assume that (H1), (H2) hold. Assume moreover that all the critical points of H are non-degenerate and that A_∞ is invertible. Let $T > 0$. Assume that the τ-positive critical points $x_1, \dots, x_k$ and the τ negative ones $y_1, \dots, y_k$ are ordered in such a way that*

$$\mu_T(x_1) \le \mu_T(x_2) \le \cdots \le \mu_T(x_k); \quad \mu_T(y_1) \le \mu_T(y_2) \le \cdots \le \mu_T(y_k).$$

If all the T-periodic solutions are T-non-resonant in the autonomous sense, the following estimate hold:

$$n(T) \ge \frac{1}{2} \sum_{j=1}^{k} |\mu_T(x_j) - \mu_T(y_j)| - \frac{k}{2} + \frac{1}{2}.$$

PROOF. Set

$$W(\lambda)=\sum_{j=1}^{h}\lambda^{\mu_T(u_j)}$$

where u_j, $j=1,\dots,h$, are the non constant T-periodic solutions. We must estimate $n(T)=W(1)$.

Let $z_1,\dots,z_{2k-1}$ be the true critical points of H.

Arguing as in the proof of Theorem 4.4.2 and using the results of Proposition 4.7.1, it is easy to see that the Morse relations for the functional f can be written as

$$\sum_{j=1}^{2k-1}\lambda^{\mu_T(z_j)}+(1+\lambda)W(\lambda)=\lambda^{\mu_T(\infty)}+(1+\lambda)Q(\lambda). \tag{4.39}$$

Here Q is a Laurent polynomial with positive coefficients.

Assume that ∞ is τ-negative, the other case being analogous. Then $\infty=y_s$ for a certain s. Setting $B=Q-W$ and dividing by $(1+\lambda)$, equation (4.39) can be rewritten as

$$\sum_{\substack{j=1\\ j\neq s}}^{k}\frac{\lambda^{\mu_T(x_j)}+\lambda^{\mu_T(x_j)}}{1+\lambda}+\frac{\lambda^{\mu_T(x_s)}-\lambda^{\mu_T(\infty)}}{1+\lambda}=B(\lambda). \tag{4.40}$$

Set

$$Q(\lambda)=\sum_j q_j\lambda^j,\quad W(\lambda)=\sum_j w_j\lambda^j,\quad B(\lambda)=\sum_j b_j\lambda^j=\sum_j(q_j-w_j)\lambda^j.$$

Since $q_j\geq 0$ and $w_j\geq 0$,

$$n(T)=\sum_j w_j\geq\sum_{b_j<0}w_j=\sum_{b_j<0}(q_j-b_j)\geq-\sum_{b_j<0}b_j.$$

Call B^- the last number: B^- is the sum of the absolute values of the negative coefficients of the Laurent polynomial B. We will estimate B^-.

Set

$$X=\{j\neq s\mid \mu_T(x_j)>\mu_T(y_j)\},\quad Y=\{j\neq s\mid \mu_T(x_j)<\mu_T(y_j)\}.$$

Since, for $j\neq s$, $\mu_T(x_j)$ is even and $\mu_T(y_j)$ is odd, they can not be equal and $X\cup Y=\{1,\dots,k\}\setminus\{s\}$. Set

$$C(\lambda)=\sum_{j\in X}\frac{\lambda^{\mu_T(x_j)}+\lambda^{\mu_T(x_j)}}{1+\lambda},\quad D(\lambda)=\sum_{j\in Y}\frac{\lambda^{\mu_T(x_j)}+\lambda^{\mu_T(x_j)}}{1+\lambda}.$$

Dividing C and D by $1+\lambda$ we get

$$C(\lambda)=\sum_{j\in X}\sum_{i=\mu_T(y_j)}^{\mu_T(x_j)-1}(-1)^{i+1}\lambda^i,\quad D(\lambda)=\sum_{j\in Y}\sum_{i=\mu_T(x_j)}^{\mu_T(y_j)-1}(-1)^i\lambda^i.$$

Set

$$S(\lambda) = \frac{\lambda^{\mu_T(x_s)} - \lambda^{\mu_T(\infty)}}{1+\lambda} = \sum_{i\in G} s_i \lambda^i,$$

where $G = \{\mu_T(x_s) - 1, \dots, \mu_T(\infty)\}$, $s_i = (-1)^{i+1}$ if $\mu_T(x_s) > \mu_T(\infty)$, and $G = \{\mu_T(\infty) - 1, \dots, \mu_T(x_s)\}$, $s_i = (-1)^i$ if $\mu_T(x_s) < \mu_T(\infty)$. Of course $S = 0$ if $\mu_T(x_s) = \mu_T(\infty)$. Then

$$B(\lambda) = C(\lambda) + D(\lambda) + S(\lambda)$$
$$= \sum_{j\in X} \frac{\lambda^{\mu_T(x_j)} + \lambda^{\mu_T(x_j)}}{1+\lambda} + \sum_{j\in Y} \frac{\lambda^{\mu_T(x_j)} + \lambda^{\mu_T(x_j)}}{1+\lambda} + \sum_{i\in G} s_i\lambda^i.$$

C and D cannot contain monomials of the same power. Moreover C and S cannot contain monomials of the same power whose coefficients have opposite signs. The same holds for D and S. Therefore

$$B^- = C^- + D^- + S^-.$$

A direct computation shows that

$$C^- + D^- = \sum_{\substack{j=1\\ j\neq s}}^{k} \frac{|\mu_T(x_j) - \mu_T(y_j)|}{2} - \frac{k-1}{2},$$

$$S^- = \frac{|\mu_T(x_s) - \mu_T(\infty)|}{2},$$

which concludes the proof. □

We can reformulate the above theorem in terms of the mean winding numbers. To do this assume that the τ-positive critical points $x_1, \dots, x_k$ and the τ-negative ones $y_1, \dots, y_k$ are ordered in such a way that

$$\tau(x_1) \le \tau(x_2) \le \dots \le \tau(x_k), \quad \tau(y_1) \le \tau(y_2) \le \dots \le \tau(y_k). \tag{4.41}$$

Definition 4.7.3 *The global twist of the system* (4.35) *is the non-negative number*

$$\Theta = \sum_{j=1}^{k} |\tau(x_j) - \tau(y_j)|$$

where $x_1, \dots, x_k, y_1, \dots, y_k$ are the critical points of H (true and virtual) ordered as in (4.41).

Corollary 4.7.5 *Assume that (H1), (H2) hold and that the system is autonomous. Assume moreover that all the critical points of H are non-degenerate and that A_∞ is invertible. If T is large enough and if all the non constant T-periodic solutions are T-non-resonant (in the autonomous sense) then the following estimate hold*

$$n(T) \ge \frac{1}{2}\Theta T - (N + \frac{1}{2})k + \frac{1}{2}.$$

PROOF. By (3.3) we can choose T_0 so large that

$$\mu_T(x_1) \leq \mu_T(x_2) \leq \cdots \leq \mu_T(x_k); \quad \mu_T(y_1) \leq \mu_T(y_2) \leq \cdots \leq \mu_T(y_k)$$

for every $T \geq T_0$. Then by Theorem 4.7.4 and estimate (3.3) we have

$$n(T) \geq \frac{1}{2} \sum_{j=1}^{k} |\mu_T(x_j) - \mu_T(j_j)| - \frac{k}{2} + \frac{1}{2} \geq \frac{1}{2}\Theta T - (N + \frac{1}{2})k + \frac{1}{2}$$

for every $T \geq T_0$. □

4.8 Some bibliography and further remarks

The variational principle

$$f(u) = \int_0^T J\dot{u}(t) \cdot u(t)\, dt + \int_0^T H(u(t), t)\, dt$$

has been known for a long time. However, because of its strong indefiniteness, it was not used to prove existence results of periodic orbits until the works of Rabinowitz, [Rab78]. An approach which overcomes the analytical difficulties with the functional f, is a finite dimensional reduction developed by Amann and Zehnder, [AZ80a], [AZ80b]. This was the tool which allowed Conley and Zehnder to produce a More-type index theory for f, [CZ84]. They also realized that the Morse index is essentially equal to the Maslov index.

A different, and more powerful, method to work with the functional f was developed by Floer: his idea was to consider only the gradient flow lines which connect critical points. These flow lines satisfy a non-linear perturbation of the Cauchy-Riemann equations, and Gromov theory of pseudo-holomorphic curves (see [Gro85]) allows to prove that the lines which go from one critical point to another form a compact manifold, whose dimension equals the difference of the Maslov indices. Floer applied these ideas to Hamiltonian systems on compact symplectic manifolds, in order to give a positive answer to a conjecture of Arnold (see [Arn78]) about fixed points of symplectic diffeomorphisms (see [Flo89], references therein and, for review expositions, [McD90], [Sal90], [SZ92], [HZ94]).

A common point of these different approaches, as well as of our approach, is the equivalence between the Morse index and the Maslov index. Floer ideas have been used to build an important symplectic invariant called symplectic homology (see [HZ94]). It would be interesting to understand if the E^+-cohomology, or some modification of it, could be used in order to give an alternative definition of symplectic homology.

The problem of finding non-trivial periodic solutions for systems which are resonant at infinity has been investigated by many authors (Chang, Liu and Liu, [CLL97], have the most general statements, together with an up-dated bibliography, see also [BL97] for a more abstract approach). Basically, a positive answer to the problem was given when one of the following conditions on the non-quadratic part G of the Hamiltonian holds: a strong resonance condition, meaning that $G(\xi, t) \to 0$ as $|\xi| \to \infty$, or a Landesman-Lazer condition, namely $G(\xi, t) \to \pm\infty$. On the contrary, our result makes no use of such conditions. Theorem 4.5.1 generalizes a result of Benci and Fortunato, [BF94], for second order systems.

It has been observed that this sort of results can be seen as a multi-dimensional generalization of the celebrated Poincaré's last geometric theorem: an area-preserving homeomorphism of the annulus, which twists the inner and outer boundaries into opposite directions, must have two fixed points (this theorem was presented by Poincaré in [Poi12] and was later proved by Birkhoff in [Bir13], [Bir25]; Brown and Neumann give a modern exposition of Birkhoff's proof in [BN77]). The analogy between these results is based on the fact that the two-dimensional Hamiltonian flows are area-preserving and that the Maslov index of an equilibrium solution z_0 gives a measure of the twist of the flow around z_0.

A somewhat astonishing result of this kind has been recently discovered by Franks, [Fra92]: an area-preserving homeomorphism of the open disc with at least two fixed points must have infinitely many periodic points. Our Theorem 4.6.1 provides an analogue statement for Hamiltonian systems. Notice that Frank's proof, as well as Birkhoff's proof of Poincaré's Theorem, uses tools from the theory of two-dimensional dynamical systems. It seems interesting to have a variational proof of an analogue result, in order to understand if there might be multi-dimensional generalizations.

Finally we should mention that Corollary 4.7.5, about the number of T-periodic orbits of an autonomous system, generalizes an analogous statement of Benci and Fortunato, [BF97], for second order systems.

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