CHAPTER II

Holomorphic Functions, Cauchy's Integral

1. Curvilinear Integrals

1. General theory

We shall revise some of the elementary ideas in the theory of curvilinear integrals in the plane $\mathbf{R}^2$. Let x and y denote the coordinates in $\mathbf{R}^2$.

A *differentiable path is a mapping*

$$t \to \gamma(t) \tag{1.1}$$

of the segment $[a, b]$ into the plane $\mathbf{R}^2$, such that the coordinates $x(t)$ and $y(t)$ of the point $\gamma(t)$ are continuously differentiable functions. We shall always suppose that $a < b$. The *initial point* of γ is $\gamma(a)$ and its *end point* is $\gamma(b)$. If D is an open set of the plane, we say that γ is a differentiable path of the open set D if the function γ takes its values in D.

A *differential form* in an open set D is an expression

$$\omega = P\,dx + Q\,dy$$

whose coefficients P and Q are (real- or complex-valued) continuous functions in D.

If γ is a differentiable path of D and ω a differential form in D, we define the integral $\int_\gamma \omega$ by the formula

$$\int_\gamma \omega = \int_a^b \gamma^*(\omega),$$

where $\gamma^*(\omega)$ denotes the differential form $f(t)\,dt$ defined by

$$f(t) = P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t);$$

in other words, $\gamma^*(\omega)$ is the differential form deduced from ω by the change of variables $x = x(t)$, $y = y(t)$. Thus,

$$\int_\gamma \omega = \int_a^b f(t)\, dt.$$

Consider now a continuously differentiable function $t = t(u)$ for $a_1 \leqslant u \leqslant b_1$ (with $a_1 < b_1$), whose derivative $t'(u)$ is always > 0 and which is such that $t(a_1) = a$, $t(b_1) = b$. The composed mapping of $u \to t(u)$ and the mapping (1.1) is

$$(1.2) \qquad u \to \gamma(t(u)).$$

It defines a differentiable path γ_1. We say that γ_1 is deduced from γ by *change of parameter*. The differential form $f_1(u)\, du$ deduced from ω by the mapping (1.2) is equal to

$$f(t(u))t'(u)\, du,$$

by virtue of the formula giving the derivative of a composed function. The formula for change of variable in an ordinary integral thus gives the equation

$$\int_\gamma \omega = \int_{\gamma_1} \omega.$$

In other words, the curvilinear integral $\int_\gamma \omega$ does not change its value if the differentiable path γ is replaced by another which is deduced from γ by change of parameter. We can, then, denote paths deduced from one another by change of parameter by the same symbol.

Take now a continuously differentiable function $t = t(u)$ defined for $a_1 \leqslant u \leqslant b_1$, but such that $t'(u) < 0$, $t(a_1) = b$, $t(b_1) = a$ (the description of the segment is reversed). We then see that $\int_{\gamma_1} \omega = -\int_\gamma \omega$. We say therefore that we have made a change of parameter in γ which *changes the orientation* of γ; the effect of this is to multiply $\int_\gamma \omega$ by -1.

Subdivide the interval $[a, b]$ described by the parameter t into a finite number of sub-intervals

$$[a, t_1], \quad [t_1, t_2], \quad \ldots, \quad [t_{n-1}, t_n], \quad [t_n, b],$$

where $a < t_1 < t_2 < \ldots < t_{n-1} < t_n < b$. Let γ_i be the restriction of the mapping γ to the i-th of these intervals; it is clear that

$$\int_\gamma \omega = \sum_{i=1}^{n} \left(\int_{\gamma_i} \omega \right).$$

This result leads to a generalization of the idea of a differentiable path. *A piecewise differentiable path is defined to be a continuous mapping*

$$\gamma : [a, b] \to \mathbf{R}^2,$$

such that there exists a subdivision of the interval $[a, b]$ into a finite number of sub-intervals as above, with the property that the restriction of γ to each sub-interval is continuously differentiable. We define

$$\int_\gamma \omega = \sum_{i=1}^{n+1} \left(\int_{\gamma_i} \omega \right).$$

The sum on the right hand side is independent of the decomposition. The initial point of γ_1 is called the initial point of γ and the final point of γ_{n+1} is called the final point of γ. We say that a path is *closed* if its initial and final points coincide.

A closed path γ can also be defined by taking, instead of a real parameter t varying from a to b, a parameter θ which describes the unit circle.

Example. Consider, in the plane $\mathbf{R}^2$, the perimeter (or 'boundary') of a rectangle A whose sides are parallel to the coordinate axes. The rectangle is the set of points (x, y) satisfying

$$a_1 \leqslant x \leqslant a_2, \qquad b_1 \leqslant y \leqslant b_2.$$

Its boundary consists of the four line segments

$$\begin{aligned} x &= a_2, & b_1 &\leqslant y \leqslant b_2, \\ y &= b_2, & a_1 &\leqslant x \leqslant a_2, \\ x &= a_1, & b_1 &\leqslant y \leqslant b_2, \\ y &= b_1, & a_1 &\leqslant x \leqslant a_2. \end{aligned}$$

For this boundary to define a piecewise differentiable closed path γ, it is necessary to stipulate the sense of description chosen. We agree always take the following sense of description:

y increases from b_1 to b_2, along the side $x = a_2$,
x decreases from a_2 to a_1, along the side $y = b_2$,
y decreases from b_2 to b_1, along the side $x = a_1$,
x increases from a_1 to a_2, along the side $y = b_1$.

Thus the integral $\int_\gamma \omega$ is well-defined; it does not depend on the choice of the initial point of γ because it is always equal to the sum of integrals along the four sides, each described in the sense indicated.

2 Primitive of a differential form

LEMMA. *Let* D *be a connected open set of the plane. Any two points* $a \in D$ *and* $b \in D$ *are the initial and final points, respectively, of some piecewise differentiable path in* D. (Briefly this says that a and b can be joined by a piecewise differentiable path).

Proof. Each point $c \in D$ is the centre of a disc contained in D and can be joined to each point of this disc by a piecewise differentiable path contained in D, for instance, a radius. Suppose that $a \in D$ is a given point; if c can be joined to a, then any point sufficiently near to c can also be joined to a because of the previous remark; thus the set E of points of D which can be joined to a is *open*. On the other hand, E is closed in D; because, if $c \in D$ is in the closure of E, c can be joined to some point of E because of previous remarks, so c can be joined to a. By hypothesis, D is connected; the subset E of D is non-empty (as $a \in E$) and is both open and closed, so it must be the whole of D. This completes the proof.

Let D again be a connected open set in the plane and let γ be a piecewise differentiable path contained in D with initial point a and final point b. Let F be a continuously differentiable function in D and consider the differential form $\omega = dF$; then we have the obvious relation

$$\int_\gamma dF = F(b) - F(a). \tag{2.1}$$

It follows from this and the lemma that, *if the differential* dF *is identically zero in* D, *the function* F *is constant in* D.
Given a differential form ω in a connected open set D, we investigate whether or not there is a continuously differentiable function $F(x, y)$ in D such that $dF = \omega$. If $\omega = P\,dx + Q\,dy$, the relation $dF = \omega$ is equivalent to

$$\frac{\partial F}{\partial x} = P, \quad \frac{\partial F}{\partial y} = Q. \tag{2.2}$$

Such a function F, if it exists, is called *a primitive* of the form ω. In this case, any other primitive G is obtained by adding a constant to F since $d(F - G) = 0$.

PROPOSITION 2. 1. *A necessary and sufficient condition that a differential form* ω *has a primitive in* D *is that* $\int_\gamma \omega = 0$ *for any piecewise differentiable closed path* γ *contained in* D.

Proof. 1. The condition is necessary because, if $\omega = dF$, relation (2. 1) shows that $\int_\gamma \omega = 0$ whenever the initial and final points of γ coincide.

2. The condition is sufficient. For, choose a point $(x_0, y_0) \in D$; any point $(x, y) \in D$ can be joined to (x_0, y_0) by a piecewise continuously differentiable path γ contained in D (by the lemma); the integral $\int_\gamma \omega$ does not depend on the choice of γ because the integral of ω round any closed path is zero by hypothesis. Let $F(x, y)$ be the common value of the integrals $\int_\gamma \omega$ along paths γ in D with initial point (x_0, y_0) and final point (x, y). We shall show that the function F so defined in D satisfies relations (2. 2).
Give x a small increment h; the difference

$$F(x + h, y) - F(x, y)$$

is equal to the integral $\int \omega$ along any path contained in D starting at (x, y) and ending at $(x + h, y)$. In particular, let us integrate along the line segment parallel to the x-axis (which is possible if $|h|$ is small enough):

$$F(x + h, y) - F(x, y) = \int_x^{x+h} P(\xi, y)\, d\xi,$$

and consequently, if $h \neq 0$,

$$\frac{F(x + h, y) - F(x, y)}{h} = \frac{1}{h} \int_x^{x+h} P(\xi, y)\, d\xi.$$

As h tends to 0, the right hand side tends to $P(x, y)$ because of the continuity of the function P. Hence we indeed have

$$\frac{\partial F}{\partial x} = P(x, y).$$

We could prove $\frac{\partial F}{\partial y} = Q(x, y)$ similarly. This completes the proof of proposition 2. 1.
Consider in particular the rectangles contained in D whose sides are parallel to the axes (we mean that the rectangle must be entirely contained in D, both its interior and its frontier). If γ is the boundary of such a rectangle, we must have $\int_\gamma \omega = 0$ for the differential form ω to have a primitive in D. This necessary condition is not always sufficient as we shall see later. Nevertheless, it is sufficient when D is 'simply connected' (cf. no. 7). For the moment we shall confine ourselves to proving following:

PROPOSITION 2. 2. *Let* D *be an open disc. If* $\int_\gamma \omega = 0$ *whenever* γ *is the boundary of a rectangle contained in* D *with sides parallel to the axes, then* ω *has a primitive in* D.

Proof. Let (x_0, y_0) be the centre of the disc D and let (x, y) be a general point of D. There are two paths γ_1 and γ_2 starting at (x_0, y_0) and ending at (x, y), each of which is composed of two sides of the rectangle (with sides parallel to the axes) whose opposite corners are (x_0, y_0) and (x, y) [see figure 1]. Thus this rectangle is contained in D and $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$. Let

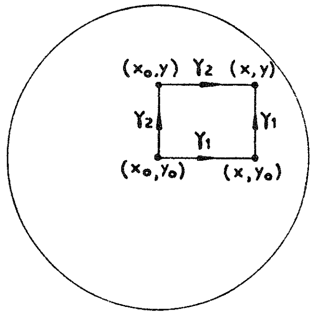

Fig. 1.

$F(x, y)$ be the common value of these two integrals; then we can show as above, that $\frac{\partial F}{\partial x} = P$, $\frac{\partial F}{\partial y} = Q$, which proves the proposition.

3. The Green-Riemann formula

This formula, in some sense, generalizes relation (2. 1) : instead of relating the value of an ordinary integral to values of a function, it relates the value of a double integral to that of a curvilinear one. Let A be a rectangle with sides parallel to the axes, let γ be its boundary and let $P(x, y)$ and $Q(x, y)$ be continuous functions defined in a neighbourhood D of A, the functions having continuous partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$.

The Green-Riemann formula can then be written

(3. 1) $$\int_\gamma P\,dx + Q\,dy = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Proof. We shall prove for instance that

$$\int_\gamma Q\,dy = \iint_A \frac{\partial Q}{\partial x}\,dx\,dy.$$

We know that the double integral of the continuous function $\frac{\partial Q}{\partial x}$ can be calculated as follows :

$$\iint_A \frac{\partial Q}{\partial x}\,dx\,dy = \int_{b_1}^{b_2} dy \left(\int_{a_1}^{a_2} \frac{\partial Q}{\partial x}\,dx\right).$$

However, $\int_{a_1}^{a_2} \frac{\partial Q}{\partial x}\,dx = Q(a_2, y) - Q(a_1, y)$; integrating this with respect to y gives

$$\int_{b_1}^{b_2} Q(a_2, y)\,dy - \int_{b_1}^{b_2} Q(a_1, y)\,dy$$

which is precisely equal to $\int_\gamma Q\,dy$.

This completes the proof.

The Green-Riemann formula is valid for more general domains than rectangles, but we shall leave this question aside for the moment.

Proposition 3. 1. *Let $\omega = P\,dx + Q\,dy$ be a differential form in a connected open set* D, *and suppose that the partial derivatives $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ exist and are continuous in* D. *Then the relation*

(3. 2) $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

is a necessary condition for ω to have a primitive in D; *it is also sufficient if* D *is an open disc.*

Proof. From formula (3.1), condition (3.2) implies that $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle contained in D; if D is an open disc, this implies that ω has a primitive (proposition 2. 2). Conversely, if $\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle A contained in D with sides parallel to the axes, we have

(3. 3) $$\iint_A \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy = 0$$

for any such rectangle A. Moreover, this implies relation (3. 2). For, if the continuous function $\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}$ is not identically zero in D, there will be some point of D in a neighbourhood of which it is > 0, say, and consequently the integral

$$\iint_A \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dx\,dy$$

will also be > 0 for a rectangle A contained in this neighbourhood, contrary to hypothesis (3. 3). Proposition 3. 3 is thus proved.

4. Closed differential forms

Definition. We say that a form $\omega = P\,dx + Q\,dy$, with continuous coefficients P and Q in an open set D, is *closed* if any point $(x_0, y_0) \in D$ has an open neighbourhood in which ω has a primitive. We can assume that such a neighbourhood is a disc with centre (x_0, y_0). Therefore, the results of nos. 2 and 3 immediately imply :

Proposition 4. 1. *A necessary and sufficient condition for a differential form ω with continuous coefficients in D to be closed is that $\int_\gamma \omega = 0$ whenever γ is the boundary of a small rectangle contained (with its interior) in D with sides parallel to the axes. If we also assume that P and Q have continuous partial derivatives of the first order, then (3. 2) is a necessary and sufficient condition for ω to be closed.*

We know from proposition 2. 2 that any closed form in an *open disc* has a primitive. We shall now give an example of a closed form ω in a connected open set D which has no primitive in D.

Proposition 4. 2. *Let D be the open set consisting of all points $z \neq 0$ of the complex plane $\mathbf{C}$. The form $\omega = dz/z$ is closed in D but has no primitive.*

For, in a neighbourhood of each point $z_0 \neq 0$, there is a branch of $\log z$ and this branch is, in the neighbourhood of z_0, a primitive of dz/z. Hence ω is closed. To show that ω has no primitive in D, it is sufficient to find a closed path γ in D such that $\int_\gamma \frac{dz}{z} \neq 0$. In fact, let γ be the unit circle centred at the origin and described in the positive sense. To calculate $\int_\gamma \omega$, we put $z = e^{it}$ with t running from 0 to 2π; we have

$$dz = ie^{it}\,dt, \qquad \frac{dz}{z} = i\,dt,$$

and consequently

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} i\,dt = 2i\pi \neq 0. \tag{4. 1}$$

This completes the proof.

In the preceding example, the form ω is complex. Let us now take the imaginary part of ω. Since

$$\frac{dz}{z} = \frac{dx + i\,dy}{x + iy} = \frac{x\,dx + y\,dy}{x^2 + y^2} + i\frac{x\,dy - y\,dx}{x^2 + y^2},$$

the differential form

$$\varpi = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is closed in the plane with the origin excluded. It has no primitive because we have by (4. 1)

$$\int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2} = 2\pi$$

if γ is the unit circle described in the positive sense. In fact, ϖ is the differential of $\arctan \frac{y}{x}$, which is a *many-valued* function (that is to say with many branches) in the plane with the origin excluded.

5. Study of many-valued primitives

Let ω be a closed form defined in a connected open set D. Although ω has not necessarily a (single-valued) primitive in D, we shall define what is meant by a *primitive of ω along a path* γ of D. Such a path is defined by a *continuous* mapping of the segment $I = [a, b]$ into D; we do not assume differentiability in this context.

Definition. Let $\gamma : [a, b] \to D$ be a path contained in an open set D, and let ω be a closed differential form in D. A continuous function $f(t)$ (t describing $[a, b]$) is called a *primitive of ω along* γ if it satisfies the following condition :

(P) *for any $\tau \in [a, b]$ there exists primitive F of ω in a neighbourhood of the point $\gamma(\tau) \in D$ such that*

$$F(\gamma(t)) = f(t) \tag{5. 1}$$

for t near enough to τ.

Theorem 1. *Such a primitive f always exists and is unique up to addition of a constant.*

Proof. First of all, if f_1 and f_2 are two such primitives, the difference $f_1(t) - f_2(t)$ is, by (5. 1), of the form $F_1(\gamma(t)) - F_2(\gamma(t))$ in a neighbourhood of each $\tau \in [a, b]$; since the difference $F_1 - F_2$ of two primitives of ω is constant, it follows that the function $f_1(t) - f_2(t)$ is constant in a neighbourhood of each point of the segment I. We express this by saying that the function $f_1 - f_2$ is *locally constant*. However, a continuous locally constant function on a *connected* topological space (the segment $I = [q, b]$ in this case) is *constant*. Indeed, for any number u, the set of points of the space where the function takes the value u is both open and closed.

It remains to be proved that there exists a continuous function $f(t)$ satisfying conditions (P). Each point $\tau \in I$ has a neighbourhood (in I) mapped by γ into an open disc where ω has a primitive F. Since I is compact, we can find a finite sequence of points

$$a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b,$$

such that, for each integer i where $0 \leqslant i \leqslant n$, γ maps the segment $[t_i, t_{i+1}]$ into an open disc U_i in which ω has a primitive F_i. The intersection $U_i \cap U_{i+1}$ contains $\gamma(t_{i+1})$ so it is not empty; it is connected, so $F_{i+1} - F_i$ is constant in $U_i \cap U_{i+1}$. We can then, by adding a suitable constant to each F_i, arrange, step by step, that F_{i+1} coincides with F_i in $U_i \cap U_{i+1}$. Then, we let $f(t)$ be the function defined by

$$f(t) = F_i(\gamma(t)) \qquad \text{for} \qquad t \in [t_i, t_{i+1}].$$

It is obvious that $f(t)$ is continuous and satisfies condition (P); the latter is clear when τ is different from the t_i and the reader should verify it when τ is equal to one of them.

Note. Suppose that γ is piecewise differentiable, in other words, that there is a subdivision of I such that the restriction of γ to each sub-interval $[t_i, t_{i+1}]$ is continuously differentiable. Then the integral $\int_\gamma \omega$ is defined; it is by definition

$$\sum_i \left(\int_{\gamma_i} \omega \right).$$

If f is a primitive along γ, we have by formula (2. 1)

$$\int_{\gamma_i} \omega = f(t_{i+1}) - f(t_i),$$

whence, by addition,

$$\int_\gamma \omega = f(b) - f(a). \tag{5. 2}$$

This leads to a definition of $\int_\gamma \omega$ for a *continuous* path γ, without the hypothesis of differentiability of γ : we take relation (5. 2) as the *definition*, which is valid because the right hand side does not depend on the choice of primitive f along γ.

PROPOSITION 5. 1 *If γ is a closed path which does not pass through the origin, $\dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{dz}{z}$ is an integer.*

Proof. $\omega = \dfrac{dz}{z}$ is a closed form. In the proof of theorem 1, we may suppose each F_i to be a branch of $\log z$. Thus $f(b) - f(a)$ is the difference between two branches of $\log z$ at the point $\gamma(a) = \gamma(b)$, and, consequently, is of the form $2\pi i n$, where n is an integer.

COROLLARY. $\dfrac{1}{2\pi}\displaystyle\int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2}$ *is an integer* (the same integer as above).

The quantity $\displaystyle\int_\gamma \frac{x\,dy - y\,dx}{x^2 + y^2}$ is often called the *variation of the argument* of the point $z = x + iy$ when this point describes the path γ (whether γ is closed or not).

6. HOMOTOPY

For simplification, we shall only consider paths parametrized by the segment $I = [0, 1]$.

Definition. We say that two paths

$$\gamma_0 : I \to D \qquad \text{and} \qquad \gamma_1 : I \to D$$

having the same initial points and the same end points (that is to say $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$) *are homotopic* (in D) *with fixed end points*, if there exists a continuous mapping $(t, u) \to \delta(t, u)$ of $I \times I$ into D, such that

$$\begin{cases} \delta(t, 0) = \gamma_0(t), & \delta(t, 1) = \gamma_1(t), \\ \delta(0, u) = \gamma_0(0) = \gamma_1(0), & \delta(1, u) = \gamma_0(1) = \gamma_1(1). \end{cases} \tag{6. 1}$$

For fixed u, the mapping $t \to \delta(t,u)$ is a path γ_u of D with the same initial point as the common initial point of γ_0 and γ_1 and the same end point as their common end point. Intuitively, this path deforms continuously as u varies from 0 to 1, its end points remaining fixed.
There is an analogous definition for two *closed paths* γ_0 and γ_1 : we say that they are homotopic (in D) *as closed paths* if there is a continous mapping $(t, u) \to \delta(t, u)$ of $I \times I$ into D, such that

$$\begin{cases} \delta(t, 0) = \gamma_0(t), & \delta(t, 1) = \gamma_1(t), \\ \delta(0, u) = \delta(1, u) & \text{for all } u, \end{cases} \tag{6. 2}$$

(thus the path γ_u is closed for each u). In particular, we say that a closed path γ_0 is *homotopic to a point* in D if the above holds with $\gamma_1(t)$ a constant function.

Theorem 2. *If γ_0 and γ_1 are two homotopic paths of* D *with fixed end points, then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

for any closed form ω in D.

Theorem 2'. *If γ_0 and γ_1 are closed paths which are homotopic as closed paths then*

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

for any closed form ω.

These two theorems are consequences of a lemma which we shall now state. First of all, here is a definition :

Definition. Let $(t, u) \to \delta(t, u)$ be a continuous mapping of a rectangle

$$(6.3) \qquad a \leqslant t \leqslant b, \qquad a' \leqslant u \leqslant b'$$

into the open set D, and let ω be a closed form in D. A *primitive of ω following the mapping* δ is a *continuous* function $f(t, u)$ in the rectangle satisfying the following condition :
(P') *For any point (τ, υ) of the rectangle, there exists a primitive* F *of ω in a neighbourhood of $\delta(\tau, \upsilon)$ such that*

$$F(\delta(t, u)) = f(t, u)$$

at any point (t, u) sufficiently near to (τ, υ).

Lemma. *Such a primitive always exists and is unique up to addition of a constant,* This lemma is, in some sense, an extension of theorem 1. We shall prove it in an similar way. By using the compactness of the rectangle, we can quadrisect it by subdividing the interval of variation of t by points t_i and that of u by points u_j, in such a way that, for all i, j, the small rectangle, which is the product of the segments $[t_i, t_{i+1}]$, $[u_j, u_{j+1}]$, is mapped by δ into an open disc $U_{i,j}$, in which ω has a primitive $F_{i,j}$.

Keep j fixed; since the intersection $U_{i,j} \cap U_{i+1,j}$ is non-empty (and connected), we can add a constant to each $F_{i,j}$ (j fixed and i variable) in such a way that $F_{i,j}$ and $F_{i+1,j}$ coincide in $U_{i,j} \cap U_{i+1,j}$; we then obtain, for $u \in [u_j, u_{j+1}]$, a function $f_j(t, u)$ such that, for all i, we have

$$f_j(t, u) = F_{i,j}(\delta(t, u)) \qquad \text{when} \qquad t \in [t_i, t_{i+1}].$$

Hence $f_j(t, u)$ is continuous in the rectangle

$$a \leqslant t \leqslant b, \qquad u_j \leqslant u \leqslant u_{j+1},$$

and it is a primitive of ω following the mapping δ_j, the restriction of δ to this rectangle. Each function f_j is defined up to the addition of a constant; we can therefore, by induction on j, choose these additive constants in such a way that the functions $f_j(t, u)$ and $f_{j+1}(t, u)$ are equal when $u = u_{j+1}$. Finally, let $f(t, u)$ be the function defined in the rectangle (6.3) by the condition that, for all j, we have

$$f(t, u) = f_j(t, u) \qquad \text{when} \qquad u \in [u_j, u_{j+1}].$$

This is a continuous function which satisfies conditions (P') and is indeed a primitive of ω following the mapping δ. The lemma is thus proved.

Proof of theorem 2. Let δ be a continuous mapping satisfying conditions (6.1) and let f be a primitive of ω following δ. It is obvious that f is a constant on the vertical sides $t = 0$ and $t = 1$ of the rectangle I × I. Thus we have

$$f(0, 0) = f(0, 1), \qquad f(1, 0) = f(1, 1)$$

and, since

$$\int_{\gamma_0} \omega = f(1, 0) - f(0, 0), \qquad \int_{\gamma_1} \omega = f(1, 1) - f(0, 1),$$

theorem 2 is proved.

The proof of theorem 2' is completely analogous; one uses a mapping δ satisfying (6.2).

7. Primitives in a simply connected open set

Definition. We say that D is *simply connected* if it is connected and if in addition any closed path in D is homotopic to a point in D.

Theorem. 3. *Any closed differential form ω in a simply connected open set* D *has a primitive in* D.

For, from theorem 2', we have $\int_\gamma \omega = 0$ for any closed path γ contained in D, which implies by proposition 2.1 that ω has a primitive in D.

In particular, in any simply connected open set not containing 0, the closed form dz/z has a primitive; in other words, log z *has a branch in any simply connected open set which does not contain* 0.

Examples of simply connected open sets. We say that a subset E of the plane is *starred* with respect to one of its points a if, for any point $z \in E$, the line segment joining a to z lies in E.

Any open set D *which is starred with respect to one of its points* a *is simply connected* : for, D is obviously connected; moreover, for each real number u between 0 and 1, the homothety of centre a and factor u transforms D into itself; as u decreases from 1 to 0, this homothety defines a homotopy of any closed curve to a point.

In particular, a *convex* open set D is *simply connected.* For, a convex open set is starred with respect to any of its points.

In contrast, the plane with the origin excluded is *not* simply connected : for example, the circle $|z| = 1$ is not homotopic to a point in $\mathbf{C} - \{0\}$ since the integral $\int \frac{dz}{z}$ of the closed form $\frac{dz}{z}$ along this circle is not zero (cf. relation (4.1)).

The reader is invited to prove the equivalence of the following four properties (for a connected open set D) as an exercise :

a) D is simply connected;

b) any continuous mapping of the circle $|z| = 1$ into D can be extended to a continuous mapping of the disc $|z| \leqslant 1$ into D;

c) any continuous mapping of the boundary of a square into D can be extended to a continuous mapping of the square itself into D.

d) if two paths of D have the same end points, then they are homotopic with fixed end points.

8. The index of a closed path

Definition. Let γ be a closed path in the plane $\mathbf{C}$ and let a be a point of $\mathbf{C}$ which does not belong to the image of γ. The *index* of γ with respect to a, denoted by $I(\gamma, a)$, is defined to be the value of the integral

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}. \tag{8.1}$$

Proposition 5.1 gives that the index $I(\gamma, a)$ is an *integer.* By referring back to the definitions, we see that, in order to calculate the index, we must find a continuous complex-valued function $f(t)$ defined for $0 \leqslant t \leqslant 1$ and such that

$$e^{f(t)} = \gamma(t) - a;$$

then we have

$$I(\gamma, a) = \frac{f(1) - f(0)}{2\pi i}.$$

Properties of the index

1) *If the point a is fixed, the index $I(\gamma, a)$ remains constant when the closed path γ is continuously deformed without passing through the point a.* This follows directly from theorem 2′ of no. 6.

2) *If the closed path γ is fixed, the index $I(\gamma, a)$ is a locally constant function of a when a varies in the complement of the image of γ.* The proof is the same as for 1). It follows that $I(\gamma, a)$ is a function of a which is *constant* in each connected component of the complement of the image of γ.

3) *If the image of γ is contained in a simply connected open set* D *which does not contain the point a, then the index $I(\gamma, a)$ is zero.* For, the closed path γ can then be deformed to a point while remaining in D, thus it never passes through a; it is sufficient now, to use 1).

4) *If γ is a circle described in the positive sense* (i.e. *in the sense such that* $I(\gamma, 0) = +1$), *the index $I(\gamma, a)$ is equal to 0 for a outside the circle and equal to 1 for a inside the circle.* The case when a is outside the circle is covered by 3); when a is inside the circle, it is sufficient to examine the case where a is the centre of the circle because of 2); so, we apply relation (4.1).

Proposition 8.1. *Let f be a continuous mapping of the closed disc $x^2 + y^2 \leqslant r^2$ into the plane $\mathbf{R}^2$ and let γ be the restriction of f to the circle $x^2 + y^2 = r^2$. If a point a of the plane does not belong to the image of γ and if the index $I(\gamma, a)$ is $\neq 0$, then f takes the value a at least once in the open disc $x^2 + y^2 < r^2$.*

We prove this by *reductio ad absurdum* supposing that f does not take the value a. The restriction of f to concentric circles of centre 0 defines a continuous deformation of the closed path γ to a point. Consequently, the integral $\int_\gamma \frac{dz}{z-a}$ is zero, which contradicts the hypothesis.

Definition. Let γ_1 and γ_2 be two closed paths which do not pass through the origin 0. The *product* of these two paths means the closed path defined by the mapping

$$t \to \gamma_1(t) \cdot \gamma_2(t),$$

where the dot means multiplication of the complex numbers $\gamma_1(t)$ and $\gamma_2(t)$.

Proposition 8.2. *The index, with respect to the origin, of the product of two closed paths, which do not pass through 0, is equal to the sum of the indices of each of these closed paths.* In other words,

$$I(\gamma_1\gamma_2, 0) = I(\gamma_1, 0) + I(\gamma_2, 0).$$

For, let $f_1(t)$ and $f_2(t)$ be two f continuous complex-valued functions such that

$$e^{f_1(t)} = \gamma_1(t), \qquad e^{f_2(t)} = \gamma_2(t).$$

Let $\gamma(t) = \gamma_1(t).\gamma_2(t)$ be the product of the two closed curves; the function $f(t) = f_1(t) + f_2(t)$ satisfies

$$e^{f(t)} = \gamma(t)$$

and we have

$$I(\gamma, 0) = \frac{f(1) - f(0)}{2\pi i} = \frac{f_1(1) - f_1(0)}{2\pi i} + \frac{f_2(1) - f_2(0)}{2\pi i} = I(\gamma_1, 0) + I(\gamma_2, 0),$$

which completes the proof.

Proposition 8. 3. *Let γ and γ_1 be two closed paths in the plane* **C**. *If γ never takes the value 0 and if we always have $|\gamma_1(t)| < |\gamma(t)|$, then the mapping $t \to \gamma(t) + \gamma_1(t)$ never takes the value 0 and*

$$I(\gamma + \gamma_1, 0) = I(\gamma, 0).$$

For, we can write

$$\gamma(t) + \gamma_1(t) = \gamma(t).\left(1 + \frac{\gamma_1(t)}{\gamma(t)}\right);$$

the closed path $t \to 1 + \frac{\gamma_1(t)}{\gamma(t)}$ has zero index with respect to the origin because it is contained in the open disc of centre 1 and radius 1. Thus the closed path $\gamma + \gamma_1$ is the product of two closed paths γ and $1 + \frac{\gamma_1}{\gamma}$, and by applying proposition 8. 2, we obtain proposition 8.3.

9. Complements : oriented boundary of a compact set

Lemma. *If a path γ is continuously differentiable and if its derivative γ' is everywhere $\neq 0$, then, in a neighbourhood of each value of the parameter t, the mapping $t \to \gamma(t)$ is injective and its image cuts the plane (locally) into two regions.*

The exact meaning of this statement will be made clear in the proof which follows. Let $t \to \gamma(t)$ be a continuously differentiable mapping of the segment $[a, b]$ into the plane $\mathbf{R}^2$ and let the derivative $\gamma'(t)$ be $\neq 0$ for all values of t. The coordinates x, y of the point $\gamma(t)$ are then continuously differentiable functions $\gamma_1(t)$, $\gamma_2(t)$ and their derivatives $\gamma_1'(t)$, $\gamma_2'(t)$ do not vanish simultaneously. The implicit function theorem shows then that, if t_0 is an interior point of the interval (that is, if $a < t_0 < b$) and if we write $x_0 = \gamma_1(t_0)$, $y_0 = \gamma_2(t_0)$, there exists a continuously differentiable mapping $(t, \omega) \to \delta(t, u)$ of an open neighbourhood U of the point $(t_0, 0)$ onto an open neighbourhood V of point (x_0, y_0), which satisfies the following conditions :

(i) $\delta(t, 0) = \gamma(t)$;

(ii) δ is a homeomorphism of U on V whose Jacobian is > 0 at each point of U (thus δ preserves ' orientation '). Thus V is mapped homeomorphically by the inverse homeomorphism of δ onto U, the points of the path γ going onto the points of the line $u = 0$. The points of V complementary to γ are then partitioned into two open sets V^+ and V^- : that

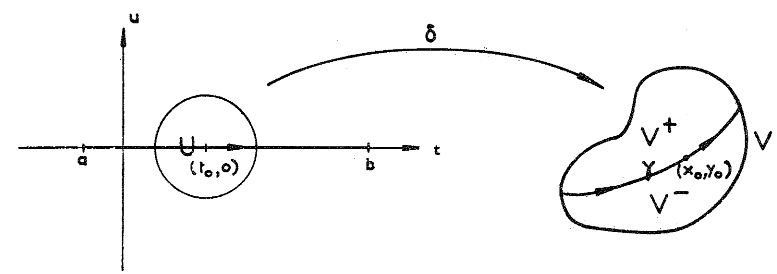

Fig. 2.

for which u is > 0 and that for which u is < 0. If we take U to be an open disc of centre $(t_0, 0)$, then the open sets V^+ and V^- are connected. Thus the path γ splits the open set V into two connected components, which completes the proof of the lemma.

Definition. Let K be a compact subset of the plane **C**, and let $\Gamma = \{\Gamma_i\}_{i \in I}$ be a finite set of *closed* piecewise differentiable paths. We say that Γ is the *oriented boundary* of the compact set K if the following conditions are satisfied :

(BO 1) each mapping $t \to \Gamma_i(t)$ takes any two distinct points into distinct points, except for the initial and final points of the defining segment, and, moreover, the images of the various Γ_i are disjoint and their union is the *frontier* of K;

(BO 2) if γ is a differentiable path of any of the Γ_i, its derivative $\gamma'(t)$ is always $\neq 0$, and, if t_0 is an interior point of the defining interval of γ and the open set V of the previous lemma is chosen to be sufficiently small, then V^- *does not meet* K *while* V^+ *is contained in the interior of* K.

Condition (BO2) is expressed intuitively by saying that, when γ is described in the direction of t increasing, the interior points of K are always *on the left*, whereas the points in the complement of K are on the right.

Example. Take K to be a (closed) rectangle whose sides are parallel to the axes, then the perimeter of this rectangle, as defined at the end of *n*° 1, is the *oriented boundary of* K.

We shall admit, without proof, that the Green-Riemann formula holds for the oriented boundary Γ of a compact set K. A precise statement of the formula is that, if $\omega = P\,dx + Q\,dy$ is a differential form with continuously differentiable coefficients in an open set containing the compact set K, then

$$(9.1)\qquad \int_\Gamma \mathrm{P}\,dx + \mathrm{Q}\,dy = \iint_{\mathrm{K}} \left(\frac{\partial \mathrm{Q}}{\partial x} - \frac{\partial \mathrm{P}}{\partial y}\right) dx\,dy.$$

(The notation $\int_\Gamma$ means $\sum_i \int_{\Gamma_i}$ where Γ_i are the closed paths of Γ).
In particular, *if the form* ω *is closed in* D, *we have the relation*

$$(9.2)\qquad \int_\Gamma \omega = 0$$

whenever Γ *is the oriented boundary of a compact subset of* D.

2. Holomorphic Functions; Fundamental Theorems

1. Revision of differentiable functions

Let D be an open set of the plane $\mathbf{R}^2$ and let $f(x, y)$ be a real- or complex-valued function defined in D. We say that f is *differentiable* at the point $(x_0, y_0) \in \mathrm{D}$ if there is a linear function $ah + bk$ of the real variables h and k, such that

$$(1.1)\qquad f(x_0 + h, y_0 + k) - f(x_0 y_0) = ah + bk + \alpha\sqrt{h^2 + k^2},$$

for all sufficiently small values of h and k; α is a (real-or complex-valued) function of h and k whose absolute value tends to 0 when $\sqrt{h^2 + k^2}$ tends to 0. If f is differentiable at the point (x_0, y_0), the (real or complex) constants a and b are uniquely determined and are equal to the partial derivatives

$$a = \frac{\partial f}{\partial x}(x_0, y_0), \qquad b = \frac{\partial f}{\partial y}(x_0, y_0).$$

Recall that the existence of the partial derivatives of f at the point (x_0, y_0) is not sufficient for the function to be differentiable at this point; but if f has partial derivatives at every point sufficiently near to (x_0, y_0) and if these partial derivatives are continuous functions at the point (x_0, y_0), then f is differentiable at this point. A function which has continuous partial derivatives in an open set D is said to be continuously differentiable in D.

2. Condition for holomorphy

Let D be an open subset of the complex plane $\mathbb{C}$ and let f be a function of the complex variable $z = x + iy$ defined in D.

Definition. We say that $f(z)$ is *holomorphic* at the point $z_0 \in \mathrm{D}$ if

$$(2.1)\qquad \lim_{\substack{u \to 0 \\ u \neq 0}} \frac{f(z_0 + u) - f(z_0)}{u} \quad \text{exists}$$

(u denotes a variable complex number). This is the same as saying that f has a *derivative* with respect to the complex variable at the point z_0. We say that f is holomorphic in the open set D if it holomorphic at each point of D.

Condition (2.1) can also be written

$$(2.2)\qquad f(z_0 + u) - f(z_0) = cu + \alpha(u)\,|u|,$$

where $\alpha(u)$ tends to 0 as u tends to 0; c is the derivative $f'(z_0)$. Since $z = x + iy$, relation (2.2) can also be written

$$(2.3)\qquad f(x_0 + h, y_0 + k) - f(x_0, y_0) = c(h + ik) + \alpha(h, k)\sqrt{h^2 + k^2}.$$

This shows that f, considered as a function of two real variables x and y, is differentiable and that

$$a = c, \qquad b = ic,$$

where a and b are the constants in relation (1.1). Thus we have $\frac{\partial f}{\partial x} = c$, and $\frac{\partial f}{\partial y} = ic$, whence

$$(2.4)\qquad \frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0.$$

Conversely, let f be a differentiable function of the real variables x and y satisfying (2.4). Then, relation (1.1) implies (2.3) with $c = a$ and $ic = b$. Thus, f is holomorphic at the point $z_0 = x_0 + iy_0$. We have, in fact, proved the following proposition :

Proposition 2.1. *For f to be holomorphic at a point, it is necessary and sufficient that f, considered as a function of the real variables x and y, is differentiable at this point and that relation* (2.4) *holds between the partial derivatives of f at this point.*

We express relation (2.4) more explicitly : if we put $f = \mathrm{P} + i\mathrm{Q}$, where P and Q are real functions, then we obtain the Cauchy conditions

$$(2.5)\qquad \frac{\partial \mathrm{P}}{\partial x} = \frac{\partial \mathrm{Q}}{\partial y}, \qquad \frac{\partial \mathrm{P}}{\partial y} = -\frac{\partial \mathrm{Q}}{\partial x}.$$

3. Introduction of the variables z and $\bar{z}$

Let f be a (real- or complex-valued) differentiable function of the real variables x and y. Consider the differential

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy. \tag{3. 1}$$

The particular functions $z = x + iy$ and $\bar{z} = x - iy$ have differentials

$$dz = dx + i\,dy, \qquad d\bar{z} = dx - i\,dy; \tag{3. 2}$$

thus we have conversely

$$dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z}). \tag{3. 3}$$

By substituting this in (3. 1) we obtain the equation

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) d\bar{z}.$$

This leads us to introduce the symbols

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{3. 4}$$

With this notation, we obtain the equation

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z}. \tag{3. 5}$$

Condition (2. 4), which expresses that f is a holomorphic function of the complex variable z, can now be written

$$\frac{\partial f}{\partial \bar{z}} = 0. \tag{3. 6}$$

In other words, a necessary and sufficient condition for f to be holomorphic is that the coefficient of $d\bar{z}$ is zero in the expression (3. 5) for the differential df. Or again: df must be proportional to dz, the coefficient of proportionality being simply the derivative $f'(z)$.

We shall apply this to prove the following result: *Let f be a holomorphic function in a connected open set D; if the real part of f is constant, then f is constant.*

For, the real part $\mathrm{Re}(f)$ is simply $\frac{1}{2}(f + \bar{f})$; by hypothesis $d(f + \bar{f}) = 0$ in D, which can be written

$$\frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar{z}} d\bar{z} + \frac{\partial \bar{f}}{\partial z} dz + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} = 0.$$

But, since f is is holomorphic, we have $\frac{\partial f}{\partial \bar{z}} = 0$; by passing to the complex conjugate, we have $\frac{\partial \bar{f}}{\partial z} = 0$. Hence,

$$\frac{\partial f}{\partial z} dz + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} = 0.$$

However, an expression $a dz + b d\bar{z}$ can only be identically zero if the coefficients a and b are zero, which gives $\frac{\partial f}{\partial z} = 0$, $\frac{\partial \bar{f}}{\partial \bar{z}} = 0$. Thus, $df = 0$, and f is constant in D.

We deduce from this that, *if f is holomorphic and $\neq 0$ in a connected open set D and if either $\log|f|$ is constant or $\arg f$ is constant, then f is constant.*

For, consider the function

$$g(z) = \log f(z) = \log|f(z)| + i \arg f(z).$$

We stay in some neighbourhood of the point z_0 and we choose a branch of the argument; g is holomorphic and its real (or imaginary) part is constant. Thus g is constant in some neighbourhood of z_0. Thus $f = e^g$ is locally constant in D and is consequently constant since D is connected.

4. Cauchy's theorem

Theorem 1. *If $f(z)$ is holomorphic in an open set D of the complex plane, then the differential form $f(z)dz$ is closed in D.*

In view of the importance of this theorem, we shall give two proofs:

First proof. This proof requires an extra hypothesis. We suppose that the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous in D. (In fact the second proof shows that this hypothesis is automatically satisfied when f is holomorphic.) To verify that the differential form $f(z)\,dz = f(z)\,dx + if(z)\,dy$ is closed, it is sufficient, by the Green-Riemann formula (§ 1, formula (3. 1)), to verify that

$$\frac{\partial f}{\partial y} = i\frac{\partial f}{\partial x}.$$

However, this is precisely condition (2. 4) expressing that f is holomorphic, and the proof is completed.

Second proof. This proof, unlike the first, does not need any additional hypothesis, but it requires a more subtle argument. To show that $f(z)dz$

is closed, we must prove that the integral $\int_\gamma f(z)\, dz$ is zero along the boundary γ of any rectangle R contained (with its interior) in D. To this end, we put *a priori*

$$\int_\gamma f(z)\, dz = \alpha(R). \tag{4.1}$$

Divide the rectangle R into four equal rectangles by dividing each side into two equal parts.

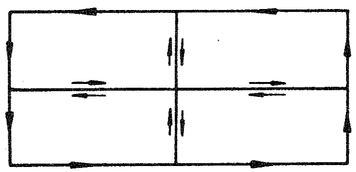

Fig. 3.

Let γ_i be the (oriented) boundaries of the four small rectangles ($i = 1, 2, 3, 4$) It is easily verified (cf. fig. 3) that

$$\int_\gamma f(z)\, dz = \sum_{i=1}^{4} \int_{\gamma_i} f(z)\, dz = \sum_{i=1}^{4} \alpha(R_i).$$

Thus among there four rectangles there is at least one such that $|\alpha(R_i)| \geqslant \frac{1}{4}|\alpha(R)|$. Call this rectangle $R^{(1)}$. Now divide the rectangle $R^{(1)}$ into four equal rectangles at least one of which, say $R^{(2)}$, will satisfy the condition $|\alpha(R^{(2)})| \geqslant \frac{1}{4^2}|\alpha(R)|$. We can repeat this operation indefinitely to obtain a sequence of rectangles each included in the previous one; the k^{th} rectangle $R^{(k)}$ will have sides 2^k times smaller than those of R and its area will then be 4^k times smaller than that of the rectangle R. If $\gamma(R^{(k)})$ denotes the oriented boundary of the rectangle $R^{(k)}$, then

$$\left|\int_{\gamma(R^{(k)})} f(z)\, dz\right| \geqslant \frac{1}{4^k}|\alpha(R)|. \tag{4.2}$$

By the Cauchy criterion of convergence, there is a unique point z_0 common to all the rectangles $R^{(k)}$. Obviously $z_0 \in D$. Thus $f(z)$ is holomorphic at the point z_0, and, consequently,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \varepsilon(z)\,|z - z_0|$$

with

$$\lim_{z \to z_0} \varepsilon(z) = 0.$$

We deduce

$$\left\{\begin{aligned} \int_{\gamma(R^{(k)})} f(z)\, dz = f(z_0) \int_{\gamma(R^{(k)})} dz + f'(z_0) \int_{\gamma(R^{(k)})} (z - z_0)\, dz \\ + \int_{\gamma(R^{(k)})} \varepsilon(z)\, |z - z_0|\, dz. \end{aligned}\right. \tag{4.3}$$

On the right hand side of (4.3), the first two integrals are zero and the third is negligible compared with the area of the rectangle $R^{(k)}$ as k increases indefinitely; it is then negligible compared with $\frac{1}{4^k}$. Comparing this with (4.2) shows that we must have $\alpha(R) = 0$; consequently by the definition of $\alpha(R)$, we have $\int_\gamma f(z)\, dz = 0$. This completes the proof.

COROLLARY 1. *A holomorphic function $f(z)$ in D has locally a primitive, which is holomorphic.*

This statement means that any point of D has an open neighbourhood in which f has a holomorphic primitive. The local existence of a primitive follows from the definition of a closed form; and this local primitive is indeed holomorphic because it has f as its derivative.

COROLLARY 2. *If $f(z)$ is holomorphic in D, then $\int_\gamma f(z)\, dz = 0$ for any closed path γ of D which is homotopic to a point in D.*

This follows from theorem 1 above and theorem 2′ of § 1, no. 6.

Generalization. We shall prove theorem 1 again with less strict conditions.

THEOREM 1′. *Let $f(z)$ be an continuous function in a open set D, which is holomorphic at every point of D except perhaps at the points of a line Δ parallel to the real axis. Then the form $f(z)\, dz$ is closed. In particular, if f is holomorphic at any point of D except perhaps at some isolated points, then the form $f(z)\, dz$ is closed.*

Proof. We must prove that the integral $\int_\gamma f(z)\, dz$ is zero for the boundary γ of any rectangle contained in D. However, this is obvious if the rectangle does not intersect the line Δ. Suppose that the rectangle has a side contained in Δ and let $u, u + a, u + ib, u + a + ib$ be the four corners of the rectangle, u and $u + a$ being on the line Δ; a and b are real, and we assume, say, that $b > 0$. Let $R(\varepsilon)$ be the rectangle with corners

$$u + i\varepsilon, \quad u + a + i\varepsilon, \quad u + ib, \quad u + a + ib,$$

ε being a very small number > 0; the integral $\int f(z)\,dz$ is zero round the boundary of $R(\varepsilon)$; however, as ε tends to 0, this integral tends to the integral round the boundary γ of the rectangle R. Thus $\int_\gamma f(z)\,dz = 0$. Finally, if the line Δ meets the rectangle without containing one of its horizontal sides, the line Δ splits R into two rectangles R′ and R″ and the integral $\int f(z)\,dz$ is zero when taken round the boundaries of either R′ or R″, because of the previous remarks; however, the sum of these integrals is equal to the integral $\int f(z)\,dz$ round the boundary of R. This completes the proof.

5. Cauchy's integral formula

Theorem 2. *Let f be a holomorphic function in an open set D. Let $a \in D$ and let γ be a closed path of D which does not pass through a and which is homotopic to a point in D. Then,*

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z-a} = I(\gamma, a)f(a), \tag{5.1}$$

where $I(\gamma, a)$ denotes the index of the closed path γ with respect ot a (cf. § 1, no. 8).

Proof. Let $g(z)$ be the function defined in D by

$$\begin{cases} g(z) = \dfrac{f(z)-f(a)}{z-a} & \text{for} \quad z \neq a, \\ g(z) = f'(a) & \text{for} \quad z = a; \end{cases}$$

this function g is continuous because of the definition of the derivative. It is holomorphic at any point of D except the point a. By theorem 1′, we have

$$\int_\gamma \frac{f(z)-f(a)}{z-a}\,dz = 0.$$

However,

$$\int_\gamma \frac{f(a)\,dz}{z-a} = 2\pi i\, I(\gamma, a)f(a),$$

by the definition of the index. This proves relation (5.1).

Example. Let f be a holomorphic function in some neighbourhood of a closed disc and let γ be the boundary of the disc described in the positive sense. Then,

$$\int_\gamma \frac{f(z)\,dz}{z-a} = \begin{cases} 2\pi i\, f(a) & \text{if } a \text{ is inside the disc,} \\ 0 & \text{if } a \text{ is outside the disc.} \end{cases}$$

6. Taylor expansion of a holomorphic function

Theorem 3. *Let $f(z)$ he a holomorphic function in the open disc $|z| < \rho$; then f can be expanded as a power series in this disc.*

This means that there exists a power series $S(X) = \sum_{n \geq 0} a_n X^n$ whose radius of convergence is $\geq \rho$ and whose sum $S(z)$ is equal to $f(z)$ for $|z| < \rho$.

Proof. Let r be $< \rho$. We shall find a power series which converges normally to $f(z)$ for $|z| \leq r$. This series will be independent of r because of the uniqueness of the power series expansion of a function in a neighbourhood of 0. The theorem will then be proved.

Choose an r_0 such that $r < r_0 < \rho$. We shall apply the integral formula of theorem 2 by taking γ to be the circle of radius r_0 centred at 0 described in the positive sense:

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(t)\,dt}{t-z} \quad \text{for} \quad |z| \leq r.$$

The function $\dfrac{1}{t-z}$ which occurs under the integral sign can be expanded as a series since $|z| < |t|$. Explicitly,

$$\frac{1}{t-z} = \frac{1}{t}\,\frac{1}{1-z/t} = \frac{1}{t}\left(1 + \frac{z}{t} + \cdots + \frac{z^n}{t^n} + \cdots\right);$$

consequently,

$$f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n \geq 0} z^n \frac{f(t)}{t^{n+1}}\,dt.$$

The series converges normally for $|z| \leq r$ and $|t| = r_0$. We can therefore |ntegrate term by term and we obtain a normally convergent series for $|z| \leq r$:

$$f(z) = \sum_{n \geq 0} a_n z^n,$$

where the coefficients are given by the integrals

$$a_n = \frac{1}{2\pi i}\int_{|t|=r_0} \frac{f(t)\,dt}{t^{n+1}}. \tag{6.1}$$

Hence we have proved theorem 3.

Comment. Theorem 3 shows that any holomorphic function in an open set D is *analytic* in D. Conversely, any analytic function in D is holomorphic in D since we know that analytic functions have derivatives. Hence, for functions of a *complex* variable, there is an equivalence between *holo-*

morphy and *analyticity*. If we apply the known results for analytic functions to holomorphic functions, we see that *a holomorphic function is infinitely differentiable* and, in particular, is continuously differentiable, and that *the derivative of a holomorphic function is holomorphic.*

7. Morera's theorem

Theorem 4. (Converse of theorem 1). *Let $f(z)$ be a continuous function in an open set* D. *If the differential form $f(z)\,dz$ is closed, then the function $f(z)$ is holomorphic in* D.

For, f has a primitive g locally. This primitive is holomorphic, and $f = g'$ is the derivative of a holomorphic function, so is itself holomorphic from the above remarks.

Corollary. *If $f(z)$ is continuous in* D *and holomorphic at all points of* D *except perhaps at the points of some line Δ, then f is holomorphic at all points of* D *without exception.*

For, we can suppose Δ to be parallel to the real axis, by rotating if necessary. By theorem 1', the form $f(z)\,dz$ is closed. Thus by theorem 4, f is holomorphic at all points of D.

We see then that theorem 1' was only an apparent generalization of theorem 1. However, we needed to establish it for technical reasons.

8. Alternative form of Cauchy's integral formula

Theorem 5. *Let Γ be the oriented boundary of a compact subset* K *of an open set* D *and let $f(z)$ be a holomorphic function in* D. *Then,*

$$\int_\Gamma f(z)\,dz = 0;$$

if, moreover, a is an interior point of K, *then*

$$\int_\Gamma \frac{f(z)\,dz}{z-a} = 2\pi i f(a). \tag{8. 1}$$

Proof. The first assertion follows from relation (9. 1) of § 1. To prove the second assertion, we consider a small open disc S centred at a whose closure is in the interior of K. The oriented boundary of the compact set K — S is composed of Γ and the frontier-circle of S described in the negative sense. We shall say that this oriented boundary is the *difference* of Γ and the frontier-circle γ of S taken in the positive sense.

By applying the first part of theorem 5 to the compact set K — S and the function $\dfrac{f(z)}{z-a}$, which is holomorphic in D — $\{a\}$, we obtain

$$\int_\Gamma \frac{f(z)\,dz}{z-a} = \int_\gamma \frac{f(z)\,dz}{z-a},$$

which, along with theorem 2, gives relation (8. 1).

9. Schwarz' principle of symmetry

We have seen (corollary to theorem 4) that, if $f(z)$ is continuous in an open set D and holomorphic at any point of D except perhaps at points on the real axis, then f is holomorphic at all points of D without exception. Consider, then, a non-empty, connected, open set D which is *symmetric with respect ot the real axis*; let D' be the intersection of D with the closed half-plane $y \geqslant 0$ and let D'' be the intersection of D with the half-plane $y \leqslant 0$. Suppose we are given a function $f(z)$ which is continuous in D', which takes real values at the points of the real axis, and which is holomorphic at points of D' where $y > 0$. We shall show that *there is a holomorphic function in* D *which extends f*; such a function is unique by the principle of analytic continuation (cf. chap. 1, § 4, no. 3).

Consider the function $g(z)$ defined in D'' by the equation

$$g(z) = \overline{f(\bar{z})}.$$

This function is continuous in D'' and it can quickly be shown that it is holomorphic at any point of D'' not lying on the real axis. The function $h(z)$ which is equal to $f(z)$ in D' and $g(z)$ in D'' is continuous in D and holomorphic at all points of D not lying on the real axis. It is therefore holomorphic at all points of D without exception.

Note that the function h takes complex conjugate values (that is, symmetric values with respect to the real axis) at pairs of points of D which are symmetric with respect to the real axis. This is why the preceding construction is called the " principle of symmetry ".

Exercises

1. a) Let γ be a piecewise differentiable path and let $\bar\gamma$ be its image under the mapping $z \to \bar{z}$ (symmetry with respect to the real axis.) Show that, if $f(z)$ is a continuous function on γ, the function $z \to \overline{f(\bar{z})}$ is continuous on $\bar\gamma$ and that

$$\overline{\int_\gamma f(z)\,dz} = \int_{\bar\gamma} \overline{f(\bar{z})}\,dz.$$

(b) In particular, if γ is the unit circle described in the positive sense, then

$$\overline{\int_\gamma f(z)\,dz} = -\int_\gamma \overline{f(z)}\,\frac{dz}{z^2}.$$

2. Let γ be a continuous path (not necessarily piecewise differentiable). Show that

$$\int_\gamma (\omega_1 + \omega_2) = \int_\gamma \omega_1 + \int_\gamma \omega_2,$$
$$\int_\gamma a\omega = a\int_\gamma \omega,$$

if ω_1, ω_2, ω are closed forms and $a \in \mathbf{C}$. (For the definition of $\int_\gamma \omega$, see *Note* § 1, no. 5.)

3. Let γ be a piecewise differentiable path, whose image is contained in an open set D, and let $\varphi(z)$ be a holomorphic function in D taking values in an open set Δ (of the plane of the complex variable w). Show that $\Gamma = \varphi \circ \gamma$ is a piecewise differentiable path and that, for any continuous function $f(\omega)$,

$$\int_\Gamma f(w)\,dw = \int_\gamma f(\varphi(z))\varphi'(z)\,dz.$$

Is this formula still true when γ is no longer necessarily differentiable?

4. Let γ be the (differentiable) path $t \to \gamma(t) = re^{it}$, $0 \leqslant t \leqslant 2\pi$, and let γ_n be the path $t \to \gamma_n(t) = (1 - 1/n)re^{it}$, with t varying over the same interval. Show that, if $f(z)$ is continuous in the closed disc $|z| \leqslant r$, then

$$\int_\gamma f(z)\,dz = \lim_{n\to\infty} \int_{\gamma_n} f(z)\,dz.$$

5. Show that, if $f(z)$ is *continuous* in the closed disc $|z| \leqslant r$ and holomorphic in the open disc $|z| < r$, then

$$f(z) = \frac{1}{2\pi i}\int_{|t|=r} \frac{f(t)}{t - z}\,dt \quad \text{for all} \quad |z| < r,$$

where the integral is taken in the positive sense.

6. Find a path $t \to \gamma(t)$ with t varying in $[0, 2\pi]$, having the ellipse $x^2/a^2 + y^2/b^2 = 1$ in the plane $\mathbf{R}^2$ $(a, b > 0)$ as image. Calculate the integral $\int_\gamma \frac{dz}{z}$ in two different ways, and deduce that

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2 t + b^2\sin^2 t} = \frac{2\pi}{ab}.$$

7. Let $P_n(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geqslant 1$ with complex coefficients and let γ_R be the image of the circle $|t| = R$ under the mapping $t \to z = P_n(t)$. Show that, if R is sufficiently large, γ_R does not pass through the origin $z = 0$ and that $I(\gamma_R, 0) = n$; deduce from this that $P_n(t) = 0$ has at least one root. (First show that, for sufficiently large R, $|t^n| > |a_{n-1}t^{n-1} + \cdots + a_0|$ for $|t| \geqslant R$. Then use proposition 8. 3 of § 1 to show that $I(\gamma_R, 0)$ is equal to the index, with respect to the origin, of the image of the circle $|t| = R$ by the mapping $t \to t^n$).

8. Let $f(z) = u(x, y) + iv(x, y)$ be a holomorphic function in a connected open set D. If

$$au(x, y) + bv(x, y) = c \quad \text{in} \quad D,$$

where a, b and c are real constants which are not all zero, then $f(z)$ is constant in D.

9. Let D be a *convex* open set in the plane and let $f(z)$ be a holomorphic function in D. Show that, for any pair of points $a, b \in D$, we can choose two points c and d on the line segment joining a and b such that

$$f(a) - f(b) = (a - b)\,(\mathrm{Re}\,(f'(c) + i\,\mathrm{Im}(f'(d))).$$

(Consider the function of a real variable r defined by

$$F(t) = f(b + (a - b)t)/(a - b),$$

and apply the mean value theorem to the real and imaginary parts of $F(t)$.)

10. Let D be a connected open set, which is symmetrical with respect to the real axis and has non-empty intersection I with it. Any holomorphic function $f(z)$ in D can be expreseed uniquely in the form

$$f(z) = g(z) + ih(z) \quad \text{for all} \quad z \in D,$$

where g and h are holomorphic functions in D which take *real* values in I. Show that, in this case,

$$\overline{g(\bar{z})} = g(z), \quad \overline{h(\bar{z})} = h(z)$$

and

$$\overline{f(\bar{z})} = g(z) - ih(z), \quad \text{for all} \quad z \in D.$$

11. Let f and g be two holomorphic functions in a connected open set D of the plane, which have no zeros in D; if there is a sequence (a_n) of points of D such that

$$\lim a_n = a, \quad a \in D \quad \text{and} \quad a_n \neq a \quad \text{for all } n,$$

and if

$$\frac{f'(a_n)}{f(a_n)} = \frac{g'(a_n)}{g(a_n)} \quad \text{for all } n,$$

show that there exists a constant c such that $f(z) = cg(z)$ in D.

12. Let $\varphi(z)$ be a *continuous* function on the oriented boundary Γ of a compact set K. Let D be open set complementary to Γ in $\mathbf{C}$, and put, for $z \in D$,

$$f(z) = \int_\Gamma \frac{\varphi(\zeta)}{\zeta - z} d\zeta.$$

(i) If $\rho = \inf_{\zeta \in \Gamma} |\zeta - a|$ for $a \in D$, show that $\frac{1}{\zeta - z}$, for $\zeta \in \Gamma$ and $|z - a| \leqslant r$ with $0 < r < \rho$, can be expanded in a series of powers of $(z - a)$ which is normally convergent; deduce that $f(z)$ is analytic in a neighbourhood of each $a \in D$. (cf. the proof of theorem 3, § 2.)

(ii) Show that

$$f^{(n)}(a) = n! \int_\Gamma \frac{\varphi(\zeta)}{(\zeta - a)^{n+1}} d\zeta,$$

for any integer $n \geqslant 1$, $a \in D$ (cf. chapter III, § 1).

13. Let $f(z)$ be holomorphic in $|z| < \rho$; show that, if $0 < r < \rho$, then

$$\lim_{\substack{h \to 0 \\ 0 < |h| < \rho - r}} \frac{f(z+h) - f(z)}{h} = f'(z)$$

uniformly for $|z| \leqslant r$. (By using 12., show that

$$\frac{f(z+h) - f(z)}{h} - f'(z) = \frac{h}{2\pi i} \int_{|t| = r'} \frac{f(t)\, dt}{(t - z - h)(t - z)^2},$$

where $r' = (\rho + r)/2$, $|h| < (r' - r)/2 = (\rho - r)/4$, say, and deduce from this that, if $M = \sup_{|t| = r'} |f(t)|$, then

$$\left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| \leqslant 4M \frac{\rho + r}{(\rho - r)^3} |h|.)$$

14. If two closed paths of $\mathbf{C} - \{0\}$ have the same index with respect to 0, show that they are homotopic as closed paths in $\mathbf{C} - \{0\}$.