CHAPTER 7

Integration on Manifolds

The topic of this chapter is integration over subsets of an r-manifold $M \subset E^n$. For this purpose we first study regular transformations from one r-manifold into another. A regular transformation from a set $S \subset M$ into E^r defines a coordinate system for S. It is not always possible to find a single coordinate system for all of M. However, from the implicit function theorem, coordinates can be introduced locally. Using this fact, together with a device called partition of unity, the integral of a continuous function f over a set $A \subset M$ is defined in Section 7–3. Next, the idea of orientation on a manifold is introduced, and integrals of differential forms of degree r are defined.

The divergence theorem states that the integral of an $(n - 1)$-form $\boldsymbol{\omega}$ over the boundary fr D of an open set D equals the integral over D of $d\boldsymbol{\omega}$. The orientations on D and fr D must be chosen consistently, and certain regularity assumptions (p. 264) are made. When $n = 2$ and 3 the divergence theorem is equivalent to theorems in vector analysis commonly attributed to Green and to Gauss.

The divergence theorem is a special case of a result which states that the integral of the differential $d\boldsymbol{\omega}$ of an $(r - 1)$-form $\boldsymbol{\omega}$ over a portion A of an oriented r-manifold M equals the integral of $\boldsymbol{\omega}$ over the suitably oriented boundary of A. This is called Stokes' formula. In the final section the idea of homotopy between two transformations is introduced and is applied to give sufficient conditions in order that a closed differential form be exact.

7–1 REGULAR TRANSFORMATIONS

In Chapter 4 an r-manifold M was defined as a subset of some euclidean E^n which can locally be described by setting equal to 0 functions $\Phi^1, \ldots, \Phi^{n-r}$ with linearly independent differentials. For the precise definition, see Section 4–7. An r-manifold M has at each $\mathbf{x} \in M$ a tangent space, denoted in the present section by $T_M(\mathbf{x})$.

For purposes of integration it is necessary to consider manifolds from a different point of view. We must show that a manifold can be locally described by a system of r coordinate functions $F^1, \ldots, F^r$. This idea will be made precise in the next section. We need first to introduce the idea of regular transformation from one r-manifold into another.

Let $\mathbf{g}$ be a transformation whose domain is a set $N \subset E^m$. As in Section 2–3, we shall say that $\mathbf{g}$ is of class $C^{(q)}$ on N if there exists a transformation $\mathbf{G}$ of class $C^{(q)}$ on some open set Δ containing N such that $\mathbf{g} = \mathbf{G}|N$. The transformation $\mathbf{G}$ is an extension of $\mathbf{g}$ of class $C^{(q)}$.

Now let $N \subset E^m$ be an r-manifold, and $\mathbf{g}$ a transformation from N into E^n, where $r \leq \min\{m, n\}$. Let $\mathbf{g}$ be of class $C^{(1)}$ on N. If $r = m$, N is itself an open subset of E^r and we shall take $N = \Delta$, $\mathbf{G} = \mathbf{g}$.

When $r < m$, different extensions of $\mathbf{g}$ may lead to different values for the differential $D\mathbf{G}(\mathbf{t})$ at a point $\mathbf{t} \in N$. However, let us now show that the restriction of $D\mathbf{G}(\mathbf{t})$ to the tangent space at $\mathbf{t}$ is the same for all extensions.

Proposition 26. *Let $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ be of class $C^{(1)}$ on some open set containing N, and let $\mathbf{G}|N = \widetilde{\mathbf{G}}|N$. Then $D\mathbf{G}(\mathbf{t})(\mathbf{k}) = D\widetilde{\mathbf{G}}(\mathbf{t})(\mathbf{k})$ for every $\mathbf{t} \in N$ and $\mathbf{k} \in T_N(\mathbf{t})$.*

Proof. Let $\mathbf{t}_0 \in N$ and $\mathbf{k} \in T_N(\mathbf{t}_0)$. By definition of tangent vector (Section 4–7), there is a function $\boldsymbol{\psi}$ from an interval $(-\delta, \delta)$ into N such that $\boldsymbol{\psi}(0) = \mathbf{t}_0$, $\boldsymbol{\psi}'(0) = \mathbf{k}$. By Corollary 2, Section 4–4, the derivative of $\mathbf{G} \circ \boldsymbol{\psi}$ at 0 is $D\mathbf{G}(\mathbf{t}_0)(\mathbf{k})$. But $\mathbf{G} \circ \boldsymbol{\psi} = \widetilde{\mathbf{G}} \circ \boldsymbol{\psi}$ since $\mathbf{G}|N = \widetilde{\mathbf{G}}|N$. ∎

By this proposition we may set without ambiguity $D\mathbf{g}(\mathbf{t}) = D\mathbf{G}(\mathbf{t})|T_N(\mathbf{t})$, and may then write $D\mathbf{g}(\mathbf{t})(\mathbf{k})$ in place of $D\mathbf{G}(\mathbf{t})(\mathbf{k})$ if $\mathbf{k}$ is any tangent vector to N at $\mathbf{t}$. Let us next show that if the values of $\mathbf{g}$ lie in an r-manifold M, then $D\mathbf{g}(\mathbf{t})$ takes the tangent space at $\mathbf{t}$ into the tangent space at $\mathbf{g}(\mathbf{t})$.

Proposition 27. *Let $\mathbf{g}$ be a transformation of class $C^{(1)}$ from N into an r-manifold M. If $\mathbf{k} \in T_N(\mathbf{t}_0)$ and $\mathbf{h} = D\mathbf{g}(\mathbf{t}_0)(\mathbf{k})$, then $\mathbf{h} \in T_M[\mathbf{g}(\mathbf{t}_0)]$.*

Proof. Let $\mathbf{G}$ be an extension of $\mathbf{g}$ of class $C^{(1)}$. Let $\boldsymbol{\psi}$ be as in the proof of Proposition 26, and let $\boldsymbol{\theta} = \mathbf{G} \circ \boldsymbol{\psi} = \mathbf{g} \circ \boldsymbol{\psi}$. Then $\boldsymbol{\theta}'(0) = \mathbf{h}$.

Let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$, U be a neighborhood of $\mathbf{x}_0$, and $\boldsymbol{\Phi} = (\Phi^1, \ldots, \Phi^{n-r})$ be the same as in the definition of manifold. Then $\boldsymbol{\Phi}[\boldsymbol{\theta}(s)] = \mathbf{0}$ for every s in some interval about 0. Calculating the derivative at 0 of $\boldsymbol{\Phi} \circ \boldsymbol{\theta}$ by Corollary 2, Section 4–4,

$$D\boldsymbol{\Phi}(\mathbf{x}_0)[\boldsymbol{\theta}'(0)] = D\boldsymbol{\Phi}(\mathbf{x}_0)(\mathbf{h}) = \mathbf{0}.$$

By Theorem 10, $\mathbf{h} \in T_M(\mathbf{x}_0)$. ∎

When $r = m = n$, $\mathbf{g}$ is a transformation from an open set $\Delta \subset E^r$ into E^r. In this chapter we call such a transformation *flat*. A flat transformation $\mathbf{g}$ has at each $\mathbf{t} \in \Delta$ a Jacobian $J\mathbf{g}(\mathbf{t})$. We recall that the factor $|J\mathbf{g}(\mathbf{t})|$ appears in the formula (5–38) for transforming integrals.

For arbitrary r, m, and n let us now introduce a nonnegative number $\mathcal{J}\mathbf{g}(\mathbf{t})$, which for flat transformations becomes $|J\mathbf{g}(\mathbf{t})|$. Let $\{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$ be any basis for the tangent space $T_N(\mathbf{t})$. Let

$$\mathcal{J}\mathbf{g}(\mathbf{t}) = \frac{|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|}{|\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r|}, \tag{7–1}$$

where

$$\mathbf{h}_l = D\mathbf{g}(\mathbf{t})(\mathbf{k}_l), \qquad l = 1, \ldots, r.$$

By Theorem 19 the denominator is not 0. Stated geometrically, $\mathcal{J}\mathbf{g}(\mathbf{t}) = V_r(K')/V_r(K)$, where K and K' are r-parallelepipeds with $\mathbf{0}$ as vertex spanned respectively by $\mathbf{k}_1, \ldots, \mathbf{k}_r$ and by $\mathbf{h}_1, \ldots, \mathbf{h}_r$.

We must show that $\mathcal{J}\mathbf{g}(\mathbf{t})$ does not depend on the particular basis chosen for $T_N(\mathbf{t})$.

Let $\mathbf{L} = D\mathbf{g}(\mathbf{t})$, $\boldsymbol{\beta} = \mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r$, $\boldsymbol{\alpha} = \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r$. Then $\boldsymbol{\alpha} = \mathbf{L}_r(\boldsymbol{\beta})$, where $\mathbf{L}_r$ is the induced linear transformation defined in Section 6–4. Let $\{\mathbf{k}_1', \ldots, \mathbf{k}_r'\}$ be another basis for $T_N(\mathbf{t})$, and consider the corresponding $\mathbf{h}_l' = \mathbf{L}(\mathbf{k}_l)$, $\boldsymbol{\beta}' = \mathbf{k}_1' \wedge \cdots \wedge \mathbf{k}_r'$, $\boldsymbol{\alpha}' = \mathbf{h}_1' \wedge \cdots \wedge \mathbf{h}_r'$. By Theorem 19 $\boldsymbol{\beta}' = c\boldsymbol{\beta}$, where c is a scalar. Since $\mathbf{L}_r$ is linear, $\boldsymbol{\alpha}' = \mathbf{L}_r(c\boldsymbol{\beta}) = c\mathbf{L}_r(\boldsymbol{\beta})$. Thus $\boldsymbol{\alpha}' = c\boldsymbol{\alpha}$ and $|\boldsymbol{\alpha}'|/|\boldsymbol{\beta}'| = |\boldsymbol{\alpha}|/|\boldsymbol{\beta}|$. This shows that $\mathcal{J}\mathbf{g}(\mathbf{t})$ does not depend on the particular basis chosen for the tangent space at $\mathbf{t}$.

The condition $\mathcal{J}\mathbf{g}(\mathbf{t}) > 0$ means that $\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r \neq \mathbf{0}$, which by Theorem 19 is equivalent to linear independence of the set $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$. Proposition 27 then has the following corollary.

Corollary. *If $\mathcal{J}\mathbf{g}(\mathbf{t}_0) > 0$, then $D\mathbf{g}(\mathbf{t}_0)$ takes $T_N(\mathbf{t}_0)$ onto $T_M[\mathbf{g}(\mathbf{t}_0)]$.*

Proof. Since $D\mathbf{g}(\mathbf{t}_0)$ is linear, it takes $T_N(\mathbf{t}_0)$ onto a vector subspace P of $T_M[\mathbf{g}(\mathbf{t}_0)]$. If $\{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$ is a basis for $T_N(\mathbf{t}_0)$, then the set $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ is linearly independent and each $\mathbf{h}_l \in P$. Since the vector spaces P and $T_M[\mathbf{g}(\mathbf{t}_0)]$ have the same dimension r and $P \subset T_M[\mathbf{g}(\mathbf{t}_0)]$, they are the same. ∎

If $r = m$ and $N = \Delta$ is an open subset of E^r, then $T_N(\mathbf{t}) = E^r$ and for $\mathbf{k}_1, \ldots, \mathbf{k}_r$ we may take the standard basis vectors $\boldsymbol{\epsilon}_1, \ldots, \boldsymbol{\epsilon}_r$. In this case, $\mathbf{h}_l = \mathbf{g}_l(\mathbf{t})$ is the lth partial derivative of $\mathbf{g}$ at $\mathbf{t}$. Since $|\boldsymbol{\epsilon}_1 \wedge \cdots \wedge \boldsymbol{\epsilon}_r| = |\boldsymbol{\epsilon}_{1\ldots r}| = 1$, we have

$$\mathcal{J}\mathbf{g}(\mathbf{t}) = |\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})| \qquad \text{if } r = m. \tag{7–2}$$

If $r = n = m$, then the right-hand side equals the absolute value of the determinant of $D\mathbf{g}(\mathbf{t})$; and thus $\mathcal{J}\mathbf{g}(\mathbf{t}) = |J\mathbf{g}(\mathbf{t})|$ if $\mathbf{g}$ is a flat transformation.

Definition. A transformation $\mathbf{g}$ from an r-manifold N into an r-manifold M is *regular* if:

(1) $\mathbf{g}$ is of class $C^{(1)}$ on N;
(2) $\mathbf{g}$ is univalent; and
(3) $\mathcal{J}\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in N$.

A regular transformation $\mathbf{g}$ may distort shapes. However, if $\mathbf{g}$ is regular, then the image $\mathbf{g}(B)$ of any set $B \subset N$ is "qualitatively" the same as B. We shall prove (Theorem 20) that the inverse $\mathbf{g}^{-1}$ is regular. In particular, $\mathbf{g}$ and $\mathbf{g}^{-1}$ are continuous, which implies that B and $\mathbf{g}(B)$ are the same topologically [to use the correct technical term, B and $\mathbf{g}(B)$ are homeomorphic]. Conditions (1) and (3) insure that $\mathbf{g}$ is properly behaved from the viewpoint of differential calculus. For instance, we have just shown that the differential takes tangent spaces to N onto the corresponding tangent spaces to M.

A regular transformation is called by many authors a *diffeomorphism* of class $C^{(1)}$.

Note that we have *assumed* that $\mathbf{g}(N) \subset M$ for some r-manifold M. One might guess that conditions (1), (2), and (3) imply that $\mathbf{g}(N)$ lies in an r-manifold; but Problem 3 shows that this is false. Later in the section we shall find some additional conditions under which the guess is correct (see Theorem 21 and its Corollary 1).

Example 1. Let $\Delta \subset E^1$ be an open interval, and $\mathbf{g}$ a transformation from Δ into a 1-manifold $M \subset E^n$. Let us assume (1), (2), and the condition $\mathbf{g}'(t) \neq \mathbf{0}$. Since $r = 1$, by (7–2) $\mathcal{J}\mathbf{g}(t) = |\mathbf{g}'(t)| > 0$. The vector $\mathbf{g}'(t)$ is a tangent vector to M at $\mathbf{g}(t)$, and $T_M[\mathbf{g}(t)]$ consists of all scalar multiples of it. The set $\mathbf{g}(\Delta)$ is called an open simple arc. If $J \subset \Delta$ is a closed interval, then $\mathbf{g}|J$ represents a simple arc with endpoints included (see Section 3–2).

Proposition 28. *Let* $\mathbf{g}$ *be a regular transformation from* N *into* M, *and* $\boldsymbol{\phi}$ *be a transformation of class* $C^{(1)}$ *from* M *into* E^p. *Then*

$$\mathcal{J}(\boldsymbol{\phi} \circ \mathbf{g})(\mathbf{t}) = \mathcal{J}\boldsymbol{\phi}(\mathbf{x})\mathcal{J}\mathbf{g}(\mathbf{t}), \qquad \textit{if } \mathbf{x} = \mathbf{g}(\mathbf{t}). \tag{7–3}$$

Proof. Let $\mathbf{k}_l$, $\mathbf{h}_l$ be as above, and let $\boldsymbol{\eta}_l = D\boldsymbol{\phi}(\mathbf{x})(\mathbf{h}_l)$, $l = 1, \ldots, r$. By the composite function theorem, $\boldsymbol{\eta}_l = D(\boldsymbol{\phi} \circ \mathbf{g})(\mathbf{t})(\mathbf{k}_l)$. If $\mathcal{J}\boldsymbol{\phi}(\mathbf{x}) > 0$, then

$$\frac{|\boldsymbol{\eta}_1 \wedge \cdots \wedge \boldsymbol{\eta}_r|}{|\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r|} = \frac{|\boldsymbol{\eta}_1 \wedge \cdots \wedge \boldsymbol{\eta}_r|}{|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|}\,\frac{|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r|}{|\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r|},$$

which is just (7–3). If $\mathcal{J}\boldsymbol{\phi}(\mathbf{x}) = 0$, then the set $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ is linearly dependent. This implies that $\{\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_r\}$ is linearly dependent, and therefore $\mathcal{J}(\boldsymbol{\phi} \circ \mathbf{g})(\mathbf{t}) = 0.$ ▮

Corollary. *If* $\boldsymbol{\phi}$ *and* $\mathbf{g}$ *are regular, then their composite* $\boldsymbol{\phi} \circ \mathbf{g}$ *is regular.*

Proof. Since $\boldsymbol{\phi}$ and $\mathbf{g}$ are of class $C^{(1)}$ and univalent, so is $\boldsymbol{\phi} \circ \mathbf{g}$. Since $\mathcal{J}\boldsymbol{\phi}(\mathbf{x}) > 0$ and $\mathcal{J}\mathbf{g}(\mathbf{t}) > 0$, by (7–3) $\mathcal{J}(\boldsymbol{\phi} \circ \mathbf{g})(\mathbf{t}) > 0.$ ▮

Example 2. Let $S \subset E^3$ be a set such that

$$S = \{(x, y, \phi(x, y)) : (x, y) \in R\},$$

where R is an open subset of E^2 and ϕ is of class $C^{(1)}$ on R. The set S is a 2-manifold. To see this, let $\Phi(x, y, z) = z - \phi(x, y)$, $D = \{(x, y, z) : (x, y) \in R\}$. Then $d\Phi(\mathbf{x}) \neq 0$

Let $$\mathbf{g}(x, y) = x\mathbf{e}_1 + y\mathbf{e}_2 + \phi(x, y)\mathbf{e}_3.$$

Then $\mathbf{g}$ is of class $C^{(1)}$ from R onto S and is univalent. The vectors $\partial\mathbf{g}/\partial x$ and $\partial\mathbf{g}/\partial y$ give a basis for the tangent space $T_S[\mathbf{g}(x, y)]$. By (7–2), $\mathcal{J}\mathbf{g}(x, y) = |\partial\mathbf{g}/\partial x \wedge \partial\mathbf{g}/\partial y|$. See Fig. 7–1.

Calculating these partial derivatives, we get

$$\frac{\partial\mathbf{g}}{\partial x} = \mathbf{e}_1 + \frac{\partial\phi}{\partial x}\mathbf{e}_3, \qquad \frac{\partial\mathbf{g}}{\partial y} = \mathbf{e}_2 + \frac{\partial\phi}{\partial y}\mathbf{e}_3,$$

$$\frac{\partial\mathbf{g}}{\partial x} \wedge \frac{\partial\mathbf{g}}{\partial y} = \mathbf{e}_{12} - \frac{\partial\phi}{\partial y}\mathbf{e}_{31} - \frac{\partial\phi}{\partial x}\mathbf{e}_{23},$$

$$\left|\frac{\partial\mathbf{g}}{\partial x} \wedge \frac{\partial\mathbf{g}}{\partial y}\right| = \left[1 + \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]^{1/2}$$
$$= [1 + |d\phi|^2]^{1/2}.$$

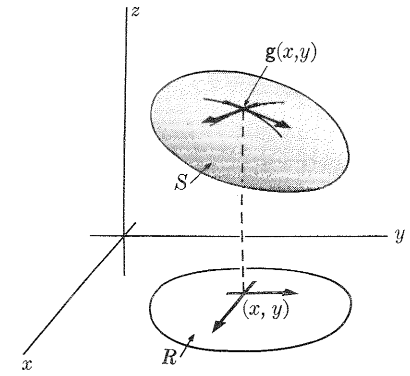


FIGURE 7–1

Since the last line is positive, $\mathcal{J}\mathbf{g}(x, y) > 0$. Thus $\mathbf{g}$ is a regular transformation.

Let X, Y, and Z be the standard cartesian coordinate functions for E^3; and let

$$\mathbf{F} = (X|S, Y|S).$$

Since X and Y are of class $C^{(1)}$, $\mathbf{F}$ is of class $C^{(1)}$ on S. Moreover, $\mathbf{F}$ is univalent; in fact, $\mathbf{F} = \mathbf{g}^{-1}$. Since $\mathbf{F} \circ \mathbf{g}$ is the identity transformation, $\mathcal{J}(\mathbf{F} \circ \mathbf{g}) = 1$. By (7–3),

$$\mathcal{J}\mathbf{F}(\mathbf{x}) = \frac{1}{\mathcal{J}\mathbf{g}(x, y)} > 0, \qquad \text{if } \mathbf{x} = \mathbf{g}(x, y).$$

Thus $\mathbf{F}$ is also a regular transformation.

Example 3. Generalizing Example 2, let $\lambda = (i_1, \ldots, i_r)$ be an increasing r-tuple of integers, $1 \leq i_k \leq n$ for $k = 1, \ldots, r$; and let $(j_1, \ldots, j_{n-r})$ be the increasing $(n - r)$-tuple complementary to λ. Let R be an open subset of E^r, and $\phi^1, \ldots, \phi^{n-r}$ of class $C^{(1)}$ on R. Let $\mathbf{g}$ be the transformation from R into E^n such that

$$g^{i_k}(\mathbf{x}^\lambda) = x^{i_k}, \qquad k = 1, \ldots, r$$
$$g^{j_l}(\mathbf{x}^\lambda) = \phi^l(\mathbf{x}^\lambda), \qquad l = 1, \ldots, n - r,$$

where

$$\mathbf{x}^\lambda = (x^{i_1}, \ldots, x^{i_r}).$$

Then $\mathbf{g}$ is of class $C^{(1)}$ and univalent. The explicit formula for $\mathcal{J}\mathbf{g}(\mathbf{x}^\lambda)$ is complicated. However, we can show that $\mathcal{J}\mathbf{g}(\mathbf{x}^\lambda) > 0$ as follows. Since

$$\frac{\partial\mathbf{g}}{\partial x^{i_k}} = \mathbf{e}_{i_k} + \sum_{l=1}^{n-r} \frac{\partial\phi^l}{\partial x^{i_k}}\mathbf{e}_{j_l}, \qquad \frac{\partial\mathbf{g}}{\partial x^{i_1}} \wedge \cdots \wedge \frac{\partial\mathbf{g}}{\partial x^{i_r}} = \mathbf{e}_\lambda + \text{other terms.}$$

This r-vector is not $\mathbf{0}$ since its λth component (the coefficient of $\mathbf{e}_\lambda$) is 1. Hence

Let $S = \mathbf{g}(R)$ and $\mathbf{F} = \mathbf{X}^\lambda|S$, where $\mathbf{X}^\lambda = (X^{i_1}, \dots, X^{i_r})$ and $X^1, \dots, X^n$ are the standard cartesian coordinate functions for E^n. As in Example 2, S is an r-manifold and $\mathbf{F} = \mathbf{g}^{-1}$ is regular. In the next section $\mathbf{F}$ will be called a cartesian coordinate system for S.

The importance of Example 3 lies in the implicit function theorem. If M is an r-manifold, then for every $\mathbf{x}_0 \in M$ there exist an increasing r-tuple λ and a neighborhood U_1 of $\mathbf{x}_0$ such that $S = M \cap U_1$ has a cartesian coordinate system $\mathbf{F}$ of the type just described. See p. 120.

Let us return from these examples to establish some further properties of regular transformations. We recall that a set $S \subset M$ is relatively open if $S = M \cap D$, where D is an open subset of E^n (Section A–6). A relatively open subset of an r-manifold is itself an r-manifold.

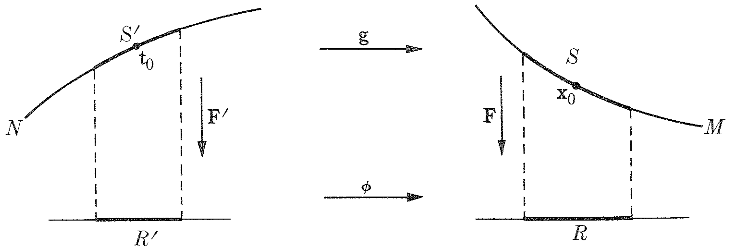
Theorem 20. *Let* $\mathbf{g}$ *be a regular transformation from an* r*-manifold* N *into an* r*-manifold* M. *Then* $\mathbf{g}(N)$ *is a relatively open subset of* M *and* $\mathbf{g}^{-1}$ *is a regular transformation.*

Proof. We know already that the theorem is true in the following two particular cases:

(a) If $\boldsymbol{\phi}$ is a flat regular transformation from an open set $\Delta \subset E^r$ into E^r, then by the inverse function theorem $\boldsymbol{\phi}(\Delta)$ is open and $\boldsymbol{\phi}^{-1}$ is regular. (b) If $\mathbf{g}$ is as in Example 3 and $S = \mathbf{g}(R) = M \cap U_1$, where U_1 is a neighborhood of some $\mathbf{x}_0 \in M$, then $\mathbf{g}(R)$ is open relative to M and $\mathbf{F} = \mathbf{g}^{-1}$ is regular.

In the general case, let $\mathbf{g}$ be regular from N into M. Let $\mathbf{t}_0 \in N$ and $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. By the implicit function theorem, there exist an increasing r-tuple λ and a relative neighborhood S of $\mathbf{x}_0$ such that $\mathbf{F} = \mathbf{X}^\lambda|S$ is regular. The set $R = \mathbf{F}(S)$ is open and $\mathbf{F}^{-1}$ is regular from R onto S. (In Example 3, $\mathbf{F}^{-1}$ was denoted by $\mathbf{g}$.) Similarly, there exist a relative neighborhood S' of $\mathbf{t}_0$ and $\mathbf{F}'$ regular from S' onto an open set $R' \subset E^r$ such that $(\mathbf{F}')^{-1}$ is also regular. Since $\mathbf{g}$ is continuous, we may arrange that $\mathbf{g}(S') \subset S$. Consider the transformation $\boldsymbol{\phi} = \mathbf{F} \circ \mathbf{g} \circ (\mathbf{F}')^{-1}$ from R' into R. By the corollary to Proposition 28, $\boldsymbol{\phi}$ is regular. Since $\boldsymbol{\phi}$ is flat, $\boldsymbol{\phi}(R')$ is open and $\boldsymbol{\phi}^{-1}$ is continuous. Let S_1 be a relative neighborhood of $\mathbf{x}_0$ such that $\mathbf{F}(S_1) \subset \boldsymbol{\phi}(R')$. If $\mathbf{x} \in S_1$, then $\mathbf{x} = \mathbf{g}(\mathbf{t})$ for $\mathbf{t} = ((\mathbf{F}')^{-1} \circ \boldsymbol{\phi}^{-1} \circ \mathbf{F})(\mathbf{x})$. Therefore $S_1 \subset \mathbf{g}(N)$. Moreover (see Fig. 7–2),

$$\mathbf{g}^{-1}|S_1 = (\mathbf{F}')^{-1} \circ \boldsymbol{\phi}^{-1} \circ (\mathbf{F}|S_1).$$



Since each of the three transformations on the right-hand side is regular, $\mathbf{g}^{-1}|S_1$ is regular.

Since every $\mathbf{x}_0 \in M$ has such a neighborhood S_1, this proves Theorem 20. ∎

In the next theorem we drop the assumption that $\mathbf{g}(N)$ is contained in an r-manifold. Instead we deduce it from conditions (1), (2), and (3) in the definition of regularity and the additional assumption that $\mathbf{g}$ is an open transformation.

Definition. A transformation $\mathbf{g}$ from a set $N \subset E^m$ into E^n is *open* if $\mathbf{g}(B)$ is open relative to $\mathbf{g}(N)$ for every set B open relative to N.

Theorem 21. *Let* $\mathbf{g}$ *be a transformation from an* r*-manifold* N *into* E^n, *such that* $\mathbf{g}$ *is open and satisfies* (1), (2), *and* (3). *Then* $\mathbf{g}(N)$ *is an* r*-manifold, and* $\mathbf{g}$ *is regular.*

Proof. We must show that $\mathbf{g}(N)$ is an r-manifold. Once this is proved, the regularity of $\mathbf{g}$ follows from the definition. Let us first prove the theorem in case N is an open set $\Delta \subset E^r$. For each increasing r-tuple $\lambda = (i_1, \dots, i_r)$ let $\mathbf{g}^\lambda$ be the flat transformation from Δ into E^r with components $g^{i_1}, \dots, g^{i_r}$. By formula (6–13a) the λth component of the r-vector $\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})$ is the Jacobian $J\mathbf{g}^\lambda(\mathbf{t})$.

Let $S = \mathbf{g}(\Delta)$, and let $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$ be any point of S. By (7–2), $\mathbf{g}_1(\mathbf{t}_0) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t}_0) \neq \mathbf{0}$. Hence there is some λ such that $J\mathbf{g}^\lambda(\mathbf{t}_0) \neq 0$. By the inverse function theorem there is a neighborhood Ω of $\mathbf{t}_0$ such that $\mathbf{g}^\lambda|\Omega$ is regular. The set $R = \mathbf{g}^\lambda(\Omega)$ is open and contains $\mathbf{x}_0^\lambda$. Let $\boldsymbol{\phi} = (\mathbf{g}^\lambda|\Omega)^{-1}$ and $\mathbf{G} = \mathbf{g} \circ \boldsymbol{\phi}$. Since $G^{i_k}(\mathbf{x}^\lambda) = x^{i_k}$ for each $k = 1, \dots, r$, $\mathbf{G}$ is of the type considered in Example 3. Therefore $\mathbf{G}(R)$ is an r-manifold. But $\mathbf{x}_0 \in \mathbf{G}(R)$ and $\mathbf{G}(R) = \mathbf{g}(\Omega)$ is a relatively open subset of S, since $\mathbf{g}$ is an open transformation. Since any $\mathbf{x}_0 \in S$ lies in a relatively open subset of S which is an r-manifold, S is an r-manifold. This proves Theorem 21 in the case $N \subset E^r$.

In the general case, let $\mathbf{t}_0 \in N$. Then $\mathbf{t}_0$ has a relative neighborhood S' with which is associated by the implicit function theorem a transformation $\mathbf{F}'$ as in the proof of Theorem 20. Let $\tilde{\mathbf{g}} = \mathbf{g} \circ (\mathbf{F}')^{-1}$. Since $\mathbf{F}'$ and $(\mathbf{F}')^{-1}$ are regular, $\tilde{\mathbf{g}}$ also satisfies the hypotheses of Theorem 21. Its domain $R' = \mathbf{F}'(S')$ is an open subset of E^r. By what has already been proved, $\mathbf{g}(S') = \tilde{\mathbf{g}}(R')$ is an r-manifold. Since every $\mathbf{t}_0 \in N$ has such a neighborhood S', $\mathbf{g}(N)$ is an r-manifold. ∎

Corollary 1. *If* $\mathbf{g}$ *satisfies* (1), (2), *and* (3) *and* $\mathbf{g}^{-1}$ *is continuous, then* $\mathbf{g}(N)$ *is an* r*-manifold and* $\mathbf{g}$ *is regular.*

Proof. Let $B \subset N$ be relatively open. Then $\mathbf{g}(B) = (\mathbf{g}^{-1})^{-1}(B)$ is open relative to $\mathbf{g}(N)$ by Proposition A–6. Therefore $\mathbf{g}$ is an open transformation. ∎

Corollary 2. *Let* $\mathbf{g}$ *be regular from an* r*-manifold* N *into an* r*-manifold* M. *If* $Q \subset N$ *is a* p*-manifold,* $p \leq r$, *then* $\mathbf{g}(Q)$ *is a* p*-manifold and* $\mathbf{g}|Q$ *is*

Proof. Clearly the restriction $\mathbf{g}|Q$ is of class $C^{(1)}$ and univalent since $\mathbf{g}$ has these properties. If $\mathbf{t} \in Q$, then there is a basis $\{\mathbf{k}_1, \ldots, \mathbf{k}_r\}$ for $T_N(\mathbf{t})$ such that $\{\mathbf{k}_1, \ldots, \mathbf{k}_p\}$ is a basis for $T_Q(\mathbf{t})$. Since $\mathbf{g}$ is regular, the set of images $\{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ of these basis elements under $D\mathbf{g}(\mathbf{t})$ is linearly independent. Therefore $\{\mathbf{h}_1, \ldots, \mathbf{h}_p\}$ is a linearly independent set. This shows that $\mathcal{J}(\mathbf{g}|Q)(\mathbf{t}) > 0$ for every $\mathbf{t} \in Q$.

Now $(\mathbf{g}|Q)^{-1} = \mathbf{g}^{-1}|\mathbf{g}(Q)$. By Theorem 20, $\mathbf{g}^{-1}$ is regular; and in particular $\mathbf{g}^{-1}$ is continuous. Hence its restriction to $\mathbf{g}(Q)$ is continuous. The conclusion follows from Corollary 1. ▮

Note: If M and N are manifolds of class $C^{(q)}$, $q > 1$, then we may consider regular transformations of class $C^{(q)}$ from N into M. In that case all transformations which appear in the proof of Theorem 20 are of class $C^{(q)}$, and hence $\mathbf{g}^{-1}$ is of class $C^{(q)}$. Similarly, in Theorem 21 and its corollaries, $\mathbf{g}(N)$ and $\mathbf{g}(Q)$ are of class $C^{(q)}$ if $\mathbf{g}$, N, and Q are of class $C^{(q)}$.

All of the results of this chapter are correct if one assumes merely that $q = 1$. However, to simplify the proof of the divergence theorem and Stokes' formula, we shall later take $q = 2$.

PROBLEMS

1. For each of the following transformations from $\Delta \subset E^2$ into E^3, find $\mathcal{J}\mathbf{g}(s, t)$ and $\mathbf{g}(\Delta)$. Show that $\mathbf{g}$ is univalent.
   (a) $\mathbf{g}(s, t) = (s + t)\mathbf{e}_1 + (s - 3t)\mathbf{e}_2 + (-2s + 2t + 2)\mathbf{e}_3$, $\Delta = \{(s, t) : 0 < s + t < 1, s > 0, t > 0\}$.
   (b) $\mathbf{g}(\rho, \theta) = (\rho \cos \alpha)\mathbf{e}_1 + (\rho \sin \alpha \cos \theta)\mathbf{e}_2 + (\rho \sin \alpha \sin \theta)\mathbf{e}_3$, $0 < \alpha < \pi/2$ (α fixed), $\Delta = (0, \infty) \times (0, 2\pi)$.
   (c) $\mathbf{g}(s, t) = st\mathbf{e}_1 + s\mathbf{e}_2 + t\mathbf{e}_3$, $\Delta = E^2$.
2. Let $\mathbf{G}(s, t, u) = as\mathbf{e}_1 + bt\mathbf{e}_2 + cu\mathbf{e}_3$, where a, b, and c are positive. Let N be the sphere $s^2 + t^2 + u^2 = 1$ and M the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.
   (a) Find $T_N(\mathbf{t})$, $T_M[\mathbf{G}(\mathbf{t})]$, $\mathbf{t} = (s, t, u)$. Verify that the image of $T_N(\mathbf{t})$ under $D\mathbf{G}(\mathbf{t})$ is $T_M[\mathbf{G}(\mathbf{t})]$.
   (b) Let $\mathbf{g} = \mathbf{G}|N$. Calculate $\mathcal{J}\mathbf{g}(\mathbf{t})$ from (7-1) and show that $\mathbf{g}$ is regular from N onto M.
3. Let $\mathbf{g}(t) = (\cos t)\mathbf{e}_1 + (\sin 2t)\mathbf{e}_2$, $\Delta = (0, 3\pi/2)$. Sketch $\mathbf{g}(\Delta)$. Show that $\mathbf{g}$ is univalent, and that $\mathcal{J}\mathbf{g}(t) > 0$, but that $\mathbf{g}(\Delta)$ is not a 1-manifold. Why does this not contradict Theorem 21?
4. Let $n = 4, r = 2, \lambda = (1, 2)$. Show that in Example 3, $\mathcal{J}\mathbf{g}(x^1, x^2) = [1 + |d\phi^1|^2 + |d\phi^2|^2 + (\phi_1^1\phi_2^2 - \phi_1^2\phi_2^1)^2]^{1/2}$.
5. Let $\Delta \subset E^r$ be open and bounded. Let $\mathbf{g}$ be continuous and univalent on $\mathrm{cl}\,\Delta$. Show that if $\mathbf{g}|\Delta$ is of class $C^{(1)}$ and $\mathcal{J}\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in \Delta$, then $\mathbf{g}(\Delta)$ is an r-manifold and $\mathbf{g}|\Delta$ is regular. [*Hint:* Problem 8(d), Section A-8, and Corollary 1.]
6. Let $\mathbf{g}$ be of class $C^{(1)}$ from an r-manifold N into E^n, and let $\mathcal{J}\mathbf{g}(\mathbf{t}_0) > 0$. Show that $\mathbf{t}_0$ has a neighborhood Ω such that $\mathbf{g}(N \cap \Omega)$ is an r-manifold and $\mathbf{g}|(N \cap \Omega)$ is regular. [*Hints:* First consider the case $N \subset E^r$. By generalizing Proposition 14, Section 4-3, find a neighborhood Ω_0 such that $\mathbf{g}$ is univalent on $N \cap \mathrm{cl}\,\Omega_0$. Use Problem 5.]

7-2 COORDINATE SYSTEMS ON MANIFOLDS

Let M be an r-manifold. Since M is r-dimensional it should be possible to find, at least locally on M, r functions $F^1, \ldots, F^r$ such that the numbers $F^1(\mathbf{x}), \ldots, F^r(\mathbf{x})$ will serve as coordinates of a point $\mathbf{x} \in M$. When $r = n$, M is an open subset of E^n. In that case we saw already in Section 5-9 that any regular transformation $\mathbf{F} = (F^1, \ldots, F^n)$ will serve to coordinate M.

The definition of coordinate system on an r-manifold is similar.

Definition. Let S be a nonempty, relatively open subset of an r-manifold $M \subset E^n$. Any regular transformation $\mathbf{F}$ from S into E^r is a *coordinate system* for S. The coordinates of a point $\mathbf{x} \in S$ in this system are $F^1(\mathbf{x}), \ldots, F^r(\mathbf{x})$.

By Theorem 20, the set $\Delta = \mathbf{F}(S)$ is an open subset of E^r and the transformation $\mathbf{g} = \mathbf{F}^{-1}$ is regular from Δ onto S. It will be $\mathbf{g}$ rather than $\mathbf{F}$ which is ordinarily used for calculations in the sections to follow.

Example 1. Let us return to the three examples in Section 7-1. In the first of them, the function $F = \mathbf{g}^{-1}$ is a coordinate for the open simple arc $S = \mathbf{g}(\Delta)$. The coordinate of a point $\mathbf{x} = \mathbf{g}(t)$ is t in this system.

In the second of those examples, $\mathbf{F}$ is a coordinate system for S. The coordinates of a point $(x, y, \phi(x, y)) \in S$ in this system are x, y. In the third example, $\mathbf{F} = \mathbf{X}^{\lambda}|S$. The coordinates of a point $\mathbf{x}$ in the coordinate system $\mathbf{F}$ are $x^{i_1}, \ldots, x^{i_r}$. Such a coordinate system will be called *cartesian.*

Example 2. Let $\mathbf{F} = (F^1, \ldots, F^r)$ be a coordinate system for S, and let $S_c = \{\mathbf{x} \in S : F^1(\mathbf{x}) = c\}$. Assume that S_c is not empty. Then S_c is an $(r - 1)$-manifold and $\mathbf{\Psi}_c = (F^2|S_c, \ldots, F^r|S_c)$ is a coordinate system for S_c. The proof is left to the reader (Problem 4). This is illustrated in Fig. 7-3 when $r = n$ and S is an open set D.

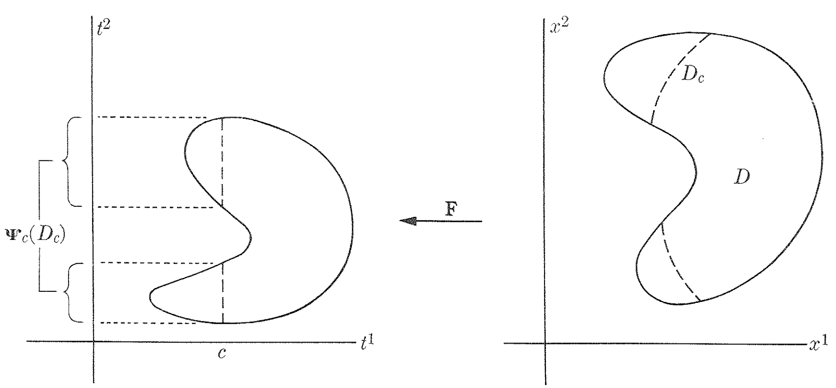


FIGURE 7-3

Example 3. Let $(R, \Theta^1, \ldots, \Theta^{n-1})$ be the spherical coordinate system for the open set D in Example 5, p. 181. The complement of D is a null set. Setting $R(\mathbf{x}) = 1$ we get, according to Example 2, a spherical coordinate system for the intersection of D with the unit $(n-1)$-sphere S^{n-1}. It turns out that $S^{n-1} - D$ is null in dimension $n-1$, in the sense to be explained in the next section. Consequently, this coordinate system can be used to evaluate integrals over S^{n-1}.

It is not ordinarily possible to find a single coordinate system for all of a manifold M. If $S \subset M$ has a coordinate system $\mathbf{F}$, then by Theorem 20 both $\mathbf{F}$ and $\mathbf{F}^{-1}$ are continuous. Therefore S is homeomorphic with an open set $\Delta \subset E^r$. Since Δ cannot be both open and compact, S is not compact. In particular, a compact manifold M (for instance, a sphere or torus) cannot be coordinatized by a single system.

Definition. A relatively open set $S \subset M$ which has a coordinate system is a *coordinate patch* on M.

By the implicit function theorem every point of M lies in some coordinate patch S. Actually, each point of M lies in an infinite number of coordinate patches. Let us now show how different coordinate systems are related in overlapping patches.

Coordinate changes. If S is a coordinate patch and $\mathbf{F}$ a coordinate system for S, then another coordinate system $\mathbf{F}'$ for S can be obtained as follows. Let $\boldsymbol{\phi}$ be a regular flat transformation from $\Delta = \mathbf{F}(S)$ into E^r, and let $\mathbf{F}' = \boldsymbol{\phi} \circ \mathbf{F}$. Since $\boldsymbol{\phi}$ and $\mathbf{F}$ are regular, $\mathbf{F}'$ is regular. Hence $\mathbf{F}'$ is a coordinate system for S.

Now let S and $\tilde{S}$ be coordinate patches, $\mathbf{F}$ a coordinate system for S, and $\tilde{\mathbf{F}}$ a coordinate system for $\tilde{S}$. Suppose that $S \cap \tilde{S}$ is not empty. Let us show that in $S \cap \tilde{S}$, $\tilde{\mathbf{F}}$ can be obtained from $\mathbf{F}$ by a regular flat transformation $\boldsymbol{\phi}$. Let

$$\Delta_0 = \mathbf{F}(S \cap \tilde{S}), \qquad \tilde{\Delta}_0 = \tilde{\mathbf{F}}(S \cap \tilde{S}),$$

and $\mathbf{g} = \mathbf{F}^{-1}$, $\tilde{\mathbf{g}} = \tilde{\mathbf{F}}^{-1}$. By Theorem 20, $\mathbf{g}$ is regular. Its restriction to Δ_0 is also regular, and hence the composite $\boldsymbol{\phi} = \tilde{\mathbf{F}} \circ (\mathbf{g}|\Delta_0)$ is regular. Moreover, $\tilde{\mathbf{F}}(\mathbf{x}) = \boldsymbol{\phi}[\mathbf{F}(\mathbf{x})]$ for every $\mathbf{x} \in S \cap \tilde{S}$. See Fig. 7–4.

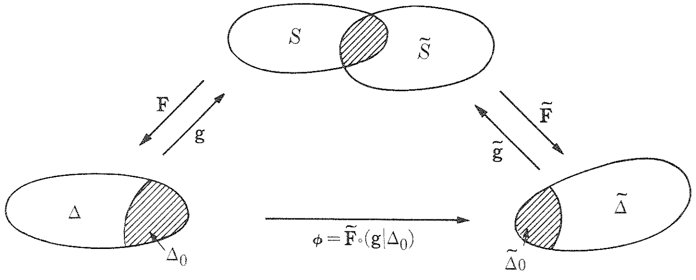


FIGURE 7–4

Example 1 (continued). If $r = 1$ and $\Delta \subset E^1$ is an open interval, then a coordinate change is determined by a real-valued ϕ such that $\phi'(t) \neq 0$. If $\phi'(t) > 0$ for every $t \in \Delta$, then ϕ was called in Section 3–2 a "parameter change." The condition $\phi'(t) > 0$ insures that ϕ does not reverse the orientation (see Section 7–4) of the 1-manifold M

***Manifolds defined by coordinate systems.** We have seen that an r-manifold $M \subset E^n$ is covered by coordinate patches. Conversely, let M be a subset of E^n with the following property: There is a collection $\mathcal{S}$ of relatively open subsets of M which cover M; and with each $S \in \mathcal{S}$ there is associated a homeomorphism $\mathbf{F}$ from S onto an open set $\Delta \subset E^r$ such that $\mathbf{F}^{-1}$ is of class $C^{(1)}$ and $\mathcal{J}\mathbf{F}^{-1}(\mathbf{t}) > 0$ for each $\mathbf{t} \in \Delta$. By Corollary 1 to Theorem 21, each $S \in \mathcal{S}$ is an r-manifold; and hence M is an r-manifold. This shows that instead of the approach in Section 4–7 we could have defined manifolds in terms of coordinate systems.

We have taken a rather concrete approach to the idea of manifold, considering only manifolds given as subsets of some euclidean space. However, the manifolds encountered in practice often are not given in this way. The approach via coordinate systems allows one to take a more abstract point of view. From this viewpoint the definition of manifold is as follows: An r-manifold of class $C^{(1)}$ is a Hausdorff topological space Z provided with a collection of open subsets S (called coordinate patches) covering Z and for each coordinate patch a homeomorphism $\mathbf{F}$ from S onto an open set $\Delta \subset E^r$. The regularity of the flat transformation $\boldsymbol{\phi}$ in Fig. 7–4 is now imposed as an axiom.

If Z is an r-manifold according to this more abstract definition and Z is separable, then Z can be realized as a submanifold of some euclidean space, in fact as a submanifold of E^{2r+1}. See [21], Chap. IV. A result of this type is called an embedding theorem. By separable we mean that Z has a covering either by finitely many coordinate patches or by a sequence of coordinate patches.

PROBLEMS

1. Let $M = \{(x, y, z) : x^2 + 2y + z^2 = 3, z > 0, y > |x|\}$. Let $\mathbf{F} = (X|M, Y|M)$ and $\tilde{\mathbf{F}} = (X|M, Z|M)$. Describe Δ, $\tilde{\Delta}$, $\mathbf{g}$, $\tilde{\mathbf{g}}$, and $\boldsymbol{\phi}$ (see Fig. 7–4).
2. Let $M = \{(y^2 + z^2, y, z) : y > 0\}$ and let $\mathbf{F}(x, y, z) = (y + z, \exp z)$ for $(x, y, z) \in M$. Show that $\mathbf{F}$ is a coordinate system for M and find $\mathbf{F}(M)$. [*Hint:* First take y and z as coordinates on M and then find a suitable coordinate change $\boldsymbol{\phi}$ giving the system $\mathbf{F}$.]
3. *Stereographic projection.* Let M be the sphere $x^2 + y^2 + (z-1)^2 = 1$. For each $\mathbf{x} = (x, y, z) \in M$ except the "north pole" $2\mathbf{e}_3$, let $(s, t, 0)$ be the point where the line through $2\mathbf{e}_3$ and $\mathbf{x}$ meets the plane $z = 0$. Let $\mathbf{F}(\mathbf{x}) = (s, t)$.
   (a) Show that $\mathbf{F}$ is a coordinate system for $M - \{2\mathbf{e}_3\}$.
   (b) Let $\mathbf{h}_1$, $\mathbf{h}_2$ be tangent vectors to M at $\mathbf{x}$, and let $\mathbf{k}_l = D\mathbf{F}(\mathbf{x})(\mathbf{h}_l)$, $l = 1, 2$. Show that the angle between $\mathbf{k}_1$ and $\mathbf{k}_2$ equals the angle between $\mathbf{h}_1$ and $\mathbf{h}_2$.

4. In Example 2 prove that S_c is an $(r-1)$-manifold and $\mathbf{\Psi}_c$ is a coordinate system for it. [*Hints:* Let $\Delta = \mathbf{F}(S)$. $\mathbf{F}(S_c)$, being the intersection of Δ with the hyperplane $t^1 = c$, is an $(r-1)$-manifold. See Corollary 2 to Theorem 21.]

5. Let $(F^1, \dots, F^r)$ be a coordinate system for S. Let $s < r$ and $S_c = \{\mathbf{x} : F^1(\mathbf{x}) = c^1, \dots, F^s(\mathbf{x}) = c^s\}$. Show that if S_c is not empty, then S_c is an $(r-s)$-manifold and $(F^{s+1}|S_c, \dots, F^r|S_c)$ a coordinate system for it.

6. Let $f^1, \dots, f^r, \Phi^1, \dots \Phi^{n-r}$ be functions of class $C^{(1)}$ on an open set D. Suppose that $\mathbf{F} = (f^1|S, \dots, f^r|S)$ is a coordinate system for S, that $S = \{\mathbf{x} \in D : \mathbf{\Phi}(\mathbf{x}) = \mathbf{0}\}$, and that $D\mathbf{\Phi}(\mathbf{x})$ has rank $n - r$ for every $\mathbf{x} \in D$. Show that each $\mathbf{x}_0 \in S$ has a neighborhood U such that $(f^1, \dots, f^r, \Phi^1, \dots, \Phi^{n-r})$ restricted to U is a coordinate system for U.

7. Let $1 \leq r < n$, and let $\mathfrak{M}(n, r) = \{\boldsymbol{\alpha} \in E_r^n : |\boldsymbol{\alpha}| = 1, \boldsymbol{\alpha} \text{ is decomposable}\}$. Identify E_r^n with $E^{\binom{n}{r}}$, and show that $\mathfrak{M}(n, r)$ is a manifold of dimension $r(n-r)$. [*Hint:* Given $\boldsymbol{\alpha}_0 \in \mathfrak{M}(n, r)$ let $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an orthonormal frame for E^n such that $\boldsymbol{\alpha}_0 = \mathbf{v}_1 \wedge \cdots \wedge \mathbf{v}_r$. Show that if $\boldsymbol{\alpha}$ is in a small enough neighborhood of $\boldsymbol{\alpha}_0$, then $\boldsymbol{\alpha}$ can be uniquely written in the form

$$\boldsymbol{\alpha} = c\left(\mathbf{v}_1 + \sum_{k=r+1}^{n} t_{1k}\mathbf{v}_k\right) \wedge \cdots \wedge \left(\mathbf{v}_r + \sum_{k=r+1}^{n} t_{rk}\mathbf{v}_k\right).$$

The $r(n-r)$ numbers t_{jk} can be taken as coordinates of $\boldsymbol{\alpha}$.]

7-3 MEASURE AND INTEGRATION ON MANIFOLDS

Let us now define r-dimensional measure for subsets of an r-manifold M and integrals with respect to it. The r-dimensional measure of a set $A \subset M$ will be denoted by $V_r(A)$, and the integral of a function f over A by $\int_A f\, dV_r$ or by $\int_A f(\mathbf{x})\, dV_r(\mathbf{x})$. When $r = n$, these turn out to have the same meaning as in Chapter 5.

For simplicity, the integral will be defined only for continuous functions. Moreover, it is assumed throughout this section that A is a σ-compact set (Section 5-6). By these assumptions, we avoid some slightly tedious discussion of measurability which for present purposes is irrelevant. We recall from p. 175 that if A is σ-compact and $\mathbf{F}$ is continuous on A, then $\mathbf{F}(A)$ is also σ-compact.

Let us first consider the case when A is contained in some coordinate patch S. Let $\mathbf{F}$ be a coordinate system for S, and let $\mathbf{g} = \mathbf{F}^{-1}$. The following discussion is intended to motivate the definition of $\int_A f\, dV_r$. Let us consider a "small" r-cube I, of side length a and vertex t_0, as indicated in Fig. 7-5.

Let us set $\mathbf{k}_l = a\boldsymbol{\epsilon}_l$, $\mathbf{h}_l = a\mathbf{g}_l(\mathbf{t}_0)$, $l = 1, \dots, r$. Then $|\mathbf{k}_1 \wedge \cdots \wedge \mathbf{k}_r| = V_r(I)$, and by (6-16) $|\mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_r| = V_r(K)$, where K is the r-parallelepiped with vertex $\mathbf{x}_0$ spanned by $\mathbf{h}_1, \dots, \mathbf{h}_r$. By (7-1) the ratio of these two numbers is $\mathcal{J}\mathbf{g}(\mathbf{t}_0)$. Thus

$$V_r(K) = \mathcal{J}\mathbf{g}(\mathbf{t}_0)V_r(I).$$

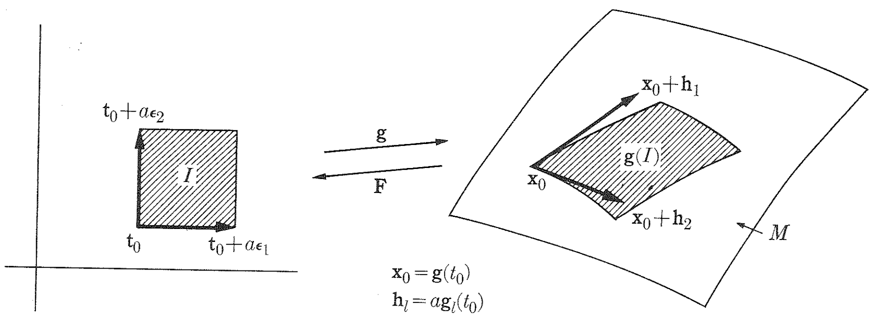


FIGURE 7-5

Moreover, $f[\mathbf{g}(\mathbf{t}_0)]V_r(K)$ should furnish a good approximation to $\int_{\mathbf{g}(I)} f\, dV_r$. If Z is a figure composed of small nonoverlapping r-cubes $I_1, \dots, I_m$, then $\int_{\mathbf{g}(Z)} f\, dV$ should be approximately the corresponding sum

$$\sum_{k=1}^{m} f[\mathbf{g}(\mathbf{t}_k)]\mathcal{J}\mathbf{g}(\mathbf{t}_k)V_r(I_k).$$

In the exact formula Z is replaced by the set $B = \mathbf{g}^{-1}(A)$, and the sum by an integral.

Definition. Let A be a σ-compact subset of a coordinate patch S, and let f be continuous on A. Then

$$\int_A f(\mathbf{x})\, dV_r(\mathbf{x}) = \int_B f[\mathbf{g}(\mathbf{t})]\mathcal{J}\mathbf{g}(\mathbf{t})\, dV_r(\mathbf{t}), \qquad A = \mathbf{g}(B), \tag{7-4}$$

provided the function $(f \circ \mathbf{g})\mathcal{J}\mathbf{g}$ is integrable over B.

The integral over B is taken in the sense of Section 5-6. By (7-2), $\mathcal{J}\mathbf{g}(\mathbf{t}) = |\mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})|$. Since $\mathbf{g}$ is of class $C^{(1)}$, $\mathcal{J}\mathbf{g}$ is continuous. Hence $(f \circ \mathbf{g})\mathcal{J}\mathbf{g}$ is continuous. If $f \geq 0$, then the integral over B either exists or diverges to $+\infty$. When the latter occurs we agree that the integral of f over A also diverges to $+\infty$.

We must show that the integral does not depend on the particular choice of coordinate system. Let $\tilde{S}$ be another coordinate patch such that $A \subset \tilde{S}$, and let $\tilde{\mathbf{F}}$ be a coordinate system for $\tilde{S}$. Let us adopt the notation of Fig. 7-4. Then $\mathbf{g} = \tilde{\mathbf{g}} \circ \boldsymbol{\phi}$, and by (7-3)

$$\mathcal{J}\mathbf{g}(\mathbf{t}) = \mathcal{J}\tilde{\mathbf{g}}[\boldsymbol{\phi}(\mathbf{t})]\mathcal{J}\boldsymbol{\phi}(\mathbf{t}) = \frac{\mathcal{J}\tilde{\mathbf{g}}(\boldsymbol{\tau})}{\mathcal{J}\boldsymbol{\phi}^{-1}(\boldsymbol{\tau})}, \qquad \text{if } \boldsymbol{\tau} = \boldsymbol{\phi}(\mathbf{t}).$$

Let $\tilde{B} = \tilde{\mathbf{F}}(A)$. Then $\tilde{B} = \boldsymbol{\phi}(B)$ and by the transformation formula for integrals (Theorem 17)

$$\int_B f[\mathbf{g}(\mathbf{t})]\mathcal{J}\mathbf{g}(\mathbf{t})\, dV_r(\mathbf{t}) = \int_{\tilde{B}} f[\tilde{\mathbf{g}}(\boldsymbol{\tau})]\, \frac{\mathcal{J}\tilde{\mathbf{g}}(\boldsymbol{\tau})}{\mathcal{J}\boldsymbol{\phi}^{-1}(\boldsymbol{\tau})}\, |J\boldsymbol{\phi}^{-1}(\boldsymbol{\tau})|\, dV_r(\boldsymbol{\tau}).$$

Since $\boldsymbol{\phi}^{-1}$ is a flat transformation, $|J\boldsymbol{\phi}^{-1}| = \mathcal{J}\boldsymbol{\phi}^{-1}$. Therefore

$$\int_B f[\mathbf{g}(\mathbf{t})]\mathcal{J}\mathbf{g}(\mathbf{t})\,dV_r(\mathbf{t}) = \int_{\tilde{B}} f[\tilde{\mathbf{g}}(\boldsymbol{\tau})]\mathcal{J}\tilde{\mathbf{g}}(\boldsymbol{\tau})\,dV_r(\boldsymbol{\tau}),$$

as required.

If we take $f(\mathbf{x}) = 1$, then (7–4) defines the *r-dimensional measure* of A:

$$V_r(A) = \int_B \mathcal{J}\mathbf{g}(\mathbf{t})\,dV_r(\mathbf{t}), \qquad A = \mathbf{g}(B). \tag{7–5}$$

Example 1. Let $M \subset E^3$ be a 2-manifold, and S a relatively open subset of M on which x and y can be taken as coordinates, as in Example 2, Section 7–1. Since $\mathcal{J}\mathbf{g} = [1 + |d\phi|^2]^{1/2}$, we have

$$\int_A f\,dV_2 = \int_B f[x, y, \phi(x, y)](1 + |d\phi(x, y)|^2)^{1/2}\,dV_2(x, y).$$

Example 2. Let M be a 1-manifold and B be a closed interval $[a, b]$. Using the terminology of Section 3–2, A is the trace of the simple arc γ represented on $[a, b]$ by $\mathbf{g}$. Formula (7–4) becomes

$$\int_A f(\mathbf{x})\,dV_1(\mathbf{x}) = \int_a^b f[\mathbf{g}(t)]\,|\mathbf{g}'(t)|\,dt.$$

The right-hand side is $\int_\gamma f\,ds$, as defined on p. 85.

**Note.* The line integral $\int_\gamma f\,ds$ was defined in Section 3–2 without requiring that γ be simple. If γ is not simple, then its trace $A = \mathbf{g}([a, b])$ need not be contained in a 1-manifold. There is a more general notion of r-dimensional measure and integral for sets which are not necessarily subsets of an r-manifold. The general formula which becomes for simple arcs the one in Example 2 is

$$\int_A f(\mathbf{x})N(\mathbf{x})\,dV_1(\mathbf{x}) = \int_a^b f[\mathbf{g}(t)]\,|\mathbf{g}'(t)|\,dt,$$

where $N(\mathbf{x})$ is the multiplicity of the point $\mathbf{x}$. For any $r \geq 1$ there is a similar formula

$$\int_A f(\mathbf{x})N(\mathbf{x})\,dV_r(\mathbf{x}) = \int_B f[\mathbf{g}(\mathbf{t})]\mathcal{J}\mathbf{g}(\mathbf{t})\,dV_r(\mathbf{t}), \tag{$*$}$$

where B is a σ-compact subset of E^r, $\mathbf{g}$ is of class $C^{(1)}$ on B, $A = \mathbf{g}(B)$, and again $N(\mathbf{x})$ (= number of points $\mathbf{t} \in B$ such that $\mathbf{g}(\mathbf{t}) = \mathbf{x}$) is the multiplicity of $\mathbf{x}$. [See p. 144, H. Federer, "The (ϕ, k) rectifiable subsets of n space," *Trans. Amer. Math. Soc.* **62** (1947), 114–192.]

In formula $(*)$ it is not necessary to assume that $D\mathbf{g}(\mathbf{t})$ has maximum rank r. If $B' = \{\mathbf{t} \in B : \text{rank } D\mathbf{g}(\mathbf{t}) < r\}$, then $\mathcal{J}\mathbf{g}(\mathbf{t}) = 0$ for every $\mathbf{t} \in B'$. Therefore B' contributes nothing to the integral on the right-hand side of $(*)$. It turns out that $\mathbf{g}(B')$ has r-dimensional measure 0, and therefore contributes nothing to the integral on the left-hand side of $(*)$.

Example 3. Let H be the hemisphere $x^2 + y^2 + z^2 = 1$, $z > 0$. Introducing spherical coordinates on H, let

$$\mathbf{g}(\phi, \theta) = (\sin\phi\cos\theta)\mathbf{e}_1 + (\sin\phi\sin\theta)\mathbf{e}_2 + (\cos\phi)\mathbf{e}_3.$$

The image of a small square $[\phi, \phi + a] \times [\theta, \theta + a]$ in the (ϕ, θ) plane is a small sector of the hemisphere which is approximately a rectangle of side lengths a and $a \sin\phi$. Since $|\partial\mathbf{g}/\partial\phi \wedge \partial\mathbf{g}/\partial\theta|a^2$ is approximately the area of the sector, this suggests that $\mathcal{J}\mathbf{g}(\phi, \theta) = |\partial\mathbf{g}/\partial\phi \wedge \partial\mathbf{g}/\partial\theta| = \sin\phi$. The reader should check this formula (Problem 3). If we take $B = (0, \pi/2) \times (0, 2\pi)$ then $H - \mathbf{g}(B)$ is an arc of a great circle corresponding to $\theta = 0$. This arc is 2-dimensionally null in the sense defined below, and hence

$$\int_H f(\mathbf{x})\,dV_2(\mathbf{x}) = \int_{\mathbf{g}(B)} f(\mathbf{x})\,dV_2(\mathbf{x}) = \int_0^{\pi/2} d\phi \int_0^{2\pi} f[\mathbf{g}(\phi, \theta)]\sin\phi\,d\theta.$$

Let us turn to the general case when A is not necessarily contained in some coordinate patch. To simplify matters let us at first assume that M is a compact manifold and f is continuous everywhere on M. The traditional way to proceed is to dissect M into a finite number of nonoverlapping pieces $S_1, \ldots, S_m$ each of which has a coordinate system, with fr $S_k \cap$ fr S_l contained in a finite union of $(r - 1)$-manifolds and $M = \text{cl } S_1 \cup \cdots \cup \text{cl } S_m$. Then

$$\int_A f\,dV_r = \sum_{k=1}^{m} \int_{A \cap S_k} f\,dV_r. \tag{7–6}$$

In simple examples it is easy to find such dissections of M. However, the theorem that every compact r-manifold M has such a dissection is a difficult one to prove. See [21], Chap. IV. Nor is it evident that the integral is independent of the particular dissection chosen.

The same result can be achieved by a simpler device called partition of unity. The basic difficulty with dissections is that S_k and S_l cannot overlap. With partitions of unity this problem is avoided.

Partition of unity. Let us recall from Section 5–3 that the support of a function ψ is the smallest closed set outside of which $\psi(\mathbf{x}) = 0$. Let us first find for every $\mathbf{x}_0$ and $r > 0$ a function ψ of class $C^{(\infty)}$ on E^n such that $\psi(\mathbf{x}) > 0$ on the open r-ball with center $\mathbf{x}_0$ and the support of ψ is the closed r-ball with center $\mathbf{x}_0$. In fact, let

$$h(x) = \exp\left(\frac{-1}{1 - x^2}\right), \qquad -1 < x < 1,$$

$$h(x) = 0, \qquad |x| \geq 1.$$

From the example at the end of Section 2–3 and the composite function theorem, h is of class $C^{(\infty)}$ on E^1. Let

$$\psi(\mathbf{x}) = h\left(\frac{|\mathbf{x} - \mathbf{x}_0|}{r}\right).$$

Definition. Let M be a compact manifold. A collection of functions $\{\phi_1, \ldots, \phi_m\}$ is a *partition of unity* for M if:

(1) ϕ_k is of class $C^{(\infty)}$ on M and $\phi_k \geq 0$, $k = 1, \ldots, m$;

(2) The support of ϕ_k is a compact subset of some coordinate patch, $k = 1, \ldots, m$; and

(3) $\sum_{k=1}^{m} \phi_k(\mathbf{x}) = 1$ for every $\mathbf{x} \in M$.

Proposition 29. *Any compact manifold M has a partition of unity.*

Proof. Every $\mathbf{x} \in M$ is contained in some coordinate patch S. Since S is relatively open there is a neighborhood U of $\mathbf{x}$ such that $M \cap \operatorname{cl} U \subset S$. Since M is compact, a finite collection $\{U_1, \ldots, U_m\}$ of such neighborhoods covers M. Let $\mathbf{x}_k$ be the center of U_k, r_k the radius, and ψ_k the function of class $C^{(\infty)}$ constructed above with $\mathbf{x}_0 = \mathbf{x}_k$, $r = r_k$. The collection of functions $\{\psi_1, \ldots, \psi_m\}$ satisfies (1) and (2) of the definition, but not necessarily (3). However, by construction $\psi_1(\mathbf{x}) + \cdots + \psi_m(\mathbf{x}) > 0$ for every $\mathbf{x} \in M$. Let

$$\phi_k(\mathbf{x}) = \frac{\psi_k(\mathbf{x})}{\psi_1(\mathbf{x}) + \cdots + \psi_m(\mathbf{x})}, \qquad k = 1, \ldots, m, \quad \mathbf{x} \in M.$$

The collection $\{\phi_1, \ldots, \phi_m\}$ is a partition of unity for M. ▮

Let f be continuous on M. Since M is compact, f is bounded on M. This insures that all of the following integrals exist. If the support of f is a compact subset K of a coordinate patch, then we let

$$\int_A f\,dV_r = \int_{A \cap K} f\,dV_r.$$

In particular, if $\{\phi_1, \ldots, \phi_m\}$ is a partition of unity, then for any f the support of $f\phi_k$ is compact and lies in some coordinate patch. Hence the integral of $f\phi_k$ is defined.

Definition. Let A be a σ-compact subset of a compact manifold M, and $\{\phi_1, \ldots, \phi_m\}$ be a partition of unity for M. Then for any f continuous on M

$$\int_A f\,dV_r = \sum_{k=1}^{m} \int_A f\phi_k\,dV_r. \tag{7-7}$$

In case A is contained in some coordinate patch S, this agrees with the earlier definition (7-4), since $\sum \phi_k(\mathbf{x}) = 1$ for every $\mathbf{x} \in A$. We must show that the integral does not depend on the particular partition of unity chosen for M. Let $\{\chi_1, \ldots, \chi_p\}$ be another partition of unity for M. Then for every $\mathbf{x} \in M$

$$f(\mathbf{x})\chi_l(\mathbf{x}) = \sum_{k=1}^{m} f(\mathbf{x})\chi_l(\mathbf{x})\phi_k(\mathbf{x}), \qquad l = 1, \ldots, p.$$

Since the support of $f\chi_l$ is contained in some coordinate patch, its integ over A can be written according to (7-4) as an integral over a set $B \subset$ By Theorem 13, the integral over B of a finite sum is the sum of the integra Hence

$$\int_A f\chi_l\,dV_r = \sum_{k=1}^{m} \int_A f\chi_l\phi_k\,dV_r, \qquad l = 1, \ldots, p,$$

$$\sum_{l=1}^{p} \int_A f\chi_l\,dV_r = \sum_{l=1}^{p} \sum_{k=1}^{m} \int_A f\chi_l\phi_k\,dV_r.$$

In the same way

$$\sum_{k=1}^{m} \int_A f\phi_k\,dV_r = \sum_{k=1}^{m} \sum_{l=1}^{p} \int_A f\phi_k\chi_l\,dV_r.$$

Since the right-hand sides are equal, the integral of f over A does not depe on which partition of unity is chosen.

If $f(\mathbf{x}) = 1$ for every $\mathbf{x} \in M$, then the integral gives the *r-dimension measure*

$$V_r(A) = \sum_{k=1}^{m} \int_A \phi_k\,dV_r.$$

When A is a subset of some coordinate patch, this agrees with the previo definition. If $V_r(A) = 0$, then A is called an *r-null* set. The integral has t same elementary properties listed in Theorem 13 for $r = n$ (Problem 9 Moreover, V_r is countably additive (Problem 10).

Measure and integration on noncompact manifolds. If M is not con pact then the discussion is somewhat more complicated. Partitions of unit consisting of infinite collections $\{\phi_1, \phi_2, \ldots\}$ must be considered. To (1)–(3 must be added:

(4) If K is any compact subset of M, then the support of ϕ_k meets K fo only finitely many k.

The sum in (3) is now an infinite series. However, on any compact se only finitely many terms are different from 0. Every manifold has a partitio of unity. This can be proved by an elaboration of the proof of Proposition 29 which we shall not give.

Let f be continuous on A. Then f is called integrable over A $\sum_{k=1}^{\infty} \int_A |f|\phi_k\,dV_r$ is finite. If f is integrable over A, then its integral i $\sum_{k=1}^{\infty} \int_A f\phi_k\,dV_r$.

The following results are true whether M is compact or not. However we shall give the proof only for the compact case.

Proposition 30. *If A is a subset of M which is an $(r-1)$-manifold, the A is an r-null set.*

Proof. Suppose first that $A \subset S$, where S is a coordinate patch. Let $\mathbf{F}$ be a coordinate system for S. By Theorem 21 the set $B = \mathbf{F}(A)$ is an $(r-1)$-manifold. By Corollary 3, p. 178, $V_r(B) = 0$. Therefore, from (7–5) $V_r(A) = 0$.

In the general case, let $\{\phi_1, \ldots, \phi_m\}$ be a partition of unity for M, and let K_k be the support of ϕ_k. Then $V_r(A \cap K_k) = 0$ and hence

$$\int_A \phi_k\, dV_r = \int_{A \cap K_k} \phi_k\, dV_r = 0.$$

Summing from 1 to m, $V_r(A) = 0$. ▮

Corollary. *If A is contained in a countable union of $(r-1)$-manifolds, then A is an r-null set.*

Formula (5–38) about change of variables in an integral by a regular flat transformation has the following generalization.

Theorem 22. *Let $\mathbf{g}$ be a regular transformation from an r-manifold N into an r-manifold M. Let f be continuous on M, and A be any σ-compact subset of M. Then*

$$\int_A f(\mathbf{x})\, dV_r(\mathbf{x}) = \int_{\mathbf{g}^{-1}(A)} f[\mathbf{g}(\mathbf{t})]\mathcal{J}\mathbf{g}(\mathbf{t})\, dV_r(\mathbf{t}), \qquad (7\text{–}7)$$

provided either integral exists.

Proof. By using a partition of unity, it suffices to consider the case when f has compact support contained in some coordinate patch S and $A \subset S$. Let $\mathbf{F}$ be a coordinate system for S. Then $\mathbf{F} \circ \mathbf{g}$ is a coordinate system for $S' = \mathbf{g}^{-1}(S)$. Let $\mathbf{G} = \mathbf{F}^{-1}$, $\mathbf{G}' = (\mathbf{F} \circ \mathbf{g})^{-1} = \mathbf{g}^{-1} \circ \mathbf{G}$. Let $A = \mathbf{G}(B)$. Then $\mathbf{g}^{-1}(A) = \mathbf{G}'(B)$. By (7–4) the left-hand side of (7–7) equals $\int_B (f \circ \mathbf{G})\mathcal{J}\mathbf{G}\, dV_r$, while the right-hand side equals $\int_B [f \circ \mathbf{g} \circ \mathbf{G}'][(\mathcal{J}\mathbf{g}) \circ \mathbf{G}']\mathcal{J}\mathbf{G}'\, dV_r$. But $\mathbf{G} = \mathbf{g} \circ \mathbf{G}'$, and by (7–3), $\mathcal{J}\mathbf{G} = (\mathcal{J}\mathbf{g}) \circ \mathbf{G}'\mathcal{J}\mathbf{G}'$. Therefore both sides of (7–7) are the same. ▮

PROBLEMS

1. Find the area of $\{(x, y, xy) : x^2 + y^2 \leq 1\}$.
2. Let $n = 2$, $r = 3$. Show that (7–5) becomes

$$V_2(A) = \int_B \left\{\left[\frac{\partial(g^2, g^3)}{\partial(s, t)}\right]^2 + \left[\frac{\partial(g^3, g^1)}{\partial(s, t)}\right]^2 + \left[\frac{\partial(g^1, g^2)}{\partial(s, t)}\right]^2\right\}^{1/2} dV_2(s, t).$$

3. In Example 3 calculate $\partial\mathbf{g}/\partial\phi \wedge \partial\mathbf{g}/\partial\theta$ and verify that its norm is $\sin\phi$.

Moments, centroids. These are defined in the same way as for $r = n$. For example, if A has positive r-dimensional measure then the components of its centroid are

$$\bar{x}^i = \frac{1}{V_r(A)} \int_A x^i\, dV_r(\mathbf{x}), \quad i = 1, \ldots, n.$$

If $r = 2$, $n = 3$, and A is thought of as a surface with continuous density $\rho($ (mass per unit of area), then the mass is $\int_A \rho(\mathbf{x})\, dV_2(\mathbf{x})$ and the components the center of mass are

$$\bar{\bar{x}}^i = \int_A x^i \rho(\mathbf{x})\, dV_2(\mathbf{x}) \Big/ \int_A \rho(\mathbf{x})\, dV_2(\mathbf{x}), \quad i = 1, 2, 3.$$

4. Find the second moment about the z-axis of:
   (a) The sphere $x^2 + y^2 + z^2 = 1$.
   (b) The triangle with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$.
5. Show that $\frac{1}{2}\mathbf{e}_3$ is the centroid of the hemisphere H in Example 3. Use spheric coordinates on H.
6. (*Surfaces of revolution*). Let γ be a simple arc (or simple closed curve) lying the half $y > 0$ of the (x, y) plane. From Section 3–2, γ has a standard repr sentation $\mathbf{G}$ on $[0, l]$, where l is the length and $|\mathbf{G}'(s)| = 1$ for $0 \leq s \leq l$. L $\mathbf{g}(s, t) = G^1(s)\mathbf{e}_1 + G^2(s)\,[(\cos t)\mathbf{e}_2 + (\sin t)\mathbf{e}_3]$, and let $M = \mathbf{g}((0, l) \times [0, 2\pi]$
   (a) Prove *Pappus' theorem:* $V_2(M) = 2\pi\bar{y}l$, where $(\bar{x}, \bar{y})$ is the centroid of
   (b) Find the area of a torus (doughnut) of major radius r_1 and minor radius r
7. Let S be the unit $(n-1)$-sphere in E^n. Show that the $(n-1)$-measure of th "zone" $\{\mathbf{x} \in S : a < x^n < b\}$ depends only on the difference $b - a$ when $n =$ but this is false when $n \neq 3$. Assume that $-1 \leq a < b \leq 1$.
8. Let $v(r) = \alpha_n r^n$ be the n-dimensional measure of a spherical n-ball of radius Show that $v'(r)$ is the $(n-1)$-measure of its boundary. [*Note:* α_n was calculate on p. 183. If $\beta_{n-1} = V_{n-1}$ [unit $(n-1)$-sphere], then $\beta_{n-1} = n\alpha_n$.]
9. Prove that the statements (1)–(7) obtained by replacing n by r everywhere Theorem 13 are true for integrals over σ-compact subsets of a compact r-manifol
10. Let M be a compact r-manifold. Let $A = A_1 \cup A_2 \cup \cdots$, where $A_1, A_2, \ldots$ are disjoint σ-compact subsets of M. Show that $V_r(A) = V_r(A_1) + V_r(A_2) + \cdots$ [*Hint:* Use a partition of unity and Theorem 12.]

7–4 ORIENTATIONS; INTEGRALS OF r-FORMS

Let M be an r-manifold. For each $\mathbf{x} \in M$ the tangent space $T(\mathbf{x})$ is a r-dimensional vector subspace of E^n. According to Section 6–3 $T(\mathbf{x})$ has tw possible orientations, each of which is an r-vector of norm 1. If one of thes orientations is denoted by $\mathbf{o}(\mathbf{x})$, then the other is $-\mathbf{o}(\mathbf{x})$. We would like t choose the orientation for $T(\mathbf{x})$ consistently on M; in other words we want th function $\mathbf{o}$ whose value at $\mathbf{x}$ is $\mathbf{o}(\mathbf{x})$ to be continuous on M.

Definition. M is an *orientable* manifold if there exists a continuous r-vecto valued function $\mathbf{o}$ such that $\mathbf{o}(\mathbf{x})$ is an orientation for the tangent spac $T(\mathbf{x})$ for every $\mathbf{x} \in M$. The function $\mathbf{o}$ is an *orientation* for M.

It can be shown that a connected manifold has at most two orientations Let us find out what orientability means in the extreme dimensions $r = 1$ $n - 1, n$.

$r = 1$. If M is a 1-manifold, then the two orientations for the 1-dimensional vector space $T(\mathbf{x})$ are unit tangent vectors at $\mathbf{x}$ pointing in opposite directions. M is oriented by assigning a unit tangent vector $\mathbf{v}(\mathbf{x})$ continuously on M (Fig. 7–6). It can be shown that every 1-manifold is orientable.

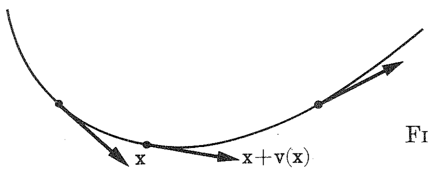


FIGURE 7–6

$r = n$. In this case the n-manifold M is an open subset of E^n. The possible values for $\mathbf{o}(\mathbf{x})$ are $\pm\mathbf{e}_{1\ldots n}$. If M is connected, then $\mathbf{o}(\mathbf{x})$ must be constant. If $\mathbf{o}(\mathbf{x}) = \mathbf{e}_{1\ldots n}$ for every $\mathbf{x} \in M$, then M is *positively oriented;* and if $\mathbf{o}(\mathbf{x}) = -\mathbf{e}_{1\ldots n}$ for every $\mathbf{x} \in M$, then M is *negatively oriented.*

$r = n - 1$. If M is an $(n-1)$-manifold in E^n, then the adjoint $\mathbf{n}(\mathbf{x}) = {*}\mathbf{o}(\mathbf{x})$ is a unit normal vector to M at $\mathbf{x}$. The condition that M be orientable is that a unit normal can be chosen continuously on M. If D is an open set which is on one side of its boundary fr D (see Section 7–5), then the exterior unit normal orients fr D. If M is not the boundary of an open set, then M may not be orientable. This is shown by the following famous surface.

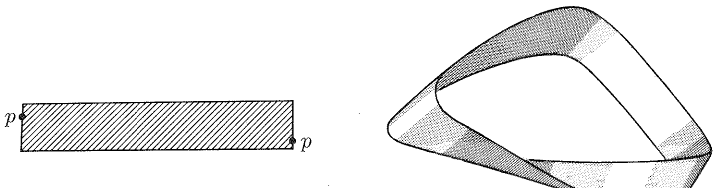
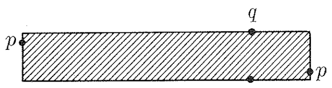


FIGURE 7–7

Example 1. The *Möbius strip.* This is a 2-manifold $M \subset E^3$ which is not orientable. It may be visualized by twisting a strip of paper and pasting together the ends (Fig. 7–7). The edge of the strip must be omitted in order that M be locally like E^2. The fact that a unit normal cannot be chosen continuously may be expressed more picturesquely by saying that the Möbius strip is a surface with "only one side."

Example 2. The Möbius strip is not a compact 2-manifold. An example of a compact, nonorientable 2-manifold is the *Klein bottle*, or *twisted torus.* It is obtained by also joining together the lateral edges of the rectangle used to make the Möbius strip, as indicated in Fig. 7–8. The Klein bottle cannot be realized as a submanifold of E^3, since it can be proved that any compact $(n-1)$-manifold $M \subset E^n$ is the boundary of an open set and hence is orientable. However, the Klein bottle can be realized as a submanifold of E^4.

FIGURE 7–8



Integrals of r-forms. Let M be an r-manifold with orientation $\mathbf{o}$, and $\boldsymbol{\omega}$ an r-form continuous on M. For each $\mathbf{x} \in M$ consider $\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})$, the scalar product of the r-covector $\boldsymbol{\omega}(\mathbf{x})$ and the r-vector $\mathbf{o}(\mathbf{x})$. Since $\boldsymbol{\omega}$ and $\mathbf{o}$ are continuous functions, $\boldsymbol{\omega} \cdot \mathbf{o}$ is a continuous real-valued function. Let A be a σ-compact subset of M.

Definition. The integral of $\boldsymbol{\omega}$ over A with the orientation $\mathbf{o}$ is

$$\int_{A^{\mathbf{o}}} \boldsymbol{\omega} = \int_A \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})\, dV_r(\mathbf{x}), \tag{7–8}$$

provided $\boldsymbol{\omega} \cdot \mathbf{o}$ is integrable over A.

In particular, if M is compact, then $\boldsymbol{\omega} \cdot \mathbf{o}$ is continuous, bounded, and hence integrable over any σ-compact subset of M. The integral has the following elementary properties:

(1) $\int_{A^{\mathbf{o}}} (\boldsymbol{\omega}^1 + \boldsymbol{\omega}^2) = \int_{A^{\mathbf{o}}} \boldsymbol{\omega}^1 + \int_{A^{\mathbf{o}}} \boldsymbol{\omega}^2$.

(2) $\int_{A^{\mathbf{o}}} c\boldsymbol{\omega} = c\int_{A^{\mathbf{o}}} \boldsymbol{\omega}$, *for any scalar* c.

(3) $\int_{A^{-\mathbf{o}}} \boldsymbol{\omega} = -\int_{A^{\mathbf{o}}} \boldsymbol{\omega}$.

(4) *If* $|\boldsymbol{\omega}(\mathbf{x})| \leq C$ *for every* $\mathbf{x} \in A$, *then* $\left|\int_{A^{\mathbf{o}}} \boldsymbol{\omega}\right| \leq CV_r(A)$.

(5) $\int_{A^{\mathbf{o}}} \boldsymbol{\omega} = \int_{A_1^{\mathbf{o}}} \boldsymbol{\omega} + \int_{A_2^{\mathbf{o}}} \boldsymbol{\omega}$ *if* $A = A_1 \cup A_2$ *and* $A_1 \cap A_2$ *is empty.*

These follow at once from corresponding elementary properties of the right-hand side of (7–8). See Problem 9, Section 7–3.

For instance, in (3)

$$\int_A \boldsymbol{\omega}(\mathbf{x}) \cdot [-\mathbf{o}(\mathbf{x})]\, dV_r(\mathbf{x}) = -\int_A \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})\, dV_r(\mathbf{x}).$$

Since $|\mathbf{o}(\mathbf{x})| = 1$, $|\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})| \leq |\boldsymbol{\omega}(\mathbf{x})|$. Then in (4)

$$\left|\int_A \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})\, dV_r(\mathbf{x})\right| \leq \int_A |\boldsymbol{\omega}(\mathbf{x})|\, dV_r(\mathbf{x}) \leq CV_r(A).$$

In (1), one can take more generally a finite number of r-forms $\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^m$, or more generally an infinite sequence $\boldsymbol{\omega}^1, \boldsymbol{\omega}^2, \ldots$ provided $\sum_{k=1}^{\infty} \int_A |\boldsymbol{\omega}^k(\mathbf{x})|\, dV_r(\mathbf{x})$ converges. Similarly, the generalization of (5) is still true if $A = A_1 \cup A_2 \cup \cdots$, where $A_1, A_2, \ldots$ are disjoint σ-compact sets and $\sum_{k=1}^{\infty} \int_{A_k} |\boldsymbol{\omega}(\mathbf{x})|\, dV_r(\mathbf{x})$ converges.

The case $r = n$. Let A^+ denote A with the positive orientation $\mathbf{e}_{1\ldots n}$ of E^n. Let $\boldsymbol{\omega} = f\, dx^1 \wedge \cdots \wedge dx^n$ be an n-form. Then

$$\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{e}_{1\ldots n} = \omega_{1\ldots n}(\mathbf{x}) = f(\mathbf{x}),$$

and (7–8) becomes

$$\int_{A^+} f\,dx^1 \wedge \cdots \wedge dx^n = \int_A f\,dV_n. \tag{7–9}$$

The left-hand side of (7–9) changes sign if either the orientation of A is reversed or two differentials dx^i and dx^j are interchanged. For instance, if $n = 2$ then

$$\int_{A^+} f\,dx \wedge dy = \int_A f\,dV_2,$$

$$\int_{A^-} f\,dx \wedge dy = \int_{A^+} f\,dy \wedge dx = -\int_A f\,dV_2.$$

Orientation induced by a regular transformation. Let N be an r-manifold which is oriented by an orientation $\mathbf{O}$; and let $\mathbf{g}$ be regular from N into M. Let $\mathbf{t} \in N$, and let $\mathbf{L}_r$ be the linear transformation induced by $D\mathbf{g}(\mathbf{t})$, as in the discussion on page 241. Let $\boldsymbol{\alpha}(\mathbf{x}) = \mathbf{L}_r[\mathbf{O}(\mathbf{t})]$. Since $|\mathbf{O}(\mathbf{t})| = 1$, from (7–1) we have

$$|\boldsymbol{\alpha}(\mathbf{x})| = \mathcal{J}\mathbf{g}(\mathbf{t})|\mathbf{O}(t)| = \mathcal{J}\mathbf{g}(\mathbf{t}), \quad \mathbf{x} = \mathbf{g}(\mathbf{t}).$$

Let $\mathbf{o}(\mathbf{x}) = |\boldsymbol{\alpha}(\mathbf{x})|^{-1}\boldsymbol{\alpha}(\mathbf{x})$. Then $\mathbf{o}$ is an orientation for $\mathbf{g}(N)$, called the orientation *induced* from $\mathbf{O}$ by $\mathbf{g}$.

If N is a positively oriented open set $\Delta \subset E^r$, then as in the derivation of formula (7–2), $\boldsymbol{\alpha}(\mathbf{x}) = \mathbf{g}_1(\mathbf{t}) \wedge \cdots \wedge \mathbf{g}_r(\mathbf{t})$.

Example 3. Let $M \subset E^3$ be an orientable 2-manifold, and $\mathbf{o}$ a given orientation for M. Let $\mathbf{g}$ be regular from $\Delta \subset E^2$ into M. We must determine whether the orientation induced from the positive orientation of E^2 agrees with $\mathbf{o}$. If $\mathbf{x} = \mathbf{g}(s, t)$, then

$$\boldsymbol{\alpha}(\mathbf{x}) = \frac{\partial \mathbf{g}}{\partial s} \wedge \frac{\partial \mathbf{g}}{\partial t} = \alpha^{23}(\mathbf{x})\mathbf{e}_{23} + \alpha^{31}(\mathbf{x})\mathbf{e}_{31} + \alpha^{12}(\mathbf{x})\mathbf{e}_{12},$$

whereby (6–13a) $\alpha^{ij}(\mathbf{x}) = \partial(g^i, g^j)/\partial(s, t)$. Then $\mathbf{o}(\mathbf{x}) = c(\mathbf{x})\boldsymbol{\alpha}(\mathbf{x})$, where $c(\mathbf{x}) = \pm|\boldsymbol{\alpha}(\mathbf{x})|^{-1}$. The induced orientation is the given one, provided $c(\mathbf{x}) > 0$.

Example 4. Let H be the hemisphere in Example 3, p. 253, oriented so that $\mathbf{o}^{12}(\mathbf{x}) > 0$ for every $\mathbf{x} \in H$. The vector $\mathbf{n}(\mathbf{x}) = *\mathbf{o}(\mathbf{x})$ is normal to H, and its third component $n^3(\mathbf{x})$ equals $o^{12}(\mathbf{x})$. We have oriented H so that the normal "points upward." If (ϕ, θ) are spherical coordinates of $\mathbf{x}$, and $\mathbf{g}$ is as on p. 253, then

$$\alpha^{12}(\mathbf{x}) = \frac{\partial(g^1, g^2)}{\partial(\phi, \theta)} = \sin\phi\cos\phi > 0.$$

Therefore the induced orientation is $\mathbf{o}$.

Let $\boldsymbol{\omega}$ be an r-form which is continuous on M, and let $\boldsymbol{\omega}^{\#}$ be the r-form defined in Section 6–5. It has the property that

$$\boldsymbol{\omega}^{\#}(\mathbf{t}) \cdot \mathbf{O}(\mathbf{t}) = \boldsymbol{\omega}(\mathbf{x}) \cdot \boldsymbol{\alpha}(\mathbf{x}), \qquad \text{if } \mathbf{x} = \mathbf{g}(\mathbf{t}).$$

Since $\mathbf{g}$ is of class $C^{(1)}$, $\boldsymbol{\omega}^{\#}$ is continuous on N.

Proposition 31. *Let $A = \mathbf{g}(B)$, where B is a σ-compact subset of N. Let $\mathbf{o}$ be the orientation induced by $\mathbf{g}$ from the orientation $\mathbf{O}$ on N. Then*

$$\int_{A^{\mathbf{o}}} \boldsymbol{\omega} = \int_{B\mathbf{O}} \boldsymbol{\omega}^{\#}, \tag{7–10}$$

provided either integral exists.

Proof. From the discussion above, $\boldsymbol{\alpha}(\mathbf{x}) = \mathcal{J}\mathbf{g}(\mathbf{t})\mathbf{o}(\mathbf{x})$ if $\mathbf{x} = \mathbf{g}(\mathbf{t})$. Dividing by $\mathcal{J}\mathbf{g}(\mathbf{t})$, we have

$$\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) = \frac{\boldsymbol{\omega}^{\#}(\mathbf{t}) \cdot \mathbf{O}(\mathbf{t})}{\mathcal{J}\mathbf{g}(\mathbf{t})}.$$

By Theorem 22,

$$\int_A \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x})\,dV_r(\mathbf{x}) = \int_B \frac{\boldsymbol{\omega}^{\#}(\mathbf{t}) \cdot \mathbf{O}(\mathbf{t})}{\mathcal{J}\mathbf{g}(\mathbf{t})} \mathcal{J}\mathbf{g}(\mathbf{t})\,dV_r(\mathbf{t}).$$

Canceling $\mathcal{J}\mathbf{g}(\mathbf{t})$ on the right-hand side, we get (7–10). ∎

An important particular case of Proposition 31 is obtained by taking for N an open set $\Delta \subset E^r$. This proposition, together with a judicious choice of coordinate systems on M, furnishes a tool for evaluating integrals of r-forms.

Example 5. Let $r = 1$, $\Delta \subset E^1$. Then $\boldsymbol{\omega}^{\#}(t) = \boldsymbol{\omega}[\mathbf{g}(t)] \cdot \mathbf{g}'(t)$. If B is an interval, then $\int_{B^+} \boldsymbol{\omega}^{\#}$ is the line integral of $\boldsymbol{\omega}$ along the curve in E^n represented parametrically on B by $\mathbf{g}$.

Example 6. Let $\boldsymbol{\omega} = f\,dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, and let $\Delta \subset E^r$. From Section 6–5

$$(dx^{i_1} \wedge \cdots \wedge dx^{i_r})^{\#} = dg^{i_1} \wedge \cdots \wedge dg^{i_r}$$
$$= \frac{\partial(g^{i_1}, \ldots, g^{i_r})}{\partial(t^1, \ldots, t^r)}\,dt^1 \wedge \cdots \wedge dt^r.$$

By formulas (7–9) and (7–10),

$$\int_{A^{\mathbf{o}}} f\,dx^{i_1} \wedge \cdots \wedge dx^{i_r} = \int_B f \circ \mathbf{g}\,\frac{\partial(g^{i_1}, \ldots, g^{i_r})}{\partial(t^1, \ldots, t^r)}\,dV_r,$$

provided $\mathbf{o}$ is the orientation induced by $\mathbf{g}$.

Continuing Example 4, we have for instance

$$\int_{H^{\mathbf{o}}} f\,dx \wedge dy = \int_{B^+} f \circ \mathbf{g}\,\frac{\partial(g^1, g^2)}{\partial(\phi, \theta)}\,d\phi \wedge d\theta$$
$$= \int_B f(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\sin\phi\cos\phi\,dV_2(\phi, \theta),$$

where $B = (0, \pi/2) \times (0, 2\pi)$.

PROBLEMS

1. Let $\Delta \subset E^n$ have the positive orientation and let $\mathbf{g}$ be a regular flat transformation from Δ into E^n.
   (a) Show that $\mathbf{g}$ induces the positive orientation on $\mathbf{g}(\Delta)$ if and only if $J\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in \Delta$.
   (b) Show that (7–10) becomes
$$\int_{A^+} f\, dx^1 \wedge \cdots \wedge dx^n = \int_{B^+} f \circ \mathbf{g}\, J\mathbf{g}\, dt^1 \wedge \cdots \wedge dt^n,$$
provided $J\mathbf{g}(\mathbf{t}) > 0$ for every $\mathbf{t} \in \Delta$.

2. Let $A = \{(x, y, z) : y = x^2 + z^2, y \leq 4\}$, oriented so that $\mathbf{o}^{31}(\mathbf{x}) > 0$. Evaluate:
   (a) $\int_{A^{\mathbf{o}}} z\, dx \wedge dy$.   (b) $\int_{A^{\mathbf{o}}} \exp y\, dz \wedge dx$.

   [*Hint:* Use polar coordinates in the (x, z)-plane.]

3. Let A be the triangle in E^3 with vertices $\mathbf{e}_1$, $-\mathbf{e}_2$, $2\mathbf{e}_3$.
   (a) Show that $\mathbf{o} = \frac{1}{3}(2\mathbf{e}_{23} - 2\mathbf{e}_{31} + \mathbf{e}_{12})$ is an orientation for the plane containing A.
   (b) Evaluate $\int_{A^{\mathbf{o}}} x\, dy \wedge dz$. [*Hint:* Take $\mathbf{g}$ affine such that $\mathbf{g}(\Sigma) = A$, where Σ is the standard 2-simplex.]

4. Let $A = \{(x, y, z) : x^2 + y^2 = z^2, x > 0, 0 < z < 1\}$, oriented so that $o^{12}(\mathbf{x}) < 0$. Evaluate
$$\int_{A^{\mathbf{o}}} z^2\, dy \wedge dz.$$

5. Let $n = 4$ and $M = \{\mathbf{x} : (x^1)^2 + (x^2)^2 = 1, (x^3)^2 + (x^4)^2 = 1\}$. Let $\mathbf{g}(s, t) = (\cos s)\mathbf{e}_1 + (\sin s)\mathbf{e}_2 + (\cos t)\mathbf{e}_3 + (\sin t)\mathbf{e}_4$, $0 \leq s, t \leq 2\pi$.
   (a) Find the orientation $\mathbf{o}$ induced by $\mathbf{g}$ from the positive orientation of E^2.
   (b) Evaluate
$$\int_{M^{\mathbf{o}}} dx^3 \wedge dx^4 + x^1 x^3\, dx^2 \wedge dx^4.$$

6. Suppose that M is the r-manifold determined by $\mathbf{\Phi}$, in the sense that M satisfies (4–26), p. 123. Show that M is orientable.

7–5 THE DIVERGENCE THEOREM

This is an n-dimensional generalization of the fundamental theorem of calculus and has numerous applications in geometry and in physics. We shall first state the theorem in two different ways and derive some corollaries of it. A proof is given later in the section.

The divergence theorem equates the integral of an $(n - 1)$-form $\boldsymbol{\omega}$ over the boundary of an open set D and the integral of $d\boldsymbol{\omega}$ over D. The integral of a differential form depends on an orientation. We must assign on the boundary fr D an orientation corresponding to the positive orientation on D.

For this purpose we must assume that D lies on one side of its boundary. This is expressed precisely as follows.

Definition. Let $\mathbf{x}_0 \in \text{fr } D$, and let U be a neighborhood of $\mathbf{x}_0$. Then D *is on one side of its boundary in* U if there exists a function Φ of class $C^{(1)}$ on U such that $d\Phi(\mathbf{x}) \neq \mathbf{0}$ for every $\mathbf{x} \in U$ and
$$(\text{fr } D) \cap U = \{\mathbf{x} \in U : \Phi(\mathbf{x}) = 0\},$$
$$D \cap U = \{\mathbf{x} \in U : \Phi(\mathbf{x}) < 0\}.$$

If every $\mathbf{x}_0 \in \text{fr } D$ has such a neighborhood U, then D *is on one side of its boundary.*

Example 1. Let $D = \{\mathbf{x} : |\mathbf{x}| < 1 \text{ or } 1 < |\mathbf{x}| < 2\}$. Then fr D is the union of two concentric $(n - 1)$-spheres of radii 1 and 2. However, D is on both sides of the inner $(n - 1)$-sphere.

Actually this example is rather artificial. If D is the interior of its closure, then using the implicit function theorem it can be shown that D is on one side of fr D.

Definition. Let $\mathbf{x} \in \text{fr } D$, and $\mathbf{n} \neq \mathbf{0}$ be a vector normal to fr D at $\mathbf{x}$. Then $\mathbf{n}$ is an *exterior normal* at $\mathbf{x}$ if there exists $\delta > 0$ such that $\mathbf{x} + t\mathbf{n} \in D$ for $-\delta < t < 0$ and $\mathbf{x} + t\mathbf{n} \in (\text{cl } D)^c$ for $0 < t < \delta$.

From the definition, all exterior normals at $\mathbf{x}$ are positive scalar multiples of any particular one. We shall be principally concerned with the unit exterior normal, which will be denoted by $\boldsymbol{\nu}(\mathbf{x})(|\boldsymbol{\nu}(\mathbf{x})| = 1)$.

Let D be on one side of its boundary in U. The vector $\mathbf{n}(\mathbf{x}) = \text{grad } \Phi(\mathbf{x})$ is normal to fr D at $\mathbf{x} \in (\text{fr } D) \cap U$.

Let $\psi(t) = \Phi(\mathbf{x} + t\mathbf{n}(\mathbf{x}))$. Then $\psi(0) = 0$ and
$$\psi'(0) = \text{grad } \Phi(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = |\text{grad } \Phi(\mathbf{x})|^2 > 0.$$

There exists $\delta > 0$ such that $\psi(t) < 0$ for $-\delta < t < 0$ and $\psi(t) > 0$ for $0 < t < \delta$. Therefore grad $\Phi(\mathbf{x})$ is an exterior normal at $\mathbf{x}$. The vector
$$\boldsymbol{\nu}(\mathbf{x}) = |\text{grad } \Phi(\mathbf{x})|^{-1} \text{ grad } \Phi(\mathbf{x})$$
is the unit exterior normal to D at $\mathbf{x}$. Since Φ is of class $C^{(1)}$, $\boldsymbol{\nu}$ is a continuous function on $(\text{fr } D) \cap U$. See Fig. 7–9.

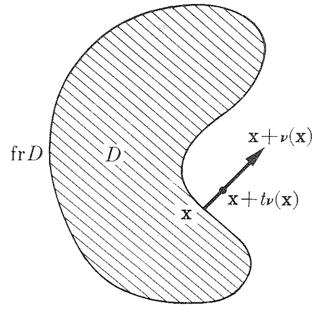


FIGURE 7–9

Let $\mathbf{o}(\mathbf{x})$ be the $(n - 1)$-vector such that $\boldsymbol{\nu}(\mathbf{x}) = *\mathbf{o}(\mathbf{x})$. The $(n - 1)$-space of $\mathbf{o}(\mathbf{x})$ is the tangent space $T(\mathbf{x})$, and $|\mathbf{o}(\mathbf{x})| = |\boldsymbol{\nu}(\mathbf{x})| = 1$. Hence $\mathbf{o}(\mathbf{x})$ is an orientation for $T(\mathbf{x})$. A frame $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ for $T(\mathbf{x})$ has this orientation if and only if $(\boldsymbol{\nu}(\mathbf{x}), \mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ is a positively oriented frame for E^n. Since $\boldsymbol{\nu}$ is continuous on $(\text{fr } D) \cap U$ and the components of $\boldsymbol{\nu}$ and $\mathbf{o}$ are related by (6–32a), the function $\mathbf{o}$ is continuous there. Thus $\mathbf{o}$ is an orientation for $(\text{fr } D) \cap U$, which we call the *positive orientation.*

The preceding discussion was local. However, let fr D be an $(n-1)$-manifold, and let D lie on one side of it. There is a (unique) exterior unit normal $\boldsymbol{\nu}(\mathbf{x})$ at each $\mathbf{x} \in \operatorname{fr} D$, and the orientation $\mathbf{o}(\mathbf{x})$ such that $*\mathbf{o}(\mathbf{x}) = \boldsymbol{\nu}(\mathbf{x})$ is defined for every $\mathbf{x} \in \operatorname{fr} D$. Since every $\mathbf{x}_0 \in \operatorname{fr} D$ has a relative neighborhood in which $\mathbf{o}$ is continuous, the function $\mathbf{o}$ is continuous on fr D.

This defines the *positive orientation* $\mathbf{o}$ on fr D. Let us write ∂D^+ instead of $(\operatorname{fr} D)^{\mathbf{o}}$ for fr D with this orientation. (The symbol ∂ is widely used to denote a boundary.)

Let us state and prove the divergence theorem for the following class of open sets, which will be called regular domains.

Definition. An open set $D \subset E^n$ is a *regular domain* if: (1) D is bounded; (2) fr D is an $(n-1)$-manifold of class $C^{(2)}$; and (3) D is on one side of its boundary.

Divergence theorem. *Let D be a regular domain and $\boldsymbol{\omega}$ an $(n-1)$-form of class $C^{(1)}$ on* cl D. *Then*

$$\int_{\partial D^+} \boldsymbol{\omega} = \int_{D^+} d\boldsymbol{\omega}. \tag{7–11a}$$

Let us defer the proof until later in the section. The last assumption means that $\boldsymbol{\omega}$ is the restriction to cl D of a form of class $C^{(1)}$ on some open set D_0 containing cl D. The somewhat restrictive assumption (2) about fr D is made to simplify the proof. The theorem is still true if fr D is not a manifold but instead consists of a finite number of pieces of class $C^{(1)}$ intersecting in sets of dimension $n-2$. For example, if D is the interior of an n-cube then the pieces are the faces, which are cubes of dimension $n-1$ and intersect in $(n-2)$-dimensional cubes. This more general form of the divergence theorem will be precisely stated at the end of the section. For certain special kinds of sets D there is an easy proof of the theorem (Problems 5, 6).

The case n = 2. Suppose that fr $D = C_1 \cup \cdots \cup C_m$ where each C_k is the trace of a simple closed curve γ_k, and $C_1, \ldots, C_m$ are disjoint. The orientation is chosen by selecting the unit tangent vector $\mathbf{v}(x, y)$ so that $(\boldsymbol{\nu}(x, y), \mathbf{v}(x, y))$ is a positively oriented orthonormal frame for E^2. Intuitively speaking this means that as the boundary is traversed, D is always on the left. Then

$$\int_{\partial D^+} \boldsymbol{\omega} = \int_{\gamma_1} \boldsymbol{\omega} + \cdots + \int_{\gamma_m} \boldsymbol{\omega}.$$

If we write $\boldsymbol{\omega} = M\,dx + N\,dy$, then (7–11a) becomes

$$\sum_{k=1}^{m} \int_{\gamma_k} M\,dx + N\,dy = \int_{D^+} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge dy. \tag{7–12}$$

This is known as *Green's theorem*. (See Fig. 7–10.)

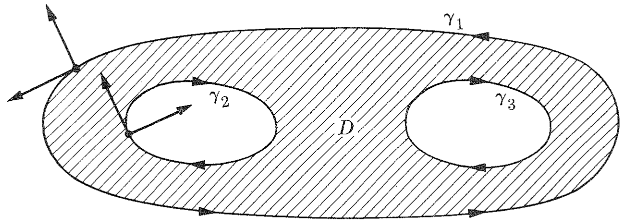


FIGURE 7–10

Example 2. Let $\boldsymbol{\omega} = \frac{1}{2}(x\,dy - y\,dx)$. Then $d\boldsymbol{\omega} = dx \wedge dy$ and $V_2(D) = \int_{D^+} dx \wedge dy$. Hence the area of D can be written as an integral over the boundary:

$$V_2(D) = \tfrac{1}{2}\int_{\partial D^+} x\,dy - y\,dx.$$

The divergence theorem is often stated in a different way which does not involve integrals of differential forms. Let $\boldsymbol{\zeta}$ be a 1-form of class $C^{(1)}$ on cl D. Its divergence

$$\operatorname{div} \boldsymbol{\zeta} = \sum_{i=1}^{n} \frac{\partial \zeta_i}{\partial x^i}$$

is continuous. Let $\boldsymbol{\omega}$ be the $(n-1)$-form such that $*\boldsymbol{\omega} = \boldsymbol{\zeta}$. By formulas (6–32) and (6–34),

$$\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) = *\boldsymbol{\omega}(\mathbf{x}) \cdot *\mathbf{o}(\mathbf{x}) = \boldsymbol{\zeta}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}),$$
$$d\boldsymbol{\omega} = \operatorname{div} \boldsymbol{\zeta}\, dx^1 \wedge \cdots \wedge dx^n.$$

Therefore

$$\int_{\partial D^+} \boldsymbol{\omega} = \int_{\operatorname{fr} D} \boldsymbol{\omega} \cdot \mathbf{o}\, dV_{n-1} = \int_{\operatorname{fr} D} \boldsymbol{\zeta} \cdot \boldsymbol{\nu}\, dV_{n-1},$$
$$\int_{D^+} d\boldsymbol{\omega} = \int_{D^+} \operatorname{div} \boldsymbol{\zeta}\, dx^1 \wedge \cdots \wedge dx^n = \int_D \operatorname{div} \boldsymbol{\zeta}\, dV_n.$$

The conclusion (7–11a) of the divergence theorem can be restated as:

$$\int_{\operatorname{fr} D} \boldsymbol{\zeta}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_{n-1}(\mathbf{x}) = \int_D \operatorname{div} \boldsymbol{\zeta}(\mathbf{x})\, dV_n(\mathbf{x}). \tag{7–11b}$$

The number $\boldsymbol{\zeta}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})$ is called the (exterior) *normal component* of the covector $\boldsymbol{\zeta}(\mathbf{x})$.

Note: In (7–11b) the distinction between vectors and covectors which we have maintained is not customary. If the covector $\boldsymbol{\zeta}(\mathbf{x})$ is replaced by the vector with the same components, then $\cdot$ means the standard euclidean inner product (see remarks in Section 3–3). In Green's formulas below, df should then be replaced by grad f.

For $n = 3$, the divergence theorem is often called Gauss theorem or Ostrogradsky's theorem. It has various interesting physical interpretations.

Let ζ be a force field acting in some open set $D_0 \subset E^3$. For each $\mathbf{x} \in D_0$, $\zeta(\mathbf{x})$ is the force covector acting at $\mathbf{x}$. For notational simplicity, let us set $M = \operatorname{fr} D$ throughout the discussion to follow. The number $\int_M \zeta(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_2(\mathbf{x})$ is called the *outward flux* across the boundary M. The divergence theorem expresses the outward flux as a volume integral over D. If D has small diameter and contains $\mathbf{x}_0$, then the outward flux is approximately $V_3(D) \operatorname{div} \zeta(\mathbf{x}_0)$. To make this statement more precise let us state the following.

Lemma. *If f is continuous on an open set D_0 containing $\mathbf{x}_0$, then*

$$f(\mathbf{x}_0) = \lim_{\operatorname{diam} D \to 0} \frac{1}{V_n(D)} \int_D f(\mathbf{x})\, dV_n(\mathbf{x}).$$

In words, this formula says that given $\epsilon > 0$ there exists $\delta > 0$ such that if D is any open set of diameter less than δ with $\mathbf{x}_0 \in D$, then

$$\left| V_n(D) f(\mathbf{x}_0) - \int_D f(\mathbf{x})\, dV_n(\mathbf{x}) \right| < \epsilon V_n(D).$$

Proof. Given $\epsilon > 0$ let $\delta > 0$ be such that $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon$ whenever $|\mathbf{x} - \mathbf{x}_0| < \delta$. If $\mathbf{x}_0 \in D$ and $\operatorname{diam} D < \delta$, then

$$\left| V_n(D) f(\mathbf{x}_0) - \int_D f(\mathbf{x})\, dV_n \right| = \left| \int_D [f(\mathbf{x}_0) - f(\mathbf{x})]\, dV_n(\mathbf{x}) \right|$$
$$\leq \int_D |f(\mathbf{x}_0) - f(\mathbf{x})|\, dV_n(\mathbf{x}) < \epsilon V_n(D). \blacksquare$$

If in the lemma we take D regular and $f = \operatorname{div} \zeta$, then for any n and $\mathbf{x}_0$ in the domain of ζ

$$\operatorname{div} \zeta(\mathbf{x}_0) = \lim_{\operatorname{diam} D \to 0} \frac{1}{V_n(D)} \int_M \zeta(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_{n-1}(\mathbf{x}). \qquad (7\text{–}13)$$

As another physical interpretation, consider a fluid flowing in an open set $D_0 \subset E^3$. Let t denote time and $\mathbf{x} = (x, y, z)$. Let $\rho(\mathbf{x}, t)$ be the density and $\mathbf{v}(\mathbf{x}, t)$ the velocity at $\mathbf{x}$ and time t. Let $\zeta = \rho\mathbf{v}$. Suppose that D is regular and $\operatorname{cl} D \subset D_0$. The left-hand side of (7–11b) represents the rate at which mass is flowing out of D. Therefore, if $m(t)$ is the mass of the fluid in D at time t, then from the divergence theorem

$$-\frac{dm}{dt} = \int_D \operatorname{div} (\rho\mathbf{v})\, dV_3.$$

On the other hand,

$$m(t) = \int_D \rho(\mathbf{x}, t)\, dV_3(\mathbf{x}).$$

Differentiating under the integral sign (Section 5–11),

$$\frac{dm}{dt} = \int_D \frac{\partial \rho}{\partial t}\, dV_3.$$

For each t_0 the functions $-\operatorname{div} (\rho\mathbf{v})$ and $\partial\rho/\partial t$ have the same integral ov every regular D with $\operatorname{cl} D \subset D_0$. By the lemma, for every $\mathbf{x}_0 \in D_0$ the functions have the same value at $(\mathbf{x}_0, t_0)$. In other words,

$$\frac{\partial \rho}{\partial t} = -\operatorname{div} (\rho\mathbf{v}).$$

If the density ρ is constant, then the fluid is *incompressible*. Thus for incon pressible fluids $\operatorname{div} \mathbf{v} = 0$ at every time t.

If $\operatorname{div} \zeta(\mathbf{x}) = 0$ for every $\mathbf{x}$ in its domain D_0, then ζ is called *divergenc free* (or *solenoidal*). The divergence theorem has the following corollary.

Corollary. *Let ζ be of class $C^{(1)}$ on an open set D_0. Then ζ is divergen free if and only if*

$$\int_M \zeta(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_{n-1}(\mathbf{x}) = 0 \qquad (\text{}$$

for every regular domain D such that $\operatorname{cl} D \subset D_0$.

Proof. If ζ is divergence free, then the equation (*) is immediate fro (7–11b). Conversely if (*) holds for every such D, then by (7–13) $\operatorname{div} \zeta(\mathbf{x}_0) =$ for every $\mathbf{x}_0 \in D_0$. $\blacksquare$

Green's formulas. Let f be of class $C^{(2)}$ on $\operatorname{cl} D$. Let $f_\nu(\mathbf{x})$ denote th derivative of f in the direction of the exterior normal at $\mathbf{x} \in M$, namely,

$$f_\nu(\mathbf{x}) = df(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}).$$

Let ϕ be another function of class $C^{(2)}$ on $\operatorname{cl} D$, and let $\zeta(\mathbf{x}) = \phi(\mathbf{x})\, df(\mathbf{x}$ Then

$$\zeta(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x}) = \phi(\mathbf{x}) f_\nu(\mathbf{x}),$$

$$\operatorname{div} \zeta = \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\phi \frac{\partial f}{\partial x^i}\right) = d\phi \cdot df + \phi \operatorname{Lapl} f.$$

Hence we get the *first Green's formula:*

$$\int_M \phi f_\nu\, dV_{n-1} = \int_D [d\phi \cdot df + \phi \operatorname{Lapl} f]\, dV_n. \qquad (7\text{–}14$$

In the same way

$$\int_M f \phi_\nu\, dV_{n-1} = \int_D [df \cdot d\phi + f \operatorname{Lapl} \phi]\, dV_n.$$

Subtracting, we get the *second Green's formula*

$$\int_M [\phi f_\nu - f \phi_\nu]\, dV_{n-1} = \int_D [\phi \operatorname{Lapl} f - f \operatorname{Lapl} \phi]\, dV_n. \qquad (7\text{–}15$$

Example 3. A function f is called *harmonic* if $\operatorname{Lapl} f = 0$. Let f be harmonic, and apply the first Green's formula with $\phi = f$. Then

$$\int_M f f_{\mathbf{v}}\, dV_{n-1} = \int_D |df|^2\, dV_n. \tag{7-16}$$

When $n = 3$ the right-hand side often has (except for a suitable multiplicative constant) the physical interpretation of energy.

If f is harmonic and $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in M$, then from (7-16) the integral of the nonnegative continuous function $|df|^2$ is 0. Hence $df(\mathbf{x}) = 0$ for every $\mathbf{x} \in \operatorname{cl} D$. Given $\mathbf{x}_0 \in D$ let $\mathbf{x}_1$ be a point of M nearest $\mathbf{x}_0$. The line joining $\mathbf{x}_0$ and $\mathbf{x}_1$ lies in $\operatorname{cl} D$, and from the mean value theorem f is constant on it. Since $f(\mathbf{x}_1) = 0$, we must have $f(\mathbf{x}_0) = 0$. Thus $f(\mathbf{x}) = 0$ for every $\mathbf{x} \in \operatorname{cl} D$.

Suppose that f and g are both of class $C^{(2)}$ on $\operatorname{cl} D$ and harmonic, and that $f(\mathbf{x}) = g(\mathbf{x})$ for every $\mathbf{x} \in M$. Then $\phi(\mathbf{x}) = f(\mathbf{x}) - g(\mathbf{x}) = 0$ for $\mathbf{x} \in M$ and ϕ is harmonic. Hence $\phi(\mathbf{x}) = 0$, and $f(\mathbf{x}) = g(\mathbf{x})$, for every $\mathbf{x} \in \operatorname{cl} D$. This shows that there is at most one harmonic function of class $C^{(2)}$ on $\operatorname{cl} D$ with given values on the boundary M. It is more difficult to show that there is in fact a harmonic function f with given boundary values. This is called *Dirichlet's problem*. If the boundary data $f|M$ are merely continuous, then f is continuous on $\operatorname{cl} D$ and of class $C^{(2)}$ and harmonic on D. See [14], Chap. XI. If the boundary data are smooth enough, then f is of class $C^{(2)}$ and harmonic on $\operatorname{cl} D$. For instance this is true if M is of class $C^{(3)}$ and f is of class $C^{(3)}$ on M.

Let us now turn to the proof of the divergence theorem. The proof will proceed by first proving the theorem when D is either E^n or a half-space and $\boldsymbol{\omega}$ has compact support. The general case will then be reduced to these two by introducing local coordinates on fr D and a partition of unity. As before, we may let $\boldsymbol{\zeta} = *\boldsymbol{\omega}$ and may prove either of the two equivalent formulas (7-11a) and (7-11b). As in Chapter 5, $\int f\, dV_n$ denotes the integral of f over all of E^n.

Lemma 1. *Let $\boldsymbol{\zeta}$ be a 1-form of class $C^{(1)}$ on E^n such that $\boldsymbol{\zeta}$ has compact support. Then $\int \operatorname{div} \boldsymbol{\zeta}\, dV_n = 0$.*

Proof. Let $1 \le i \le n$. By the iterated integrals theorem

$$\int \frac{\partial \zeta_i}{\partial x^i}\, dV_n = \int \left\{ \int \frac{\partial \zeta_i}{\partial x^i}\, dx^i \right\} dV_{n-1}(\mathbf{x}^{i'}),$$

where $\mathbf{x}^{i'} = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^n)$. Since ζ_i has compact support, the inner integral is 0 by the fundamental theorem of calculus. Therefore $\int \partial\zeta_i/\partial x^i\, dV_n = 0$. Summing from 1 to n we get the lemma. ∎

In the next lemma we write (as in Section 5-5) $\mathbf{x}' = (x^1, \ldots, x^{n-1})$ instead of $\mathbf{x}^{n'}$.

Lemma 2. *Let H be the half-space $\{\mathbf{x} : x^n < 0\}$, and let $\boldsymbol{\zeta}$ be as in Lemma 1. Then*

$$\int_H \operatorname{div} \boldsymbol{\zeta}\, dV_n = \int \zeta_n(\mathbf{x}', 0)\, dV_{n-1}(\mathbf{x}').$$

Proof. If $i < n$, then $\int_H \partial\zeta_i/\partial x^i\, dV_n = 0$ as in the proof of Lemma 1. For $i = n$ we have

$$\int_H \frac{\partial \zeta_n}{\partial x^n}\, dV_n = \int \left\{ \int_{-\infty}^{0} \frac{\partial \zeta_n}{\partial x^n}\, dx^n \right\} dV_{n-1}(\mathbf{x}').$$

By the fundamental theorem of calculus the inner integral is $\zeta_n(\mathbf{x}', 0)$, since ζ_n has compact support. ∎

Proposition 32. *Let $\mathbf{f}$ be a regular flat transformation from an open set $D_1 \subset E^n$ onto an open set $\Delta_1 \subset E^n$. Let D be a regular domain such that $D \cap D_1$ is not empty, and let $\Delta = \mathbf{f}(D \cap D_1)$, $S = (\operatorname{fr} D) \cap D_1$, $N = \mathbf{f}(S)$. Then:*

- (a) *Δ is open and $N = (\operatorname{fr} \Delta) \cap \Delta_1$.*
- (b) *Δ is on one side of its boundary in a neighborhood of each point of N.*
- (c) *If $J\mathbf{f}(\mathbf{x}) > 0$ for every $\mathbf{x} \in D_1$, then the positive orientation for D is induced by $\mathbf{f}^{-1}$ from the positive orientation for Δ, and the positive orientation for S from the positive orientation for N.*

Proof. Let $\mathbf{g} = \mathbf{f}^{-1}$. The first assertion (a) follows from the fact that a regular transformation $\mathbf{f}$ is a homeomorphism. Let $\mathbf{t}_0 \in N$, and $\mathbf{x}_0 = \mathbf{g}(\mathbf{t}_0)$. Let U and Φ be as in the definition, p. 263. We may assume that $U \subset D_1$. Let $\Psi(\mathbf{t}) = \Phi[\mathbf{g}(\mathbf{t})]$ for $\mathbf{t} \in \mathbf{f}(U)$. Since $D\mathbf{g}(\mathbf{t})$ has maximum rank n and $d\Phi(\mathbf{x}) \neq \mathbf{0}$, the chain rule implies that $d\Psi(\mathbf{t}) \neq \mathbf{0}$. The open set $\mathbf{f}(U)$ contains a neighborhood Ω of $\mathbf{t}_0$, and

$$(\operatorname{fr} \Delta) \cap \Omega = \{\mathbf{t} \in \Omega : \Psi(\mathbf{t}) = 0\},$$
$$\Delta \cap \Omega = \{\mathbf{t} \in \Omega : \Psi(\mathbf{t}) < 0\}.$$

Therefore Δ is on one side of its boundary in Ω. This proves (b).

Since $J\mathbf{g}(\mathbf{t}) = 1/J\mathbf{f}(\mathbf{x}) > 0$, $\mathbf{g}$ preserves the positive orientation of E^n. We must show that the orientation induced on S from the positive orientation is positive.

Let $\mathbf{k}_0 = \operatorname{grad} \Psi(\mathbf{t}_0)$ and $\mathbf{n}_0 = \operatorname{grad} \Phi(\mathbf{x}_0)$. They are exterior normals to Δ and to D respectively. Let $\mathbf{h}_0 = \mathbf{L}(\mathbf{k}_0)$, where $\mathbf{L} = D\mathbf{g}(\mathbf{t}_0)$. From the chain rule, $\mathbf{k}_0 = \mathbf{L}^t(\mathbf{n}_0)$ where $\mathbf{L}^t$ is the transpose of $\mathbf{L}$. By formula (4-8)

$$\mathbf{h}_0 \cdot \mathbf{n}_0 = [\mathbf{L} \circ \mathbf{L}^t(\mathbf{n}_0)] \cdot \mathbf{n}_0 = \mathbf{L}^t(\mathbf{n}_0) \cdot \mathbf{L}^t(\mathbf{n}_0),$$
$$\mathbf{h}_0 \cdot \mathbf{n}_0 = |\mathbf{L}^t(\mathbf{n}_0)|^2 > 0.$$

Let $(\mathbf{k}_1, \ldots, \mathbf{k}_{n-1})$ be a positively oriented frame for the tangent space to fr Δ at $\mathbf{t}_0$, and let $\mathbf{h}_l = \mathbf{L}(\mathbf{k}_l)$, $l = 1, \ldots, n-1$. Then $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ is a frame for the tangent space $T(\mathbf{x}_0)$ to fr D at $\mathbf{x}_0$. Since $(\mathbf{k}_0, \mathbf{k}_1, \ldots, \mathbf{k}_{n-1})$ is a positively oriented frame and $\mathbf{g}$ preserves the orientation of E^n, $(\mathbf{h}_0, \mathbf{h}_1, \ldots, \mathbf{h}_n)$ is a positively oriented frame.

Now $\mathbf{h}_0 = c\mathbf{n}_0 + \mathbf{h}$, where $c = (\mathbf{h}_0 \cdot \mathbf{n}_0)/(\mathbf{n}_0 \cdot \mathbf{n}_0)$ and $\mathbf{h} \in T(\mathbf{x}_0)$. From this,

$$\mathbf{n}_0 \wedge \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_{n-1} = c\mathbf{h}_0 \wedge \mathbf{h}_1 \wedge \cdots \wedge \mathbf{h}_{n-1}.$$

Since $\mathbf{h}_0 \cdot \mathbf{n}_0 > 0$, $c > 0$. Therefore the frame $(\mathbf{n}_0, \mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ has positive orientation, which implies that $(\mathbf{h}_1, \ldots, \mathbf{h}_{n-1})$ orients S positively at $\mathbf{x}_0$. ∎

If $Jf(\mathbf{x}) < 0$ for every $\mathbf{x} \in D_1$, then $\mathbf{f}^{-1}$ induces the negative orientation (corresponding to the interior normal) on S.

Proof of divergence theorem. Let us show that each $\mathbf{x}_0 \in$ cl D has a neighborhood U_0 such that (7–11a) holds provided $\boldsymbol{\omega}$ has compact support contained in cl U_0. If $\mathbf{x}_0 \in D$, let U_0 be a small enough neighborhood that cl $U_0 \subset D$. Then $\int_{\partial D^+} \boldsymbol{\omega} = 0$, and by Lemma 1, $\int_{\partial D^+} d\boldsymbol{\omega} = 0$.

Let $\mathbf{x}_0 \in$ fr D, and let H be as in Lemma 2. Let us find a neighborhood D_1 of $\mathbf{x}_0$ and a regular transformation $\mathbf{f}$ with domain D_1 such that

$$\mathbf{f}(D \cap D_1) \subset H, \qquad \mathbf{f}[(\text{fr } D) \cap D_1] \subset \text{fr } H,$$

and $J\mathbf{f}(\mathbf{x}) > 0$ for every $\mathbf{x} \in D_1$. For this purpose let us first suppose that $\nu^n(\mathbf{x}_0) > 0$, where $\boldsymbol{\nu}(\mathbf{x}_0)$ is the unit exterior normal at $\mathbf{x}_0$. Let Φ be as on p. 263. Then grad $\Phi(\mathbf{x}) = c\boldsymbol{\nu}(x)$, where $c > 0$. Taking nth components, $\Phi_n(\mathbf{x}_0) = c\nu^n(\mathbf{x}_0)$. For D_1 take a neighborhood of $\mathbf{x}_0$ in which $\Phi_n(\mathbf{x}) > 0$, and let

$$f^i(\mathbf{x}) = x^i, \qquad i = 1, \ldots, n-1, \qquad f^n(\mathbf{x}) = \Phi(\mathbf{x}).$$

Then $J\mathbf{f}(\mathbf{x}) = \Phi_n(\mathbf{x}) > 0$ for every $\mathbf{x} \in D_1$. If the condition $\nu^n(\mathbf{x}_0) > 0$ is not satisfied, then for $\mathbf{f}$ we take $\tilde{\mathbf{f}} \circ \mathbf{L}$, where $\mathbf{L}$ is a rotation of E^n such that $\mathbf{L}[\boldsymbol{\nu}(\mathbf{x}_0)]$ is a vector whose last component is positive and $\tilde{\mathbf{f}}$ is of the type just described.

Let U_0 be a neighborhood of $\mathbf{x}_0$ such that cl $U_0 \subset D_1$, and let $\boldsymbol{\omega}$ have compact support contained in U_0. Let $\mathbf{g} = \mathbf{f}^{-1}$. Since $\mathbf{g}$ is of class $C^{(2)}$ and preserves the positive orientation of E^n, by Proposition 31 and (c) of Proposition 32,

$$\int_{D^+} d\boldsymbol{\omega} = \int_{H^+} (d\boldsymbol{\omega})^{\#} = \int_{H^+} d\boldsymbol{\omega}^{\#}. \tag{*}$$

$$\int_{\partial D^+} \boldsymbol{\omega} = \int_{\partial H^+} \boldsymbol{\omega}^{\#}. \tag{**}$$

But by Lemma 2, the right-hand sides of (*) and (**) are equal.

Since cl D is a compact set, a finite number of such neighborhoods $U_1, \ldots, U_m$ cover cl D. Let $\psi_k(\mathbf{x})$ and $\phi_k(\mathbf{x})$ be defined as in the proof of Proposition 29, for $\mathbf{x} \in$ cl D. Since $\phi_k\boldsymbol{\omega}$ has compact support contained in cl U_k,

$$\int_{\partial D^+} \phi_k \boldsymbol{\omega} = \int_{D^+} d(\phi_k \boldsymbol{\omega}), \qquad k = 1, \ldots, m. \tag{*}$$

By the product rule,

$$d(\phi_k \boldsymbol{\omega}) = d\phi_k \wedge \boldsymbol{\omega} + \phi_k \, d\boldsymbol{\omega}.$$

Since $\sum \phi_k = 1$, $\sum d\phi_k = 0$. Summing from 1 to m in (*), we have

$$\int_{\partial D^+} \left(\sum_{k=1}^{m} \phi_k\right) \boldsymbol{\omega} = \int_{D^+} \left(\sum_{k=1}^{m} \phi_k\right) d\boldsymbol{\omega},$$

which is precisely (7–11a). ∎

The assumption that fr D is a manifold of class $C^{(2)}$ can be considerably weakened. Let us state without proof a somewhat more general version of the divergence theorem. Let D be an open, bounded set. Assume that

$$\text{fr } D = A_1 \cup \cdots \cup A_p \cup B,$$

where: (a) A_k is a relatively open subset of fr D and cl A_k is a compact subset of an $(n-1)$-manifold M_k, $k = 1, \ldots, p$; and (b) B is a compact set contained in a finite union of $(n-2)$-manifolds, and (cl A_k) $\cap$ (cl A_l) $\subset B$ whenever $k \neq l$. Moreover, assume that D is on one side of its boundary in a neighborhood of each point of (fr D) $-$ B. On each A_k we assign the positive orientation, determined by the exterior normal. Then

$$\sum_{k=1}^{p} \int_{A_k^+} \boldsymbol{\omega} = \int_{D^+} d\boldsymbol{\omega},$$

provided $\boldsymbol{\omega}$ is of class $C^{(1)}$ on cl D. (See Fig. 7–11.)

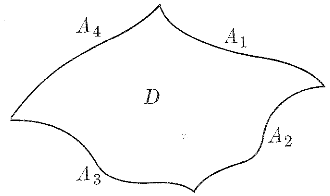


FIGURE 7–11

Let us say that such a set D has a boundary which is *piecewise of class* $C^{(1)}$.

Example 4. Let D be an n-simplex. Let $A_0, \ldots, A_n$ be its (open) $(n-1)$-dimensional faces, let M_k be the hyperplane containing A_k, and let B be the union of the $(n-2)$-dimensional faces of D.

PROBLEMS

Unless otherwise indicated, assume that D is a regular domain.

1. Let $n = 2$. Show that:
   (a) $V_2(D) = -\int_{\partial D^+} y \, dx.$
   (b) $\int_D (x^2 + y^2) \, dV_2 = \frac{1}{3} \int_{\partial D^+} x^3 \, dy - y^3 \, dx.$
2. Evaluate $\int_{\partial\Sigma^+} y^2 \, dx \wedge dz$, where Σ is the standard 3-simplex.
3. Let D be the disk $x^2 + y^2 < 1$ and $\boldsymbol{\omega} = (x \, dy - y \, dx)/(x^2 + y^2)$. Then $\int_{\partial D^+} \boldsymbol{\omega} = 2\pi$ while $\int_{D^+} d\boldsymbol{\omega} = 0$. Why does this not contradict Green's theorem?

4. Let $n = 4$ and $D = \{\mathbf{x} : (x^1)^2 + (x^2)^2 + (x^3)^2 < (x^4)^2, 0 < x^4 < 1\}$.
Evaluate:

(a) $\int_{\partial D^+} (x^2 + x^4)\, dx^1 \wedge dx^2 \wedge dx^3$. (b) $\int_{\partial D^+} |\mathbf{x}|^2\, dx^1 \wedge dx^2 \wedge dx^3$.

5. Suppose that $D = \{(x, y) : f(x) < y < g(x), a < x < b\} = \{(x, y) : \phi(y) < x < \psi(y), c < y < d\}$. Show directly from the fundamental theorem of calculus and properties of line integrals that

$$\int_{\partial D^+} N\, dy = \int_D \frac{\partial N}{\partial x}\, dV_2,$$

$$\int_{\partial D^+} M\, dx = -\int_D \frac{\partial M}{\partial x}\, dV_2.$$

Adding, we get the Green's theorem for regular domains of this special type. (See Fig. 7-12.)

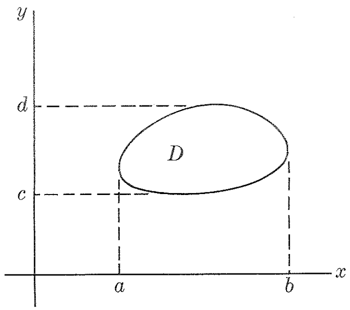


FIGURE 7-12

6. Prove the divergence theorem directly from the fundamental theorem of calculus when D is:
   - (a) The unit n-cube $\{\mathbf{x} : 0 < x^i < 1, i = 1, \ldots, n\}$.
   - (b) The standard n-simplex.

7. For each t in some interval $(-a, a)$ let $\mathbf{T}_t$ be a regular flat transformation with domain D_0. Assume that $\mathbf{T}_0(\mathbf{x}) = \mathbf{x}$ for every $\mathbf{x} \in D_0$ and that $\mathbf{T}$ is of class $C^{(2)}$ as a function of $(\mathbf{x}, t)$ on $D_0 \times (-a, a)$. Let $v(t) = V_n[\mathbf{T}_t(D)]$ where cl $D \subset D_0$. Prove that $v'(0) = \int_D \operatorname{div} \mathbf{W}_0\, dV_n$, where $\mathbf{W}_t = \partial \mathbf{T}_t / \partial t$. [*Hint:* Show that the integrand is $(\partial/\partial t) J\mathbf{T}_t(\mathbf{x})$ evaluated at $t = 0$.]

In Problems 8 and 9 let $M = \operatorname{fr} D$.

8. Show that:

(a) $\int_M \nu^i(\mathbf{x})\, dV_{n-1}(\mathbf{x}) = 0$. (b) $\int_M \mathbf{x} \cdot \boldsymbol{\nu}(\mathbf{x})\, dV_{n-1}(\mathbf{x}) = nV_n(D)$.

(c) $\int_M f_{\boldsymbol{\nu}}(\mathbf{x})\, dV_{n-1}(\mathbf{x}) = \int_D \operatorname{Lapl} f(\mathbf{x})\, dV_n(\mathbf{x})$.

9. Let D be connected, f harmonic, and $f_{\boldsymbol{\nu}}(\mathbf{x}) = 0$ for every $\mathbf{x} \in M$. Show that $f(\mathbf{x})$ is constant on D.

10. Let $D = \{\mathbf{x} : a < |\mathbf{x}| < b\}$, where $0 < a < b$.
   - (a) Show that if $f(\mathbf{x}) = \psi(|\mathbf{x}|)$, then $f_{\boldsymbol{\nu}}(\mathbf{x}) = \psi'(|\mathbf{x}|)$ when $|\mathbf{x}| = b$ and $f_{\boldsymbol{\nu}}(\mathbf{x}) = -\psi'(|\mathbf{x}|)$ when $|\mathbf{x}| = a$.
   - (b) Let $\psi(r) = -[(n-2)\beta_{n-1}]^{-1} r^{n-2}$, where $n > 2$ and β_{n-1} is the $(n-1)$-measure of the unit $(n-1)$-sphere. Let f be as in (a). Show that f is harmonic.
   - (c) Let ϕ be harmonic on the n-ball $B = \{\mathbf{x} : |\mathbf{x}| \leq b\}$. Show that $\phi(\mathbf{0}) = (\beta_{n-1} b^{n-1})^{-1} \int_{\operatorname{fr} B} \phi\, dV_{n-1}$. [*Hint:* Apply the second Green's formula with D and f as above and let $a \to 0^+$.]

7-6 STOKES' FORMULA

The divergence theorem is a special case of a result which is nowaday called Stokes' formula. Let $\boldsymbol{\omega}$ be an $(r-1)$-form. Stokes' formula equate the integral of $d\boldsymbol{\omega}$ over a portion A of an oriented r-manifold M and the integra of $\boldsymbol{\omega}$ over the (suitably oriented) boundary of A.

Let us begin with the following particular case and afterward generalize Let $B \subset E^r$ be a regular domain, and let $A = \mathbf{g}(B)$ where $\mathbf{g}$ is a regular trans formation of class $C^{(2)}$ from some open set containing cl B into M. Let $\mathbf{o}$ b the orientation induced on A from the positive orientation of E^r. The $(r-1)$ manifold $K = \mathbf{g}(\operatorname{fr} B)$ is the boundary of A relative to M. Let $\partial A^{\mathbf{o}}$ denot K with the orientation induced from the positive orientation of fr B.

Let $\boldsymbol{\omega}$ be an $(r-1)$-form of class $C^{(1)}$ on cl A. Then

$$\int_{A^{\mathbf{o}}} d\boldsymbol{\omega} = \int_{B^+} (d\boldsymbol{\omega})^{\#} = \int_{B^+} d\boldsymbol{\omega}^{\#},$$

$$\int_{\partial A^{\mathbf{o}}} \boldsymbol{\omega} = \int_{\partial B^+} \boldsymbol{\omega}^{\#}$$

By the divergence theorem, the right-hand sides are equal. Therefore we hav

Stokes' formula

$$\int_{\partial A^{\mathbf{o}}} \boldsymbol{\omega} = \int_{A^{\mathbf{o}}} d\boldsymbol{\omega}. \tag{7-17}$$

The case $r = 1$, $n = 3$. Then $\boldsymbol{\omega} = P\,dx + Q\,dy + R\,dz$ is a 1-forn and $d\boldsymbol{\omega}$ is a 2-form. The 1-form $*d\boldsymbol{\omega}$ is called curl $\boldsymbol{\omega}$, and the vector $\mathbf{n}(\mathbf{x}) = *\mathbf{o}(\mathbf{x}$ is a unit normal to A. Since $d\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{o}(\mathbf{x}) = \operatorname{curl} \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$, formula (7-17 becomes

$$\int_{\partial A^{\mathbf{o}}} \boldsymbol{\omega} = \int_A \operatorname{curl} \boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\, dV_2(\mathbf{x}). \tag{7-18}$$

The name Stokes' formula was traditionally applied to (7-18), and not its generalization (7-17).

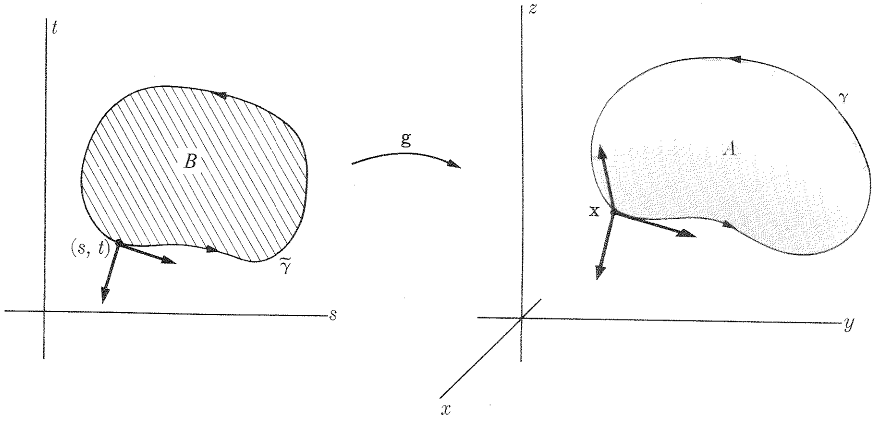


FIGURE 7-13

Example. Let ∂B^+ consist of a single simple closed curve $\tilde{\gamma}$ in E^2. Then ∂A^+ consists of a simple closed curve γ in E^3.

The normal $\mathbf{n}(\mathbf{x})$ varies continuously on A. At a boundary point $\mathbf{x}$ of A, $\mathbf{n}(\mathbf{x})$ can be visualized in the following way. Let $\mathbf{x} = \mathbf{g}(s, t)$, where $(s, t) \in \text{fr } B$. Let $\boldsymbol{\nu}$ be the exterior normal and $\mathbf{v}$ the positively oriented unit tangent vector to $\tilde{\gamma}$ at (s, t). The vector $\mathbf{h} = D\mathbf{g}(s, t)(\mathbf{v})$ is a tangent vector to γ at $\mathbf{x}$. If $\mathbf{h}_0 = D\mathbf{g}(s, t)(\boldsymbol{\nu})$, then $(\mathbf{h}_0, \mathbf{h})$ is a frame for the tangent space to M at $\mathbf{x}$ and has the required orientation $\mathbf{o}(\mathbf{x})$. Hence $(\mathbf{n}(\mathbf{x}), \mathbf{h}_0, \mathbf{h})$ is a positively oriented frame for E^3. (See Fig. 7–13.)

If P, Q, R are regarded as the components of a velocity field, then $\int_\gamma \boldsymbol{\omega}$ represents the circulation along the boundary γ. Stokes' formula expresses the circulation as the integral over A of the normal component curl $\boldsymbol{\omega}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})$ of the curl.

In particular, let A lie in a plane Π, oriented by a unit vector $\mathbf{n}_0$ normal to Π. Then $\mathbf{n}(\mathbf{x}) = \mathbf{n}_0$ for every $\mathbf{x} \in A$. Let $\mathbf{x}_0$ be a point of the domain of $\boldsymbol{\omega}$. If A contains $\mathbf{x}_0$ and A has small diameter, then the right-hand side of (7–18) is approximately curl $\boldsymbol{\omega}(\mathbf{x}_0) \cdot \mathbf{n}_0 V_2(A)$. More precisely,

$$\operatorname{curl} \boldsymbol{\omega}(\mathbf{x}_0) \cdot \mathbf{n}_0 = \lim_{\operatorname{diam} A \to 0} \frac{1}{V_2(A)} \int_\gamma \boldsymbol{\omega}.$$

This is proved using a lemma similar to the one for the proof of the corresponding formula (7–12) for the divergence.

Some generalizations. Let M be an orientable manifold of class $C^{(2)}$. We proved Stokes' formula above in case cl A is contained in some coordinate patch. By using partitions of unity, this restriction can be removed.

Proposition 33. *Let M be compact and $\mathbf{o}$ an orientation for M. Then*

$$\int_{M^\mathbf{o}} d\boldsymbol{\omega} = 0$$

for every $(r - 1)$-form of class $C^{(1)}$ on M.

Proof. Let $\{\phi_1, \ldots, \phi_m\}$ be a partition of unity for M. Let $\mathbf{g}^{(k)}$ be a regular transformation of class $C^{(2)}$ from an open set $\Delta_k \subset E^r$ onto a coordinate patch S_k containing the support of ϕ_k. Then

$$\int_{M^\mathbf{o}} d(\phi_k\boldsymbol{\omega}) = \pm \int_{\Delta_k^+} [d(\phi_k\boldsymbol{\omega})]^\# = \pm \int_{\Delta_k^+} d(\phi_k\boldsymbol{\omega})^\# = 0,$$

by Lemma 1 of the last section. Since $\sum \phi_k = 1$, $\sum d\phi_k = 0$, we get, as in the proof of the divergence theorem,

$$\int_{M^\mathbf{o}} d\boldsymbol{\omega} = \sum_{k=1}^{m} \int_{M^\mathbf{o}} d(\phi_k\boldsymbol{\omega}) = 0. \blacksquare$$

Since M has empty boundary relative to itself, one would expect to obtain 0 on the left-hand side of (7–17) when $A = M$. Proposition 33 states that this is correct.

Now let M be any orientable r-manifold of class $C^{(2)}$. Let us call a relatively open set $A \subset M$ a *regular domain on M* if:

(1) cl A is a compact subset of M;
(2) the boundary K of A relative to M is an $(r - 1)$-manifold of class $C^{(2)}$;
(3) A is on one side of K.

By condition (3) we mean that if $\mathbf{F}$ is any coordinate system for $S \subset M$, then $\mathbf{F}(A \cap S)$ is on one side of $\mathbf{F}(K \cap S)$ in a neighborhood of each point of $\mathbf{F}(K \cap S)$.

Let $\mathbf{o}$ be an orientation for M. Then $\mathbf{o}$ determines an orientation $\mathbf{o}'$ on K as follows. Let S be a coordinate patch and $\mathbf{F}$ a coordinate system for S. If the orientation induced by $\mathbf{F}^{-1}$ from the positive orientation of E^r is $\mathbf{o}$, then for $\mathbf{x} \in K \cap S$, $\mathbf{o}'(\mathbf{x})$ is the orientation induced from the positive orientation of $\mathbf{F}(K \cap S)$. Otherwise, $\mathbf{o}'(\mathbf{x})$ is the orientation opposite to this one. From part (c) of Proposition 32, Section 7–5, it can be shown that $\mathbf{o}'(\mathbf{x})$ is independent of the particular coordinate system chosen (Problem 3).

Let K with the orientation $\mathbf{o}'$ be denoted by $\partial A^\mathbf{o}$.

Theorem 23. *Let A be a regular domain on M, and let $\boldsymbol{\omega}$ be an $(r - 1)$-form of class $C^{(1)}$ on cl A. Then*

$$\int_{\partial A^\mathbf{o}} \boldsymbol{\omega} = \int_{A^\mathbf{o}} d\boldsymbol{\omega}. \tag{7–17}$$

This theorem can be proved using the divergence theorem and a partition of unity in much the same way as for Proposition 33. We shall not give the details.

We have assumed that M is of class $C^{(2)}$, but Theorem 23 is still true for manifolds of class $C^{(1)}$. Moreover, the relative boundary K may be piecewise of class $C^{(1)}$ in the sense explained at the end of Section 7–5. For instance, if M is an r-plane and A an r-simplex contained in M, then the boundary of A relative to M is piecewise of class $C^{(1)}$.

PROBLEMS

1. Let $\boldsymbol{\omega} = yz\,dx + x\,dy + dz$. Let γ be the unit circle in the xy-plane, oriented in the counterclockwise direction. Calculate $\int_\gamma \boldsymbol{\omega}$ and $\int_{A^\mathbf{o}} d\boldsymbol{\omega}$ and verify that they are equal, where the orientation $\mathbf{o}$ is chosen so that $\partial A^\mathbf{o} = \gamma$ and:
   (a) A is the disk $x^2 + y^2 < 1$ in the xy-plane.
   (b) $A = \{(x, y, 1 - x^2 - y^2) : x^2 + y^2 < 1\}$.
2. Let $\boldsymbol{\omega} = z \exp(-y)\,dx + z\,dy + y\,dz$. Evaluate $\int_{A^\mathbf{o}} d\boldsymbol{\omega}$ when A is:
   (a) The ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ oriented by the exterior normal.
   (b) The square with vertices $\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2, \sqrt{2}\,\mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \sqrt{2}\,\mathbf{e}_3$, oriented so that $o^{23}(\mathbf{x}) > 0$.
   (c) The paraboloid $y = x^2 + z^2$ oriented so that $o^{31}(\mathbf{x}) > 0$.
3. Show that the orientation $\mathbf{o}'$ for K does not depend on the particular choice of coordinate systems for M used in its definition.

4. Let $M = \text{fr } D$, where D is a regular domain in E^n. Show that $\int_M (*d\boldsymbol{\omega}) \cdot \boldsymbol{\nu} \, dV_{n-1} = 0$ if $\boldsymbol{\omega}$ is an $(n - 2)$-form of class $C^{(1)}$ on M.
5. Prove the following:

$$d\boldsymbol{\omega}(\mathbf{x}_0) \cdot \boldsymbol{\alpha}_0 = \lim_{\text{diam A} \to 0} [V_r(A)]^{-1} \int_{\partial A^{\alpha_0}} \boldsymbol{\omega},$$

where $\mathbf{x}_0 \in A$ and A lies in an r-plane Π oriented by $\boldsymbol{\alpha}_0$.

6. Let $\boldsymbol{\alpha}$ be the r-vector of an r-simplex S_0 and $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_r$ the $(r - 1)$-vectors of its oriented faces (Problem 12, Section 6–3). Show that

$$d\boldsymbol{\omega}(\mathbf{x}_0) \cdot \boldsymbol{\alpha} = \sum_{i=0}^{r} (-1)^i \boldsymbol{\omega}(\mathbf{x}_0) \cdot \boldsymbol{\beta}_i.$$

[*Hint:* Consider simplexes S similar to S_0 and containing $\mathbf{x}_0$. Apply Problem 5 with $A = S$.]

7–7 CLOSED AND EXACT DIFFERENTIAL FORMS

Any exact differential form $\boldsymbol{\omega} = d\boldsymbol{\eta}$ is closed, provided $\boldsymbol{\eta}$ is of class $C^{(2)}$. This is a consequence of the formula in Section 6–5 $d(d\boldsymbol{\eta}) = \mathbf{0}$. Whether, conversely, every closed form $\boldsymbol{\omega}$ is exact depends on the topological nature of the domain D of $\boldsymbol{\omega}$. In this section we shall give two sufficient conditions that every closed r-form with domain D be exact. The first is that D be simply connected and applies when $r = 1$. The second is that D be star-shaped and applies for any degree r.

Homotopies. Let $\mathbf{f}$ and $\mathbf{g}$ be transformations of class $C^{(2)}$ from a set $B \subset E^m$ into a set $A \subset E^n$. We are interested in whether it is possible to smoothly interpolate in A between $\mathbf{f}$ and $\mathbf{g}$. If this is possible then $\mathbf{f}$ and $\mathbf{g}$ are called homotopic in A. To state this more precisely, let us consider the subset $[0, 1] \times B$ of E^{m+1}.

Definition. If there is a transformation $\mathbf{H}$ of class $C^{(2)}$ on $[0, 1] \times B$ such that $\mathbf{H}(s, \mathbf{t}) \in A$ for every $(s, \mathbf{t}) \in [0, 1] \times B$ and $\mathbf{H}(0, \mathbf{t}) = \mathbf{f}(\mathbf{t}), \mathbf{H}(1, \mathbf{t}) = \mathbf{g}(\mathbf{t})$ for every $\mathbf{t} \in B$, then $\mathbf{f}$ and $\mathbf{g}$ are *homotopic in* A.

In the usual definition of homotopy in topology, $\mathbf{H}$ is required to be merely continuous. What we call homotopy is then called a homotopy of class $C^{(2)}$.

Example 1. Let A be convex. Then we may take

$$\mathbf{H}(s, \mathbf{t}) = s\mathbf{g}(\mathbf{t}) + (1 - s)\mathbf{f}(\mathbf{t}).$$

Therefore any two transformations $\mathbf{f}$ and $\mathbf{g}$ of class $C^{(2)}$ with values in a convex set A are homotopic in A. In particular, this is true when $A = E^n$.

To define simple connectedness one may take B to be a circle. However, instead of a circle it is more convenient to let B be an interval $[a, b]$ with the endpoints identified. Let $\mathbf{f}$ and $\mathbf{g}$ be transformations from $[a, b]$ into A such that $\mathbf{f}(a) = \mathbf{f}(b)$ and $\mathbf{g}(a) = \mathbf{g}(b)$. Then $\mathbf{f}$ and $\mathbf{g}$ are *strictly homotopic in* A if the homotopy $\mathbf{H}$ in the definition above can be chosen so that $\mathbf{H}(s, a) = \mathbf{H}(s, b)$ for every $s \in [0, 1]$.

If $\partial\mathbf{H}/\partial t \neq 0$, then for each s the transformation $\mathbf{H}(s, \ )$ represents on $[a, b]$ a closed curve γ_s of class $C^{(2)}$ in the sense of Section 3–2. Intuitively, one may regard a homotopy as a smooth interpolation by the curves γ_s between the curve γ_0 represented by $\mathbf{f}$ and the curve γ_1 represented by $\mathbf{g}$. However, for technical reasons it is disadvantageous to include the conditions $\partial\mathbf{H}/\partial t \neq 0$ in the definition of homotopy.

Definition. If $\mathbf{g}$ is strictly homotopic in A to a constant transformation $\mathbf{f}$, then $\mathbf{g}$ is *null homotopic* in A.

If $\mathbf{f}(t) = \mathbf{x}_0$ for every $t \in [a, b]$, then one should think intuitively that γ_s shrinks to the point $\mathbf{x}_0$ as $s \to 0^+$. When A is an open subset of E^2 this is possible roughly speaking provided γ_s does not loop around any holes which may be present in A. In Fig. 7–14, A has two holes and the curves γ_s in the figure are not null homotopic in A.

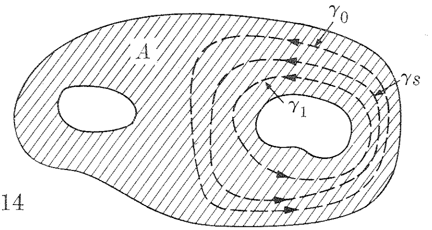


FIGURE 7–14

Let D be an open set, and $\boldsymbol{\omega}$ a 1-form with domain D. Let us set

$$\langle \mathbf{g}, \boldsymbol{\omega} \rangle = \int_a^b \boldsymbol{\omega}[\mathbf{g}(t)] \cdot \mathbf{g}'(t) \, dt.$$

In case $\mathbf{g}'(t) \neq \mathbf{0}$, $\langle \mathbf{g}, \boldsymbol{\omega} \rangle$ is just another notation for the line integral of $\boldsymbol{\omega}$ along the curve represented by $\mathbf{g}$.

Proposition 34. *Let $\boldsymbol{\omega}$ be closed. If $\mathbf{f}$ and $\mathbf{g}$ are strictly homotopic in D, then* $\langle \mathbf{f}, \boldsymbol{\omega} \rangle = \langle \mathbf{g}, \boldsymbol{\omega} \rangle$.

Proof. Let $\boldsymbol{\omega}^\#$ be the 1-form on the rectangle $R = [0, 1] \times [a, b]$ induced by the transformation $\mathbf{H}$. Since $d\boldsymbol{\omega} = \mathbf{0}$, $d\boldsymbol{\omega}^\# = (d\boldsymbol{\omega})^\# = \mathbf{0}$. By Green's theorem

$$\int_{\partial R^+} \boldsymbol{\omega}^\# = \int_{R^+} d\boldsymbol{\omega}^\# = 0.$$

The integral over ∂R^+ is the sum of the integrals over the four segments $\lambda_1, \ldots, \lambda_4$ indicated in Fig. 7-15. Now

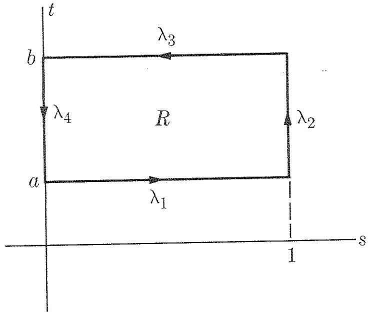


FIGURE 7-15

$$\omega^{\#} = \sum_{i=1}^{n} \omega_i \circ \mathbf{H}(dx^i)^{\#}$$

$$= \sum_{i=1}^{n} \omega_i \circ \mathbf{H}\left(\frac{\partial H^i}{\partial s}\, ds + \frac{\partial H^i}{\partial t}\, dt\right),$$

$$\int_{\lambda_2} \omega^{\#} = \sum_{i=1}^{n} \int_a^b \omega_i \circ \mathbf{H}\, \frac{\partial H^i}{\partial t}\, dt,$$

$\mathbf{H}$ and $\partial H^i/\partial t$ being evaluated at $(1, t)$. Since $\mathbf{H}(1, t) = \mathbf{g}(t)$, the right-hand side is just $\langle \mathbf{g}, \omega \rangle$. Similarly, since $\mathbf{H}(0, t) = \mathbf{f}(t)$

$$\int_{\lambda_4} \omega^{\#} = -\langle \mathbf{f}, \omega \rangle.$$

Since $\mathbf{H}(s, a) = \mathbf{H}(s, b)$,

$$\int_{\lambda_1} \omega^{\#} = -\int_{\lambda_3} \omega^{\#}. \blacksquare$$

Example 2. Let $n = 2$ and let D be the plane with $(0, 0)$ removed. Let $\omega = (x\,dy - y\,dx)/(x^2 + y^2)$. Formally, $\omega = d\Theta$, where $\Theta(x, y)$ is the angle from the positive x-axis to (x, y), $0 < \Theta(x, y) < 2\pi$. However, Θ is defined only in the plane with a slit removed even though ω is defined and of class $C^{(\infty)}$ in D. For each integer $m \neq 0$ let $\mathbf{g}_m(t) = (\cos mt)\mathbf{e}_1 + (\sin mt)\mathbf{e}_2$, $0 \leq t \leq 2\pi$. Then $\langle \mathbf{g}_m, \omega \rangle = 2m\pi$, which shows that $\mathbf{g}_m$ and $\mathbf{g}_l$ are not strictly homotopic in D when $m \neq l$. The transformation $\mathbf{g}_m$ represents the unit circle traversed $|m|$ times, counterclockwise if $m > 0$ and clockwise if $m < 0$.

Proposition 34 has the following corollaries.

Corollary 1. *If* $\mathbf{g}$ *is null homotopic in* D, *then* $\langle \mathbf{g}, \omega \rangle = 0$.

Proof. If $\mathbf{f}$ is constant, then $\langle \mathbf{f}, \omega \rangle = 0$. ∎

Definition. An open set D is *simply connected* if every transformation $\mathbf{g}$ of class $C^{(2)}$ from an interval $[a, b]$ into D, satisfying $\mathbf{g}(a) = \mathbf{g}(b)$, is null homotopic in D.

Roughly speaking, D is simply connected if every closed curve in D can be shrunk in D to a point. When $D \subset E^2$ this amounts to saying that D "has no holes." Removal of a single point, as in Example 2, must be counted as introducing a hole.

If $D = \{\mathbf{x} \in E^3 : |\mathbf{x}| > 1\}$, then D is simply connected, yet D has a "hole."

Corollary 2. *If D is a simply connected open subset of E^n, then every closed 1-form with domain D is exact.*

Proof. By Theorem 7, Section 3-3, it suffices to show that $\int_\gamma \omega = 0$ for every piecewise smooth closed curve γ lying in D. Let $\mathbf{g}$ be a representation of such a curve γ on $[0, 1]$, such that $\mathbf{g}$ is piecewise of class $C^{(1)}$. There is a sequence $\mathbf{g}_1, \mathbf{g}_2, \ldots$ of transformations of class $C^{(\infty)}$ on $[0, 1]$ such that: (1) $\mathbf{g}_m(0) = \mathbf{g}_m(1)$ for $m = 1, 2, \ldots$; (2) $\mathbf{g}_m(t) \to \mathbf{g}(t)$ for every $t \in [0, 1]$, and $\mathbf{g}'_m(t) \to \mathbf{g}'(t)$ except at the (finitely many) points of discontinuity of $\mathbf{g}'$, as $m \to \infty$; (3) $|\mathbf{g}_m(t)|$ and $|\mathbf{g}'_m(t)|$ are bounded by some number C. Such a sequence can be found by a standard smoothing technique (Problem 5). By Lebesgue's dominated convergence theorem, $\langle \mathbf{g}_m, \omega \rangle \to \langle \mathbf{g}, \omega \rangle$ as $m \to \infty$. By Corollary 1, $\langle \mathbf{g}_m, \omega \rangle = 0$ for each $m = 1, 2, \ldots$ Therefore $\langle \mathbf{g}, \omega \rangle = \int_\gamma \omega = 0$. ∎

Let us turn to the question of finding a condition on D which insures that any closed form of arbitrary degree r is exact. For this purpose, let B be an open subset of E^m. Let us introduce an operation which changes any r-form $\boldsymbol{\eta}$ of class $C^{(1)}$ on $[0, 1] \times B$ into an $(r - 1)$-form of class $C^{(1)}$ on B. The latter form is denoted by $\int_0^1 \boldsymbol{\eta}$. If $r = 1$, then $\boldsymbol{\eta} = f\,ds + \boldsymbol{\eta}^1$, where $\boldsymbol{\eta}^1$ involves the differentials $dt^1, \ldots, dt^m$. In this case $\int_0^1 \boldsymbol{\eta} = \int_0^1 f(s, \ )\,ds$, which is of class $C^{(1)}$ on B. Next, if $\boldsymbol{\eta} = ds \wedge \boldsymbol{\theta} = \sum_{[\mu]} \theta_\mu\, ds \wedge dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}}$, then we set

$$\int_0^1 \boldsymbol{\eta} = \sum_{[\mu]} \left(\int_0^1 \theta_\mu\, ds\right) dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}}. \tag{7-19}$$

Finally, any r-form $\boldsymbol{\eta}$ on $[0, 1] \times B$ can be written $\boldsymbol{\eta} = ds \wedge \boldsymbol{\theta} + \boldsymbol{\eta}^1$, where $\boldsymbol{\eta}^1$ involves only the differentials $dt^1, \ldots, dt^m$:

$$\boldsymbol{\eta}^1 = \sum_{[\lambda]} \eta^1_\lambda\, dt^{i_1} \wedge \cdots \wedge dt^{i_r}.$$

We set $\int_0^1 \boldsymbol{\eta} = \int_0^1 ds \wedge \boldsymbol{\theta}$. Using the rules for exterior differentiation, we find that

$$d\boldsymbol{\eta} = d(ds \wedge \boldsymbol{\theta}) + d\boldsymbol{\eta}^1 = -ds \wedge d'\boldsymbol{\theta} + ds \wedge \frac{\partial \boldsymbol{\eta}^1}{\partial s} + d'\boldsymbol{\eta}^1,$$

where d' denotes the differential with respect to $\mathbf{t}$ of a form on $[0, 1] \times B$ and the components of the r-form $\partial \boldsymbol{\eta}^1/\partial s$ are the partial derivatives $\partial \eta^1_\lambda/\partial s$. Therefore

$$\int_0^1 d\boldsymbol{\eta} = -\int_0^1 ds \wedge d'\boldsymbol{\theta} + \int_0^1 ds \wedge \frac{\partial \boldsymbol{\eta}^1}{\partial s} \tag{*}$$

$$= -\int_0^1 ds \wedge d'\boldsymbol{\theta} + \boldsymbol{\eta}^1(1) - \boldsymbol{\eta}^1(0),$$

where $\boldsymbol{\eta}^1(s)$ is the r-form on B with coefficients $\eta^1_\lambda(s,\ )$. Differentiating under the integral sign,

$$\frac{\partial}{\partial t^j}\int_0^1 f\,ds = \int_0^1 \frac{\partial f}{\partial t^j}\,ds, \qquad j = 1, \ldots, m,$$

provided f is of class $C^{(1)}$. Hence

$$d\int_0^1 f\,ds = \sum_{j=1}^m \left(\frac{\partial}{\partial t^j}\int_0^1 f\,ds\right)dt^j = \int_0^1 ds \wedge d'f.$$

Applying this in (7–19), if $\boldsymbol{\eta}$ is of class $C^{(1)}$ we get

$$d\int_0^1 \boldsymbol{\eta} = \sum_{[\mu]}\int_0^1 ds \wedge d'\theta_\mu \wedge dt^{j_1} \cdots \wedge dt^{j_{r-1}} \tag{**}$$

$$= \int_0^1 ds \wedge \left(\sum_{[\mu]} d'\theta_\mu \wedge dt^{j_1} \wedge \cdots \wedge dt^{j_{r-1}}\right) = \int_0^1 ds \wedge d'\boldsymbol{\theta}.$$

From (*) and (**) we get

$$\int_0^1 d\boldsymbol{\eta} + d\int_0^1 \boldsymbol{\eta} = \boldsymbol{\eta}^1(1) - \boldsymbol{\eta}^1(0). \tag{7–20}$$

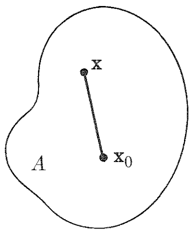


FIGURE 7–16

Now let $\boldsymbol{\omega}$ be an r-form of class $C^{(1)}$ on A. Let $\mathbf{H}$ be a homotopy between transformations $\mathbf{f}$ and $\mathbf{g}$, and let $\boldsymbol{\omega}^\#_{\mathbf{f}}$, $\boldsymbol{\omega}^\#_{\mathbf{g}}$, $\boldsymbol{\omega}^\#_{\mathbf{H}}$ denote the r-forms induced respectively by $\mathbf{f}$, $\mathbf{g}$, and $\mathbf{H}$. Let $\boldsymbol{\eta} = \boldsymbol{\omega}^\#_{\mathbf{H}}$. Then $\boldsymbol{\eta}^1(1) = \boldsymbol{\omega}^\#_{\mathbf{g}}$ and $\boldsymbol{\eta}^1(0) = \boldsymbol{\omega}^\#_{\mathbf{f}}$. Therefore

$$\int_0^1 d\boldsymbol{\omega}^\#_{\mathbf{H}} + d\int_0^1 \boldsymbol{\omega}^\#_{\mathbf{H}} = \boldsymbol{\omega}^\#_{\mathbf{g}} - \boldsymbol{\omega}^\#_{\mathbf{f}}. \tag{7–21}$$

With this formula we can readily deduce a result about closed forms which is called Poincaré's lemma.

Definition. A set A is *star-shaped* if there is a point $\mathbf{x}_0 \in A$ such that for every $\mathbf{x} \in A$ the line segment joining $\mathbf{x}_0$ and $\mathbf{x}$ is contained in A (Fig. 7–16).

Poincaré's lemma. *Let D be a star-shaped open set and let $1 \le r \le n$. Then every closed r-form with domain D is exact.*

Proof. Let $\mathbf{x}_0$ be a point with respect to which D is star-shaped. Let $\mathbf{f}(\mathbf{x}) = \mathbf{x}_0$, $\mathbf{g}(\mathbf{x}) = \mathbf{x}$, $\mathbf{H}(s, \mathbf{x}) = \mathbf{x}_0 + s(\mathbf{x} - \mathbf{x}_0)$, $B = D$. [This homotopy merely shrinks D radially to the point $\mathbf{x}_0$.] Then $\boldsymbol{\omega}^\#_{\mathbf{g}} = \boldsymbol{\omega}$; and since $r > 0$ and $df^i = 0$, $\boldsymbol{\omega}^\#_{\mathbf{f}} = \mathbf{0}$. Since $\boldsymbol{\omega}$ is closed, $d\boldsymbol{\omega}^\#_{\mathbf{H}} = (d\boldsymbol{\omega})^\#_{\mathbf{H}} = \mathbf{0}$. Let $\boldsymbol{\zeta} = \int_0^1 \boldsymbol{\omega}^\#_{\mathbf{H}}$. Then by (7–21), $d\boldsymbol{\zeta} = \boldsymbol{\omega}$. ∎

**Note:* Poincaré's lemma gives only a sufficient condition on D that every closed form be exact. A necessary and sufficient condition can be obtained from DeRham's theorem ([21], Chap. IV or [17], Chap. IV).

Let us state without proof the following version of the theorem. Let $\mathcal{Z}^r(D)$ denote the set of all closed r-forms of class $C^{(\infty)}$ on D. If $\boldsymbol{\omega}$ and $\boldsymbol{\zeta}$ are closed, then $\boldsymbol{\omega} + \boldsymbol{\zeta}$ is closed and $c\boldsymbol{\omega}$ is closed for any scalar c. Thus $\mathcal{Z}^r(D)$ is a vector space over E^1. Similarly, let $\mathcal{E}^r(D)$ denote the vector space consisting of all exact r-forms of the type $\boldsymbol{\omega} = d\boldsymbol{\zeta}$ where $\boldsymbol{\zeta}$ is of class $C^{(\infty)}$ on D. Then $\mathcal{E}^r(D) \subset \mathcal{Z}^r(D)$. According to DeRham's theorem, the quotient vector space $\mathcal{H}^r(D) = \mathcal{Z}^r(D)/\mathcal{E}^r(D)$ is isomorphic to the r-dimensional cohomology group of D with real coefficients. (The homology and cohomology groups of a space are defined in algebraic topology. They contain a great deal of topological information about the space.) In particular, every closed r-form is exact if and only if $\mathcal{H}^r(D) = 0$.

PROBLEMS

1. Let D be the solid torus obtained by rotating the circular disk $(y - a)^2 + z^2 < b^2$, $0 < b < a$, about the z-axis. Let γ be the circular path traversed by the center of the disk. Show that $\int_\gamma (x\,dy - y\,dx)/(x^2 + y^2) \neq 0$. Hence by Corollary 1, γ is not null homotopic in D.
2. Let S be the sphere $x^2 + y^2 + z^2 = a^2$, oriented by the unit exterior normal. Let

$$\boldsymbol{\omega} = \rho^{-3}(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy),$$

$$\rho^2 = x^2 + y^2 + z^2,$$

the domain of $\boldsymbol{\omega}$ being $E^3 - \{\mathbf{0}\}$. Show that:

(a) $\boldsymbol{\omega}$ is closed.
(b) $\int_{S^0} \boldsymbol{\omega} = 4\pi$. [*Hint:* Find $*\boldsymbol{\omega}(\mathbf{x}) \cdot \boldsymbol{\nu}(\mathbf{x})$, where $\boldsymbol{\nu}(\mathbf{x})$ is the exterior normal.]
(c) $E^3 - \{\mathbf{0}\}$ is simply connected.

3. Let $\widetilde{D}$ be star-shaped and let $D = \mathbf{g}(\widetilde{D})$, where $\mathbf{g}$ is a regular flat transformation of class $C^{(2)}$. Show that every closed form with domain D is exact.
4. The *winding number* of a closed curve γ in E^2 about a point (x_0, y_0) not in the trace of γ is

$$w(x_0, y_0) = \frac{1}{2\pi}\int_\gamma \frac{(x - x_0)\,dy - (y - y_0)\,dx}{(x - x_0)^2 + (y - y_0)^2}.$$

Let γ be the positively oriented boundary of a regular domain D.

(a) Show that $w(x_0, y_0) = 1$ if $(x_0, y_0) \in D$. [*Hint:* Apply Green's theorem to

$$D_\epsilon = \{(x, y) \in D : (x - x_0)^2 + (y - y_0)^2 \ge \epsilon\}$$

where $\epsilon < \operatorname{dist}[(x_0, y_0), \operatorname{fr} D]$. Note that $m = 2$ in formula (7–12).]

(b) Show that $w(x_0, y_0) = 0$ if $(x_0, y_0) \notin \operatorname{cl} D$.

5. For $m = 1, 2, \dots$ let h_m be a function of class $C^{(\infty)}$ on E^1 such that $h_m \geq 0$, $\int_{-\infty}^{\infty} h_m\, dx = 1$, $h_m(x) = 0$ whenever $|x| \geq 1/m$. [For instance, we may take $h_m(x) = mh(mx)$, where h is as on p. 253.] Let ψ be a piecewise continuous function on E^1 which is periodic of period 1. Let $\psi_m(x) = \int_{-\infty}^{\infty} \psi(y) h_m(x - y)\, dy = \int_{-\infty}^{\infty} \psi(x + z) h_m(z)\, dz$. Show that:
   - (a) If $|\psi(x)| \leq C$ for every x, then $|\psi_m(\mathbf{x})| \leq C$ for every x and $m = 1, 2, \dots$
   - (b) ψ_m is of class $C^{(\infty)}$ and of period 1, $m = 1, 2, \dots$
   - (c) If $\int_0^1 \psi\, dx = 0$, then $\int_0^1 \psi_m\, dx = 0$, $m = 1, 2, \dots$
   - (d) At each point x_0 of continuity of ψ, $\psi_m(x_0) \to \psi(x_0)$ as $m \to \infty$. [*Hint:* $\psi_m(x_0) - \psi(x_0) = \int_{-1/m}^{1/m} [\psi(x_0 + z) - \psi(x_0)] h_m(z)\, dz$.]

   [*Note:* In the proof of Corollary 2, let ψ be a periodic extension of $g^{i\prime}$, and let $g_m^i(t) = g^i(0) + \int_0^t \psi_m(x)\, dx$, $i = 1, \dots, n$.]