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# *Integral Representation of Functionals Defined on Sobolev Spaces*

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**Abstract:** We give an integral representation result for functionals defined on Sobolev spaces; more precisely, for a functional $F$, we find necessary and sufficient conditions that imply the integral representation formula

$$F(u, B) = \int_B f(x, Du)\, dx.$$


## 1. – Introduction

The problem of representing in an integral form a given functional defined on a function space and satisfying suitable "abstract" conditions, has been considered by several authors in different frameworks (see References). One of the reasons is that it is the key point in many problems of relaxation and Γ-convergence (see for instance [3], [6], [8], [10], [11], [16], [20], [25]); in fact, the relaxed functionals (or the Γ-limits of a sequence of functionals) are merely lower semicontinuous mappings defined on a function space, and the first step in order to get their complete characterization, is just to represent them in a suitable integral form.

The most classical integral representation result is the well-known Riesz theorem which states that every linear continuous map $F : L^p(\Omega;\mathbb{R}^m) \to \mathbb{R}$ can be written in the form

$$F(u) = \int_\Omega f(x) \cdot u(x)\, dx \tag{1.1}$$

for a suitable $f \in L^q(\Omega;\mathbb{R}^m)$ (with $1/p + 1/q = 1$).

A nonlinear version of the Riesz representation theorem has been also proved (see for instance [14], [25], [28]); it states that every lower semicontinuous map $F : L^p(\Omega;\mathbb{R}^m) \to ]-\infty, +\infty]$ which is disjointly additive in the sense that

$$F(u + v) = F(u) + F(v) \qquad \text{whenever } u \cdot v = 0 \text{ a.e. on } \Omega$$

can be represented in the form

$$F(u) = \int_\Omega f\big(x, u(x)\big)\, dx \tag{1.2}$$

for a suitable Borel function $f(x,s)$ lower semicontinuous in $s$ and such that

$$f(x,s) \geq -\left[a(x) + b|s|^p\right] \qquad \text{for all } (x,s) \in \Omega \times \mathbb{R}^m$$

with $a \in L^1(\Omega)$ and $b \geq 0$.

Other integral representation results for functionals defined on the space of measures, have also been proved (see [3], [6], [7], [8], [9], [10], [11], [20], [40]).

In this paper, we deal with functionals $F(u,B)$ defined for every $u$ belonging to a Sobolev space $W^{1,p}(\Omega;\mathbb{R}^m)$ and every $B$ belonging to the class $\mathscr{B}(\Omega)$ of all Borel subsets of $\Omega$, and we look for an integral representation of $F$ in the form

$$F(u,B) = \int_B f\big(x, Du(x)\big)\, dx \tag{1.3}$$

for a suitable integrand $f(x,z)$. When $F$ satisfies growth conditions as

$$|F(u,B)| \leq \int_B \left[a(x) + b|Du|^p\right]\, dx \tag{1.4}$$

with $a \in L^1(\Omega)$ and $b \geq 0$, the integral representation formula (1.3) has been obtained by Buttazzo & Dal Maso in [15], [16] under the following additional hypotheses:

(i) $F$ is local, that is

$$u = v \text{ a.e. on } B \in \mathscr{B}(\Omega) \;\Rightarrow\; F(u,B) = F(v,B)\ ;$$

(ii) for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ the set function $F(u,\cdot)$ is a measure on $\mathscr{B}(\Omega)$;

(iii) for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$, $c \in \mathbb{R}^m$, and $B \in \mathscr{B}(\Omega)$ we have

$$F(u+c,B) = F(u,B)\ ;$$

(iv) for every $B \in \mathscr{B}(\Omega)$ the function $F(\cdot,B)$ is sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$.

In this case the integrand $f(x,z)$ in (1.3) turns out to be quasi-convex with respect to $z$ in the sense of Morrey [36].

Here we follow a different approach based on a recent result by Alberti (see [1]) concerning a Lusin type property for $L^p$-functions (Theorem 2.7). This will enable us to obtain the integral representation (1.3) even if the growth condition (1.4) is dropped and condition (iv) is substituted by the weaker one:

(iv') for every $B \in \mathscr{B}(\Omega)$ the function $F(\cdot, B)$ is lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ with respect to the strong topology.

## 2. – Notation and Statement of the Result

In this section we fix the notation we shall use in the following and we state our main result. We also recall some other results which will be used in the proofs.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, let $m \geq 1$ be an integer, and let $p \in [1, +\infty]$; we denote by $W^{1,p}(\Omega;\mathbb{R}^m)$ the usual Sobolev space with norm

$$\|u\|_{W^{1,p}(\Omega;\mathbb{R}^m)} = \|u\|_{L^p(\Omega;\mathbb{R}^m)} + \|Du\|_{L^p(\Omega;\mathbb{R}^{mn})}.$$

For every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and for a.e. $x \in \Omega$ the gradient $Du(x)$ will be the $m \times n$ matrix defined by $\big(Du(x)\big)_{i,j} = D_j u_i(x)$ for $i = 1, \dots, m$ and $j = 1, \dots, n$.

We shall consider functionals $F : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathscr{B}(\Omega) \to [0, +\infty]$ where $\mathscr{B}(\Omega)$ denotes the class of all Borel subsets of $\Omega$. For this kind of functionals we introduce the following definitions.

Definition 2.1. *We say that a functional $F : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathscr{B}(\Omega) \to [0, +\infty]$ is*

(i) *local, if $F(u, B) = F(v, B)$ whenever $B \in \mathscr{B}(\Omega)$ and $u, v \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $u = v$ a.e. on $B$;*

(ii) *D-local, if $F(u, B) = F(v, B)$ whenever $B \in \mathscr{B}(\Omega)$ and $u, v \in W^{1,p}(\Omega;\mathbb{R}^m)$ with $Du = Dv$ a.e. on $B$;*

(iii) *a measure, if for every $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ the set function $F(u, \cdot)$ is countably additive on $\mathscr{B}(\Omega)$.*

We are now in a position to state our integral representation result.

Theorem 2.2. *Let $p \in [1, +\infty[$, and let $F : W^{1,p}(\Omega;\mathbb{R}^m) \times \mathscr{B}(\Omega) \to [0, +\infty]$ be a functional such that:*

(i) *$F$ is D-local;*

(ii) *$F$ is a measure;*

(iii) *for every $B \in \mathscr{B}(\Omega)$ the function $F(\cdot, B)$ is lower semicontinuous on $W^{1,p}(\Omega;\mathbb{R}^m)$ with respect to the strong topology;*

(iv) *there exists $\bar{u} \in W^{1,p}(\Omega;\mathbb{R}^m)$ such that $F(\bar{u}, \cdot)$ is a bounded measure which is absolutely continuous with respect to the Lebesgue measure.*

*Then there exists a Borel function* $f : \Omega \times \mathbb{R}^{mn} \to [0, +\infty]$ *such that*
(a) *for every* $x \in \Omega$ *the function* $f(x, \cdot)$ *is lower semicontinuous on* $\mathbb{R}^{mn}$*;*
(b) *for every* $(u, B) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{B}(\Omega)$ *it is*

$$F(u, B) = \int_B f\big(x, Du(x)\big)\, dx\ .$$

*Moreover, the integrand* $f$ *is uniquely determined in the following sense: if* $g$ *is a Borel function so that* (b) *holds with* $g$ *instead of* $f$*, then there exists a negligible set* $N \in \mathscr{B}(\Omega)$ *such that* $f(x, s) = g(x, s)$ *for all* $x \in \Omega \setminus N$ *and* $s \in \mathbb{R}^{mn}$*.*

Remark 2.3. Note that hypotheses (i) and (iv) of Theorem 2.2 yield $F(u, B) = F(\bar{u}, B) = 0$ for all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and all $B \in \mathscr{B}(\Omega)$ with $|B| = 0$.

Remark 2.4. For simplicity we consider only the case $p < +\infty$; in the case $p = +\infty$ the same result (with the same proof) holds, provided condition (iii) is substituted by the following one:

(iii') *for every* $B \in \mathscr{B}(\Omega)$ *the function* $F(\cdot, B)$ *is lower semicontinuous on* $W^{1,\infty}(\Omega; \mathbb{R}^m)$ *with respect to the* $\tau_\infty$*-convergence, where we say that* $u_h$ *is* $\tau_\infty$*-convergent to* $u$ *if* $u_h$ *is bounded in* $W^{1,\infty}(\Omega; \mathbb{R}^m)$*,* $u_h$ *converges to* $u$ *uniformly on compact subsets of* $\Omega$*, and* $Du_h$ *converges to* $Du$ a.e. *in* $\Omega$*.*

Remark 2.5. The hypothesis that $F$ is positive can be easily weakened by requiring that for suitable $a \in L^1(\Omega)$ and $b \geq 0$

$$F(u, B) \geq -\int_B [a(x) + b|Du|^p]\, dx$$

for all $(u, B) \in W^{1,p}(\Omega; \mathbb{R}^m) \times \mathscr{B}(\Omega)$.

Remark 2.6. By Definition 2.1 and by the well-known locality of the gradient (see for instance Gilbarg & Trudinger [27], Lemma 7.7), it follows immediately that $D$-locality implies locality. The converse implication can be also proved (we refer to Alberti [2] for the proof) provided $F$ satisfies condition (ii) of Theorem 2.2 and the invariance condition (iii) stated in the Introduction. Finally, when $F$ satisfies conditions (ii) and (iii) of Theorem 2.2 and the growth condition (1.4), then the locality of $F$ follows from the locality on open sets (see Buttazzo & Dal Maso [15], Lemma 2.8), that is

$F(u, A) = F(v, A)$ *whenever* $A$ *is an open subset of* $\Omega$*, and* $u, v \in W^{1,p}(\Omega; \mathbb{R}^m)$ *with* $u = v$ *a.e. on* $A$*.*

The main tools in the proof of the integral representation Theorem 2.2 are the following results.

THEOREM 2.7.(See Alberti [1], Theorem 1)
*For every* $v \in L^p(\Omega;\mathbb{R}^{mn})$ *and every* $\varepsilon > 0$ *there exist a function* $u \in C_0^1(\Omega;\mathbb{R}^m)$ *and a closed set* $B \subset \Omega$ *such that*

(i) $|\Omega \setminus B| \le \varepsilon|\Omega|$;
(ii) $v = Du$ a.e. *in* $B$;
(iii) $\|Du\|_p \le C\varepsilon^{1/p-1}\|f\|_p$ *where* $C$ *is a constant which depends only on* $n$.

THEOREM 2.8.(See Buttazzo & Dal Maso [14])
*Let* $m \ge 1$ *be a given integer and let* $F : L^p(\Omega;\mathbb{R}^m) \times \mathscr{B}(\Omega) \to [0,+\infty]$ *be a functional such that*

(i) *for all* $v \in L^p(\Omega;\mathbb{R}^m)$, *the function* $F(v,\cdot)$ *is a measure which is absolutely continuous with respect to the Lebesgue measure;*
(ii) $F$ *is local, that is* $F(v,B) = F(v',B)$ *whenever* $v = v'$ a.e. *in* $B$;
(iii) *for all* $B \in \mathscr{B}(\Omega)$ *the function* $F(\cdot,B)$ *is lower semicontinuous with respect to strong topology of* $L^p(\Omega;\mathbb{R}^m)$;
(iv) *there exists* $\bar{v} \in L^p(\Omega;\mathbb{R}^m)$ *such that* $F(\bar{v},\Omega) < +\infty$.

*Then there exists a Borel function* $f : \Omega \times \mathbb{R}^m \to [0,+\infty]$ *such that*

(a) *for every* $x \in \Omega$ *the function* $f(x,\cdot)$ *is lower semicontinuous on* $\mathbb{R}^m$;
(b) *for every* $(v,B) \in L^p(\Omega;\mathbb{R}^m) \times \mathscr{B}(\Omega)$ *it is*

$$F(v,B) = \int_B f\big(x, v(x)\big)\,dx\ .$$

## 3. – Proof of the Result

For the sake of simplicity we always refer to $W^{1,p}(\Omega;\mathbb{R}^m)$ as $W^{1,p}$, to $C_0^1(\Omega;\mathbb{R}^m)$ as $C_0^1$ and to $L^p(\Omega;\mathbb{R}^{mn})$ as $L^p$.

Let $F$ be a functional on $W^{1,p}$ satisfying conditions (i), (ii), (iii), (iv) of Theorem 2.2. We shall apply Theorem 2.7 to find a functional $G$ on $L^p$ which satisfies hypotheses (i), (ii), (iii), (iv) of Theorem 2.8 and such that

$$F(u,B) = G(Du,B) \qquad \text{for all } (u,B) \in W^{1,p} \times \mathscr{B}(\Omega).$$

DEFINITION 3.1. *Let* $v \in L^p$ *and let* $(u_h,B_h) \in W^{1,p} \times \mathscr{B}(\Omega)$ *for every* $h \in \mathbb{N}$. *We say that* $(u_h,B_h)$ *is a local partition of* $v$ *if*

*(3.1)* *the sets* $B_h$ *are pairwise disjoint and cover almost all of* $\Omega$;
*(3.2)* $Du_h = v$ a.e. *in* $B_h$ *for every* $h \in \mathbb{N}$.

Proposition 3.2. *For every $v \in L^p$ and every $\varepsilon > 0$ there exists a local partition $(u_h, B_h)$ of $v$ such that*

$$|\Omega \setminus B_0| < \varepsilon \qquad \text{and} \qquad \|u_0\|_{W^{1,p}} \le C\varepsilon^{1/p-1}\|v\|_p$$

*where $C$ is a constant which does not depend on $v$.*

Proof. Fix $v \in L^p$ and $\varepsilon > 0$. By Theorem 2.7 there exist functions $u_h \in C_0^1$ and closed sets $A_h \subset \Omega$ such that

(i) $|\Omega \setminus A_h| < \varepsilon 2^{-h}$ for every $h \in \mathbb{N}$;
(ii) $Du_h = v$ a.e. in $A_h$;
(iii) $\|Du_h\|_p \le C\left(\varepsilon|\Omega|^{-1}2^{-h}\right)^{1/p-1}\|v\|_p$,

where $C$ is the constant of Theorem 2.7.
Setting for every $h \in \mathbb{N}$

$$B_h = A_h \setminus \bigcup_{j=0}^{h-1} A_j$$

it is easy to verify that $|\Omega \setminus \bigcup_0^\infty B_h| = 0$. Moreover, by Poincaré inequality, we get

$$\|u_0\|_{W^{1,p}} \le C'\|Du_0\|_p \le C'C|\Omega|^{1-1/p}\varepsilon^{1/p-1}\|v\|_p$$

where $C'$ is a constant which does not depend on $v$. Hence Proposition 3.2 is proved. □

Proposition 3.3. *Let $v \in L^p$ and let $(u_h, A_h)$ and $(u'_h, A'_h)$ be two local partitions of $v$. Then it is*

$$F(u_h, B) = F(u'_k, B)$$

*for all $h, k \in \mathbb{N}$ and all Borel sets $B \subset A_h \cap A'_k$. In particular, for all $B \in \mathscr{B}(\Omega)$ we have*

$$\sum_{h\in\mathbb{N}} F(u_h, B \cap A_h) = \sum_{k\in\mathbb{N}} F(u'_k, B \cap A'_k)\ . \tag{3.3}$$

Proof. Since $v = Du_h = Du'_k$ a.e in $A_h \cap A'_k$, by hypothesis (i) of Theorem 2.2 we obtain that $F(u_h, B) = F(u'_k, B)$ for all integers $h, k$ and all Borel sets $B \subset A_h \cap A'_k$. Taking into account (3.1) and Remark 2.3, this yields

$$\begin{aligned}\sum_{h\in\mathbb{N}} F(u_h, B \cap A_h) &= \sum_{h,k\in\mathbb{N}} F(u_h, B \cap A_h \cap A'_k)\\ &= \sum_{h,k\in\mathbb{N}} F(u'_k, B \cap A_h \cap A'_k) = \sum_{k\in\mathbb{N}} F(u'_k, B \cap A'_k)\ . \ \square\end{aligned}$$

LEMMA 3.4. *For all* $(v, B) \in L^p \times \mathscr{B}(\Omega)$ *define*

$$G(v, B) = \sum_{h \in \mathbb{N}} F(u_h, B \cap A_h) \tag{3.4}$$

*where* $(u_h, A_h)$ *is a local partition of* $v$. *We have:*

(i) $G$ *is well-defined, in the sense that* $G(v, B)$ *does not depend on the choice of the local partition of* $v$*;*

(ii) *for all* $(u, B) \in W^{1,p} \times \mathscr{B}(\Omega)$ *it is* $F(u, B) = G(Du, B)$.

PROOF. The fact that $G$ is well-defined follows from Proposition 3.3. In order to prove (ii), set $(u_0, A_0) = (u, \Omega)$ and $(u_h, A_h) = (0, \emptyset)$ for $h \geq 1$ and note that this is a local partition for $Du$. By the definition of $G$ we have

$$G(Du, B) = \sum_{h \in \mathbb{N}} F(u_h, B \cap A_h) = F(u, B) \qquad \text{for all } B \in \mathscr{B}(\Omega). \ \square$$

LEMMA 3.5. *For all* $v \in W^{1,p}$ *the function* $G(v, \cdot)$ *is a positive measure which is absolutely continuous with respect to the Lebesgue measure.*

PROOF. Let $v$ be a function in $L^p$, let $B \in \mathscr{B}(\Omega)$, and let $(B_k)_{k \in \mathbb{N}}$ be a partition of $B$ into Borel sets. If $(u_h, A_h)$ is a local partition of $v$, by using the definition of $G$, hypothesis (i) and (ii) of Theorem 2.2, and Remark 2.3, we get

$$\begin{aligned} \sum_{k \in \mathbb{N}} G(v, B_k) &= \sum_{k \in \mathbb{N}} \Big[ \sum_{h \in \mathbb{N}} F(u_h, B_k \cap A_h) \Big] \\ &= \sum_{h \in \mathbb{N}} \Big[ \sum_{k \in \mathbb{N}} F(u_h, B_k \cap A_h) \Big] \\ &= \sum_{h \in \mathbb{N}} F(u_h, B \cap A_h) = G(v, B) \ . \end{aligned}$$

Therefore $G(v, \cdot)$ is a measure. The fact that $G(v, B) = 0$ whenever $|B| = 0$ is obvious. $\square$

LEMMA 3.6. $G$ *is local, that is* $G(v, B) = G(v', B)$ *for all* $v, v' \in L^p$ *and all* $B \in \mathscr{B}(\Omega)$ *such that* $v = v'$ *a.e in* $B$.

PROOF. Let $v, v' \in L^p$ and let $B \in \mathscr{B}(\Omega)$ such that $v = v'$ a.e in $B$. If $(u_h, A_h)$ and $(u'_h, A'_h)$ are local partitions of $v$ and $v'$ respectively, by (3.2) we get $Du_h = Du'_k$ a.e. in $B \cap A_h \cap A'_k$ for all integers $h, k$. Hence, taking into account hypothesis (i) of Theorem 2.2,

$$F(u_h, B \cap A_h \cap A'_k) = F(u'_k, B \cap A_h \cap A'_k)$$

for all $h, k \in \mathbb{N}$. Arguing as in the proof of Proposition 3.3 we obtain

$$\begin{aligned} G(v, B) &= \sum_{h\in\mathbb{N}} F(u_h, B \cap A_h) \\ &= \sum_{h,k\in\mathbb{N}} F(u_h, B \cap A_h \cap A'_k) = \sum_{h,k\in\mathbb{N}} F(u'_k, B \cap A_h \cap A'_k) \\ &= \sum_{k\in\mathbb{N}} F(u'_k, B \cap A'_k) = G(v', B) \ . \square \end{aligned}$$

LEMMA 3.7. *For all $B \in \mathscr{B}(\Omega)$ the function $G(\cdot, B)$ is lower semicontinuous in the strong topology of $L^p$.*

PROOF. An easy computation shows that it is enough to prove that

$$G(v, B) \leq \liminf_{h\to\infty} G(v + v_h, B) \tag{3.5}$$

whenever $v_h$ are functions in $L^p$ such that $\|v_h\|_p \leq 4^{-h}$ for every $h \in \mathbb{N}$. By Proposition 3.2, for every $h \in \mathbb{N}$ we may choose a local partition $(u_{h,j}, A_{h,j})$ of $v_h$ such that

$$|\Omega \setminus A_{h,0}| < 2^{-h} \qquad \text{and} \qquad \|u_{h,0}\|_{W^{1,p}} \leq C\, 2^{(1-1/p)h} \|v_h\|_p \leq C\, 2^{-h} \ , \tag{3.6}$$

where $C$ is a constant which does not depend on $h$.
Fix an integer $k$. Choose a local partition $(u_h, A_h)$ of $v$ such that $|\Omega \setminus A_0| \leq 2^{-k}$ (cf. Proposition 3.2) and set

$$C_k = A_0 \cap \Big( \bigcap_{h\geq k} A_{h,0} \Big) \ .$$

By the definition of $G$, for all $h, k \in \mathbb{N}$ we get

$$\begin{aligned} &G(v, B \cap C_k) = F(u_0, B \cap C_k) \\ &G(v + v_h, B \cap C_k) = F(u_0 + u_{h,0}, B \cap C_k) \qquad \text{whenever} h \geq k, \end{aligned}$$

and taking into account that $u_0 + u_{h,0}$ converge to $u_0$ by (3.6), and that $F$ is lower semicontinuous (hypothesis (iii) of Theorem 2.2), we obtain

$$\begin{aligned} G(v, B \cap C_k) &= F(u_0, B \cap C_k) \\ &\leq \liminf_{h\to\infty} F(u_0 + u_{h,0}, B \cap C_k) \\ &= \liminf_{h\to\infty} G(v_0 + v_{h,0}, B \cap C_k) \ . \end{aligned} \tag{3.7}$$

Note that by definition of $C_k$ and by (3.6)

$$|\Omega \setminus C_k| \leq |\Omega \setminus A_0| + \sum_{h \geq k} |\Omega \setminus A_{h,0}| \leq 2^{-k} + \sum_{h \geq k} 2^{-h} = 3 \cdot 2^{-k} ,$$

so that $|\Omega \setminus C_k|$ converge to 0 as $k \to \infty$. Hence, for every $t < G(v, B)$ there exists an integer $k$ such that $t \leq G(v, B \cap C_k)$ and inequality (3.7) and the fact that $G$ is positive yield

$$\begin{aligned} t &\leq G(v, B \cap C_k) \\ &\leq \liminf_{h \to \infty} G\left(v_0 + v_{h,0}, B \cap C_k\right) \\ &\leq \liminf_{h \to \infty} G\left(v_0 + v_{h,0}, B\right) \ . \end{aligned}$$

Therefore (3.5) is satisfied because $t$ is any real number less than $G(v, B)$. □

PROOF OF THEOREM 2.2

By Lemmas 3.5, 3.6 and 3.7 we have that $G$ satisfies hypotheses (i), (ii), (iii) of Theorem 2.8. Lemma 3.4 and (iv) of Theorem 2.2 imply that hypothesis (iv) of Theorem 2.8 holds with $\bar{v} = D\bar{u}$. Then there exists a Borel function $f : \Omega \times \mathbb{R}^{mn} \to [0, +\infty]$ which is lower semicontinuous in the second variable and such that

$$G(v, B) = \int_B f\big(x, v(x)\big)\, dx$$

for every $(v, B) \in L^p \times \mathscr{B}(\Omega)$. Lemma 3.4 again implies

$$F(u, B) = G(Du, B) = \int_B f\big(x, Du(x)\big)\, dx$$

for every $(u, B) \in W^{1,p} \times \mathscr{B}(\Omega)$, and then we have proved (a) and (b). The uniqueness of the integrand $f$ follows for instance from Corollary 6 of Alberti [1]. □

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