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On the singularities of convex functions*

G. Alberti, L. Ambrosio, P. Cannarsa

Abstract. Given a (semi)-convex function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and an integer $k \in [0, n]$, we show that the set Σ^k defined by

$$\Sigma^k := \big\{x \in \Omega : \dim\big(\partial u(x)\big) \geq k\big\}$$

is countably H^{n-k}-rectifiable, i.e., it is contained (up to a H^{n-k}-negligible set) in a countable union of C^1 hypersurfaces of dimension $(n-k)$. Moreover, if u is convex in Ω, we show that

$$\int\limits_{\Omega' \cap \Sigma^k} \mathscr{H}^k(\partial u(x))\, d\mathscr{H}^{n-k}(x) < +\infty$$

for any open set $\Omega' \subset\subset \Omega$.

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1. Introduction

This paper originated from our interest in the following question about the singularities of a convex functions:

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Problem: given a convex function $u : \mathbb{R}^n \to \mathbb{R}$, and an integer $k \in [0, n]$, how to estimate the size of the k-th singular set of u, i.e., of the set

$$\Sigma^k(u) := \big\{x \in \mathbb{R}^n : \dim\big(\partial u(x)\big) \geq k\big\}?$$

Of course, the problem has a trivial answer if $k = 0$ or $k = n$, as $\Sigma^0(u) = \mathbb{R}^n$, whereas $\Sigma^n(u)$ is at most countable.

Moreover, if $k = 1$, a solution to our problem could be given noting that ∇u has locally bounded first variation in $\mathbb{R}^n$ (see e.g. [13] and [18]). Indeed, the jump set of such a function is known to be countably $\mathscr{H}^{n-1}$-rectifiable (see [9] and [20]), where $\mathscr{H}^m$ denotes the m-dimensional Hausdorff measure in $\mathbb{R}^n$. Equivalently, $\mathscr{H}^{n-1}$-almost all of $\Sigma^1(u)$ can be covered with a sequence of C^1 hypersurfaces.

In this paper we show that, for any $k \in \{0, 1, \ldots, n\}$, $\Sigma^k(u)$ is countably $\mathscr{H}^{n-k}$-rectifiable (Theorem 4.1). Consequently, $\Sigma^k(u)$ is σ-finite with respect to $\mathscr{H}^{n-k}$ and, in particular, its Hausdorff dimension does not exceed $(n - k)$. Very simple examples show that $\Sigma^k(u)$ may well be a $(n - k)$-dimensional set, for instance, a plane.

Another result contained in Theorem 4.1 of this paper is the estimate

$$\int\limits_{\Sigma^k(u)\cap\Omega} \mathscr{H}^k(\partial u(x))\, d\mathscr{H}^{n-k}(x) \leq C(n)\left(\big[u\big]_{\mathrm{Lip}(\Omega)} + \mathrm{diam}(\Omega)\right)^n,$$

that holds true for any integer $k \in [0, n]$. Such bound provides a quantitative information on the “measure” of the set $\Sigma^k(u)$.

At this point, a brief description of our techniques is in order. The main idea of our approach is to connect the $\mathscr{H}^m$-rectifiability of a set S with an upper bound on the dimension of the contingent cone $\mathrm{T}(S, x)$ to S at any point $x \in S$ (Theorem 3.1). Then, the rectifiability of $\Sigma^k(u)$ follows by splitting $\Sigma^k(u)$ as a countable union of sets $\Sigma^k_\alpha(u)$ for which we are able to prove an upper bound on the dimension of the contingent cone. Such a bound is obtained showing $\mathrm{T}\big(\Sigma^k_\alpha(u), x\big)$ is orthogonal to $\partial u(x)$ (Proposition 2.2), and recalling that $\dim\big(\partial u(x)\big) \geq k$.

Although we have stated the problem for a convex function, the method we propose in this work also applies to semi-convex functions (see §2 for notation). Therefore our results are stated in this more general setup.

Semi-convexity – or, better, semi-concavity – properties are well known for solutions of nonlinear partial differential equations such as Hamilton-Jacobi-Bellmann equations of first or second order, see e.g. [16], [15].

Hence, the results of this paper provide *upper* bounds on the singular sets of solutions to these equations, and somehow complement the singularity propagation results of [6].

Finally, an interesting problem in this research is to provide lower bounds on the singular set of a solution in the neighborhood of a fixed singular point. Bounds of this kind are false for a general semi concave (or even concave) function, see Remark 2.4. However, they will be obtained in a forthcoming paper [3], using additional information derived from the equation.

2. Properties of semi-convex functions

We fix a bounded, convex, open set $\Omega \subset \mathbb{R}^n$, and we denote by $B_\rho(x)$ the open ball in $\mathbb{R}^n$ centered at x with radius ρ.

For any $S \subset \mathbb{R}^n$ we denote by $S^\perp$ the plane

$$\big\{p \in \mathbb{R}^n : q \mapsto \langle q, p\rangle \text{ is constant on } S\big\}.$$

For any integer $m = 0, \dots, n$ we denote by $\mathscr{H}^m$ the Hausdorff m-dimensional measure in $\mathbb{R}^n$, defined by

$$\mathscr{H}^m(B) := \frac{\omega_m}{2^m} \sup_{\delta>0} \inf\Big\{\sum_i \operatorname{diam}^m(B_i) \; : \; B \subset \bigcup_i B_i, \; \operatorname{diam}(B_i) < \delta\Big\}, \tag{2.1}$$

where ω_m is the Lebesgue measure of the unit ball in $\mathbb{R}^m$ if $m \geq 1$ and $\omega_m = 1$ if $m = 0$. In particular, $\mathscr{H}^0$ is the so-called counting measure.

If u is a Lipschitz function in Ω, we set

$$[u]_{\mathrm{Lip}(\Omega)} := \sup\left\{ \frac{|u(x) - u(y)|}{|y - x|} \,:\, x,\, y \in \Omega,\ x \neq y \right\}.$$

Definition. We say that u is semi-convex in Ω, and we write $u \in SC(\Omega)$, if we can find a non decreasing upper semicontinuous function $\omega : [0, +\infty[\to [0, +\infty[$ such that $\omega(0) = 0$ and

(2.2)
$$t\, u(x_1) + (1-t)\, u(x_0) - u(x_t) \geq -t(1-t)\, |x_1 - x_0|\, \omega(|x_1 - x_0|)$$
$$\text{for all } x_0, x_1 \in \Omega,\ t \in [0,1] \text{ and } x_t := t x_1 + (1-t) x_0.$$

If $u \in SC(\Omega)$, we denote by $\omega_{u,\Omega}$ the least function ω satisfying (2.2).

For any $x \in \Omega$ and any $u : \Omega \to \mathbb{R}$, the subdifferential $\partial u(x)$ of u at x is defined by

$$\partial u(x) := \left\{ p \in \mathbb{R}^n :\ \liminf_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \geq 0 \right\}.$$

The subdifferential is a closed convex set, possibly empty.

If u is a convex function, the above set coincides with the well-known subdifferential of convex analysis, which captures all the relevant differential properties of convex functions. In particular, the subdifferential of a convex function is non-empty at every point (see for instance [8]). In the following proposition we list analogous properties of subdifferentials of semi-convex functions (see also [4] and [6]). We give a fairly detailed proof for the reader's convenience.

Proposition 2.1. *Let $u \in SC(\Omega)$. Then, u is locally Lipschitz continuous in Ω, the sets $\partial u(x)$ are non-empty, compact, and $p \in \partial u(x)$, if and only if*

$$u(y) - u(x) - \langle p, y - x \rangle \geq -|y - x|\, \omega_{u,\Omega}(|y - x|) \qquad \forall y \in \Omega. \tag{2.3}$$

Finally, the map $x \to \partial u(x)$ is upper semi-continuous, i.e.,

$$x_h \to x,\ p_h \to p,\ p_h \in \partial u(x_h) \quad \Longrightarrow \quad p \in \partial u(x). \tag{2.4}$$

Proof. Let x_0, x_1, x_2, x_3 be an ordered set of points lying on the same line contained in Ω. By using (2.2), it is not difficult to see that

$$\frac{u(x_3)-u(x_1)}{|x_3-x_1|}-\frac{u(x_2)-u(x_1)}{|x_2-x_1|}\geq -\omega_{u,\Omega}(|x_3-x_1|). \tag{2.5}$$

Similarly,

$$\frac{u(x_2)-u(x_1)}{|x_2-x_1|}-\frac{u(x_1)-u(x_0)}{|x_1-x_0|}\geq -\omega_{u,\Omega}(|x_2-x_0|). \tag{2.6}$$

Hence

$$\begin{aligned} -\omega_{u,\Omega}(|x_2-x_0|)+\frac{u(x_1)-u(x_0)}{|x_1-x_0|}\leq \frac{u(x_2)-u(x_1)}{|x_2-x_1|} &\leq \\ \leq \frac{u(x_3)-u(x_1)}{|x_3-x_1|}+\omega_{u,\Omega}(|x_3-x_1|). \end{aligned}$$

This shows that u is locally Lipschitz continuous on lines. Moreover, if x_1 and x_2 belong to $\Omega' \subset\subset \Omega$, the above provides a uniform estimate of the Lipschitz constant, and therefore shows that u is a Lipschitz function in Ω'.

Since u is locally Lipschitz continuous, $\partial u(x)$ is compact.

Any vector $p \in \mathbb{R}^n$ satisfying (2.3) trivially belongs to $\partial u(x)$. Conversely, let $p \in \partial u(x)$, and let $x_1 = x$, $x_3 = y$, $x_2 = x_1 + t(y-x)$ in (2.5) with $0 < t \leq 1$:

$$\frac{u(y)-u(x)}{|y-x|}\geq \frac{u(x+t(y-x))-u(x)}{t|y-x|}-\omega_{u,\Omega}(|y-x|).$$

By letting $t \to 0^+$ we obtain that p fulfils (2.3).

By using (2.2), (2.5) and (2.6) it can be seen that the function

$$v(y)=\lim_{t\to 0^+}\frac{u(x+ty)-u(x)}{t}$$

is well defined, and convex. Therefore $\partial u(x)$ is not empty because it coincides, by (2.3), with $\partial v(0)$.

Finally, the upper semicontinuity of $x \to \partial u(x)$ is a straightforward consequence of (2.3). ■

In this paper we are interested in the properties of the singular sets of semi-convex functions.

Definition. For any integer $k \in [0, n]$ we define

$$\Sigma^k(u) := \big\{ x \in \Omega : \ \dim\big(\partial u(x)\big) \geq k \big\},$$

and for any $\alpha > 0$ we denote by $\Sigma^k_\alpha(u)$ the set of points $x \in \Sigma^k(u)$ such that $\partial u(x)$ contains some k-dimensional ball B^k_α of diameter 2α, i.e.,

$$\Sigma^k_\alpha(u) := \big\{ x \in \Sigma^k(u) : \exists B^k_\alpha \subset \partial u(x) \text{ with } \operatorname{diam}\big(B^k_\alpha\big) = 2\alpha \big\}. \tag{2.7}$$

We define now the contingent cone $\mathrm{T}(S, x)$ to a set $S \subset \mathbb{R}^n$ at a point x (see [4], [8], and [11], 3.1.21).

Definition. Let $x \in S$. We define

$$\mathrm{T}(S, x) := \Big\{ r\theta : \ r \geq 0, \theta = \lim_{h \to +\infty} \frac{x_h - x}{|x_h - x|} \text{ with } x_h \in S \setminus \{x\},\ x_h \to x \Big\}.$$

We denote by $\mathrm{Tan}(S, x)$ the vector space generated by $\mathrm{T}(S, x)$.

In the following lemma we investigate the properties of $\Sigma^k_\alpha(u)$.

Proposition 2.2. *For any $u \in SC\big(\Omega\big)$, the set $\Sigma^k_\alpha(u)$ is closed in Ω and*

$$\mathrm{Tan}\big(\Sigma^k_\alpha(u), x\big) \subset \big[\partial u(x)\big]^\perp \tag{2.8}$$

for any $x \in \Sigma^k_\alpha(u) \setminus \Sigma^{k+1}(u)$. In particular, the dimension of $\mathrm{Tan}(\Sigma^k_\alpha(u), x)$ is not greater than $(n-k)$ for any $x \in \Sigma^k_\alpha(u) \setminus \Sigma^{k+1}(u)$.

Proof. Let us prove that $\Sigma^k_\alpha(u)$ is closed. Let $\{x_i\} \subset \Sigma^k_\alpha(u)$ be converging to $x \in \Omega$, and let $B^k_\alpha(p_i) \subset \partial u(x_i)$ be k-dimensional balls centered at p_i with radius α. Possibly passing to subsequences, we can assume with no loss of generality that there is a k-dimensional ball B^k_α with radius α such that each point $p \in B^k_\alpha$ can be approximated by points in $B^k_\alpha(p_i)$. By the upper semicontinuity of the differential (see (2.4)) we get $B^k_\alpha \subset \partial u(x)$, hence $x \in \Sigma^k_\alpha(u)$.

In order to show (2.8), we only need to prove that the map $p \to \langle \eta, p\rangle$ is constant on $\partial u(x)$ for any $\eta \in \mathrm{T}(\Sigma^k_\alpha(u), x)$ with $|\eta| = 1$. Let $\{x_h\} \subset \Sigma^k_\alpha(u) \setminus \{x\}$ be a sequence converging to x such that

$$\lim_{h\to+\infty} \frac{x_h - x}{|x_h - x|} = \eta.$$

Possibly extracting a subsequence, we can assume with no loss of generality that there is a k-dimensional ball B^k_α with radius α such that each $p \in B^k_\alpha$ can be approximated by vectors in $\partial u(x_h)$. By (2.4), $B^k_\alpha \subset \partial u(x)$. Since $\partial u(x)$ is a k-dimensional set, we only need to know that $p \mapsto \langle \eta, p\rangle$ is constant on B^k_α. Let p, $p' \in B^k_\alpha$, and let $p_h \in \partial u(x_h)$ be converging to p'; by adding the inequalities

$$\begin{aligned}\frac{u(x_h) - u(x) - \langle p, x_h - x\rangle}{|x_h - x|} &\geq -\omega_{u,\Omega}(|x_h - x|)\\ \frac{u(x) - u(x_h) - \langle p_h, x - x_h\rangle}{|x_h - x|} &\geq -\omega_{u,\Omega}(|x_h - x|),\end{aligned}$$

and passing to the limit as $h \to +\infty$, we get

$$\langle \eta, p'\rangle \leq \langle \eta, p\rangle.$$

Since p and p' are arbitrary, (2.8) follows. ■

Remark 2.3. Proposition 2.2 yields $\mathrm{Tan}\big(\Sigma^n_\alpha(u), x\big) = \{0\}$ for any $x \in \Sigma^n_\alpha(u)$. Hence, $\Sigma^n_\alpha(u)$ is a discrete set in Ω and $\Sigma^n(u)$ is at most countable.

Remark 2.4. One may wonder whether the inclusion in (2.8) is indeed an equality. This fact could be regarded as a "singularity propagation" phenomenon. Now, Theorem 3.1 below shows that the set of points $S \subset \Sigma^k_\alpha(u)$ at which the inclusion is strict is countably $\mathscr{H}^{n-k-1}$-rectifiable, hence σ-finite with respect to $\mathscr{H}^{n-k-1}$. Indeed, since $\mathrm{Tan}(S,x) \subset \mathrm{Tan}(\Sigma^k_\alpha(u),x)$, by the definition of S it follows that

$$\dim\bigl(\mathrm{Tan}(S,x)\bigr) \le n-k-1$$

for any $x \in S$. However, the following example shows that equality (2.8) may fail at some point. Let $n=2$, $k=1$, and let $u(x,y) := \sqrt{x^2+y^4}$. It is easy to check that u is continuously differentiable in $\mathbb{R}^2 \setminus \{0\}$, and convex in $\mathbb{R}^2$. Moreover, $\partial u(0) = [-1,1]\times\{0\}$, so that $\dim\bigl[\partial u(0)\bigr]^\perp = 1$. On the other hand, $\mathrm{T}\bigl(\Sigma^1(u)\bigr) = \emptyset$. Based on the above, it is not hard to construct an example of function $u : \mathbb{R}^2 \to [0,+\infty[$ such that the exceptional set S is countable.

3. A rectifiability criterion

Let us first give a definition.

Definition. We say that $S \subset \mathbb{R}^p$ is countably $\mathscr{H}^m$-rectifiable if there is a countable family of C^1 hypersurfaces $\Gamma_h \subset \mathbb{R}^p$ of dimension m such that

$$\mathscr{H}^m\left(S \setminus \bigcup_{h=1}^{\infty} \Gamma_h\right) = 0. \tag{3.1}$$

If $D \subset \mathbb{R}^m$ and $f : D \to \mathbb{R}^p$ is a Lipschitz function, then a Lusin-type argument shows that $f(D)$ is countably $\mathscr{H}^m$-rectifiable (see [19], Lemma 11.1).

We can now state a sufficient condition for rectifiability.

Theorem 3.1. *Let $S \subset \mathbb{R}^n$, and let us assume that* $\mathrm{Tan}(S,x)$ *has dimension not greater than m for any $x \in S$. Then, S is countably $\mathscr{H}^m$-rectifiable.*

Proof. Let us denote by $\varphi(x)$ the function $x/|x|$, defined for all $x \in \mathbb{R}^n \setminus \{0\}$. Then, the following two properties are satisfied for any $x \in S$ and any $\epsilon > 0$:

(i) there exists $r > 0$ such that

$$\forall y \in S \cap B_r(x) \setminus \{x\},\ \exists v \in \mathrm{T}(S,x) \text{ s.t. } |\varphi(y-x) - v| < \epsilon; \tag{3.2}$$

(ii) for any $r > 0$ there exists $\rho < r/2$ such that

$$\forall v \in \mathrm{T}(S,x),\ \exists y \in S \cap B_{r/2}(x) \setminus B_\rho(x) \text{ s.t. } |\varphi(y-x) - v| < \epsilon. \tag{3.3}$$

Both these properties can be proved arguing by contradiction.

Let us fix $\epsilon < 1/7$, and for $0 < \rho < r/2$ define

$$S_{r,\rho} := \big\{x \in S : \ (3.2) \text{ and } (3.3) \text{ hold}\big\}$$

We claim that $S_{r,\rho}$ is locally contained in the graph of a Lipschitz function. More precisely, let $x \in S_{r,\rho}$, let M be the set $\mathrm{Tan}(S,x)$ and let us denote by $\pi : \mathbb{R}^n \to M$ the orthogonal projection on M. We will show that there is a set $D \subset M$ such that $\pi : S_{r,\rho} \cap B_{\epsilon\rho}(x) \to D$ is one to one and $f = \pi^{-1}$ is Lipschitz continuous.

Possibly replacing $S_{r,\rho}$ by $S_{r,\rho} - x$, it is not restrictive to assume that $x = 0$. Let $y,\, z \in S_{r,\rho} \cap B_{\epsilon\rho}(0)$ with $y \neq z$. Since $|y - z| < 2\epsilon\rho < r/2$, by (3.2) we get

$$\exists v \in \mathrm{T}(S,y) \text{ such that } |\varphi(y-z) - v| < \epsilon. \tag{3.4}$$

Similarly, (3.3) yields

$$\exists \bar z \in S \cap B_{r/2}(y) \setminus B_\rho(y) \text{ such that } |\varphi(y-\bar z) - v| < \epsilon. \tag{3.5}$$

By using the inequality $|\nabla\varphi(x)(y)| \leq 2|y|/|x|$, and

$$|\bar z - ty| \geq |\bar z - y| - (1-t)|y| \geq \rho - \epsilon\rho \geq \rho/2 \qquad \forall t \in [0,1],$$

we have

$$|\varphi(\bar z - y) - \varphi(\bar z)| \leq \int_0^1 |\nabla\varphi(\bar z - ty)||y|\, dt \leq \frac{2|y|}{\rho/2} \leq 4\epsilon. \tag{3.6}$$

Moreover, since $\bar z \in B_{r/2}(y)$, we get by (3.2) $w \in \mathrm{T}(S,0)$ such that

$$|\varphi(\bar z) - w| < \epsilon. \tag{3.7}$$

Putting together (3.3), (3.4), (3.5) and (3.6) we obtain

$$\begin{aligned} |\varphi(z-y) - w| \le {} & |\varphi(z-y) - v| + |v - \varphi(\bar z - y)| + \\ & + |\varphi(\bar z - y) - \varphi(\bar z)| + |\varphi(\bar z) - w| \le 7\epsilon. \end{aligned} \tag{3.8}$$

By (3.8) we infer

$$|\varphi(z-y) - \pi\big(\varphi(z-y)\big)| \le |\varphi(z-y) - w| \le 7\epsilon < 1.$$

This shows that $\pi(z-y) \neq 0$ if $z \neq y$, hence π is one to one in $S_{r,\rho} \cap B_{\epsilon\rho}(0)$. Moreover,

$$\big|\pi\big(\varphi(z-y)\big)\big| \ge \sqrt{1-(7\epsilon)^2},$$

and

$$|\pi(z) - \pi(y)| \ge \sqrt{1-(7\epsilon)^2}\,|y-z|. \tag{3.9}$$

Let $D = \pi\big(S_{r,\rho} \cap B_{\epsilon\rho}(0)\big)$, and let $f : D \to \mathbb{R}^n$ be the inverse function of π. By (3.9), f is a Lipschitz function, and $f(D) = S_{r,\rho} \cap B_{\epsilon\rho}(0)$.

This shows that $S_{r,\rho}$ is countably $\mathscr{H}^m$-rectifiable. Since any point $x \in S$ belongs to $S_{1/n,1/p}$ for sufficiently large natural numbers n, p with $p > 2n$, also S is countably $\mathscr{H}^m$-rectifiable. ■

4. Estimates on singularities and rectifiability

Let u be a semi-convex function, and let us denote by $\Gamma(u)$ the graph of the subdifferential, i.e.

$$\Gamma(u) = \big\{(x,p) \in \mathbb{R}^n \times \mathbb{R}^n : p \in \partial u(x)\big\}.$$

In the following we apply the rectifiability criterion of §3 to the problem described in the introduction.

Theorem 4.1. *Let $u : \Omega \to \mathbb{R}$ be semi-convex and Lipschitz continuous. Then, for any integer $k \in [0, n]$, the set*

$$\Sigma^k(u) := \big\{x \in \Omega : \dim\big(\partial u(x)\big) \geq k\big\}$$

is countably $\mathscr{H}^{n-k}$-rectifiable. Moreover, if $\omega_{u,\Omega}(t) \leq Ct$ for some $C \geq 0$, then $\Gamma(u)$ is countably $\mathscr{H}^n$-rectifiable in $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\mathscr{H}^n\big(\Gamma(u)\big) \leq C(n)\bigg(1 + (C+1)^2\bigg)^{n/2}\big[u\big]^n_{\mathrm{Lip}(\Omega)}. \tag{4.1}$$

Moreover,

$$\int\limits_{\Sigma^k(u)} \mathscr{H}^k(\partial u(x))\, d\mathscr{H}^{n-k}(x) \leq \mathscr{H}^n\big(\Gamma(u)\big). \tag{4.2}$$

Proof. By Theorem 3.1 and Proposition 2.2, the sets $\Sigma^k_\alpha(u)$ are countably $\mathscr{H}^{n-k}$-rectifiable. Since

$$\Sigma^k(u) = \bigcup_{p \in \mathbb{N}} \Sigma^k_{1/p}(u),$$

also $\Sigma^k(u)$ is countably $\mathscr{H}^{n-k}$-rectifiable.

Let us assume now that $\omega_{u,\Omega}(t) \leq Ct$ for some $C \geq 0$. Given any $x_0 \in \Omega$ we define

$$v(x) = u(x) + \frac{C}{2}|x - x_0|^2.$$

It is not hard to see that v is convex and

$$\langle p - q, x - y\rangle \geq |x - y|^2 \quad \forall x,\, y \in \Omega,\, p \in \partial u(x),\, q \in \partial u(y). \tag{4.3}$$

In addition, we have

$$\Gamma(u) = \Phi_C\big(\Gamma(v)\big)$$

where $\Phi_C : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by

$$\Phi_C(x,p) := \big(x, p - (C+1)(x - x_0)\big).$$

Since the Lipschitz constant of Φ_C equals $\sqrt{1+(C+1)^2}$, by well-known properties of Hausdorff measures (see for instance [11], paragraph 2.10.11) we infer the inequality

$$\mathscr{H}^n\big(\Gamma(u)\big) \leq \Big(1 + (C+1)^2\Big)^{n/2} \mathscr{H}^n\big(\Gamma(v)\big) \tag{4.4}$$

Let $D \subset \mathbb{R}^n$ be the projection of $\Gamma(v)$ on the second factor, (a similar idea is also used in [13]) and let $\varphi : D \to \mathbb{R}^n$ be the function which assigns to each $p \in D$ the unique (by (4.3)) $x \in \Omega$ such that $p \in \partial v(x)$. By (4.3) we get

$$|\varphi(p) - \varphi(q)|^2 \; leq \langle p - q, \varphi(p) - \varphi(q)\rangle \leq |p-q|\,|\varphi(p) - \varphi(q)|,$$

so that φ is a contraction. Since $\Gamma(v)$ coincides with the graph of φ, by the area formula for Lipschitz functions (see [11], 3.2.1) we obtain

$$\mathscr{H}^n\big(\Gamma(v)\big) = \int_D \psi\big(\nabla\varphi(p)\big)\, dp,$$

where

$$\psi\big(A\big) = \sqrt{1 + \sum_{B \subset A} \det{}^2\big(B\big)}$$

for any $n \times n$ matrix A. In particular,

$$\mathscr{H}^n\big(\Gamma(v)\big) \leq C(n)\, \mathscr{H}^n\big(D\big) \leq \omega_n C(n) \big[v\big]^n_{\mathrm{Lip}(\Omega)}. \tag{4.5}$$

Hence, (4.1) follows by (4.4) and (4.5). Finally, (4.2) follows by general properties of products of Hausdorff measures ([11], 2.10,.27) and of Lipschitz mappings between rectifiable sets. In fact, denoting by $\pi : \mathbb{R}^{2n} \to \mathbb{R}^n$ the projection on the first factor, by [11], 3.2.22 we infer

$$\mathscr{H}^n\big(\Gamma(v)\big) \geq \int_\Omega \mathscr{H}^k\big(\pi^{-1}(x) \cap \Gamma(u)\big)\, d\mathscr{H}^{n-k}(x) \ .$$

Since $\pi^{-1}(x) \cap \Gamma(u) = \{(x,p) :\ p \in \partial u(x)\}$, (4.6) is equivalent to (4.2). ■

Remark 4.2. Let $M \subset \mathbb{R}^p$ be a countably $\mathscr{H}^m$-rectifiable set, and let $\pi : M \to \mathbb{R}^n$ be a Lipschitz function. In [11], 3.2.31 Federer shows that the set

$$\big\{z \in \mathbb{R}^n : \mathscr{H}^k\big(\pi^{-1}(z)\big) > 0\big\}$$

is countably $\mathscr{H}^{m-k}$-rectifiable. Hence, the rectifiability of $\Sigma^k(u)$ follows by the rectifiability of $\Gamma(u)$ by applying Federer's proposition with $M = \Gamma(u)$, $p = 2n$, $m = n$ and π equal to the projection on the first variable.

A similar approach is followed by Baldo and Ossanna in [5]. However, this method does not apply to a general semi-convex function, like a function with Hölder continuous gradient. Therefore, using Theorem 3.1 to derive the rectifiability of $\Sigma^k(u)$ is more powerful. Moreover, we believe it is more direct as well, because it minimizes the application of sophisticated techniques from Geometric Measure Theory.

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G. Alberti
Scuola Normale Superiore
P.za Cavalieri 7
56126 Pisa, Italy

L. Ambrosio and P. Cannarsa
Dipartimento di Matematica
II Università di Roma
Tor Vergata, 00173 Roma, Italy