CHAPTER 6

Exterior Algebra and Differential Calculus

In this chapter we shall introduce the calculus of differential forms, which also goes by the name "exterior differential calculus." We recall from Section 2–6 that a differential form of degree 1 is a covector-valued function. In order to define differential forms of higher degree $r > 1$ we first introduce multicovectors of degree r. For brevity, they are called r-covectors. An r-covector is an alternating, multilinear function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$. It turns out that the r-covectors form a vector space of dimension $\binom{n}{r}$, which is denoted by $(E_r^n)^*$.

Dually, an alternating, multilinear function with domain the r-fold cartesian product $(E^n)^* \times \cdots \times (E^n)^*$ is called an r-vector. The r-vectors form a vector space E_r^n, whose dual space turns out to be $(E_r^n)^*$.

There is a natural product for multicovectors called the exterior product and denoted by the symbol $\wedge$. If $\boldsymbol{\omega}$ is an r-covector and $\boldsymbol{\zeta}$ an s-covector, then $\boldsymbol{\omega} \wedge \boldsymbol{\zeta}$ is a certain $(r + s)$-covector. Dually, the exterior product of an r-vector $\boldsymbol{\alpha}$ with an s-vector $\boldsymbol{\beta}$ is an $(r + s)$-vector $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$. The exterior product is associative, and it is commutative except for a possible sign change (Proposition 20).

Certain multivectors, called decomposable, have an interesting geometric interpretation. An r-vector $\boldsymbol{\alpha}$ is decomposable if there are 1-vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ such that $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. It turns out (Theorem 19) that if $\boldsymbol{\alpha} \neq \mathbf{0}$, then $\mathbf{h}_1, \ldots, \mathbf{h}_r$ span an r-dimensional vector subspace P of E^n. With $\boldsymbol{\alpha}$ is associated an orientation of P. If two of the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are interchanged, then $\boldsymbol{\alpha}$ changes sign and the orientation of P changes. The norm $|\boldsymbol{\alpha}|$ of a decomposable r-vector $\boldsymbol{\alpha}$ equals the r-dimensional measure of a certain r-parallelepiped.

A differential form of degree r (called for brevity an r-form) is defined as an r-covector-valued function.

Every r-form ω of class $C^{(1)}$ has an exterior differential $d\omega$, which is a form of degree $r+1$. The usual formulas for the differentials of sums and products remain true except for a possible sign change in the product rule. Another important fact is that $d(d\omega) = \mathbf{0}$ for any form ω of class $C^{(2)}$. Besides its differential, ω has a codifferential $\tilde{d}\omega$ which is a form of degree $r-1$. In the next chapter the codifferential is used only for $r = 1$, in which case it becomes the divergence. In the last section of the chapter the basic formulas of vector analysis in E^3 are derived.

6-1 ALTERNATING MULTILINEAR FUNCTIONS

Let us call a real-valued function L with domain E^n *1-linear* if L is linear. For any integer $r > 1$ we shall now consider functions called r-linear. For simplicity let us first consider $r = 2$. Let B be a real-valued function with domain the cartesian product $E^n \times E^n$. The elements of $E^n \times E^n$ are pairs of vectors, denoted by $(\mathbf{h}, \mathbf{k})$. We recall from p. 29 that the function B is *bilinear* if $B(\mathbf{h},\ )$ and $B(\ ,\mathbf{k})$ are linear functions for every $(\mathbf{h}, \mathbf{k})$. It was shown there that if B is bilinear and

$$\omega_{ij} = B(\mathbf{e}_i, \mathbf{e}_j), \qquad i, j = 1, \ldots, n, \tag{6-1a}$$

then for every $(\mathbf{h}, \mathbf{k})$

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i,j=1}^{n} \omega_{ij} h^i k^j. \tag{6-2a}$$

In this chapter we are interested in a special class of r-linear functions, called alternating. For $r = 2$, B is *alternating* if $B(\mathbf{h}, \mathbf{k}) = -B(\mathbf{k}, \mathbf{h})$ for every $(\mathbf{h}, \mathbf{k})$. If B is bilinear and alternating, then $\omega_{ij} = -\omega_{ji}$, and in particular $\omega_{ii} = 0$. Formula (6-2a) can be rewritten

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i<j} (\omega_{ij} h^i k^j + \omega_{ji} h^j k^i),$$

or

$$B(\mathbf{h}, \mathbf{k}) = \sum_{i<j} \omega_{ij} (h^i k^j - h^j k^i). \tag{6-3}$$

Conversely, given $n(n-1)/2$ numbers ω_{ij}, $i < j$, formula (6-3) defines an alternating bilinear function.

Similarly, for any $r \geq 2$ let M be a real-valued function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$. The elements of $E^n \times \cdots \times E^n$ are r-tuples of vectors, denoted by $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$.

Definition. The function M is *multilinear of degree r* if for each $l = 1, \ldots, r$ and $\mathbf{h}_1, \ldots, \mathbf{h}_{l-1}, \mathbf{h}_{l+1}, \ldots, \mathbf{h}_r$ the function $M(\mathbf{h}_1, \ldots, \mathbf{h}_{l-1},\ , \mathbf{h}_{l+1}, \ldots, \mathbf{h}_r)$ is linear.

For brevity we write *r-linear* instead of multilinear of degree r. When $r = 2$ we wrote $\mathbf{h}_1 = \mathbf{h}$, $\mathbf{h}_2 = \mathbf{k}$. The new definition agrees for $r = 2$ with the definition of bilinear function. The formula which generalizes (6-2a) to multilinear functions is

$$M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \sum_{i_1, \ldots, i_r = 1}^{n} \omega_{i_1 \cdots i_r} h_1^{i_1} \cdots h_r^{i_r}, \tag{6-2b}$$

where

$$\omega_{i_1 \cdots i_r} = M(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}). \tag{6-1b}$$

This is proved by induction on r.

Interchanges. Let S be some set. For our purposes we shall take either $S = E^n$ or $S = \{1, 2, \ldots, n\}$. If $(p_1, \ldots, p_r)$ and $(p'_1, \ldots, p'_r)$ are r-tuples of elements of S, let us say that the second r-tuple is obtained from the first by interchanging p_s and p_t if $p'_s = p_t$, $p'_t = p_s$, and $p'_l = p_l$ for $l \neq s, t$.

Examples. The triple of vectors $(\mathbf{h}_3, \mathbf{h}_2, \mathbf{h}_1)$ is obtained from $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ by interchanging $\mathbf{h}_3$ and $\mathbf{h}_1$. The 4-tuple of integers $(1, 5, 3, 7)$ is obtained from the 4-tuple $(1, 7, 3, 5)$ by interchanging 5 and 7.

Definition. An r-linear function M is *alternating* if $M(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ changes sign whenever two vectors in an r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ are interchanged.

We know that the sum of two linear functions is a linear function. From this fact and the definition of multilinear function, the sum $M + N$ of two r-linear functions M and N is r-linear. If M and N are alternating, then $M + N$ is alternating. Similarly, if c is a scalar then cM is r-linear when M is r-linear and alternating when M is alternating.

Let $(E_r^n)^*$ denote the set of all alternating, r-linear functions with domain $E^n \times \cdots \times E^n$. By the remarks just made, $(E_r^n)^*$ satisfies the axioms for a vector space. Let us now prove two propositions which enable us to find the dimension of $(E_r^n)^*$ and a basis for it.

An r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is called *linearly dependent* if there exist scalars $c^1, \ldots, c^r$, not all 0, such that $c^1\mathbf{h}_1 + \cdots + c^r\mathbf{h}_r = \mathbf{0}$.

Proposition 18. *Let M be r-linear and alternating. If $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is a linearly dependent r-tuple, then $M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = 0$.*

Proof. First of all, the conclusion is true if some vector in the r-tuple is repeated. For instance, suppose that $\mathbf{h}_1 = \mathbf{h}_2$. Since M is alternating,

$$M(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3, \ldots, \mathbf{h}_r) = -M(\mathbf{h}_2, \mathbf{h}_1, \mathbf{h}_3, \ldots, \mathbf{h}_r).$$

Then $M(\mathbf{h}_1, \mathbf{h}_1, \mathbf{h}_3, \ldots, \mathbf{h}_r)$ is its own negative, and must be 0.

Suppose for instance that $\mathbf{h}_r$ is a linear combination of the vectors preceding it,

$$\mathbf{h}_r = c^1\mathbf{h}_1 + \cdots + c^{r-1}\mathbf{h}_{r-1}.$$

Since $M(\mathbf{h}_1, \ldots, \mathbf{h}_{r-1},\ )$ is a linear function,

$$M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \sum_{l=1}^{r-1} c^l M(\mathbf{h}_1, \ldots, \mathbf{h}_{r-1}, \mathbf{h}_l).$$

In the lth term on the right-hand side, the vector $\mathbf{h}_l$ is repeated, and hence each term is 0. Thus $M(\mathbf{h}_1, \ldots, \mathbf{h}_r) = 0$. ∎

For any $r \geq 2$ there is the trivial alternating r-linear function **0**, which has the value 0 for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. If $r > n$, then $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ must be linearly dependent and from the proposition we get the following.

Corollary. *If $r > n$, then* **0** *is the only alternating r-linear function.*

Therefore let us suppose that $r \leq n$. It is now convenient to introduce some more notation. The letter λ will denote an r-tuple of integers,

$$\lambda = (i_1, \ldots, i_r),$$

where $1 \leq i_k \leq n$ for each $k = 1, \ldots, r$. There are n^r such r-tuples of integers. If $i_1 < \cdots < i_r$, then λ is called an *increasing* r-tuple. There are $\binom{n}{r}$ increasing r-tuples, where $\binom{n}{r} = n!/r!(n-r)!$ is a binomial coefficient. We write $\sum_\lambda$ for a sum over all r-tuples and $\sum_{[\lambda]}$ for a sum over all increasing r-tuples.

The following generalization of the Kronecker symbol δ^i_j will be used. Let $\lambda = (i_1, \ldots, i_r)$, $\mu = (j_1, \ldots, j_r)$ be r-tuples of integers. Then $\delta^{i_k}_{j_l}$ is an element of an $r \times r$ matrix; and is 1 if $i_k = j_l$, 0 otherwise. Let

$$\delta^\lambda_\mu = \det(\delta^{i_k}_{j_l}).$$

The important properties of δ^λ_μ are:

(1) *If no integer is repeated in the r-tuple λ and $\mu = \lambda$, then $\delta^\lambda_\mu = 1$.*

In this case $i_k = j_l$ if and only if $k = l$. Hence $\delta^\lambda_\mu = \det(\delta^k_l) = 1$.

(2) *If no integer is repeated in the r-tuple μ and λ is obtained from μ by p interchanges, then $\delta^\lambda_\mu = (-1)^p$.*

Each interchange of elements of μ interchanges two column vectors of the matrix $(\delta^{i_k}_{j_l})$ and changes the sign of the determinant. Therefore (2) follows from (1).

(3) *In all other cases, $\delta^\lambda_\mu = 0$.*

If some integer is repeated in μ, then two column vectors of the matrix are the same and the determinant is 0. If the integers $j_1, \ldots, j_r$ are distinct and some i_k does not appear among them, then the kth row covector of the matrix is **0** and the determinant is 0.

Now let M be an alternating r-linear function. For brevity let us set

$$\omega_\lambda = \omega_{i_1 \cdots i_r}.$$

Sometimes we will still write $\omega_{i_1 \cdots i_r}$ rather than ω_λ, particularly when $r \leq 3$ or $r = n$. If λ is obtained from μ by one interchange, then $\omega_\lambda = -\omega_\mu$. In particular, $\omega_\lambda = 0$ if any integer is repeated. If λ is obtained from μ by p interchanges, then $\omega_\mu = (-1)^p \omega_\lambda = \delta^\lambda_\mu \omega_\lambda$.

If μ has no repetitions, then exactly one increasing λ is obtained from μ by interchanges. Hence for every μ,

$$\omega_\mu = \sum_{[\lambda]} \omega_\lambda \delta^\lambda_\mu, \tag{6–4}$$

where at most one term on the right-hand side is different from 0.

Examples. Let $n = 5$, $r = 4$. Then $\omega_{1231} = 0$ since 1 is repeated in the 4-tuple $\lambda = (1, 2, 3, 1)$. Since $(2, 3, 4, 5)$ is obtained from $(5, 4, 2, 3)$ by an odd number of interchanges, $\omega_{2345} = -\omega_{5423}$.

Let us now consider some particular elements of the space $(E^n_r)^*$. For each r-tuple $\lambda = (i_1, \ldots, i_r)$ let $\mathbf{e}^\lambda$ be the function such that

$$\mathbf{e}^\lambda(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \det(h^{i_k}_l) \tag{6–5a}$$

for every r-tuple of vectors $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. Note that the $r \times r$ matrix $(h^{i_k}_l)$ is formed from rows $i_1, \ldots, i_r$ of the $n \times r$ matrix (h^i_l) which has $\mathbf{h}_1, \ldots, \mathbf{h}_r$ as column vectors. By properties of determinants, $\mathbf{e}^\lambda$ is r-linear and alternating. Thus $\mathbf{e}^\lambda$ belongs to $(E^n_r)^*$.

Taking in particular $\mathbf{h}_1, \ldots, \mathbf{h}_r$ to be standard basis vectors, $\mathbf{h}_l = \mathbf{e}_{j_l}$ for $l = 1, \ldots, r$, we obtain in (6–5a) the matrix $(\delta^{i_k}_{j_l})$ whose determinant is δ^λ_μ. Thus

$$\mathbf{e}^\lambda(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \delta^\lambda_\mu. \tag{6–6}$$

If λ is obtained from μ by an interchange, then two row covectors of the matrix $(h^{i_k}_l)$ in (6–5a) are interchanged. The determinant changes sign. Hence

$$\mathbf{e}^\lambda(\mathbf{h}_1, \ldots, \mathbf{h}_r) = -\mathbf{e}^\mu(\mathbf{h}_1, \ldots, \mathbf{h}_r)$$

for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, which means that $\mathbf{e}^\lambda = -\mathbf{e}^\mu$. In particular, $\mathbf{e}^\lambda = \mathbf{0}$ if λ has any repetitions. If λ is obtained from μ by p interchanges, then $\mathbf{e}^\lambda = (-1)^p \mathbf{e}^\mu$.

Let us make the convention that $\mathbf{e}^\lambda = \mathbf{0}$ in case $r > n$. This is useful in defining the exterior product in the next section.

When $r = 2$ and $\lambda = (i, j)$, $\mathbf{e}^{ij}(\mathbf{h}, \mathbf{k}) = h^i k^j - h^j k^i$. If B is bilinear and alternating, then

$$B = \sum_{i<j} \omega_{ij} \mathbf{e}^{ij},$$

since by formula (6–3) both sides have the same value for each pair of vectors $(\mathbf{h}, \mathbf{k})$. This is a particular case of the following.

Proposition 19. ($r \leq n$.) *Let M be r-linear and alternating. Then*

$$M = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda, \tag{6–7}$$

where the numbers ω_λ are given by (6–1b).

Proof. Let $\tilde{M}$ equal the right-hand side of (6–7). For each $\mu = (j_1, \ldots, j_r)$,

$$\tilde{M}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}).$$

From (6–4) and (6–6),

$$\tilde{M}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \sum_{[\lambda]} \omega_\lambda \delta_\mu^\lambda = \omega_\mu.$$

But M and $\tilde{M}$ are r-linear and have the same value ω_μ at $(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r})$ for each μ. By (6–2b) $M = \tilde{M}$. ∎

PROBLEMS

1. Let $n = 5$. Find

$$\delta_{51}^{15}, \quad \delta_{51}^{25}, \quad \delta_{214}^{142}, \quad \delta_{255}^{525}, \quad \delta_{425}^{345}.$$

2. Let $n = 4$, $r = 3$, $\omega_{123} = 2$, $\omega_{134} = -1$, and $\omega_\lambda = 0$ for every other increasing triple λ. Find $M(\mathbf{e}_4, \mathbf{e}_1 - \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3)$.
3. Show that:

(a) $\delta_\mu^\lambda = \delta_\lambda^\mu$. (b) $\delta_\nu^\lambda = \sum_{[\mu]} \delta_\mu^\lambda \delta_\nu^\mu = \frac{1}{r!} \sum_\mu \delta_\mu^\lambda \delta_\nu^\mu$.

(c) $\mathbf{e}^\lambda = \sum_{[\mu]} \delta_\mu^\lambda \mathbf{e}^\mu = \frac{1}{r!} \sum_\mu \delta_\mu^\lambda \mathbf{e}^\mu$.

[*Hint for* (c): Use (b) and (6–6).]

4. Let M be r-linear, not necessarily alternating. Let ω_λ be as in (6–1b) and $\tilde{\omega}_\mu = (1/r!) \sum_\lambda \omega_\lambda \delta_\mu^\lambda$. The function $M_1 = \sum_{[\lambda]} \tilde{\omega}_\lambda \mathbf{e}^\lambda$ is r-linear and alternating.

(a) Show that $M_1 = (1/r!) \sum_\mu \tilde{\omega}_\mu \mathbf{e}^\mu$. [*Hint:* $\tilde{\omega}_\mu = \sum_{[\lambda]} \tilde{\omega}_\lambda \delta_\mu^\lambda$; use Problem 3(c).]

(b) Show that if M is alternating, then $\tilde{\omega}_\mu = \omega_\mu$ and hence $M_1 = M$.

6–2 MULTICOVECTORS

Let us now introduce a different name and a different notation for alternating, multilinear functions.

Definition. A *multicovector of degree r* is an alternating r-linear function with domain the r-fold cartesian product $E^n \times \cdots \times E^n$.

For brevity, multicovectors of degree r are called *r-covectors.* From now on multicovectors will ordinarily be denoted by the Greek letters $\boldsymbol{\omega}$ or $\boldsymbol{\zeta}$ rather than M as in the previous section.

We observed in the last section that the set $(E_r^n)^*$ of all r-covectors satisfies the axioms for a vector space. When $r > n$, its only element is $\mathbf{0}$ by Proposition 18. When $1 \leq r \leq n$, Proposition 19 states that if $\boldsymbol{\omega}$ is any r-covector, then

$$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda. \tag{6–8}$$

Therefore the r-covectors $\mathbf{e}^\lambda$ with λ increasing span $(E_r^n)^*$. These r-covectors form a linearly independent set (Problem 7), which is therefore a basis for $(E_r^n)^*$. It is called the *standard basis.* The number ω_λ is the *component* of $\boldsymbol{\omega}$ with respect to the basis element $\mathbf{e}^\lambda$. Since there are $\binom{n}{r}$ increasing r-tuples of integers between 1 and n, $(E_r^n)^*$ has dimension $\binom{n}{r}$.

Every 1-linear function is alternating. Thus a 1-covector is just a covector, and $(E_1^n)^* = (E^n)^*$ is the dual space of E^n. If we identify the 1-tuple (i) with i, then the standard basis 1-covectors $\mathbf{e}^1, \ldots, \mathbf{e}^n$ are just those introduced in Section 1–3. As in previous chapters we shall use the letters $\mathbf{a}, \mathbf{b}$ to denote 1-covectors.

If $r = n$, then the n-covector $\mathbf{e}^{1 \cdots n}$ is essentially the determinant function. Its value at $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is $\det(h_l^k)$, which is the determinant of the $n \times n$ matrix with column vectors $\mathbf{h}_1, \ldots, \mathbf{h}_n$. Since $(E_n^n)^*$ is one-dimensional, every n-covector has the form $\boldsymbol{\omega} = c\mathbf{e}^{1 \cdots n}$ where $c = \omega_{1 \cdots n}$.

Example. Let $n = 5$, $r = 3$, and $\boldsymbol{\omega} = 6\mathbf{e}^{145} - 2\mathbf{e}^{431} - \mathbf{e}^{514}$. The increasing triple (1, 4, 5) is obtained from (5, 1, 4) by an even number of interchanges. Hence $\mathbf{e}^{514} = \mathbf{e}^{145}$. The increasing triple (1, 3, 4) is obtained from (4, 3, 1) by one interchange. Hence $\mathbf{e}^{431} = -\mathbf{e}^{134}$, and $\boldsymbol{\omega} = 2\mathbf{e}^{134} + 5\mathbf{e}^{145}$. This expresses $\boldsymbol{\omega}$ as a linear combination of the standard basis 3-covectors. The components of $\boldsymbol{\omega}$ are $\omega_{134} = 2$, $\omega_{145} = 5$, and $\omega_\lambda = 0$ for every other increasing triple λ.

Products. In $(E_r^n)^*$ we define the *euclidean inner product*

$$\boldsymbol{\omega} \cdot \boldsymbol{\zeta} = \sum_{[\lambda]} \omega_\lambda \zeta_\lambda$$

and set $|\boldsymbol{\omega}|^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$. The standard basis elements are orthonormal with respect to this inner product.

Another important product is the exterior product, denoted by the symbol $\wedge$. The exterior product of an r-covector and an s-covector is an $(r + s)$-covector, defined as follows: If

$$\lambda = (i_1, \ldots, i_r), \qquad \nu = (j_1, \ldots, j_s),$$

let us write λ, ν for the $(r + s)$-tuple

$$(i_1, \ldots, i_r, j_1, \ldots, j_s).$$

Definition. Let $1 \leq r \leq n$, $1 \leq s \leq n$. If λ and ν are increasing, then

$$\mathbf{e}^\lambda \wedge \mathbf{e}^\nu = \mathbf{e}^{\lambda,\nu}. \tag{6–9}$$

If ω is an r-covector and ζ is an s-covector, with respective components ω_λ, ζ_ν, then

$$\omega \wedge \zeta = \sum_{[\lambda][\nu]} \omega_\lambda \zeta_\nu \mathbf{e}^\lambda \wedge \mathbf{e}^\nu.$$

Note that if $r + s > n$ then $\omega \wedge \zeta$, being an $(r + s)$-covector, must be $\mathbf{0}$.

Examples. Let $n = 4$. Then $\mathbf{e}^{12} \wedge \mathbf{e}^{34} = \mathbf{e}^{1234}$.

$$\mathbf{e}^3 \wedge \mathbf{e}^{124} = \mathbf{e}^{3124} = \mathbf{e}^{1234}, \qquad \mathbf{e}^{14} \wedge \mathbf{e}^{24} = \mathbf{e}^{1424} = \mathbf{0},$$

since the integer 4 is repeated.

Proposition 20. *The exterior product has the following properties:*

(1) $(\omega + \zeta) \wedge \eta = (\omega \wedge \eta) + (\zeta \wedge \eta)$.

(2) $(c\omega) \wedge \zeta = c(\omega \wedge \zeta)$.

(3) $\zeta \wedge \omega = (-1)^{rs}\omega \wedge \zeta$, *if ω has degree r and ζ has degree s.*

(4) $(\zeta \wedge \omega) \wedge \eta = \zeta \wedge (\omega \wedge \eta)$.

Proof. The proof of (1) and (2) is almost immediate from the definition and is left to the reader (Problem 8). To prove (3),

$$\nu, \lambda = (j_1, \ldots, j_s, i_1, \ldots, i_r).$$

By s interchanges we may bring i_1 to the left past $j_1, \ldots, j_s$. Similarly, s interchanges bring each of $i_2, \ldots, i_r$ in turn past $j_1, \ldots, j_s$. Thus λ, ν is obtained from ν, λ by rs interchanges, and $\mathbf{e}^{\nu,\lambda} = (-1)^{rs}\mathbf{e}^{\lambda,\nu}$. Hence

$$\zeta \wedge \omega = \sum_{[\nu][\lambda]} \zeta_\nu \omega_\lambda \mathbf{e}^{\nu,\lambda} = (-1)^{rs} \sum_{[\lambda][\nu]} \omega_\lambda \zeta_\nu \mathbf{e}^{\lambda,\nu},$$

which proves (3).

Let us first prove the associative law (4) for basis elements. Let $\lambda = (i_1, \ldots, i_r)$, $\nu = (j_1, \ldots, j_s)$, and $\rho = (k_1, \ldots, k_t)$ be increasing r-, s-, and t-tuples, respectively. Let

$$\lambda, \nu, \rho = (i_1, \ldots, i_r, j_1, \ldots, j_s, k_1, \ldots, k_t).$$

Let us show that

$$\mathbf{e}^{\lambda,\nu,\rho} = (\mathbf{e}^\lambda \wedge \mathbf{e}^\nu) \wedge \mathbf{e}^\rho.$$

If some integer is repeated in the $(r + s)$-tuple λ, ν, then both sides are $\mathbf{0}$. If no integer is repeated, then

$$(\mathbf{e}^\lambda \wedge \mathbf{e}^\nu) \wedge \mathbf{e}^\rho = \mathbf{e}^{\lambda,\nu} \wedge \mathbf{e}^\rho = (-1)^p \mathbf{e}^\tau \wedge \mathbf{e}^\rho = (-1)^p \mathbf{e}^{\tau,\rho},$$

where τ is an increasing $(r + s)$-tuple obtained from λ, ν by p interchanges.

These same p interchanges change the $(r + s + t)$-tuple λ, ν, ρ into τ, ρ. Hence

$$\mathbf{e}^{\lambda,\nu,\rho} = (-1)^p \mathbf{e}^{\tau,\rho} = (\mathbf{e}^\lambda \wedge \mathbf{e}^\nu) \wedge \mathbf{e}^\rho.$$

Similarly $\mathbf{e}^{\lambda,\nu,\rho} = \mathbf{e}^\lambda \wedge (\mathbf{e}^\nu \wedge \mathbf{e}^\rho)$, and hence

$$(\mathbf{e}^\lambda \wedge \mathbf{e}^\nu) \wedge \mathbf{e}^\rho = \mathbf{e}^\lambda \wedge (\mathbf{e}^\nu \wedge \mathbf{e}^\rho). \tag{6–10}$$

From this formula it is a straightforward matter to obtain (4) (Problem 9). ∎

If either r or s is even, then the exterior product is commutative. If $r = s = 1$, we have $\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$.

The exterior product of any finite number of multicovectors is defined by induction. Using (6–9) repeatedly, we find that if λ is increasing,

$$\mathbf{e}^\lambda = \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}. \tag{6–11}$$

Since both sides of (6–9) change sign under interchanges in λ or ν, formula (6–9) is also true for nonincreasing r-tuples. Thus (6–11) is valid whether λ is increasing or not.

Examples. Let $n = 5$. Then

$$(\mathbf{e}^1 + 3\mathbf{e}^4) \wedge (\mathbf{e}^{24} - 2\mathbf{e}^{15}) = \mathbf{e}^{124} + 3\mathbf{e}^{424} - 2\mathbf{e}^{115} - 6\mathbf{e}^{415} = \mathbf{e}^{124} + 6\mathbf{e}^{145}.$$

$$\mathbf{e}^2 \wedge (3\mathbf{e}^1 - 2\mathbf{e}^3) \wedge \mathbf{e}^5 \wedge \mathbf{e}^3 = (3\mathbf{e}^{21} - 2\mathbf{e}^{23}) \wedge \mathbf{e}^{53} = 3\mathbf{e}^{2153} = 3\mathbf{e}^{1235}.$$

***Remarks.** The exterior product has been defined in terms of the standard bases. It is not clear that it is "coordinate free," in other words, that the same exterior product would be obtained starting from different bases. However, let us add one additional property to the list (1)—(4):

(5) *If* $\omega = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$, *then* $\omega(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \det(\mathbf{a}^k \cdot \mathbf{h}_l)$ *for every r-tuple* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$.

This property of the exterior product will be proved in Section 6–3 [see (6–12), (6–14)]. Formula (6–11) is a special case of (5). This is seen by taking $\mathbf{a}^k = \mathbf{e}^{i_k}$ and recalling (6–5a). Moreover, (6–9) is a consequence of (6–11) and the associative law (4). Once the product is known for basis elements, Properties (1) and (2) determine it in general. Thus $\wedge$ is the only product with Properties (1)—(5). In fact, (3) can be omitted from the list since it follows from the other four. Since none of these five properties refers to bases, the exterior product is coordinate free.

***Note about terminology.** A multilinear function M of degree r and domain $E^n \times \cdots \times E^n$ is often called a *covariant tensor of rank r*. An r-covector is then called an *alternating covariant tensor of rank r*.

The sum of an r-covector and an s-covector has been defined only when $r = s$. However, one may form the direct sum

$$(A^n)^* = (E_0^n)^* \oplus (E_1^n)^* \oplus \cdots \oplus (E_r^n)^* \oplus \cdots,$$

where we agree that $(E_0^n)^*$ is the scalar field. The exterior product induces a product in $(A^n)^*$, which is then an algebra over the real numbers. This algebra is called the *exterior algebra* of $(E^n)^*$. See reference [4]. $(A^n)^*$ is sometimes called the *covariant Grassmann algebra* or the *covariant alternating tensor algebra* of E^n.

PROBLEMS

1. Write down the standard basis for $(E_r^4)^*$ for each $r = 1, 2, 3, 4$. Find all products $\mathbf{e}^\lambda \wedge \mathbf{e}^\nu$ where $\lambda = (i)$ and $\nu = (j, k, l)$ is an increasing triple.
2. Let $n = 3$. Simplify:
   (a) $(2\mathbf{e}^1 - \mathbf{e}^2) \wedge (3\mathbf{e}^2 + \mathbf{e}^3)$. (b) $\mathbf{e}^{21} \wedge \mathbf{e}^{23}$.
   (c) $(\mathbf{e}^1 - \mathbf{e}^2 + 3\mathbf{e}^3) \wedge \mathbf{e}^{21}$. (d) $(\mathbf{e}^{23} + \mathbf{e}^{31}) \wedge (5\mathbf{e}^1 - \mathbf{e}^2)$.
3. Let $n = 5$. Simplify:
   (a) $\mathbf{e}^{253} \wedge (\mathbf{e}^{14} + \mathbf{e}^{42})$. (b) $(\mathbf{e}^2 + \mathbf{e}^5) \wedge \mathbf{e}^{31} \wedge (\mathbf{e}^5 - \mathbf{e}^4)$.
4. Let $\mathbf{a}$ and $\mathbf{b}$ be 1-covectors and $\boldsymbol{\omega} = \mathbf{a} \wedge \mathbf{b}$. Show that $\omega_{ij} = a_ib_j - a_jb_i$.
5. Show that if $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are 1-covectors, then
$$\mathbf{a} \wedge \mathbf{b} + \mathbf{b} \wedge \mathbf{c} + \mathbf{c} \wedge \mathbf{a} = (\mathbf{a} - \mathbf{b}) \wedge (\mathbf{b} - \mathbf{c}).$$
6. Show that if $\boldsymbol{\omega} = \mathbf{a} \wedge \mathbf{b}$, then $\omega_{ij}\omega_{kl} + \omega_{ik}\omega_{lj} + \omega_{il}\omega_{jk} = 0$ for $i, j, k, l = 1, \ldots, n$. [*Hint:* Using Problem 4,
$$\det\begin{pmatrix} \omega_{ij} & \omega_{ik} & \omega_{il} \\ a_j & a_k & a_l \\ b_j & b_k & b_l \end{pmatrix} = 0$$
since the first row is a linear combination of the second and third rows.]
7. Show that if $\sum_{[\lambda]} c_\lambda \mathbf{e}^\lambda = \mathbf{0}$, then $c_\lambda = 0$ for every increasing λ. [*Hint:* See (6–6).]
8. Prove (1) and (2) of Proposition 20.
9. Prove the associative law (4) of Proposition 20, using (1), (2), and (6–10).
10. Show that $\boldsymbol{\omega} \wedge \boldsymbol{\zeta} \wedge \boldsymbol{\eta} = -\boldsymbol{\eta} \wedge \boldsymbol{\zeta} \wedge \boldsymbol{\omega}$ if $\boldsymbol{\omega}$ has degree r, $\boldsymbol{\eta}$ has degree t, and both r, t are odd.

6–3 MULTIVECTORS

If $\mathcal{V}$ is any vector space, then alternating r-linear functions on $\mathcal{V} \times \cdots \times \mathcal{V}$ can be defined just as in Section 6–1 where we took $\mathcal{V} = E^n$. Let us now take $\mathcal{V} = (E^n)^*$, the dual space to E^n.

Definition. A *multivector of degree r* is an alternating r-linear function with domain the r-fold cartesian product $(E^n)^* \times \cdots \times (E^n)^*$.

For brevity, multivectors of degree r are called *r-vectors*. They will usually be denoted by the Greek letters $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$. When $r = 1$, the 1-linear functions on $(E^n)^*$ are identified with the elements of E^n in the way explained in Section A–2. Then a 1-vector is just a vector, and will be denoted as usual by $\mathbf{x}$ or $\mathbf{h}$.

For every statement about multicovectors in Sections 6–1 and 6–2, there is a dual statement about multivectors obtained by everywhere exchanging the words "vector" and "covector." For instance, if $\boldsymbol{\alpha}$ is an r-vector and $\lambda = (i_1, \ldots, i_r)$, let

$$\alpha^\lambda = \boldsymbol{\alpha}(\mathbf{e}^{i_1}, \ldots, \mathbf{e}^{i_r}).$$

This is dual to the formula [see (6–1b) with $M = \boldsymbol{\omega}$]

$$\omega_\lambda = \boldsymbol{\omega}(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}).$$

Let $\mathbf{e}_\lambda$ be the r-vector defined by the formula dual to (6–5a):

$$\mathbf{e}_\lambda(\mathbf{a}^1, \ldots, \mathbf{a}^r) = \det(a_{i_l}^k) \tag{6–5b}$$

for every r-tuple $(\mathbf{a}^1, \ldots, \mathbf{a}^r)$ of covectors.

Let E_r^n denote the set of all r-vectors. Then E_r^n satisfies the axioms for a vector space. It consists of $\mathbf{0}$ only if $r > n$. For $1 \le r \le n$ the r-vectors $\mathbf{e}_\lambda$ with λ increasing form the *standard basis* for E_r^n. The number α^λ is the *component* of $\boldsymbol{\alpha}$ with respect to $\mathbf{e}_\lambda$.

The inner product $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}$ of two r-vectors, and the exterior product $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ of an r-vector $\boldsymbol{\alpha}$ and an s-vector $\boldsymbol{\beta}$ are defined by the formulas dual to those in Section 6–2. In each instance subscripts are replaced by superscripts and vice versa. The exterior product of multivectors has the same properties listed in Proposition 20. The scalar product $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}$ of an r-covector $\boldsymbol{\omega}$ and an r-vector $\boldsymbol{\alpha}$ is defined in the third from last line of the Table 6–1. The last two lines of the table are particular cases of the formula for $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}$.

The formulas in the second line and in the last two lines are true whether λ and μ are increasing or not, since they are known to be true for increasing r-tuples, and both sides of each formula change sign under interchanges.

The reader should compare this table with the corresponding table for $r = 1$, p. 12. According to the definition (Section A–2), the dual space of E_r^n consists of all real-valued linear functions F with domain E_r^n. The dual space may be identified with $(E_r^n)^*$ in the following way. Given an r-covector $\boldsymbol{\omega}$, let $F(\boldsymbol{\alpha}) = \boldsymbol{\omega} \cdot \boldsymbol{\alpha}$ for every $\boldsymbol{\alpha} \in E_r^n$. This establishes an isomorphism between $(E_r^n)^*$ and the dual space of E_r^n. The next to last line of the table implies that the standard bases for E_r^n and $(E_r^n)^*$ are dual.

***Note about terminology.** Multivectors of degree r are also called *alternating contravariant tensors of rank r*. The exterior algebra A^n of E^n can be introduced in the way indicated at the end of Section 6–2.

TABLE 6-1

	r-vectors	r-covectors
Elements of	E^n_r	$(E^n_r)^*$
Standard basis elements (λ increasing)	$\mathbf{e}_\lambda = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}$	$\mathbf{e}^\lambda = \mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_r}$
	$\boldsymbol{\alpha} = \sum_{[\lambda]} \alpha^\lambda \mathbf{e}_\lambda$	$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \mathbf{e}^\lambda$
Euclidean inner product	$\boldsymbol{\alpha} \cdot \boldsymbol{\beta} = \sum_{[\lambda]} \alpha^\lambda \beta^\lambda$	$\boldsymbol{\omega} \cdot \boldsymbol{\zeta} = \sum_{[\lambda]} \omega_\lambda \zeta_\lambda$
Euclidean norm	$\lvert\boldsymbol{\alpha}\rvert^2 = \boldsymbol{\alpha} \cdot \boldsymbol{\alpha}$	$\lvert\boldsymbol{\omega}\rvert^2 = \boldsymbol{\omega} \cdot \boldsymbol{\omega}$
Scalar product	$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \sum_{[\lambda]} \omega_\lambda \alpha^\lambda$	
	$\mathbf{e}^\lambda \cdot \mathbf{e}_\mu = \delta^\lambda_\mu$	
	$\mathbf{e}^\lambda \cdot \boldsymbol{\alpha} = \alpha^\lambda$	$\mathbf{e}_\lambda \cdot \boldsymbol{\omega} = \omega_\lambda$

Definition. An r-covector $\boldsymbol{\omega}$ is *decomposable* if there exist covectors $\mathbf{a}^1, \ldots, \mathbf{a}^r$ such that $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. Similarly, an r-vector $\boldsymbol{\alpha}$ is decomposable if there exist vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ such that $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$.

In the remainder of this section we shall mainly discuss decomposable r-vectors. Each statement about them has a dual which applies to decomposable r-covectors. Clearly every 1-vector is decomposable. If $\boldsymbol{\alpha}$ is an n-vector, then

$$\boldsymbol{\alpha} = c\mathbf{e}_{1\cdots n} = (c\mathbf{e}_1) \wedge \mathbf{e}_2 \wedge \cdots \wedge \mathbf{e}_n,$$

where $c = \alpha^{1\cdots n}$. Hence every n-vector is decomposable. In Section 6-6 it will be shown that any $(n-1)$-vector is decomposable. However, for $2 \le r \le n-2$ there are nondecomposable r-vectors; see Problem 9. Since $\mathbf{e}_\lambda = \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_r}$, the standard basis r-vectors are decomposable.

It is not correct to identify a decomposable r-vector $\boldsymbol{\alpha}$ with the r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ since there are many ways to write $\boldsymbol{\alpha}$ as an exterior product of vectors. The corollary to Theorem 19 below will furnish a geometric description of all possible such decompositions of $\boldsymbol{\alpha}$.

Proposition 21. *If* $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, *then for every r-covector* $\boldsymbol{\omega}$,

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega}(\mathbf{h}_1, \ldots, \mathbf{h}_r). \tag{6-12}$$

Proof. Let $\tilde{\boldsymbol{\omega}}(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \boldsymbol{\omega} \cdot (\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r)$ for every r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. Then $\tilde{\boldsymbol{\omega}}$ is an alternating r-linear function. Moreover

$$\tilde{\boldsymbol{\omega}}(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_r}) = \boldsymbol{\omega} \cdot \mathbf{e}_\mu = \omega_\mu$$

for every μ. Hence $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}$. ∎

Proposition 22. *Let* $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$ *and* $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. *Then for every r-tuple* $\lambda = (i_1, \ldots, i_r)$,

$$\alpha^\lambda = \det(h^{i_k}_l), \tag{6-13a}$$

$$\omega_\lambda = \det(a^k_{i_l}), \tag{6-13b}$$

and

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \det(\mathbf{a}^k \cdot \mathbf{h}_l). \tag{6-14}$$

Proof. Taking $\boldsymbol{\omega} = \mathbf{e}^\lambda$ in formula (6-12), we get

$$\alpha^\lambda = \mathbf{e}^\lambda \cdot \boldsymbol{\alpha} = \mathbf{e}^\lambda(\mathbf{h}_1, \ldots, \mathbf{h}_r).$$

Recalling the definition (6-5a) of $\mathbf{e}^\lambda$, we get (6-13a). The formula (6-13b) dual to (6-13a) is obtained similarly. Let $\boldsymbol{\omega}'(\mathbf{h}_1, \ldots, \mathbf{h}_r) = \det(\mathbf{a}^k \cdot \mathbf{h}_l)$ for every $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$. Then $\boldsymbol{\omega}'$ is multilinear and alternating and for every λ

$$\boldsymbol{\omega}'(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}) = \det(\mathbf{a}^k \cdot \mathbf{e}_{i_l}) = \det(a^k_{i_l}).$$

Using (6-13b), we get $\boldsymbol{\omega}' = \boldsymbol{\omega}$. Then (6-14) follows from (6-12). ∎

The formulas (6-13a), (6-13b), and (6-14) may not provide the easiest way to compute the components and the scalar product in numerical examples. For instance, see Examples (1) and (2) below. However, they are important for various other reasons.

Proposition 23. *If* $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega} \cdot \boldsymbol{\beta}$ *for every decomposable r-covector* $\boldsymbol{\omega}$, *then* $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

Proof. The standard basis r-covectors $\mathbf{e}^\lambda$ are decomposable. Hence for every increasing λ,

$$\alpha^\lambda = \mathbf{e}^\lambda \cdot \boldsymbol{\alpha} = \mathbf{e}^\lambda \cdot \boldsymbol{\beta} = \beta^\lambda. \;∎$$

The decomposable r-vectors have an important geometric significance which will be described next.

First we recall the following results from linear algebra:

(1) Any linearly independent set $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ is a basis for the vector subspace $P \subset E^n$ spanned by these vectors (definition).

(2) Given any such set there exist $\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n$ such that $\{\mathbf{h}_1, \ldots, \mathbf{h}_n\}$ is a basis for E^n.

(3) For every basis $\{\mathbf{h}_1, \ldots, \mathbf{h}_n\}$ for E^n there is a dual basis $\{\mathbf{a}^1, \ldots, \mathbf{a}^n\}$ for $(E^n)^*$, $\mathbf{a}^k \cdot \mathbf{h}_l = \delta^k_l$ for $k, l = 1, \ldots, n$ (Section A-2).

Definition. A linearly independent r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is called a *frame* for the vector subspace P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$.

The only difference between the notions of basis and frame is that the latter takes into account the order in which the basis vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are written.

Theorem 19. (a) *An r-tuple $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is linearly dependent if and only if* $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = \mathbf{0}$.

(b) *Let $P \subset E^n$ be an r-dimensional vector subspace and $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ be any two frames for P. Then there is a scalar c such that*

$$\mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r = c\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r. \tag{6–15}$$

(c) *Conversely, if $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ and $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ are frames which satisfy (6–15) for some scalar c, then they are frames for the same vector subspace P.* (See Fig. 6–1.)

Proof of (a). Let $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ be linearly dependent and $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. By Propositions 18 and 21, $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = 0$ for every $\boldsymbol{\omega}$. By Proposition 23, $\boldsymbol{\alpha} = \mathbf{0}$. On the other hand, if $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is linearly independent, let $\mathbf{a}^1, \ldots, \mathbf{a}^r$ be covectors such that $\mathbf{a}^k \cdot \mathbf{h}_l = \delta^k_l$, and let $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. By (6–14)

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \det(\delta^k_l) = 1.$$

Hence $\boldsymbol{\alpha} \neq \mathbf{0}$.

Proof of (b). Each $\mathbf{h}'_l$ is a linear combination of $\mathbf{h}_1, \ldots, \mathbf{h}_r$,

$$\mathbf{h}'_l = \sum_{m=1}^{r} c^m_l \mathbf{h}_m, \qquad l = 1, \ldots, r.$$

If $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$ is any decomposable r-covector, then

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = \det(\mathbf{a}^k \cdot \mathbf{h}'_l) = \det\left(\sum_{m=1}^{r} (\mathbf{a}^k \cdot \mathbf{h}_m) c^m_l\right)$$

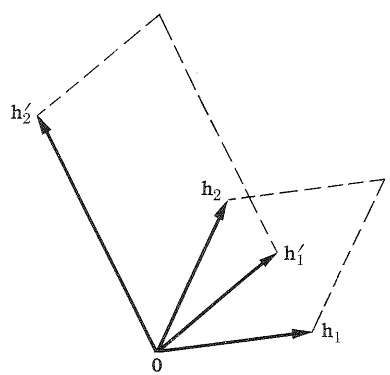

FIGURE 6–1

where $\boldsymbol{\alpha}' = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$. The matrix on the right is the product of the matrices $(\mathbf{a}^k \cdot \mathbf{h}_m)$ and (c^m_l). Hence if $c = \det(c^m_l)$,

$$\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = c \det(\mathbf{a}^k \cdot \mathbf{h}_m) = c\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = \boldsymbol{\omega} \cdot (c\boldsymbol{\alpha}).$$

This is true for every decomposable $\boldsymbol{\omega}$; hence $\boldsymbol{\alpha}' = c\boldsymbol{\alpha}$ by Proposition 23.

Proof of (c). Since $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, $\boldsymbol{\alpha}' = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$ are not $\mathbf{0}$ by hypothesis, (a) implies that $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ and $\{\mathbf{h}'_1, \ldots, \mathbf{h}'_r\}$ are linearly

independent sets. It suffices to show that each $\mathbf{h}'_l$ is a linear combination of $\mathbf{h}_1, \ldots, \mathbf{h}_r$. Suppose that this is false for some l, say for $l = 1$. Then $\{\mathbf{h}_1, \ldots, \mathbf{h}_r, \mathbf{h}'_1\}$ is a linearly independent set. Let $\mathbf{a}^1, \ldots, \mathbf{a}^{r+1}$ be such that $\mathbf{a}^k \cdot \mathbf{h}_m = \delta^k_m$ for $k, m = 1, \ldots, r+1$, where we have set $\mathbf{h}'_1 = \mathbf{h}_{r+1}$. Let $\boldsymbol{\omega} = \mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$. Then $\boldsymbol{\omega} \cdot \boldsymbol{\alpha} = 1$, but $\boldsymbol{\omega} \cdot \boldsymbol{\alpha}' = 0$ since the elements $\mathbf{a}^k \cdot \mathbf{h}'_1 = \delta^k_{r+1}$ of the first column of the $r \times r$ matrix $(\mathbf{a}^k \cdot \mathbf{h}'_l)$ are 0. This contradicts the assumption that $\boldsymbol{\alpha}' = c\boldsymbol{\alpha}$, $c \neq 0$. ∎

Example 1. Show that $2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3$, $\mathbf{e}_1 + 2\mathbf{e}_2$, $\mathbf{e}_1 - 2\mathbf{e}_3$ are linearly dependent. Their exterior product is

$$(2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3) \wedge (\mathbf{e}_1 + 2\mathbf{e}_2) \wedge (\mathbf{e}_1 - 2\mathbf{e}_3)$$
$$= (\mathbf{e}_{12} - \mathbf{e}_{31} - 2\mathbf{e}_{32}) \wedge (\mathbf{e}_1 - 2\mathbf{e}_3) = -2\mathbf{e}_{123} - 2\mathbf{e}_{321} = 0$$

Example 2. Show that $(\mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3)$ is a frame for the same 2-dimensional vector subspace of E^3 as $(2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3, 4\mathbf{e}_1 + \mathbf{e}_2 + 11\mathbf{e}_3)$. Calculating the exterior products, we get

$$(\mathbf{e}_1 + 3\mathbf{e}_3) \wedge (\mathbf{e}_2 - \mathbf{e}_3) = \mathbf{e}_{12} - \mathbf{e}_{13} - 3\mathbf{e}_{23},$$
$$(2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3) \wedge (4\mathbf{e}_1 + \mathbf{e}_2 + 11\mathbf{e}_3) = -2\mathbf{e}_{12} + 2\mathbf{e}_{13} + 6\mathbf{e}_{23}.$$

The second 2-vector is -2 times the first.

Let $\boldsymbol{\alpha} \neq \mathbf{0}$ be a decomposable r-vector. Then $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, and the vector subspace P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ is called the *r-space* of $\boldsymbol{\alpha}$. If $\boldsymbol{\alpha} = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$, then taking $c = 1$ in part (c) of Theorem 19, we see that $\mathbf{h}'_1, \ldots, \mathbf{h}'_r$ also span this same vector subspace P. Thus P depends only on $\boldsymbol{\alpha}$ and not on the particular way $\boldsymbol{\alpha}$ is written as the exterior product of vectors.

If $c \neq 0$, then $(c\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_r)$ is another frame for P and $c\boldsymbol{\alpha} = (c\mathbf{h}_1) \wedge \mathbf{h}_2 \wedge \cdots \wedge \mathbf{h}_r$. Thus $\boldsymbol{\alpha}$ and $c\boldsymbol{\alpha}$ have the same r-space P. On the other hand, if $\boldsymbol{\alpha}'$ is not a scalar multiple of $\boldsymbol{\alpha}$, then $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ have different r-spaces.

Orientations. Let P be an r-dimensional vector subspace of E^n.

Definition. A decomposable r-vector $\boldsymbol{\alpha}_0$ is an *orientation* for P if $|\boldsymbol{\alpha}_0| = 1$ and P is the r-space of $\boldsymbol{\alpha}_0$.

If $\boldsymbol{\alpha}$ is any r-vector whose r-space is P, then $c\boldsymbol{\alpha}$ is an orientation for P provided $|c\boldsymbol{\alpha}| = 1$. Since $|c\boldsymbol{\alpha}| = |c|\,|\boldsymbol{\alpha}|$, we must have $c = \pm|\boldsymbol{\alpha}|^{-1}$. P has two orientations. If $\boldsymbol{\alpha}_0$ is one of them, then $-\boldsymbol{\alpha}_0$ is the other.

Example 3. Let $\lambda = (i_1, \ldots, i_r)$ have no repetitions. The r-space of $\mathbf{e}_\lambda$ is spanned by $\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_r}$. Since $|\mathbf{e}_\lambda| = 1$, $\mathbf{e}_\lambda$ is an orientation for it.

If $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is any frame for P and $\boldsymbol{\alpha}_0$ is an orientation of P, then

$$\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = c\boldsymbol{\alpha}_0, \qquad c = \pm|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r| \neq 0.$$

Let us say that the frame $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ *has orientation* $\boldsymbol{\alpha}_0$ if $c > 0$, and orientation $-\boldsymbol{\alpha}_0$ if $c < 0$.

If two vectors in the frame $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ are interchanged, then the exterior product $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$ changes sign. Thus the orientation of a frame changes under interchanges.

In Example (2), $(11)^{-1/2}(\mathbf{e}_{12} - \mathbf{e}_{13} - 3\mathbf{e}_{23})$ is an orientation. The frame $(\mathbf{e}_1 + 3\mathbf{e}_3, \mathbf{e}_2 - \mathbf{e}_3)$ has this orientation.

Let $r = n$. Then $P = E^n$ and $\pm\mathbf{e}_{1\cdots n}$ are the two orientations. Let us call $\mathbf{e}_{1\cdots n}$ the *standard*, or *positive*, orientation of E^n and $-\mathbf{e}_{1\cdots n}$ the *negative* orientation of E^n. When $r < n$ we do not attempt to call one orientation of P positive and the other negative. If $(\mathbf{h}_1, \ldots, \mathbf{h}_n)$ is a frame for E^n, then by (6–13a)

$$\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_n = \det(h_l^k)\mathbf{e}_{1\cdots n}.$$

The frame has positive orientation if $\det(h_l^k) > 0$ and negative orientation if $\det(h_l^k) < 0$.

Measure for r-parallelepipeds. It was shown in Section 5–7 that if K is an n-parallelepiped spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_n$ with $\mathbf{x}_0$ as vertex, then

$$V_n(K) = |\det(h_l^k)| = |\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_n|.$$

More generally, if $\mathbf{x}_0, \mathbf{h}_1, \ldots, \mathbf{h}_r$ are vectors, then

$$K = \left\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{r} t^k\mathbf{h}_k,\ 0 \le t^k \le 1,\ k = 1, \ldots, r\right\}$$

is the *r-parallelepiped* spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ with $\mathbf{x}_0$ as vertex.

Definition. The *r-dimensional measure* of K is

$$V_r(K) = |\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|. \tag{6–16}$$

By part (a) of Theorem 19, $V_r(K) = 0$ if and only if $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$ is linearly dependent.

We now have a criterion which shows when two frames lead to the same r-vector.

Corollary. *Let* $(\mathbf{h}_1, \ldots, \mathbf{h}_r)$, $(\mathbf{h}'_1, \ldots, \mathbf{h}'_r)$ *be frames. Then* $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_r$ *if and only if these frames span the same r-space P, have the same orientation, and their parallelepipeds with* $\mathbf{0}$ *as vertex have the same r-measure.*

Measure for r-simplices. Let S be an r-simplex with vertices $\mathbf{x}_0, \ldots, \mathbf{x}_r$. Let $\mathbf{h}_k = \mathbf{x}_k - \mathbf{x}_0$, $k = 1, \ldots, r$. Reasoning as in Section 5–7, we have

$$S = \left\{\mathbf{x} : \mathbf{x} = \mathbf{x}_0 + \sum_{k=1}^{r} t^k\mathbf{h}_k,\ t^k \ge 0 \text{ for } k = 1, \ldots, r,\ \sum_{k=1}^{r} t^k \le 1\right\}.$$

The *r-dimensional measure* of S is defined to be

$$V_r(S) = \frac{1}{r!}|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|. \tag{6–17}$$

Both (6–16) and (6–17) are very special cases of a general formula (7–5) in Chapter 7 for r-dimensional measure.

Example 4. The area of the triangle in E^3 with vertices $\mathbf{0}$, $3\mathbf{e}_1 + \mathbf{e}_2$, $\mathbf{e}_3 - \mathbf{e}_2$ is $\frac{1}{2}|(3\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2)|$. Since

$$(3\mathbf{e}_1 + \mathbf{e}_2) \wedge (\mathbf{e}_3 - \mathbf{e}_2) = -3\mathbf{e}_{12} + 3\mathbf{e}_{13} + \mathbf{e}_{23},$$

the components are $\alpha^{12} = -3$, $\alpha^{13} = 3$, $\alpha^{23} = 1$. Since $|\boldsymbol{\alpha}|^2 = \sum_{[\lambda]}(\alpha^\lambda)^2$, the area is $\sqrt{19}/2$. The area can also be calculated from formula (6–18) below.

To show that the definition (6–16) of r-measure for parallelepipeds is reasonable, let us show that $V_r(K)$ is the product of the lengths of the vectors $\mathbf{h}_1, \ldots, \mathbf{h}_r$ in case these vectors are mutually orthogonal. If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are vectors, let $\mathbf{a}^k$ be the covector with the same components as the vector $\mathbf{v}_k$. Then $\mathbf{a}^1 \wedge \cdots \wedge \mathbf{a}^r$ is the r-covector with the same components as the r-vector $\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. By (6–14)

$$(\mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r) \cdot (\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r) = \det(\mathbf{v}_k \cdot \mathbf{h}_l),$$

where the $\cdot$ now denotes inner product. In particular, let $\mathbf{v}_k = \mathbf{h}_k$. Then

$$|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|^2 = \det(\mathbf{h}_k \cdot \mathbf{h}_l). \tag{6–18}$$

Taking square roots, we get a formula for $V_r(K)$. If $\mathbf{h}_1, \ldots, \mathbf{h}_r$ are mutually orthogonal, then $\mathbf{h}_k \cdot \mathbf{h}_l = 0$ for $k \ne l$ and $\det(\mathbf{h}_k \cdot \mathbf{h}_l) = |\mathbf{h}_1|^2 \cdots |\mathbf{h}_r|^2$. In this case $V_r(K) = |\mathbf{h}_1| \cdots |\mathbf{h}_r|$ as required.

PROBLEMS

1. Simplify ($n = 6$):
   (a) $\mathbf{e}_3 \wedge \mathbf{e}_5 \wedge \mathbf{e}_{24}$.
   (b) $\mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_{62}$.
   (c) $\mathbf{e}_1 \wedge (\mathbf{e}_{14} + \mathbf{e}_{64})$.
   (d) $(\mathbf{e}_1 + 3\mathbf{e}_4 - \mathbf{e}_6) \wedge (2\mathbf{e}_{23} + \mathbf{e}_{36}) \wedge \mathbf{e}_{45}$.
   (e) $(\mathbf{e}_{12} + \mathbf{e}_{13}) \wedge (\mathbf{e}_{34} + \mathbf{e}_{25}) \wedge (\mathbf{e}_{56} + \mathbf{e}_{46})$.
2. Evaluate the indicated scalar products ($n = 4$), using (6–14).
   (a) $(\mathbf{e}^1 + \mathbf{e}^2) \cdot (\mathbf{e}_1 + \mathbf{e}_2)$.
   (b) $\mathbf{e}^{12} \cdot \mathbf{e}_{34}$.
   (c) $\mathbf{e}^{134} \cdot (\mathbf{e}_{431} + 3\mathbf{e}_{124})$.
   (d) $(\mathbf{e}^1 - \mathbf{e}^4) \wedge (\mathbf{e}^2 + \mathbf{e}^4) \cdot (\mathbf{e}_1 + 2\mathbf{e}_4) \wedge (\mathbf{e}_2 - 2\mathbf{e}_4)$.
3. Using Theorem 19 show that $(2\mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_4, \mathbf{e}_3 + \mathbf{e}_4)$ is a frame for E^4. What is its orientation?

4. Do $\mathbf{e}_1 + \mathbf{e}_4$, $\mathbf{e}_2 + \mathbf{e}_5$, $\mathbf{e}_3 + \mathbf{e}_6$, $\mathbf{e}_1 + \mathbf{e}_5$, $\mathbf{e}_2 + \mathbf{e}_6$, $\mathbf{e}_3 + \mathbf{e}_4$ form a basis for E^6?
5. Show that $(\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3)$ and $(3\mathbf{e}_1 - \mathbf{e}_2 - 2\mathbf{e}_3, 2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$ are frames for the same vector subspace of E^3. Do their orientations agree?
6. Find the area of the triangle with vertices $2\mathbf{e}_3$, $\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{e}_1 + 3\mathbf{e}_3$.
7. Find the volume of the 3-simplex in E^4 with vertices $\mathbf{0}$, $\mathbf{e}_1 - \mathbf{e}_3$, $\mathbf{e}_2$, $\mathbf{e}_3 + 2\mathbf{e}_4$.
8. Let K be an r-parallelepiped spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_r$ with $\mathbf{0}$ as vertex. For each increasing λ let $K^\lambda = \mathbf{X}^\lambda(K)$ where $\mathbf{X}^\lambda$ is the projection onto the r-space of $\mathbf{e}_\lambda$. ($\mathbf{X}^\lambda$ leaves the components $x^{i_1}, \ldots, x^{i_r}$ of any $\mathbf{x}$ unchanged and replaces each of the other components by 0.) Let $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. Show that $|\alpha^\lambda| = V_r(K^\lambda)$ and hence $[V_r(K)]^2 = \sum_{[\lambda]} [V_r(K^\lambda)]^2$. Illustrate for $n = 3$ and $r = 1, 2$.
9. Show that:
   (a) If $\boldsymbol{\alpha}$ is decomposable, then $\boldsymbol{\alpha} \wedge \boldsymbol{\alpha} = \mathbf{0}$.
   (b) If $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are decomposable r-vectors, then $(\boldsymbol{\alpha} + \boldsymbol{\beta}) \wedge (\boldsymbol{\alpha} + \boldsymbol{\beta}) = 2\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ if r is even and is $\mathbf{0}$ if r is odd.
   (c) The 2-vector $\mathbf{e}_{12} + \mathbf{e}_{34}$ is not decomposable. [*Hint:* Use (a).]
10. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be decomposable nonzero 2-vectors, and P, Q be their respective 2-spaces. Show that if $P \cap Q = \{0\}$, then $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is not decomposable; and if $P \cap Q$ is a line through $\mathbf{0}$, then $\boldsymbol{\alpha} + \boldsymbol{\beta}$ is decomposable and $\boldsymbol{\alpha} \neq c\boldsymbol{\beta}$. [*Hints:* In the first instance $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\beta} = \mathbf{h}' \wedge \mathbf{k}'$, where $\{\mathbf{h}, \mathbf{k}, \mathbf{h}', \mathbf{k}'\}$ is a linearly independent set. In the second $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\beta} = \mathbf{h} \wedge \mathbf{k}'$, where $\mathbf{h} \in P \cap Q$.]
11. Let $\boldsymbol{\alpha} = \mathbf{h} \wedge \mathbf{k}$, $\boldsymbol{\alpha} \neq \mathbf{0}$. Show that the matrix (α^{ij}) has rank 2. [*Hint:* Show that each column vector of the matrix is a linear combination of $\mathbf{h}$ and $\mathbf{k}$.]
12. Let $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_r)$ be an $(r+1)$-tuple such that the vectors $\mathbf{x}_1 - \mathbf{x}_0, \ldots, \mathbf{x}_r - \mathbf{x}_0$ are linearly independent. Such an $(r+1)$-tuple defines an *oriented* r-simplex. Its *r-vector* is $1/r!(\mathbf{x}_1 - \mathbf{x}_0) \wedge \cdots \wedge (\mathbf{x}_r - \mathbf{x}_0)$. Let $\boldsymbol{\beta}_i$ be the $(r-1)$-vector of the ith oriented face $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r)$. Show that

$$\sum_{i=0}^{r} (-1)^i \boldsymbol{\beta}_i = \mathbf{0}.$$

6–4 INDUCED LINEAR TRANSFORMATIONS

Let m and n be positive integers. With any linear transformation $\mathbf{L}$ from E^m into E^n is associated for each $r = 1, 2, \ldots$ a linear transformation $\mathbf{L}_r$ from E^m_r into E^n_r with the following property. If $(\mathbf{k}_1, \ldots, \mathbf{k}_r)$ is any r-tuple of vectors in E^m, then we require that

$$\mathbf{L}_r(\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r) = \mathbf{L}(\mathbf{k}_1) \wedge \cdots \wedge \mathbf{L}(\mathbf{k}_r). \tag{6–19}$$

For $r = 2$ this is illustrated in Fig. 6–2.

With this in mind let us define $\mathbf{L}_r$ as follows. Let $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_m$ be the standard basis elements of E^m. Then $\mathbf{v}_j = \mathbf{L}(\boldsymbol{\epsilon}_j)$ is the jth column vector of $\mathbf{L}$. Let $\mu = (j_1, \ldots, j_r)$ be increasing. Then $\boldsymbol{\epsilon}_\mu = \boldsymbol{\epsilon}_{j_1} \wedge \cdots \wedge \boldsymbol{\epsilon}_{j_r}$, and remembering

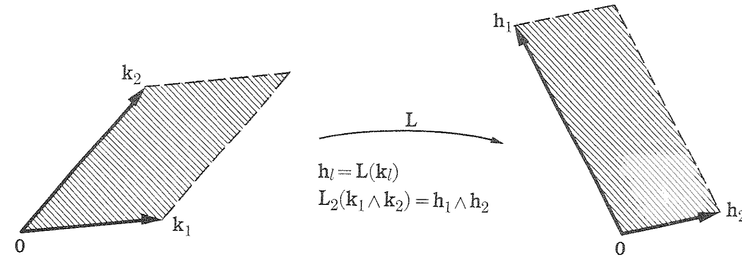

FIGURE 6–2

that we want (6–19) to be correct, we set

$$\mathbf{L}_r(\boldsymbol{\epsilon}_\mu) = \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}. \tag{6–20a}$$

Since $\mathbf{L}_r$ is to be linear, its value at any $\boldsymbol{\beta}$ is determined once the values at the basis elements $\boldsymbol{\epsilon}_\mu$ are known. For any $\boldsymbol{\beta} = \sum_{[\mu]} \beta^\mu \boldsymbol{\epsilon}_\mu$

$$\mathbf{L}_r(\boldsymbol{\beta}) = \sum_{[\mu]} \beta^\mu \mathbf{L}_r(\boldsymbol{\epsilon}_\mu). \tag{6–21}$$

The linear transformations $\mathbf{L}_r$ are said to be *induced* by $\mathbf{L}$. Of course $\mathbf{L}_1 = \mathbf{L}$. If $r > m$, then E^m_r has the single element $\mathbf{0}$ and $\mathbf{L}_r(\mathbf{0}) = \mathbf{0}$.

Let us show that for every $\boldsymbol{\beta} \in E^m_r$, $\boldsymbol{\gamma} \in E^m_s$,

$$\mathbf{L}_{r+s}(\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) = \mathbf{L}_r(\boldsymbol{\beta}) \wedge \mathbf{L}_s(\boldsymbol{\gamma}). \tag{6–22a}$$

Let $\mu = (j_1, \ldots, j_r)$ and $\nu = (k_1, \ldots, k_s)$ be increasing. Then

$$\mathbf{L}_r(\boldsymbol{\epsilon}_\mu) \wedge \mathbf{L}_s(\boldsymbol{\epsilon}_\nu) = \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r} \wedge \mathbf{v}_{k_1} \wedge \cdots \wedge \mathbf{v}_{k_s}.$$

If any integer is repeated in the $(r+s)$-tuple (μ, ν), then this is $\mathbf{0}$. Otherwise, the right-hand side is $(-1)^p \mathbf{L}_{r+s}(\boldsymbol{\epsilon}_\tau)$ where τ is the increasing $(r+s)$-tuple obtained from (μ, ν) by p interchanges. Since $\boldsymbol{\epsilon}_{\mu,\nu} = (-1)^p \boldsymbol{\epsilon}_\tau$ and $\mathbf{L}_{r+s}$ is linear, $(-1)^p \mathbf{L}_{r+s}(\boldsymbol{\epsilon}_\tau) = \mathbf{L}_{r+s}(\boldsymbol{\epsilon}_{\mu,\nu})$. Thus

$$\mathbf{L}_r(\boldsymbol{\epsilon}_\mu) \wedge \mathbf{L}_s(\boldsymbol{\epsilon}_\nu) = \mathbf{L}_{r+s}(\boldsymbol{\epsilon}_{\mu,\nu}).$$

Therefore (6–22a) is correct for basis of elements of E^m_r and E^m_s. Since each of these transformations is linear, (6–22a) then holds in general.

By induction there is a generalization of (6–22a) for products of any number of multivectors. In particular, in this way we get the required formula (6–19) for products of vectors.

Let $\boldsymbol{\beta} \in E^m_r$ and $\boldsymbol{\alpha} = \mathbf{L}_r(\boldsymbol{\beta})$. Let us find a formula for the components α^λ in terms of the components of $\boldsymbol{\beta}$. If $\lambda = (i_1, \ldots, i_r)$, $\mu = (j_1, \ldots, j_r)$, and (c^i_j) is the matrix of $\mathbf{L}$, let

$$c^\lambda_\mu = \det(c^{i_k}_{j_l})$$

By (6–13a), c^λ_μ is the λth component of $\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}$. By (6–20) and (6–21)

$$\boldsymbol{\alpha} = \sum_{[\mu]} \beta^\mu \mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}.$$

Since both sides have the same components,

$$\alpha^\lambda = \sum_{[\mu]} c^\lambda_\mu \beta^\mu, \quad \text{if } \boldsymbol{\alpha} = \mathbf{L}_r(\boldsymbol{\beta}). \tag{6–23a}$$

When $r = 1$, this becomes (4–4a).

The dual transformation. Let $\mathbf{L}^*_r$ be the linear transformation from $(E^n_r)^*$ into $(E^m_r)^*$ which is dual to $\mathbf{L}_r$. It is defined from the formula

$$\boldsymbol{\omega} \cdot \mathbf{L}_r(\boldsymbol{\beta}) = \mathbf{L}^*_r(\boldsymbol{\omega}) \cdot \boldsymbol{\beta} \tag{6–24}$$

for every $\boldsymbol{\beta} \in E^m_r$, $\boldsymbol{\omega} \in (E^n_r)^*$. Let us prove the formula dual to (6–20a):

$$\mathbf{L}^*_r(\mathbf{e}^\lambda) = \mathbf{w}^{i_1} \wedge \cdots \wedge \mathbf{w}^{i_r}, \tag{6–20b}$$

where $\mathbf{w}^1, \ldots, \mathbf{w}^n$ are the row covectors of $\mathbf{L}$. By (6–13b) the μth component of $\mathbf{w}^{i_1} \wedge \cdots \wedge \mathbf{w}^{i_r}$ is c^λ_μ. If in (6–24) we set $\boldsymbol{\omega} = \mathbf{e}^\lambda$, $\boldsymbol{\beta} = \boldsymbol{\epsilon}_\mu$ and recall (6–20a), then

$$\mathbf{e}^\lambda \cdot (\mathbf{v}_{j_1} \wedge \cdots \wedge \mathbf{v}_{j_r}) = \mathbf{L}^*_r(\mathbf{e}^\lambda) \cdot \boldsymbol{\epsilon}_\mu.$$

The left-hand side equals c^λ_μ, and the right-hand side is the μth component of $\mathbf{L}^*_r(\mathbf{e}^\lambda)$. Since both sides of (6–20b) have the same components c^λ_μ, they are equal.

From (6–20b) the formulas dual to (6–19)—(6–23a) follow in the same way as before.

PROBLEMS

1. Let $m = 2$, $n = 3$, $\mathbf{L}(s, 2t) = (s - 2t)\mathbf{e}_1 - s\mathbf{e}_2 + (2s + 3t)\mathbf{e}_3$. Find:
   (a) $\mathbf{L}^*(\mathbf{a})$. (b) c^{ij}_{12}. (c) $\mathbf{L}_2(\boldsymbol{\beta})$.
   (d) $\mathbf{L}^*_2(\boldsymbol{\omega})$. (e) $\mathbf{L}^*_3(\mathbf{e}^{123})$.
2. Prove the dual of (6–22a):
$$\mathbf{L}^*_{r+s}(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}) = \mathbf{L}^*_r(\boldsymbol{\omega}) \wedge \mathbf{L}^*_s(\boldsymbol{\zeta}) \tag{6–22b}$$
   if $\boldsymbol{\omega} \in (E^n_r)^*$, $\boldsymbol{\zeta} \in (E^n_s)^*$.
3. Prove the dual of (6–23a):
$$\zeta_\mu = \sum_{[\lambda]} \omega_\lambda c^\lambda_\mu, \quad \text{if } \boldsymbol{\zeta} = \mathbf{L}^*_r(\boldsymbol{\omega}). \tag{6–23b}$$
4. Let $\mathbf{L}$ be an orthogonal transformation of E^n. Show that $|\mathbf{L}_r(\boldsymbol{\alpha})| = |\boldsymbol{\alpha}|$:
   (a) If $\boldsymbol{\alpha}$ is decomposable. [*Hint:* (6–18).]
   (b) For any r-vector $\boldsymbol{\alpha}$.

6–5 DIFFERENTIAL FORMS

In Section 2–6 a differential form of degree 1 was defined as a covector-valued function. It was shown that any such differential form $\boldsymbol{\omega}$ is a linear combination of $dx^1, \ldots, dx^n$,

$$\boldsymbol{\omega} = \sum_{i=1}^n \omega_i \, dx^i,$$

where the coefficients $\omega_1, \ldots, \omega_n$ are real-valued functions. For $r > 2$ a differential form of degree r is supposed to be an alternating polynomial of degree r in $dx^1, \ldots, dx^n$ with coefficients ω_λ which are real-valued functions. This idea is expressed more precisely by the following definition.

Definition. A *differential form of degree r* is a function $\boldsymbol{\omega}$ with domain $D \subset E^n$ and values in $(E^n_r)^*$. The value of $\boldsymbol{\omega}$ at $\mathbf{x}$ is denoted by $\boldsymbol{\omega}(\mathbf{x})$.

The values of $\boldsymbol{\omega}$ are r-covectors. The same Greek letters $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ used in the last section to denote r-covectors are now used to denote differential forms. The context will indicate clearly which is intended.

For brevity we say "*r-form*" instead of "differential form of degree r." It is convenient to call any real-valued function f a *0-form*. If $r > n$, then the only r-form is the one which has the value $\mathbf{0}$ for every $\mathbf{x} \in D$. We also use $\mathbf{0}$ to denote this r-form.

Let $\boldsymbol{\omega}$ be an r-form and $\boldsymbol{\zeta}$ be an s-form, with the same domain D. The *exterior product* $\boldsymbol{\omega} \wedge \boldsymbol{\zeta}$ is the $(r + s)$-form defined by

$$(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})(\mathbf{x}) = \boldsymbol{\omega}(\mathbf{x}) \wedge \boldsymbol{\zeta}(\mathbf{x})$$

for every $\mathbf{x} \in D$. Similarly, $f\boldsymbol{\omega}$ is the r-form such that

$$(f\boldsymbol{\omega})(\mathbf{x}) = f(\mathbf{x})\boldsymbol{\omega}(\mathbf{x})$$

for every $\mathbf{x} \in D$. The rules for multiplication of multicovectors described in Proposition 20 hold also for products of differential forms.

We recall that dx^i is the 1-form with constant value $\mathbf{e}^i$ for every $\mathbf{x}$. Then $dx^i \wedge dx^j$ is the 2-form with constant value $\mathbf{e}^i \wedge \mathbf{e}^j = \mathbf{e}^{ij}$ for every $\mathbf{x}$. Let us denote the components of the 2-covectors $\boldsymbol{\omega}(\mathbf{x})$ by $\omega_{ij}(\mathbf{x})$. By (6–8)

$$\boldsymbol{\omega}(\mathbf{x}) = \sum_{i<j} \omega_{ij}(\mathbf{x})\mathbf{e}^{ij}$$

for every $\mathbf{x} \in D$. If ω_{ij} is the real-valued function whose value at $\mathbf{x}$ is $\omega_{ij}(\mathbf{x})$, then

$$\boldsymbol{\omega} = \sum_{i<j} \omega_{ij} \, dx^i \wedge dx^j.$$

(Strictly speaking, the 2-form $dx^i \wedge dx^j$ has domain E^n, and we mean here its restriction to D.)

Similarly, for any r-tuple $\lambda = (i_1, \ldots, i_r)$ the r-form $dx^{i_1} \wedge \cdots \wedge dx^{i_r}$ has constant value $\mathbf{e}^\lambda$. Hence if $\boldsymbol{\omega}$ is an r-form, then

$$\boldsymbol{\omega} = \sum_{[\lambda]} \omega_\lambda \, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \tag{6–25}$$

where the value of ω_λ at $\mathbf{x}$ is $\omega_\lambda(\mathbf{x})$. Using Problem 4, Section 6–1, one can also write

$$\boldsymbol{\omega} = \frac{1}{r!} \sum_{\lambda} \omega_\lambda \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

We say that an r-form $\boldsymbol{\omega}$ is of *class* $C^{(q)}$ if the functions ω_λ in (6–25) are of class $C^{(q)}$.

We recall that if f is a 0-form of class $C^{(1)}$, then df is the 1-form

$$df = f_1 \, dx^1 + \cdots + f_n \, dx^n,$$

where $f_1, \ldots, f_n$ are the partial derivatives. In particular, if $\boldsymbol{\omega}$ is an r-form of class $C^{(1)}$, then ω_λ is a 0-form of class $C^{(1)}$ and $d\omega_\lambda$ is defined.

Definition. Let $\boldsymbol{\omega}$ be an r-form of class $C^{(1)}$. The *exterior differential* $d\boldsymbol{\omega}$ is the $(r+1)$-form defined by the formula

$$d\boldsymbol{\omega} = \sum_{[\lambda]} d\omega_\lambda \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}. \tag{6–26}$$

Example 1. Let $r = 1$, $\boldsymbol{\omega} = \omega_1 \, dx^1 + \cdots + \omega_n \, dx^n$. Then

$$d\boldsymbol{\omega} = \sum_{i=1}^n d\omega_i \wedge dx^i = \sum_{i=1}^n \left\{ \sum_{j=1}^n \frac{\partial \omega_i}{\partial x_j} dx^j \right\} \wedge dx^i.$$

From the formulas $\mathbf{e}^i \wedge \mathbf{e}^j = -\mathbf{e}^j \wedge \mathbf{e}^i$, $\mathbf{e}^i \wedge \mathbf{e}^i = 0$, we have

$$dx^i \wedge dx^j = -dx^j \wedge dx^i, \qquad dx^i \wedge dx^i = \mathbf{0}.$$

Therefore

$$d\boldsymbol{\omega} = \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j. \tag{6–27}$$

In particular, if $n = 2$ and $\boldsymbol{\omega} = M \, dx + N \, dy$, then

$$d\boldsymbol{\omega} = dM \wedge dx + dN \wedge dy = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx \wedge dy.$$

Example 2. If $r = n$, then

$$\boldsymbol{\omega} = f \, dx^1 \wedge \cdots \wedge dx^n$$

where $f = \omega_{1 \cdots n}$. Since $d\boldsymbol{\omega}$ is an $(n+1)$-form, $d\boldsymbol{\omega} = \mathbf{0}$.

Example 3. If $n = 3$ and $\boldsymbol{\omega} = 2\,dx + z^2\,dy + x^2 y\,dz$, then

$$d\boldsymbol{\omega} = d(2) \wedge dx + d(z^2) \wedge dy + d(x^2 y) \wedge dz$$
$$= 2z\,dz \wedge dy + 2xy\,dx \wedge dz + x^2\,dy \wedge dz.$$

Proposition 24. *The exterior differential has the following properties:*

(1) $d(\boldsymbol{\omega} + \boldsymbol{\zeta}) = d\boldsymbol{\omega} + d\boldsymbol{\zeta}$, *if* $\boldsymbol{\omega}$ *and* $\boldsymbol{\zeta}$ *are* r-*forms of class* $C^{(1)}$.

(2) $d(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}) = d\boldsymbol{\omega} \wedge \boldsymbol{\zeta} + (-1)^r \boldsymbol{\omega} \wedge d\boldsymbol{\zeta}$, *if* $\boldsymbol{\omega}$ *is an* r-*form and* $\boldsymbol{\zeta}$ *is an* s-*form, both of class* $C^{(1)}$.

(3) $d(d\boldsymbol{\omega}) = 0$ *if* $\boldsymbol{\omega}$ *is an* r-*form of class* $C^{(2)}$.

Note: If $r = 0$ we agree that $f \wedge \boldsymbol{\zeta} = f\boldsymbol{\zeta}$. Similarly, if $s = 0$ then $\boldsymbol{\omega} \wedge f = f\boldsymbol{\omega}$. The proposition remains true if $r = 0$ or $s = 0$.

Proof. The coefficients of $\boldsymbol{\omega} + \boldsymbol{\zeta}$ in (6–25) are $\omega_\lambda + \zeta_\lambda$ and $d(\omega_\lambda + \zeta_\lambda) = d\omega_\lambda + d\zeta_\lambda$. Therefore (1) holds. Similarly, $d(c\boldsymbol{\omega}) = c\,d\boldsymbol{\omega}$.

To prove (2) let us for brevity set

$$\mathbf{E}^\lambda = dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

Let us first show that

$$d(f\mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = df \wedge \mathbf{E}^\lambda \wedge \mathbf{E}^\nu. \tag{*}$$

If any integer is repeated in the $(r+s)$-tuple λ, ν then both sides are $\mathbf{0}$. Otherwise, $\mathbf{E}^\lambda \wedge \mathbf{E}^\nu = (-1)^p \mathbf{E}^\tau$ where τ is increasing. By definition, $d(f\mathbf{E}^\tau) = df \wedge \mathbf{E}^\tau$. Multiplying both sides by $(-1)^p$ we get (*). Now

$$\boldsymbol{\omega} \wedge \boldsymbol{\zeta} = \sum_{[\lambda][\nu]} \omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu.$$

By the ordinary product rule

$$d(\omega_\lambda \zeta_\nu) = \zeta_\nu \, d\omega_\lambda + \omega_\lambda \, d\zeta_\nu.$$

By (*) with $f = \omega_\lambda \zeta_\nu$,

$$d(\omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = (\zeta_\nu \, d\omega_\lambda + \omega_\lambda \, d\zeta_\nu) \wedge \mathbf{E}^\lambda \wedge \mathbf{E}^\nu.$$

Since $d\zeta_\nu$ has degree 1 and $\mathbf{E}^\lambda$ degree r, by (3) of Proposition 20

$$d\zeta_\nu \wedge \mathbf{E}^\lambda = (-1)^r \mathbf{E}^\lambda \wedge d\zeta_\nu.$$

The scalar-valued function ζ_ν commutes with any differential form. Hence

$$d(\omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu) = (d\omega_\lambda \wedge \mathbf{E}^\lambda) \wedge (\zeta_\nu \mathbf{E}^\nu) + (-1)^r (\omega_\lambda \mathbf{E}^\lambda) \wedge (d\zeta_\nu \wedge \mathbf{E}^\nu). \tag{**}$$

Using (1),

$$d(\boldsymbol{\omega} \wedge \boldsymbol{\zeta}) = \sum_{[\lambda][\nu]} d(\omega_\lambda \zeta_\nu \mathbf{E}^\lambda \wedge \mathbf{E}^\nu),$$

while

$$\sum_{[\lambda][\nu]} (d\omega_\lambda \wedge \mathbf{E}^\lambda) \wedge (\zeta_\nu \mathbf{E}^\nu) = \left[\sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda\right] \wedge \left[\sum_{[\nu]} \zeta_\nu \mathbf{E}^\nu\right] = d\boldsymbol{\omega} \wedge \boldsymbol{\zeta}.$$

Similarly,

$$(-1)^r \sum_{[\lambda][\nu]} (\omega_\lambda \mathbf{E}^\lambda) \wedge (d\zeta_\nu \wedge \mathbf{E}^\nu) = (-1)^r \boldsymbol{\omega} \wedge d\boldsymbol{\zeta},$$

which proves (2).

If f is of class $C^{(2)}$, then from (6–27)

$$d(df) = \sum_{i<j} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial^2 f}{\partial x^j \partial x^i} \right) dx^i \wedge dx^j = \mathbf{0}.$$

The form $\mathbf{E}^\lambda$ has constant coefficients and hence $d\mathbf{E}^\lambda = \mathbf{0}$. Using the product rule (2), $d(df \wedge \mathbf{E}^\lambda) = \mathbf{0}$. Taking $f = \omega_\lambda$ and using (1),

$$d(d\boldsymbol{\omega}) = d\left(\sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda\right) = \sum_{[\lambda]} d(d\omega_\lambda \wedge \mathbf{E}^\lambda) = \mathbf{0}. \blacksquare$$

Definition. An r-form $\boldsymbol{\omega}$ is *closed* if $d\boldsymbol{\omega} = \mathbf{0}$. If $\boldsymbol{\omega} = d\boldsymbol{\zeta}$ for some $(r-1)$-form $\boldsymbol{\zeta}$, then $\boldsymbol{\omega}$ is an *exact* r-form.

If $r = 1$, these definitions agree with the ones given in Section 2–6. If $\boldsymbol{\omega}$ is exact and $\boldsymbol{\zeta}$ can be chosen to be of class $C^{(2)}$, then $d\boldsymbol{\omega} = d(d\boldsymbol{\zeta}) = \mathbf{0}$. Hence $\boldsymbol{\omega}$ is closed. Poincaré's lemma states that if domain D is star-shaped then conversely any closed form $\boldsymbol{\omega}$ is exact. This will be proved in Section 7–7.

***Remark.** The exterior differential d is uniquely determined by Properties (1), (2), (3) and the following property.

(4) *For $r = 0$, df agrees with its definition in Section 2–6.*

Let d' also have these four properties. Then $d'\mathbf{E}^\lambda = d'(d'x^{i_1} \wedge \mathbf{E}^\mu)$, where $\mu = (i_2, \ldots, i_r)$. But $dx^i = d'x^i$ by (4) since dx^i stands for the differential of the coordinate function X^i. Using (2), (3), and induction on r, $d'\mathbf{E}^\lambda = \mathbf{0}$. Using (2) and (4), $d'(f\mathbf{E}^\lambda) = df \wedge \mathbf{E}^\lambda$. Using (1),

$$d'\boldsymbol{\omega} = d'\left(\sum_{[\lambda]} \omega_\lambda \mathbf{E}^\lambda\right) = \sum_{[\lambda]} d'(\omega_\lambda \mathbf{E}^\lambda)$$

$$= \sum_{[\lambda]} d\omega_\lambda \wedge \mathbf{E}^\lambda.$$

Thus $d'\boldsymbol{\omega} = d\boldsymbol{\omega}$ for every $\boldsymbol{\omega}$ of class $C^{(1)}$. In particular, this proves that the exterior differential d is "coordinate free."

Transformation law for differential forms. Let $\mathbf{g}$ be a transformation of class $C^{(1)}$ from an open set $\Delta \subset E^m$ into E^n. Let D be an open set containing

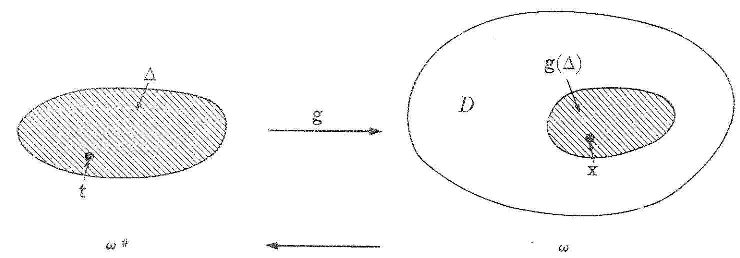

FIGURE 6–3

the image $\mathbf{g}(\Delta)$. If $\boldsymbol{\omega}$ is any r-form with domain D, then there is a correspondin
r-form denoted by $\boldsymbol{\omega}^\#$ with domain Δ. Formally, $\boldsymbol{\omega}^\#$ is obtained by merel
substituting $\mathbf{g}(\mathbf{t})$ for $\mathbf{x}$ and dg^i for dx^i. The precise definition of $\boldsymbol{\omega}^\#$ is as follows

Definition. For each $\mathbf{t} \in \Delta$, the value of $\boldsymbol{\omega}^\#$ at $\mathbf{t}$ is

$$\boldsymbol{\omega}^\#(\mathbf{t}) = \mathbf{L}_r^*[\boldsymbol{\omega}(\mathbf{x})], \quad (6\text{–}28$$

where

$$\mathbf{x} = \mathbf{g}(\mathbf{t}), \qquad \mathbf{L} = D\mathbf{g}(\mathbf{t}),$$

and $\mathbf{L}_r^*$ is the dual linear transformation induced by $\mathbf{L}$ (p. 224). In case $r =$
we agree that $f^\# = f \circ \mathbf{g}$. (The notation $^\#$ is used for brevity even though i
does not indicate the dependence on $\mathbf{g}$ and the degree r.)

Proposition 25. *The operation $^\#$ has the following properties:*

(1) $(\boldsymbol{\omega} + \boldsymbol{\zeta})^\# = \boldsymbol{\omega}^\# + \boldsymbol{\zeta}^\#$, *if $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ are of degree r.*

(2) $(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})^\# = \boldsymbol{\omega}^\# \wedge \boldsymbol{\zeta}^\#$, *if $\boldsymbol{\omega}$ is of degree r and $\boldsymbol{\zeta}$ of degree s.*

(3) $(df)^\# = d(f \circ \mathbf{g})$, *if f is of class $C^{(1)}$.*

(4) $(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^\# = dg^{i_1} \wedge \cdots \wedge dg^{i_r}$.

(5) $d\boldsymbol{\omega}^\# = (d\boldsymbol{\omega})^\#$, *if $\boldsymbol{\omega}$ is of class $C^{(1)}$ and $\mathbf{g}$ of class $C^{(2)}$.*

Proof. Since $\mathbf{L}_r^*$ is linear,

$$(\boldsymbol{\omega} + \boldsymbol{\zeta})^\#(\mathbf{t}) = \boldsymbol{\omega}^\#(\mathbf{t}) + \boldsymbol{\zeta}^\#(\mathbf{t}).$$

Since this is true for every $\mathbf{t} \in \Delta$, this proves (1). Using (6–22b), we get

$$(\boldsymbol{\omega} \wedge \boldsymbol{\zeta})^\#(\mathbf{t}) = \boldsymbol{\omega}^\#(\mathbf{t}) \wedge \boldsymbol{\zeta}^\#(\mathbf{t})$$

for every $\mathbf{t} \in \Delta$, which proves (2). By the chain rule (p. 106), $d(f \circ \mathbf{g})(\mathbf{t}) =$
$\mathbf{L}_1^*[df(\mathbf{x})]$. By (6–28) the right-hand side is $(df)^\#(\mathbf{t})$. Thus (3) holds. Recal
that dx^i stands for dX^i, where $X^i(\mathbf{x}) = x^i$ for each $\mathbf{x}$. Then $g^i = X^i \circ \mathbf{g}$ and
from (3) with $f = X^i$,

$$(dx^i)^\# = dg^i. \quad (6\text{–}29)$$

Then (4) follows from this and (2). To prove (5) we have from (1)–(4)

$$\omega^{\#} = \sum_{[\lambda]} \omega_\lambda \circ \mathbf{g}\, dg^{i_1} \wedge \cdots \wedge dg^{i_r},$$

$$d\omega^{\#} = \sum_{[\lambda]} d(\omega_\lambda \circ \mathbf{g}\, dg^{i_1} \wedge \cdots \wedge dg^{i_r}).$$

By (3), $(d\omega_\lambda)^{\#} = d(\omega_\lambda \circ \mathbf{g})$. Since $\mathbf{g}$ is of class $C^{(2)}$, $d(dg^i) = 0$. Therefore, by the product rule

$$d(dg^{i_1} \wedge \cdots \wedge dg^{i_r}) = 0.$$

Using the product rule again, we have

$$d\omega^{\#} = \sum_{[\lambda]} (d\omega_\lambda)^{\#} \wedge dg^{i_1} \wedge \cdots \wedge dg^{i_r} = (d\omega)^{\#}. \blacksquare$$

As in Chapter 4 let g^i_j denote the jth partial derivative of the component g^i. Let

$$g^\lambda_\mu = \det\,(g^{i_k}_{j_l}) = \frac{\partial(g^{i_1}, \ldots, g^{i_r})}{\partial(t^{j_1}, \ldots, t^{j_r})}.$$

The matrix of $D\mathbf{g}(\mathbf{t})$ is $(g^i_j(\mathbf{t}))$, and the row covectors are $dg^1(\mathbf{t}), \ldots, dg^n(\mathbf{t})$. By (6–13b) the μth component of $dg^{i_1} \wedge \cdots \wedge dg^{i_r}$ is g^λ_μ. Therefore

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^{\#} = \sum_{[\mu]} g^\lambda_\mu\, dt^{j_1} \wedge \cdots \wedge dt^{j_r}. \qquad (6\text{–}30)$$

In applying this formula in the next chapter we shall usually take $r = m$. In that case the only increasing r-tuple is $\mu = (1, 2, \ldots, r)$ and the right-hand side of (6–30) has just one term.

Example 4. Let $n = 3$, $r = m = 2$, $(x, y, z) = \mathbf{g}(s, t)$. Then

$$(dx \wedge dz)^{\#} = \frac{\partial(g^1, g^3)}{\partial(s, t)}\, ds \wedge dt.$$

If $\omega = f\,dx \wedge dz$, then $\omega^{\#} = f \circ \mathbf{g}\,(dx \wedge dz)^{\#}$.

Example 5. Let $m = n = r$. Then, writing $f = \omega_{1\cdots n}$, we have

$$\omega^{\#} = (f\,dx^1 \wedge \cdots \wedge dx^n)^{\#} = f \circ \mathbf{g}(dx^1 \wedge \cdots \wedge dx^n)^{\#}$$
$$= f \circ \mathbf{g}\,\frac{\partial(g^1, \ldots, g^n)}{\partial(t^1, \ldots, t^n)}\, dt^1 \wedge \cdots \wedge dt^n.$$

Example 6. Let $m = r = 1$. Then $\omega^{\#}(t) = \omega[g(t)] \cdot \mathbf{g}'(t)$, and the definition (3–8a) of the line integral can be rewritten

$$\int_\gamma \omega = \int_a^b \omega^{\#}.$$

*Note. In tensor language a differential form of degree r, being an r-covector-valued function, is called a covariant alternating *tensor field* of rank r. From (6–23b) we obtain the transformation law for the components of such a tensor field:

$$\zeta_\mu = \sum_{[\lambda]} \omega_\lambda \circ \mathbf{g}\, g^\lambda_\mu, \quad \text{if } \zeta = \omega^{\#}.$$

PROBLEMS

Assume that all forms which appear are of class $C^{(1)}$.

1. Find the exterior differential of:
 (a) $x^2y\,dy - xy^2\,dx$.
 (b) $\cos(xy^2)\,dx \wedge dz$.
 (c) $f(x, z)\,dx$.
 (d) $x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$.
2. Let P, Q, R have domain $D \subset E^3$. Show that
$$d(P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy) = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$
3. (a) Find an $(n-1)$-form ζ such that $d\zeta = dx^1 \wedge \cdots \wedge dx^n$. [*Hint:* Problem 1(d).]
 (b) Find an $(r-1)$-form ζ^λ such that $d\zeta^\lambda = \mathbf{E}^\lambda$.
 (c) Show that if the coefficients ω_λ in (6–25) are constant functions, then ω is exact.
4. (a) Show that if ω and ζ are closed differential forms, then $\omega \wedge \zeta$ is closed.
 (b) Show that if ω is closed and ζ is exact, then $\omega \wedge \zeta$ is exact.
5. Find the exterior differential of:
 (a) $d\omega \wedge \zeta - \omega \wedge d\zeta$.
 (b) $d\omega \wedge \zeta \wedge \eta + \omega \wedge d\zeta \wedge \eta + \omega \wedge \zeta \wedge d\eta$, if ω and ζ are of even degree.
6. A function f is an *integrating factor* for a 1-form ω if $f(\mathbf{x}) \neq 0$ for every $\mathbf{x} \in D$ and $f\omega$ is closed. Show that if ω has an integrating factor then $\omega \wedge d\omega = 0$.
7. Let $n = m = 2$, $r = 1$, $\omega = M\,dx + N\,dy$. Find explicitly $d\omega$ and $\omega^{\#}$ and verify that $(d\omega)^{\#} = d\omega^{\#}$.
8. Let $n = m = 3$, $\mathbf{g}(s, t, u) = (s \cos t)\mathbf{e}_1 + (s \sin t)\mathbf{e}_2 + u\mathbf{e}_3$. Find:
 (a) $(f\,dx \wedge dy \wedge dz)^{\#}$. (b) $(x\,dy \wedge dz)^{\#}$.
9. Show that if ω is a 2-form, then
$$d\omega = \sum_{i<j<k} \left(\frac{\partial \omega_{ij}}{\partial x^k} + \frac{\partial \omega_{ki}}{\partial x^j} + \frac{\partial \omega_{jk}}{\partial x^i}\right) dx^i \wedge dx^j \wedge dx^k.$$
10. Let $\omega^1, \ldots, \omega^p$ be 1-forms such that $\omega^i = \sum_{j=1}^p f^i_j\, dg^j$, $i = 1, \ldots, p$. Assume that the functions f^i_j are of class $C^{(1)}$, the g^i are of class $C^{(2)}$, and that the 1-covectors $\omega^1(\mathbf{x}), \ldots, \omega^p(\mathbf{x})$ are linearly independent for every $\mathbf{x} \in D$. Find 1-forms θ^i_j such that $d\omega^i = \sum_{j=i}^p \theta^i_j \wedge \omega^j$. [*Hint:* The $p \times p$ matrix $(f^i_j(\mathbf{x}))$ must be nonsingular.]

[*Note:* Conversely, if $d\omega^i$ is a linear combination of $\omega^1, \ldots, \omega^p$ with coefficients 1-forms θ^i_j, then locally functions f^i_j, g^j as above can be found. This result is called the *Frobenius integration theorem*, and has important applications in geometry and differential equations. See [9], p. 97.]

6–6 THE ADJOINT AND CODIFFERENTIAL

To each r-vector $\boldsymbol{\alpha}$ we shall now assign a certain $(n - r)$-vector, which is called the *adjoint* of $\boldsymbol{\alpha}$ and is denoted by $*\boldsymbol{\alpha}$. Let us begin with the special dimension $r = n - 1$, which is the only one needed in connection with the divergence theorem in the next chapter.

Let $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_{n-1}$. If $\boldsymbol{\alpha} = \mathbf{0}$, then we set $*\boldsymbol{\alpha} = \mathbf{0}$. If $\boldsymbol{\alpha} \neq \mathbf{0}$, then $*\boldsymbol{\alpha}$ will turn out to be the vector $\mathbf{h}$ with the following three properties: (1) $\mathbf{h}$ is a vector normal to the $(n-1)$-space P spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$; (2) $(\mathbf{h}, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ is a positively oriented frame for E^n; (3) $|\mathbf{h}| = |\boldsymbol{\alpha}|$. Condition (3) says that the length of $\mathbf{h}$ equals $V_{n-1}(K)$, where K is the $(n-1)$-parallelepiped vertex $\mathbf{0}$ spanned by $\mathbf{h}_1, \ldots, \mathbf{h}_{n-1}$. (See Fig. 6–4.)

With this in mind, let us define $*$ first for the standard basis $(n-1)$-vectors. Let $i' = (1, 2, \ldots, i-1, i+1, \ldots, n)$. Since $i - 1$ interchanges will change the n-tuple (i, i') into the increasing n-tuple $(1, \ldots, n)$,

$$(-1)^{i-1}\mathbf{e}_i \wedge \mathbf{e}_{i'} = \mathbf{e}_{1\cdots n}.$$

Therefore we set

$$*\mathbf{e}_{i'} = (-1)^{i-1}\mathbf{e}_i. \qquad (6\text{–}31)$$

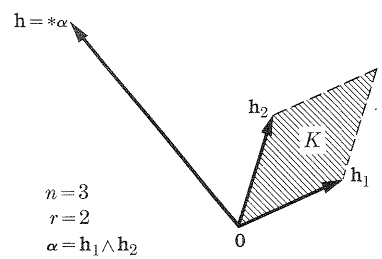

FIGURE 6–4

We want the operation $*$ to behave linearly.

For any $(n-1)$-vector $\boldsymbol{\alpha} = \sum_{i=1}^n \alpha^{i'}\mathbf{e}_{i'}$, let $*\boldsymbol{\alpha}$ be the vector $\mathbf{h} = \sum_{i=1}^n \alpha^{i'}(*\mathbf{e}_{i'})$. Its components are

$$h^i = (-1)^{i-1}\alpha^{i'}, \qquad i = 1, \ldots, n. \qquad (6\text{–}32\text{a})$$

Example 1. Let $n = 3$. In this particular dimension it is useful to consider instead of the standard basis $\{\mathbf{e}_{23}, \mathbf{e}_{13}, \mathbf{e}_{12}\}$ for E_2^3 the basis $\{\mathbf{e}_{23}, \mathbf{e}_{31}, \mathbf{e}_{12}\}$, where $\mathbf{e}_{31} = -\mathbf{e}_{13}$. Then any 2-vector $\boldsymbol{\alpha}$ can be written $\boldsymbol{\alpha} = \alpha^{23}\mathbf{e}_{23} + \alpha^{31}\mathbf{e}_{31} + \alpha^{12}\mathbf{e}_{12}$, where $\alpha^{31} = -\alpha^{13}$; and

$$*\mathbf{e}_{23} = \mathbf{e}_1, \qquad *\mathbf{e}_{31} = \mathbf{e}_2, \qquad *\mathbf{e}_{12} = \mathbf{e}_3,$$
$$\alpha^{23} = h^1, \qquad \alpha^{31} = h^2, \qquad \alpha^{12} = h^3 \quad \text{if } \mathbf{h} = *\boldsymbol{\alpha}.$$

Let us show that $*\boldsymbol{\alpha}$ has Properties (1), (2), and (3) above. Given a frame $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$, these three properties determine a vector, which we denote temporarily by $\tilde{\mathbf{h}}$. Let $(\mathbf{h}'_1, \ldots, \mathbf{h}'_{n-1})$ be an orthogonal frame for P, $\mathbf{h}'_k \cdot \mathbf{h}'_l = 0$ if $k \neq l$. Then $\boldsymbol{\alpha}$ is a scalar multiple b of $\mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_{n-1}$, and replacing

$\mathbf{h}'_1$ by $b\mathbf{h}'_1$ we may assume that $\boldsymbol{\alpha} = \mathbf{h}'_1 \wedge \cdots \wedge \mathbf{h}'_{n-1}$. By (1), $\tilde{\mathbf{h}} \cdot \mathbf{h}'_k = 0$ for each $k = 1, \ldots, n-1$. Therefore

$$|\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha}| = |\tilde{\mathbf{h}}|\,|\mathbf{h}'_1| \cdots |\mathbf{h}'_{n-1}| = |\tilde{\mathbf{h}}|\,|\boldsymbol{\alpha}|.$$

By (2), $\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = c\mathbf{e}_{1\cdots n}$ where $c > 0$. In fact, $c = |\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha}|$. Let $\mathbf{h} = *\boldsymbol{\alpha}$. From (6–32a), $|\mathbf{h}| = |\boldsymbol{\alpha}|$ and consequently $c = |\tilde{\mathbf{h}}|\,|\mathbf{h}|$.

On the other hand,

$$\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = \left(\sum_{i=1}^n \tilde{h}^i\mathbf{e}_i\right) \wedge \left(\sum_{j=1}^n \alpha^{j'}\mathbf{e}_{j'}\right) = \left[\sum_{i=1}^n \tilde{h}^i\alpha^{i'}(-1)^{i-1}\right]\mathbf{e}_{1\cdots n},$$

since $\mathbf{e}_i \wedge \mathbf{e}_{j'} = \mathbf{0}$ unless $i = j$. By (6–32a)

$$\tilde{\mathbf{h}} \wedge \boldsymbol{\alpha} = (\tilde{\mathbf{h}} \cdot \mathbf{h})\mathbf{e}_{1\cdots n},$$

and hence $\tilde{\mathbf{h}} \cdot \mathbf{h} = c = |\tilde{\mathbf{h}}|\,|\mathbf{h}|$. Equality holds in Cauchy's inequality (Section 1–1) and therefore $\tilde{\mathbf{h}}$ is a positive scalar multiple of $\mathbf{h}$. By (3), $|\tilde{\mathbf{h}}| = |\mathbf{h}|$, and hence $\tilde{\mathbf{h}} = \mathbf{h}$ as required.

We can now show that every $(n-1)$-vector $\boldsymbol{\alpha}$ is decomposable. Let $\boldsymbol{\alpha} \neq \mathbf{0}$, and let $\mathbf{h} = *\boldsymbol{\alpha}$. The vector $\mathbf{h}$ is normal to an $(n-1)$-dimensional subspace P. Let $\tilde{\boldsymbol{\alpha}}$ be an $(n-1)$-vector of P whose orientation and norm are chosen such that $\tilde{\boldsymbol{\alpha}}$ and $\mathbf{h}$ are related by (1)—(3). Then $\tilde{\boldsymbol{\alpha}}$ is decomposable and $\mathbf{h} = *\tilde{\boldsymbol{\alpha}}$. Thus $*\tilde{\boldsymbol{\alpha}} = *\boldsymbol{\alpha}$, which by (6–32a) implies that $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$.

If $\boldsymbol{\omega}$ is an $(n-1)$-covector, then $*\boldsymbol{\omega}$ is the 1-covector whose components are given by the dual to (6–32a):

$$\zeta_i = (-1)^{i-1}\omega_{i'}, \qquad i = 1, \ldots, n, \quad \text{if } \boldsymbol{\zeta} = *\boldsymbol{\omega}. \qquad (6\text{–}32\text{b})$$

If $\boldsymbol{\omega}$ is an $(n-1)$-form, then $*\boldsymbol{\omega}$ is the 1-form such that $(*\boldsymbol{\omega})(\mathbf{x}) = *\boldsymbol{\omega}(\mathbf{x})$ for each $\mathbf{x}$ in the domain D of $\boldsymbol{\omega}$. The dual to (6–31) is $*\mathbf{e}^{i'} = (-1)^{i-1}\mathbf{e}^i$, and hence

$$*(dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n) = (-1)^{i-1}\,dx^i.$$

If $\boldsymbol{\omega}$ is an $(n-1)$-form of class $C^{(1)}$, then $d\boldsymbol{\omega}$ is an n-form. Consequently, $d\boldsymbol{\omega} = f\,dx^1 \wedge \cdots \wedge dx^n$, where f is a scalar-valued function. To get a convenient expression for f, let $\boldsymbol{\zeta} = *\boldsymbol{\omega}$. Its components are given by (6–32b). Let

$$\operatorname{div} \boldsymbol{\zeta} = \frac{\partial\zeta_1}{\partial x^1} + \cdots + \frac{\partial\zeta_n}{\partial x^n}. \qquad (6\text{–}33)$$

This function is called the *divergence* of the 1-form $\boldsymbol{\zeta}$. By a short calculation (Problem 2), the desired function f is just $\operatorname{div}\boldsymbol{\zeta}$. Thus

$$d\boldsymbol{\omega} = \operatorname{div}\boldsymbol{\zeta}\,dx^1 \wedge \cdots \wedge dx^n, \quad \text{if } \boldsymbol{\zeta} = *\boldsymbol{\omega}. \qquad (6\text{–}34)$$

When $n = 3$ the divergence has an important physical significance which will be indicated in Section 7–4.

The remainder of this section will not be used in Chapter 7. Let us define $*\boldsymbol{\alpha}$ for any r-vector $\boldsymbol{\alpha}$ when $0 \le r \le n$. If $r = 0$ or n we set

$$*c = c\mathbf{e}_{1\cdots n}, \qquad *(c\mathbf{e}_{1\cdots n}) = c.$$

If $0 < r < n$, let $\lambda = (i_1, \ldots, i_r)$ be any increasing r-tuple, and let $\lambda' = (j_1, \ldots, j_{n-r})$ be the increasing $(n - r)$-tuple whose entries are those integers j_l between 1 and n which do not appear in λ. Let

$$\epsilon_\lambda = \delta^{1\cdots n}_{\lambda',\lambda}.$$

It is ± 1, depending on whether an odd or an even number of interchanges puts λ', λ in increasing order. If $\boldsymbol{\alpha}$ is any r-vector, then its *adjoint* is the $(n - r)$-vector $*\boldsymbol{\alpha}$ whose components satisfy

$$(*\alpha)^{\lambda'} = \alpha^\lambda \epsilon_\lambda. \tag{6–35a}$$

If $r = n - 1$ and $\lambda = i'$, then $\lambda' = (i)$, $\epsilon_\lambda = (-1)^{i-1}$, and (6–35a) agrees with (6–32a). From the definition (6–35a),

$$*(\boldsymbol{\alpha} + \boldsymbol{\beta}) = *\boldsymbol{\alpha} + *\boldsymbol{\beta}, \qquad *(c\boldsymbol{\alpha}) = c*\boldsymbol{\alpha}.$$

Moreover, $*\boldsymbol{\alpha} = \mathbf{0}$ if and only if $\boldsymbol{\alpha} = \mathbf{0}$. Thus the operation $*$ gives an isomorphism between E^n_r and E^n_{n-r}. This isomorphism preserves inner products. In fact, if $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are r-vectors then

$$*\boldsymbol{\alpha} \cdot *\boldsymbol{\beta} = \sum_{[\lambda]} (*\alpha)^{\lambda'}(*\beta)^{\lambda'} = \sum_{[\lambda]} (\epsilon_\lambda)^2 \alpha^\lambda \beta^\lambda.$$

Since $(\epsilon_\lambda)^2 = 1$, $*\boldsymbol{\alpha} \cdot *\boldsymbol{\beta} = \boldsymbol{\alpha} \cdot \boldsymbol{\beta}$. Taking $\boldsymbol{\alpha} = \boldsymbol{\beta}$ we have in particular $|*\boldsymbol{\alpha}| = |\boldsymbol{\alpha}|$. Since $\epsilon_{\lambda'}\epsilon_\lambda = (-1)^{r(n-r)}$,

$$**\boldsymbol{\alpha} = (-1)^{r(n-r)}\boldsymbol{\alpha}.$$

Now let ν be any increasing $(n - r)$-tuple. Then

$$\mathbf{e}_\nu \wedge \mathbf{e}_\lambda = \delta^{1\cdots n}_{\nu,\lambda}\mathbf{e}_{1\cdots n},$$

which is $\mathbf{0}$ if $\nu \ne \lambda'$ and is $\epsilon_\lambda \mathbf{e}_{1\cdots n}$ if $\nu = \lambda'$. If $\boldsymbol{\beta}$ is any $(n - r)$-vector, then

$$\boldsymbol{\beta} \wedge \boldsymbol{\alpha} = \sum_{[\nu][\lambda]} \beta^\nu \alpha^\lambda \mathbf{e}_\nu \wedge \mathbf{e}_\lambda = \left(\sum_{[\lambda]} \beta^{\lambda'}\alpha^\lambda \epsilon_\lambda\right)\mathbf{e}_{1\cdots n},$$

and

$$\boldsymbol{\beta} \wedge \boldsymbol{\alpha} = (\boldsymbol{\beta} \cdot *\boldsymbol{\alpha})\mathbf{e}_{1\cdots n}. \tag{6–36}$$

If $\boldsymbol{\alpha} \ne \mathbf{0}$ is decomposable, then $*\boldsymbol{\alpha}$ has the following geometric interpretation. Let $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$, and let $\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n$ be vectors such that: (1) $(\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n)$ is a frame for the orthogonal complement of the r-space of $\boldsymbol{\alpha}$; (2) $(\mathbf{h}_{r+1}, \ldots, \mathbf{h}_n, \mathbf{h}_1, \ldots, \mathbf{h}_r)$ is a positively oriented frame for E^n; (3) $|\mathbf{h}_{r+1} \wedge \cdots \wedge \mathbf{h}_n| = |\boldsymbol{\alpha}|$. Then $*\boldsymbol{\alpha} = \mathbf{h}_{r+1} \wedge \cdots \wedge \mathbf{h}_n$. The proof is similar to the one given above for $r = n - 1$.

If $\boldsymbol{\omega}$ is an r-covector, then $*\boldsymbol{\omega}$ is the $(n - r)$-covector such that

$$(*\omega)_{\lambda'} = \omega_\lambda \epsilon_\lambda. \tag{6–35b}$$

If $\boldsymbol{\omega}$ is a differential form of degree r, then $*\boldsymbol{\omega}$ is the $(n - r)$-form such that $(*\boldsymbol{\omega})(\mathbf{x}) = *\boldsymbol{\omega}(\mathbf{x})$ for every $\mathbf{x}$ in the domain of $\boldsymbol{\omega}$.

Example 2. $*(f\, dx^{i_1} \wedge \cdots \wedge dx^{i_r}) = \epsilon_\lambda f\, dx^{j_1} \wedge \cdots \wedge dx^{j_{n-r}}$, where $\lambda' = (j_1, \ldots, j_{n-r})$. For $r = 0$ or n,

$$*f = f\,dx^1 \wedge \cdots \wedge dx^n, \quad *(f\,dx^1 \wedge \cdots \wedge dx^n) = f.$$

Let $\boldsymbol{\omega}$ be an r-form of class $C^{(1)}$. Then $d(*\boldsymbol{\omega})$ is an $(n - r + 1)$-form, and $*d(*\boldsymbol{\omega})$ is an $(r - 1)$-form.

Definition. The *codifferential* of $\boldsymbol{\omega}$ is

$$\tilde{d}\boldsymbol{\omega} = (-1)^{r(n-r)} *d(*\boldsymbol{\omega}). \tag{6–37a}$$

Since $**\boldsymbol{\omega} = (-1)^{r(n-r)}\boldsymbol{\omega}$, substituting $*\boldsymbol{\omega}$ for $\boldsymbol{\omega}$ we get

$$\tilde{d}(*\boldsymbol{\omega}) = *d\boldsymbol{\omega}. \tag{6–37b}$$

If $r = 0$, we invent a form $\mathbf{0}$ of degree -1 and agree that $\tilde{d}f = \mathbf{0}$. If $\boldsymbol{\zeta}$ is a 1-form, consider the $(n - 1)$-form $\boldsymbol{\omega} = (-1)^{n-1} *\boldsymbol{\zeta}$. Then $*\boldsymbol{\omega} = \boldsymbol{\zeta}$ and by (6–37b) $\tilde{d}\boldsymbol{\zeta} = *d\boldsymbol{\omega}$. By (6–34) $\tilde{d}\boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\zeta}$. Thus the codifferential of a 1-form $\boldsymbol{\zeta}$ is just the 0-form div $\boldsymbol{\zeta}$.

Notes. Many authors define the adjoint so that in (2) above $(\mathbf{h}_1, \ldots, \mathbf{h}_r, \mathbf{h}_{r+1}, \ldots, \mathbf{h}_n)$ is a positively oriented frame for E^n. When $r(n - r)$ is odd, according to that definition $*\boldsymbol{\alpha}$ has opposite sign to the one here.

The definition of the adjoint involves the euclidean norm. Hence both the adjoint and the codifferential depend on the euclidean structure inherited by E^n_r and $(E^n_r)^*$ from the euclidean inner product in E^n; while the notions of $\wedge$ and d actually depend only on the vector space structure and not the inner product.

In riemannian geometry one is provided at each point $\mathbf{x}$ with an inner product $B_{\mathbf{x}}$, not necessarily the euclidean inner product. The definition of adjoint must be modified accordingly. The codifferential is again defined by (6–37a). However, the formula (6–33) for the divergence and its generalization (Problem 5) must be modified. See [9] and Chap. V of [17].

PROBLEMS

1. Let $n = 2$. Show that
   (a) $*\mathbf{h} = h^2\mathbf{e}_1 - h^1\mathbf{e}_2$.
   (b) $*(M\,dx + N\,dy) = N\,dx - M\,dy$.
   (c) $*d(N\,dx - M\,dy) = -(\partial M/\partial x + \partial N/\partial y)$.
2. (a) Let $n = 3$, and $\omega = P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy$ be a 2-form. Show that
$$*\omega = P\,dx + Q\,dy + R\,dz,\ d\omega = (\partial P/\partial x + \partial Q/\partial y + \partial R/\partial z)\,dx \wedge dy \wedge dz.$$
   (b) Let ω be an $(n-1)$-form,
$$\omega = \sum_{i=1}^{n} \omega_{i'}\,dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n,$$
   and let $\zeta = *\omega$. Show that $d\omega = \operatorname{div}\zeta\,dx^1 \wedge \cdots \wedge dx^n$.
3. Show that: (a) div (df) is the Laplacian of f. (b) $\operatorname{div}(f\omega) = f\operatorname{div}\omega + df\cdot\omega$, where $(\zeta\cdot\omega)(\mathbf{x}) = \zeta(\mathbf{x})\cdot\omega(\mathbf{x})$ for 1-forms ζ, ω.

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4. Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be r-vectors. Show that:
   (a) $(*\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} = \boldsymbol{\alpha}\cdot\boldsymbol{\beta}\,\mathbf{e}_{1\cdots n}$.
   (b) $(*\boldsymbol{\alpha}) \wedge \boldsymbol{\beta} = (-1)^{r(n-r)}\,\boldsymbol{\alpha} \wedge (*\boldsymbol{\beta})$.
   (c) $(*\omega)\cdot(*\boldsymbol{\alpha}) = \omega\cdot\boldsymbol{\alpha}$ for any r-covector ω.
5. Show that the components of $\tilde{d}\omega$ satisfy $(\tilde{d}\omega)_\nu = \sum_{i=1}^{n} \partial\omega_{\nu,i}/\partial x^i$, where $(\nu, i) = (i_1, \ldots, i_{r-1}, i)$.
6. Show that the components of $d\omega$ satisfy $(d\omega)_\lambda = \sum_{j=1}^{r+1} (-1)^{j-1}\,\partial\omega_{\lambda_j}/\partial x^{i_j}$, where λ_j is the r-tuple $(i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_{r+1})$.
7. If ζ is a 1-form and ω is an r-form let $\zeta\cdot\omega$ be the $(r-1)$-form such that $*(\zeta\cdot\omega) = (-1)^{n-1}\zeta \wedge (*\omega)$. Show that:
   (a) $\tilde{d}(f\omega) = f\,\tilde{d}\omega + df\cdot\omega$. (b) $(\zeta\cdot\omega)_\nu = \sum_{i=1}^{n} \zeta_i\omega_{\nu,i}$.

*6–7 SPECIAL RESULTS FOR $n = 3$

Vector analysis in E^3 is traditionally based on four operations besides the usual vector addition, scalar multiplication, and inner product. These operations are the cross product, triple scalar product, curl, and divergence. The last of these was defined in the previous section, for any dimension n. The other three are special to three dimensions, and can be expressed in terms of $\wedge$, $*$, and d as follows.

If $\mathbf{h}_1$ and $\mathbf{h}_2$ are vectors, their cross product is denoted by $\mathbf{h}_1 \times \mathbf{h}_2$. It is the vector
$$\mathbf{h}_1 \times \mathbf{h}_2 = *(\mathbf{h}_1 \wedge \mathbf{h}_2). \tag{6–38a}$$
See Fig. 6–4. The cross product distributes with vector addition and scalar multiplication, and $\mathbf{h}_2 \times \mathbf{h}_1 = -\mathbf{h}_1 \times \mathbf{h}_2$. However, it is not associative.

The triple scalar product of three vectors is denoted by $[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]$. It is given by
$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = *(\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3). \tag{6–39}$$
Its absolute value equals $|\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3|$, which is the volume of the parallelepiped spanned by $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ with vertex $\mathbf{0}$. The sign of the triple scalar product is positive if $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ is a positively oriented frame for E^3 and negative if this frame is negatively oriented.

When $n = 3$, $r(n-r)$ is always even and $(-1)^{r(n-r)} = 1$. Then $*(\mathbf{h}_1 \times \mathbf{h}_2) = \mathbf{h}_1 \wedge \mathbf{h}_2$. Using Problem 4(a), Section 6–6,
$$\mathbf{h}_1 \wedge \mathbf{h}_2 \wedge \mathbf{h}_3 = (\mathbf{h}_1 \times \mathbf{h}_2)\cdot\mathbf{h}_3\,\mathbf{e}_{123},$$
which gives another formula for the triple scalar product:
$$[\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3] = (\mathbf{h}_1 \times \mathbf{h}_2)\cdot\mathbf{h}_3.$$

The cross product of two covectors, or of two 1-forms, is given by
$$\omega \times \zeta = *(\omega \wedge \zeta). \tag{6–38b}$$
The curl of a 1-form ω is the 1-form curl ω given by
$$\operatorname{curl}\omega = *d\omega. \tag{6–40}$$
Its physical significance will be indicated in Section 7–6 in connection with Stokes' formula.

Example. Show that div (curl ω) $= 0$ for every 1-form ω of class $C^{(2)}$. Using the fact that $\tilde{d}* = *d$ (formula 6–37b),
$$\operatorname{div}(\operatorname{curl}\omega) = \tilde{d}(*d\omega) = *d(d\omega) = *0 = 0.$$

PROBLEMS

Assume that all forms are of class $C^{(2)}$.

1. Show that:
   (a) $\mathbf{h} \times \mathbf{k} = -\mathbf{k} \times \mathbf{h}$. (b) $\mathbf{h} \times (\mathbf{k}_1 + \mathbf{k}_2) = \mathbf{h} \times \mathbf{k}_1 + \mathbf{h} \times \mathbf{k}_2$.
2. Let $\omega = M\,dx + N\,dy + O\,dz$. Show that $\operatorname{curl}\omega = (\partial O/\partial y - \partial N/\partial z)\,dx + (\partial M/\partial z - \partial O/\partial x)\,dy + (\partial N/\partial x - \partial M/\partial y)\,dz$.
3. Find $\mathbf{e}_i \times \mathbf{e}_j$ for all pairs $i, j = 1, 2, 3$.
4. With the aid of (6–38a) and (6–40), show that:
   (a) div $(\zeta \times \omega) = 0$ if ζ and ω are closed. (b) curl $(f\omega) = f\operatorname{curl}\omega + df \times \omega$.
   (c) curl $(f\,df) = 0$. (d) curl $(\zeta \times \omega) = \tilde{d}(\zeta \wedge \omega)$.
   (e) curl (curl ω) $= d(\operatorname{div}\omega) - \operatorname{Lapl}\omega$, where $\operatorname{Lapl}(M\,dx + N\,dy + O\,dz) = (\operatorname{Lapl}M)\,dx + (\operatorname{Lapl}N)\,dy + (\operatorname{Lapl}O)\,dz$ and Lapl f is the Laplacian of the function f.
   (f) $\zeta\cdot\operatorname{curl}\omega - \omega\cdot\operatorname{curl}\zeta = \operatorname{div}(\omega \times \zeta)$. [*Hint:* By the dual to (6–36), $\zeta\cdot *d\omega = *(\zeta \wedge d\omega)$.]

5. Show that:
   (a) $\mathbf{h}_1 \times (\mathbf{h}_2 \times \mathbf{h}_3) = (\mathbf{h}_1 \cdot \mathbf{h}_3)\mathbf{h}_2 - (\mathbf{h}_1 \cdot \mathbf{h}_2)\mathbf{h}_3$. [*Hint:* Since both sides are trilinear in $(\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3)$ it suffices to prove this when $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$ are standard basis vectors. Use Problem 3.]
   (b) The cross product is not associative.
   (c) $(\mathbf{h}_1 \times \mathbf{h}_2) \times (\mathbf{h}_3 \times \mathbf{h}_4) = [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_4]\mathbf{h}_3 - [\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3]\mathbf{h}_4$.

6. Let $\boldsymbol{\omega} = (E_1\,dx^1 + E_2\,dx^2 + E_3\,dx^3) \wedge dx^4 + B_1\,dx^2 \wedge dx^3 + B_2\,dx^3 \wedge dx^1 + B_3\,dx^1 \wedge dx^2$, where the functions B_i, E_i are of class $C^{(1)}$ on an open subset of E^4. Show that $d\boldsymbol{\omega} = 0$ if and only if $\operatorname{curl} \mathbf{E} + \partial B/\partial x^4 = \mathbf{0}$, $\operatorname{div} \mathbf{B} = 0$. Here curl and div are taken in the variables (x^1, x^2, x^3). [*Note:* The equation $d\boldsymbol{\omega} = 0$ represents one-half of Maxwell's equations for an electromagnetic field in free space. The functions E_1, E_2, E_3 represent the electrical components of the field and B_1, B_2, B_3 the components of a magnetic induction vector. There is a similar equation which represents the other half of Maxwell's equations. See [9], p. 45.]