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A nonlocal anisotropic model for phase transitions

Part I: the optimal profile problem

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1. Introduction

In this paper we study some problems related to a nonlocal model in phase transitions. More precisely we consider the free energy

$$F(u) := \frac{1}{4} \iint J(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx + \int W(u(x))\,dx \ , \qquad (1.1)$$

where u is a scalar density function on a domain of $\mathbb{R}^N$ and takes values in $[-1,1]$, W is a positive double-well potential which vanishes at ± 1, and J is a positive, possibly anisotropic, interaction potential which vanishes at infinity (see paragraph 1.2 for precise definitions).

The scalar function u represents the macroscopic density profile of a system which has two equilibrium pure phases described by the profiles $u \equiv +1$ and $u \equiv -1$. The integral $\int W(u)$ at the right side of (1.1) forces a minimizer of F to take values close to $+1$ and -1 (phase separation), while the double integral represents an interaction energy which penalizes the spatial inhomogeneity of the system (surface tension).

In equilibrium Statistical Mechanics functionals of the form (1.1) arise as free energies of continuum limits of Ising spin systems on lattices; in this setting u plays the rôle of a macroscopic magnetization density and J is a ferromagnetic Kac potential (see for instance [2] and references therein).

We underline the analogy with the more familiar gradient theory for phase transition proposed in [9], where the free energy of the system is of the form

$$E(u) := \frac{1}{2} \int |\nabla u|^2 + \int W(u) \ . \qquad (1.2)$$

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Indeed the free energy in (1.1) is obtained by replacing the (local) gradient energy $\frac{1}{2}\int|\nabla u|^2$ in (1.2) with the (nonlocal) interaction energy $\frac{1}{4}\iint J(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx$.

In this paper we study the optimal profiles for the interface; in the one-dimensional case the optimal profile is a minimizer of F among all functions $u:\mathbb{R}\to[-1,1]$ which satisfy the boundary conditions

$$\lim_{x\to+\infty} u(x)=1 \quad\text{and}\quad \lim_{x\to-\infty} u(x)=-1\ . \tag{1.3}$$

In the N-dimensional case the optimal profile depends on the choice of a direction, and is a function on $\mathbb{R}^N$ which minimizes F subject to more complicated boundary conditions.

The main goal of this paper is twofold: first we prove the existence of the optimal profile both in the one and in the N-dimensional case (Theorems 2.4 and 3.3); secondly we show that in the N-dimensional case the optimal profile for a "plane" transition in the direction e is a function which varies only in the direction e, i.e., it is invariant with respect to translations orthogonal to e (Theorem 3.3). Precise statements and definitions are given in paragraphs 1.3 and 1.4. In a special isotropic case this result was already proved in [2], using completely different techniques.

A different approach to the optimal profile problem may be obtained by considering the Euler-Lagrange equation associated with the functional F: when $\int J(x)\,dx=1$ it reads as

$$J*u-u=\dot W(u)\ . \tag{1.4}$$

Equation (1.4) has been widely studied in the one-dimensional case, in particular in connection with the parabolic equation

$$u_t=J*u-u-f(u)\ , \tag{1.5}$$

where f is the derivative of a potential with two wells at possibly different depth at ±1. In a sequence of papers [10-13] De Masi et al. established existence, uniqueness and stability of the stationary solutions and travelling waves for a particular class of equations of type (1.5). These results have been proved under general assumptions in [6], and then have been extended to the N-dimensional case in [8] (we refer to [13], [6] and [8] for detailed references).

In a forthcoming paper [1] we study the asymptotic behavior as ε tends to 0 of the rescaled energies

$$F_\varepsilon(u):=\frac{1}{4\varepsilon}\iint J_\varepsilon(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx+\frac{1}{\varepsilon}\int W(u(x))\,dx\ , \tag{1.6}$$

where $J_\varepsilon(y):=\varepsilon^{-N}J(y/\varepsilon)$. We show that the functionals F_ε converge in a variational sense to a limit energy $F_0(u)$ which is finite only when $u=\pm1$ everywhere, and is given by

$$F_0(u):=\int_S \sigma(\nu)\ , \tag{1.7}$$

where S is the interface between the phases $u = +1$ and $u = -1$, ν is the unit normal to S, and for every unit vector e, $\sigma(e)$ may be computed via the optimal profile associated with the direction e. Moreover the one-homogeneous extension $x \mapsto |x|\,\sigma\big(x/|x|\big)$ is a convex function on $\mathbb{R}^N$ (see [1]). This result shows that the interface energy in the classical treatment of phase separation is recovered as the limit of the rescaled free energy F_ε, and then $\sigma(e)$ represents the surface tension with respect to the direction e.

Finally we underline that the fact that for every direction e there exists a one-dimensional optimal profile (that is, invariant under translation orthogonal to e) shows that no wrinkling instability occurs in ferromagnetic Ising systems, i.e., when the interaction potential J is non-negative, independently of the anisotropy of J; the situation is known to be quite different in the non-ferromagnetic case.

Similar conclusions have been proved in a different way by Katsoulakis and Souganidis: they showed in the recent paper [15] that in the limit $\varepsilon \to 0$ the gradient flows associated with the energies F_ε converge after a suitable rescaling to an anisotropic (local) surface flow of parabolic type. It would be interesting to understand whether a direct mathematical relation exists between Theorem 3.3 and the convergence result described in [15].

Before stating the main results we briefly introduce the notation adopted in this paper.

1.1. Notation

We use the terms *increasing* and *decreasing* in the weak sense, that is, to mean non-decreasing and non-increasing respectively. We denote by (a, b) and $[a, b]$ the open and the closed intervals respectively (an interval is not necessarily bounded). Given $a, b \in \mathbb{R}$, $a \vee b$ and $a \wedge b$ denote respectively the maximum and the minimum of $\{a, b\}$.

The first and second (distributional) derivatives of a function u defined on the real line are denoted by $\dot{u}$ and $\ddot{u}$ respectively.

Unless otherwise specified, all functions and sets are assumed Borel measurable, and we often omit explicit mention to measurability properties. Also, if it is not explicitly mentioned we do not identify functions which agree almost everywhere.

We usually consider integrals on (subsets of) $\mathbb{R}^k$ or some k-dimensional affine subspaces of $\mathbb{R}^N$: unless otherwise specified we integrate with respect to the k-dimensional Hausdorff measure (that is, the usual Lebesgue measure on $\mathbb{R}^k$) and we often omit any explicit mention to the measure, that is, we write $\int f(x)\,dx$ for the integral of a function f and $|B|$ for the measure of a set B. When we make use of a different measure, it is always a bounded or locally bounded measure on Borel sets, and we never omit to write it explicitly.

Let us define now some specific notation for our problem.

1.2. Hypotheses on J and W

Throughout this paper N is the dimension of the space. Unless differently stated (cf. Remark 2.2) the functions J and W satisfy the following assumptions:

(i) W is a *double-well potential*, that is, a non-negative continuous real function on $\mathbb{R}$ which is 0 only at ± 1 and tends to $+\infty$ at infinity;

(ii) J is a *positive interaction potential*, that is, a non-negative function on $\mathbb{R}^N$ which is even ($J(h) = J(-h)$), belongs to $L^1(\mathbb{R}^N)$, and satisfies

$$\int_{\mathbb{R}^N} J(h)\,|h|\,dh < \infty \ . \tag{1.8}$$

1.3. The optimal profile in dimension one

In Sect. 2 we study the optimal profile in the case $N = 1$. More precisely, for every function $u : \mathbb{R} \to \mathbb{R}$ we set

$$F(u) := \frac{1}{4}\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,\big(u(x') - u(x)\big)^2 dx'dx \ + \int_{\mathbb{R}} W\big(u(x)\big)\,dx \ , \tag{1.9}$$

and then we consider the minimum problem

$$\min\big\{F(u) :\ u \in X\big\} \ , \tag{1.10}$$

where

$$X := \big\{u : \mathbb{R} \to [-1,1] :\ (1.3) \text{ holds}\big\} \ . \tag{1.11}$$

Notice that F is not identically equal to $+\infty$ on X (take for instance $u(x) := 1$ for $x \geq 0$, $u(x) := -1$ for $x < 0$).

In Theorem 2.4 we show that the minimum problem (1.10) admits an *increasing* solution. Every increasing solution of (1.10) is called *optimal profile associated with* F and is denoted by γ. The value $\sigma := F(\gamma)$ of the minimum in (1.10) is called the *surface tension associated with* F. The proof of Theorem 2.4 essentially relies on a rearrangement result given in Sect. 5 (cf. paragraph 1.6).

Notice that the if u solves (1.10) so does every translation of u. Therefore the notion of optimal profile is translation invariant. A uniqueness result for the optimal profile (up to translation) is given in Theorem 4.1.

In Theorem 2.11 we show that

$$F(u) = F^\circ(u^{-1}) \quad \text{for every increasing } u \in X, \tag{1.12}$$

where $u^{-1} : (-1,1) \to \mathbb{R}$ is the inverse of u (cf. Definition 2.6) and F° is the integral functional given in Definition 2.8. The main feature of F° is convexity (Proposition 2.10); therefore an increasing function v minimizes F° if and only if the first variation of F° at v is zero, and this condition may be explicitly computed (Theorem 2.14). As a corollary we also obtain a characterization of the increasing solutions of the minimum problem (1.10) (see Corollary 2.16): in the particular case that u is strictly increasing, then it solves (1.10) if and only if

$$W(s) := -\int\limits_{\substack{s<t'<1\\ -1<t<s}} \dot{K}\big(v(t') - v(t)\big)\,dt'dt \quad \text{for every } s \in (-1,1)\,. \tag{1.13}$$

where $v := u^{-1}$ and K is given in (2.8). We remark that this characterization allows us to find the potential W once the interaction potential J and the optimal profile γ are assigned (see paragraph 2.19). This fact is used in the proof of Theorem 3.3.

1.4. The optimal profile in arbitrary dimension

In Sect. 3 we study the optimal profile in arbitrary dimension $N > 1$.

For every $\Omega \subset \mathbb{R}^N$ and every $u : \mathbb{R}^N \to \mathbb{R}$ we define the functional $F(u, \Omega)$ by setting

$$F(u,\Omega) := \frac{1}{4} \int\limits_{x\in\Omega,\, h\in\mathbb{R}^N} J(h) \big(u(x+h) - u(x)\big)^2 dx\, dh \ + \int\limits_{x\in\Omega} W\big(u(x)\big)\, dx \ . \tag{1.14}$$

In order to define the optimal profile, we must fix some notation.

We first choose a direction, that is, a unit vector $e \in \mathbb{R}^N$, and we denote by M the orthogonal complement of e. We say that a function u on $\mathbb{R}^N$ *varies only in the direction* e if there exists a function $\bar{u}$ on $\mathbb{R}$ such that $u(x) = \bar{u}(x_e)$ for every $x \in \mathbb{R}^N$; here and in the following $x_e := \langle e, x\rangle$ is the component of x in the direction e.

Then we choose a basis $\big\{e_1, \dots, e_{N-1}\big\}$ of M, and we take the $(N-1)$-dimensional rectangle $A := \big\{ \sum \alpha_i e_i : 0 \le \alpha_i \le 1,\ i = 1, \dots, N-1\big\}$. We say that a function u on $\mathbb{R}^N$ is *A-periodic* if $u(x + e_i) = u(x)$ for every $x \in \mathbb{R}^N$ and every $i = 1, \dots, N-1$.

Now we consider the minimum problem

$$\min\Big\{ \frac{F(u, A\times\mathbb{R})}{|A|} \ : \ u \in X_A^e \Big\} \ , \tag{1.15}$$

where $A \times \mathbb{R}$ is the stripe with direction e and section A, that is, $A \times \mathbb{R} := \big\{ y + te : y \in A,\ t \in \mathbb{R}\big\}$, and

$$X_A^e := \Big\{ u : \mathbb{R}^N \to [-1,1] \ : \ u \text{ is } A\text{-periodic and } \lim_{x_e\to\pm\infty} u(x) = \pm 1 \Big\} \ . \tag{1.16}$$

In other words we choose an unbounded stripe $A \times \mathbb{R}$ parallel to the direction e, and we minimize $F(u, A \times \mathbb{R})$ among all functions u which take values $+1$ and -1 at the two "ends" of the stripe and are extended periodically outside.

In Theorem 3.3 we show that the minimum problem (1.15) admits a solution which varies only in the direction e; moreover this solution is given by the optimal profile γ_e associated with a suitably defined one-dimensional functional F^e, then the value of the minimum (1.15) is equal to $F^e(\gamma_e)$ and in particular it is independent of the choice of A.

The function γ_e is called the *optimal profile associated with* F *with respect to the direction* e, while the value $\sigma(e) := F^e(\gamma_e)$ of the minimum (1.15) is called the *surface tension associated with* F *with respect to the direction* e (notice that $\sigma(e) = \sigma(-e)$). The last definition is motivated by the asymptotic behavior of the rescaled functionals F_ε given in (1.6): the function σ is in fact the same as in formula (1.7).

The fact that there exists a solution to (1.15) which varies only in the direction e is the main result of this paper.

1.5. Further remarks, generalizations and open problems

In Sect. 4 we add some remarks and we briefly discuss possible generalizations of the results of the previous sections and some open problems.

In Subsect. 4a we use the Euler-Lagrange equation associated with F and a uniqueness result proved in [6] to deduce that under certain additional assumptions on J and W the one-dimensional optimal profile is unique (up to translations).

In Subsect. 4b we consider the multi-phase extension of our model, that is, when F is defined as in (1.1) but the density function $u : \mathbb{R}^N \to \mathbb{R}^k$ is vector-valued, and the potential W is a non-negative continuous function on $\mathbb{R}^k$ which vanishes at $d+1$ affinely independent wells.

In Subsect. 4c we discuss the optimal assumptions on the interaction potential J. The hypothesis $J \geq 0$ (which in the Statistical Mechanics setting is referred to as the *ferromagnetic condition*) cannot be removed in the present setting, while the summability hypotheses on J, namely $J \in L^1(\mathbb{R}^N)$ and (1.8), can be weakened in order to allow singular potentials which do not belong $L^1(\mathbb{R}^N)$ (cf. Theorem 4.6). Indeed the main results of Sects. 2 and 3 hold whenever J is a non-negative even function on $\mathbb{R}^N$ such that $\int_{\mathbb{R}^N} J(h)\,\big(|h| \wedge |h|^2\big)\,dh < \infty$.

Eventually in Subsect. 4d we discuss the lower semicontinuity of functionals of type (1.1) and we obtain some existence results for minimization problems on bounded domains. Minimization problems related to functionals of the form (1.1) have been studied in the setting of Young measures in [16] and [19]; other examples of non-local energy functionals have been considered for instance in [17], [18], [20] and [21].

1.6. A rearrangement result

In Appendix 5 we state and prove a rearrangement result which is the cornerstone of the proof of Theorem 2.4.

Given $u \in X$ we define the increasing rearrangement u^* as follows: for every $t \in (-1,1)$ the level set $E_t := \{x : \, u(x) \geq t\}$ can be written as the disjoint union of a bounded set A_t and a half-line $(a_t, +\infty)$ (cf. (1.11)); u^* is then obtained by replacing each level set E_t with the half-line $(a_t - |A_t|, +\infty)$.

In Theorem 5.6 we show that replacing a function in X with its increasing rearrangement preserves every integral of the form $\int_{\mathbb{R}} W(u)\,dx$. In Theorem 5.8 we show that this substitution decreases every integral of the form

$$\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,L\big(u(x')-u(x)\big)\,dx'dx$$

where J is non-negative and L is convex and non-negative. Therefore also the functional F given in (1.9) decreases when u is replaced by u^* (Proposition 2.3).

2. The optimal profile problem: the one-dimensional case

In this section we study the minimum problem (1.10), which defines the optimal profile for the interface in the one-dimensional case. We adopt a notation which is slightly different from that one introduced in Sect. 1, so we begin by recalling some basic definitions.

Definition 2.1. *In the rest of this section $W : \mathbb{R} \to \mathbb{R}$ is a non-negative continuous function which vanishes at ± 1 (cf. paragraph 1.2) and $J : \mathbb{R} \to \mathbb{R}$ is a non-negative even function which satisfies*

$$\int_0^\infty J(h)\,(1+h)\,dh < \infty \ . \tag{2.1}$$

The functional F is now defined as in (1.9) *and X is given in* (1.11).

Remark 2.2. While the assumptions on J are the same as in paragraph 1.2, here the potential W may be not strictly positive out of -1 and 1. In fact in the proof of Theorem 3.3 we apply some results of this section (namely Theorem 2.14 and Corollary 2.16) to functionals of the form (1.9) where W is only non-negative and vanishes at ± 1.

The key lemma of our approach is the following:

Proposition 2.3. *If u belongs to X and u^* is the increasing rearrangement of u (see Definition 5.5), then*

$$F(u^*) \le F(u) \ . \tag{2.2}$$

Proof. Apply Theorems 5.6 and 5.8. □

We shall also need the following immediate identity: for every function $u : \mathbb{R} \to \mathbb{R}$ there holds

$$\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx = \int_{\mathbb{R}} J(h)\,\big\|\tau_h u - u\big\|_2^2\,dh \ , \tag{2.3}$$

here $\tau_h u$ denotes the translated function $u(x-h)$.

2a. Existence of the optimal profile

The aim of this subsection is to prove that F attains a minimum on X. The idea is to use the rearrangement results given in Proposition 2.3 to show that it is enough to minimize F on the subclass of all $u \in X$ which are increasing. Provided some minor additional conditions are fulfilled, it may be easily proved that in this case minimizing sequences are compact (with respect to a suitable topology).

Theorem 2.4. *Let F and X be given as before and assume that W is strictly positive in* $(-1,1)$. *Then the minimum problem*

$$\min\big\{F(u) :\, u \in X\big\}$$

has a solution. More precisely we can find a minimizer $u \in X$ which is increasing and satisfies $u(x) \geq 0$ for $x > 0$, $u(x) \leq 0$ for $x < 0$.

Remark 2.5. As we remarked in paragraph 1.3, if u minimizes F on X, so does every translation of u. Moreover one easily verifies that replacing $u : \mathbb{R} \to \mathbb{R}$ with the truncated function $(u \vee 1) \wedge -1$ decreases the value of F, and then minimizing F on X is equivalent to minimizing F on the larger class of all functions $u : \mathbb{R} \to \mathbb{R}$ which converge to ± 1 at $\pm\infty$.

Proof of Theorem 2.4. We denote by X' the class of all $u \in X$ which satisfy the following additional conditions:

(i) u is increasing,

(ii) $u(x) \leq 0$ for all $x < 0$ and $u(x) \geq 0$ for all $x > 0$.

We shall prove the claim by showing that

(a) *the infimum of F on X is equal to the infimum of F on X',*

(b) *F attains a minimum on X'.*

We point out that the hypothesis $W > 0$ on $(-1, 1)$ will be used only in the proof of statement (b).

The proof of statement (a) is divided into three steps.

Step 1. For every $u \in X$ we may consider the increasing rearrangement u^* of u given in Definition 5.5. Then u^* satisfies (i) and by Proposition 2.3 there holds $F(u^*) \leq F(u)$.

Step 2. For every $u \in X$ which satisfies (i), there exists at least one point $y \in \mathbb{R}$ such that $u(x) \leq 0$ for $x < y$ and $u(x) \geq 0$ for $x > y$ (we take $y = 0$ if u already satisfies (ii)). We set $\tau u(x) := u(x + y)$ for all $x \in \mathbb{R}$. Then τu satisfies (i), (ii), and since F is translation invariant,

$$F(\tau u) = F(u) \ . \tag{2.4}$$

Step 3. For every $u \in X$ we set $Pu := \tau(u^*)$. By steps 1 and 2 this definition is well-posed, Pu belongs to X', and by (2.2) and (2.4) we have $F(Pu) \leq F(u)$. Moreover $Pu = u$ for every $u \in X'$, and then P is a projection of X onto X' which decreases the functional F. Hence statement (a) is proved.

Let us prove (b). We remark that by Fatou's lemma F is lower semicontinuous with respect to convergence almost everywhere. Hence it is enough to prove that from every minimizing sequence $(u_n) \subset X'$ we can extract a subsequence which converges almost everywhere to some $u \in X'$.

Taking (i) into account, we have that the distributional derivative $\dot{u}_n$ of u_n is a positive measure on $\mathbb{R}$ with $\|\dot{u}_n\| = 2$ for every n. Therefore the sequence (u_n) is bounded in $BV_{\rm loc}(\mathbb{R})$ and then it is relatively compact in $L^1_{\rm loc}(\mathbb{R})$ by the compact embedding theorem, and we may extract a subsequence $(u_k) = (u_{n_k})$ which converges almost everywhere to some u in $BV_{\rm loc}(\mathbb{R})$.

Clearly, u is (a.e. equal to) a function which satisfies (i) and (ii) and then it remains to show that $u(x)$ converges to ± 1 as x tends to $\pm\infty$. Since u is increasing, there exist

$$\alpha := \lim_{x \to -\infty} u(x) \quad \text{and} \quad \beta := \lim_{x \to +\infty} u(x) \ ,$$

and taking (ii) into account, we have $-1 \leq \alpha \leq 0 \leq \beta \leq 1$. If we assume by contradiction that either $\alpha \neq -1$ or $\beta \neq 1$, and recall that W is continuous and strictly positive in $(-1,1)$, we obtain that $\int_{\mathbb{R}} W\big(u(x)\big)\,dx = +\infty$. Hence $F(u) = +\infty$, and this is impossible because $F(u) \leq \liminf_k F(u_k) < +\infty$. $\square$

2b. The conjugate functional

The functional F given in (1.9) is clearly not convex, and indeed proving the existence of a minimizer on X was not immediate. Finding conditions on a given function in X which ensure that it minimizes F is even more difficult: we can easily write the equivalent of the Euler-Lagrange equation for F and a solution of this equation is a critical point for F, but may be not a minimizer because F is not convex.

In this subsection we show that $F(u)$ may be written as $F^\circ(u^{-1})$ for every increasing function $u \in X$, where u^{-1} is the inverse of u and F° is called the *conjugate functional of* F: F° is explicitly computed, and turns out to be a convex integral functional (see Definition 2.8, Proposition 2.10 and Theorem 2.11).

Definition 2.6. *Let $u \in X$ be an increasing function. We define the inverse of u as the function $u^{-1} : (-1,1) \to \mathbb{R}$ given by*

$$u^{-1}(t) := \inf\big\{x \,:\, t \leq u(x)\big\} \qquad \textit{for all } t \in (-1,1). \tag{2.5}$$

The function u^{-1} is well-defined and increasing. Moreover it agrees with the usual left inverse of u when u is strictly increasing. For a general increasing $u \in X$ there holds $u^{-1}\big(u(x)\big) = \inf\big\{x' :\, u(x') = u(x)\big\}$ for every $x \in \mathbb{R}$.

This definition of inverse function enjoys the following essential property:

Proposition 2.7. *Let (u_n) be a sequence of increasing functions in X which converges almost everywhere to u. Then the sequence (u_n^{-1}) converges almost everywhere to u^{-1} on $(-1,1)$ (in fact everywhere except for a countable set).*

Proof. We say that u is *right strictly increasing at the point* $x \in \mathbb{R}$ if u is never constant in the interval $(x, x_1]$ for every $x_1 > x$. We prove that $u_n^{-1}(t) \to u^{-1}(t)$ for every $t \in (-1,1)$ such that u is right strictly increasing at $u^{-1}(t)$, and then we show that the set of all points t which do not verify this condition is countable.

Set $x := u^{-1}(t)$ and take $\varepsilon > 0$. Since (u_n) converges to u almost everywhere, we may find x_0, x_1 so that $x - \varepsilon \leq x_0 < x < x_1 \leq x + \varepsilon$ and (u_n) converges to u at x_0 and x_1. Since $x_0 < x = u^{-1}(t)$, (2.5) yields $u(x_0) < t$. Since $x_1 > x = u^{-1}(t)$, (2.5) yields $u(x_1) \geq t$, and $u(x_1) = t$ implies that u is constant in the interval $(x, x_1]$. Therefore if u is right strictly increasing at x we have that

$$u(x_0) < t < u(x_1)\ . \tag{2.6}$$

By (2.6) there exists $\bar{n}$ so that $u_n(x_0) < t < u_n(x_1)$ for every $n \geq \bar{n}$, and then (2.6) yields $x_0 \leq u_n^{-1}(t) \leq x_1$. Hence $x - \varepsilon \leq u_n^{-1}(t) \leq x + \varepsilon$, and since ε is arbitrary, we have proved that $u_n^{-1}(t)$ converges to $x = u^{-1}(t)$.

Finally we denote by $\{I_i\}$ the collection of all maximal open intervals where u is constant. This collection is countable, and if u is not right strictly increasing at x then x belongs to the closure of one of these intervals. Therefore if u is not right strictly increasing at $u^{-1}(t)$ then t belongs to the union of all $u(I_i)$, that is, a countable set. □

Now we can give the definition of conjugate functional F°.

Definition 2.8. *For every increasing function $v : (-1,1) \to \mathbb{R}$ we set*

$$F^\circ(v) := \int\limits_{-1<t<t'<1} K\big(v(t') - v(t)\big)\, dt' dt \ + \int\limits_{-1<t<1} W(t)\, d\dot v(t) \ , \tag{2.7}$$

where $\dot v$ stands for the distributional derivative of v (since v is an increasing function, $\dot v$ is a positive measure on $(-1,1)$) and $K : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) := \max\{-x, 0\} \ , \quad K(x) := J * f\,(x) \ . \tag{2.8}$$

Remark 2.9. Since f and J are non-negative, the convolution product which defines K is well-defined and finite by (2.1).

Since f is Lipschitz, convex, decreasing and non-negative and J is non-negative, then K is of class C^1, convex, decreasing and non-negative. Since the second order (distributional) derivative of f is the Dirac mass centered at 0, then $\ddot K = J$ (in the sense of distributions). For every $x \geq 0$ there holds

$$0 \leq K(x) \leq K(0) = \int_0^\infty J(h)\, h\, dh \ , \quad 0 \geq \dot K(x) \geq \dot K(0) = -\int_0^\infty J(h)\, dh \ . \tag{2.9}$$

Moreover

$$\lim_{x\to+\infty} K(x) = \lim_{x\to+\infty} \dot K(x) = 0 \ . \tag{2.10}$$

The proof of the following proposition is a consequence of the convexity of K and we omit it.

Proposition 2.10. *The functional F° is convex on the set of all increasing functions $v : (-1,1) \to \mathbb{R}$.*

Now we can state and prove the main result of this subsection.

Theorem 2.11. *Under the previous assumptions, for every increasing function $u \in X$ there holds*

$$F(u) = F^\circ(u^{-1}) \ . \tag{2.11}$$

Proof. Let be given an increasing function $u \in X$ and set $v := u^{-1}$. We will prove the following two equalities:

$$\int_{\mathbb{R}} W\big(u(x)\big)\, dx = \int\limits_{-1<t<1} W(t)\, d\dot v(t) \ , \tag{2.12}$$

$$\frac{1}{4}\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\, \big(u(x') - u(x)\big)^2\, dx' dx = \int\limits_{-1<t<t'<1} K\big(v(t') - v(t)\big)\, dt' dt \ . \tag{2.13}$$

The proof of equality (2.12) is divided into two steps.

Step 1. Assume that W is 0 in $[-1,-1+\varepsilon]$ and $[1-\varepsilon,1]$ for some $\varepsilon>0$.

Take a sequence of strictly increasing functions u_n with smooth inverse which converges to u almost everywhere in $\mathbb{R}$ and set $v_n := u_n^{-1}$. By Proposition 2.7 the sequence (v_n) converges to $v := u^{-1}$ almost everywhere in $(-1,1)$ and then in $BV_{\rm loc}(-1,1)$. Hence $\dot v_n \rightharpoonup \dot v$ in the sense of measures. Passing to the limit as $n\to\infty$, we obtain

$$\begin{aligned}\int_{\mathbb{R}} W\big(u(x)\big)\,dx &=^{(1)} \lim_{n\to+\infty}\int_{\mathbb{R}} W\big(u_n(x)\big)\,dx\\ &=^{(2)} \lim_{n\to+\infty}\int_{-1}^{1} W(t)\,\dot v_n(t)\,dt =^{(3)} \int_{-1}^{1} W(t)\,d\dot v(t)\ .\end{aligned}$$

Step 2. We prove (2.12) in the general case by approximating W with an increasing sequence of non-negative continuous functions W_n which are 0 in a neighborhood of -1 and 1 and then applying monotone convergence theorem.

Also the proof of equality (2.13) is divided into two steps.

Step 1. Assume first that $u:\mathbb{R}\to(-1,1)$ is a strictly increasing continuous function with smooth inverse $v:(-1,1)\to\mathbb{R}$. Then

$$\begin{aligned}\frac14\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx \ &= \\ &=^{(4)} \frac12 \int\limits_{-1<t<t'<1} \ddot K\big(v(t')-v(t)\big)\,(t'-t)^2\,\dot v(t')\,\dot v(t)\,dt'dt\ .\end{aligned}$$

Now we remark that $\ddot K\big(v(t')-v(t)\big)\ \dot v(t')\,\dot v(t)$ is the second partial derivative of the function $-K\big(v(t')-v(t)\big)$ with respect to t' and t, and then (2.13) is obtained by integrating by parts twice, first with respect to t' and then with respect to t (we use (2.10) to show that no boundary contributions arise in both integrations by parts).

Step 2. We extend (2.13) to a general increasing u by approximation. We take $u_\varepsilon := u*\rho_\varepsilon$ where $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$ and ρ is a smooth function on $\mathbb{R}$ such that $\rho>0$ everywhere and $\|\rho\|_1=1$, and we set $v_\varepsilon := u_\varepsilon^{-1}$.

Each u_ε is smooth and $\dot u_\varepsilon = \dot u*\rho_\varepsilon>0$ everywhere; hence also v_ε is smooth. The functions u_ε converge a.e. to u when $\varepsilon\to0$, and then v_ε converge a.e. to $v:=u^{-1}$ by Proposition 2.7. Moreover $\|\tau_h u_\varepsilon - u_\varepsilon\|_2 = \|(\tau_h u-u)*\rho_\varepsilon\|_2$ increases to $\|\tau_h u-u\|_2$ for every h, and using identity (2.3) we get

$$\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,\big(u(x')-u(x)\big)^2 dx'dx = \lim_{\varepsilon\to0}\int_{\mathbb{R}\times\mathbb{R}} J(x'-x)\,\big(u_\varepsilon(x')-u_\varepsilon(x)\big)^2 dx'dx\ . \tag{2.14}$$

---

(1) Apply dominated convergence theorem, recalling that W has compact support in $(-1,1)$.

(2) Apply the change of variable $x=v_n(t)$.

(3) Recall that $\dot v_n \rightharpoonup \dot v$ weakly* in $\mathcal{M}_{\rm loc}(-1,1)$ and $W\in C_c(-1,1)$.

(4) We make the change of variable $x=v(t)$ and $x'=v(t')$, using that J is even and $J=\ddot K$ (see Definition 2.8).

We remark that identity (2.13) holds for every u_ε by step 1, and passing to the limit as $\varepsilon \to 0$ we recover (2.13) for u (the left side converges by (2.14), while the right side converges by the dominated convergence theorem). $\square$

2c. Characterization of the minimizers of F°

In this subsection we characterize the functions which minimize the conjugate functional F° in the class of all increasing functions $v : (-1,1) \to \mathbb{R}$, and therefore we are able to characterize also the increasing minimizers of F in X (see Theorem 2.14 and Corollary 2.16). Roughly speaking, since F° is convex then a function v minimizes F° if and only if the first variation of F° at v is 0, and the first variation of F° can be easily computed in many cases (see Lemma 2.22).

Definition 2.12. *We denote by Y the class of all increasing functions $v : (-1,1) \to \mathbb{R}$. With every $v \in Y$ we associate the function $Hv : [-1,1] \to \mathbb{R}$ given by*

$$Hv(s) := \int\limits_{\substack{s<t'<1 \\ -1<t<s}} -\dot{K}\big(v(t') - v(t)\big)\, dt' dt \quad \textit{for every } s \in [-1,1] \tag{2.15}$$

(when $s = \pm 1$ the integration domain in (2.15) *is empty, and then $Hv(\pm 1) = 0$).*

Since all the functionals we are concerned with take the same value on functions which agree almost everywhere, for the rest of this section we shall make no difference between functions and equivalence classes. In particular we shall denote by Y also the class of all functions which agree almost everywhere with an increasing function, that is, all v whose distributional derivative is a locally finite positive measure on $(-1,1)$.

Proposition 2.13. *For every $v \in Y$ the function Hv given in* (2.15) *is continuous and satisfies*

$$0 \le Hv(s) \le \Big[\int_0^\infty J(h)\, dh \Big] (1 - s^2) \quad \textit{for every } s \in [-1,1]\,. \tag{2.16}$$

Moreover Hv is strictly positive in $(-1,1)$ if either J has unbounded support or v is continuous.

Proof. Since $-\dot{K}(x)$ is non-negative and bounded by $-\dot{K}(0) = \int_0^\infty J(h)\, dh$ when $x \ge 0$ (cf. (2.9)), the integrand in (2.15) is summable and non-negative. Therefore Hv is finite, continuous and satisfies (2.16).

Since $-\dot{K}(0) > 0$ and $-\dot{K}$ is continuous, there exists $\varepsilon > 0$ so that $-\dot{K}(x) > 0$ for all $x \le \varepsilon$. If v is continuous, then for every $s \in (-1,1)$ there exists $\delta > 0$ so that $\big|v(t') - v(t)\big| \le \varepsilon$ when $t, t' \in [s-\delta, s+\delta]$. Hence $-\dot{K}\big(v(t') - v(t)\big) > 0$ for $t' \in [s, s+\delta]$, $t \in [s-\delta, s]$, and then $Hv(s) > 0$.

If J has unbounded domain, then $-\dot{K}$ is strictly positive on $\mathbb{R}$ (cf. (2.8)), and Hv is strictly positive on $(-1,1)$. $\square$

Theorem 2.14. *A function v minimizes F° in Y if and only if $W \geq Hv$ everywhere in $[-1,1]$ and $W = Hv$ everywhere in the support of the measure derivative $\dot v$.*

Remark 2.15. Notice that the support of the measure $\dot v$ is the complement in $[-1,1]$ of the maximal open set where v is locally constant. When v is strictly increasing, the support of $\dot v$ is the interval $[-1,1]$, and v minimizes F° in Y if and only if the equality $W = Hv$ holds in $[-1,1]$. When v is not strictly increasing, it is constant on some interval, and then it belongs to the "boundary" of Y, in the sense that not all infinitesimal variations of v still belongs to Y. This explains why the minimality condition given in Theorem 2.14 sometimes becomes an inequality instead of an equality.

Corollary 2.16. *Let u be an increasing function in X and let $v := u^{-1}$. Then u minimizes F in X if and only if $W \geq Hv$ everywhere in $[-1,1]$ and $W = Hv$ everywhere in the support of the measure $\dot v$.*

Remark 2.17. Notice that the support of $\dot v$ is the essential image of u (that is, the set of all $y \in [-1,1]$ such that $u^{-1}(y-\varepsilon, y+\varepsilon)$ has positive measure for all $\varepsilon > 0$); when u is left or right-continuous the essential image is just the closure of the image.

Remark 2.18. The assumption that W is strictly positive in $(-1,1)$ is essential to prove the existence of the optimal profile (Theorem 2.4) but plays no rôle in the proofs of Theorems 2.11 and 2.14. Indeed in Sect. 3 we shall apply these results without assuming W strictly positive in $(-1,1)$.

2.19. The inverse problem

We apply the previous results to answer the following question: *given an interaction potential J and an increasing function $u \in X$, how can we find a double-well potential W so that u is the optimal profile associated with the corresponding functional F?*

Corollary 2.16 shows that it is enough to set $W := Hu^{-1}$, provided that Hu^{-1} is strictly positive in $(-1,1)$. By Proposition 2.13 this holds for instance when u^{-1} is continuous, (that is, u is strictly increasing), or when J has unbounded support. These conditions are sufficient, but not necessary. In fact if $W(t) \geq C\,(1-t^2)$ where $C := \int_0^\infty J(h)\,dh$, then the optimal profile associated with F is $\gamma(x) := \operatorname{sgn} x$, independently of the choice of J; to prove this fact it is enough to remark that $\gamma^{-1}(t) = 0$ and $(H\gamma^{-1})(t) = C\,(1-t^2) \leq W(t)$ for every $t \in (-1,1)$, and then apply Corollary 2.16.

Yet we remark that this way W is no more regular than continuous, and indeed further regularity of W necessarily implies further regularity of the optimal profile (cf. [6], Sect. 3).

The rest of this section is devoted to prove Theorem 2.14 and Corollary 2.16.

Definition 2.20. *Let $v \in Y$. We say that $w : (-1,1) \to \mathbb{R}$ is an admissible variation for v if $v+w$ belongs to Y. We denote by $\mathcal{A}(v)$ the class of all admissible* *variations for v.*

The following proposition is an immediate corollary of this definition, and we omit the proof.

Proposition 2.21. *Let $v \in Y$. The following statements hold:*
(a) *if $w \in \mathcal{A}(v)$, then $w \in BV_{\mathrm{loc}}(-1,1)$;*
(b) *if $w \in \mathcal{A}(v)$, then $hw \in \mathcal{A}(v)$ for every $h \in (0,1)$;*
(c) *if $w \in BV_{\mathrm{loc}}(-1,1)$, then $w \in \mathcal{A}(v)$ if and only if $\dot w \geq -\dot v$;*
(d) *if $w \in BV_{\mathrm{loc}}(-1,1)$ and $\dot w$ is a positive measure, then $w \in \mathcal{A}(v)$.*

Now we can compute the first variation of F°:

Lemma 2.22. *Take $v \in Y$ such that $F^\circ(v)$ is finite, $w \in \mathcal{A}(v) \cap BV(-1,1)$, and define Hv as in* (2.15)*. Then*

$$\lim_{h\to 0^+} \frac{F^\circ(v+hw) - F^\circ(v)}{h} = \int_{-1}^{1} \big(W(s) - Hv(s)\big)\, d\dot w(s) \ . \tag{2.17}$$

Remark 2.23. The limit at the left side of equality (2.17) is simply the right derivative of F° with respect to the direction w at the point v, and it exists, possibly equal to $\pm\infty$, for every v in the domain of F° and every admissible variation w because the function $h \mapsto F^\circ(v+hw)$ is convex (cf. Proposition 2.10) and finite when $h=0$. On the other hand, the integral at the right side of (2.17) makes sense when w belongs to $BV(-1,1)$ because W and Hv belong to $C_0(-1,1)$ and $\dot w$ is a bounded measure, but it may be not defined when $\dot w$ is only a locally bounded measure.

Proof of Lemma 2.22. For every $h>0$, $t,t' \in (-1,1)$ we set

$$\Phi_h(t',t) := \frac{K\big(v(t') - v(t) + hw(t') - hw(t)\big) - K\big(v(t') - v(t)\big)}{h} \ .$$

Therefore

$$\frac{F^\circ(v+hw) - F^\circ(v)}{h} = \int\limits_{-1<t<t'<1} \Phi_h(t',t)\, dt'dt + \int_{-1}^{1} W(t)\, d\dot w(t) \ . \tag{2.18}$$

We notice that $\Phi_h(t',t)$ converges to $\dot K\big(v(t') - v(t)\big)\big(w(t') - w(t)\big)$ as $h \to 0^+$ whenever $-1 < t < t' < 1$, and then we apply the dominated convergence theorem to pass to the limit in (2.18) as $h \to 0^+$ [5]:

$$\lim_{h\to 0^+} \frac{F^\circ(v+hw) - F^\circ(v)}{h} =$$

[5] In order to apply the dominated convergence theorem, we need a summable upper bound for the functions Φ_h. This may be obtained by the mean value theorem as follows

$$\big|\Phi_h(t',t)\big| \leq \sup_y \big|\dot K(y)\big|\,\big|w(t') - w(t)\big| \leq \Big(2\int_{\mathbb{R}} J(y)\, dy\Big)\|\dot w\| \quad \text{for } t < t'.$$

$$= \int\limits_{-1<t<t'<1} \dot K\big(v(t')-v(t)\big)\big(w(t')-w(t)\big)\,dt'dt \ + \int\limits_{-1<t<1} W(t)\,d\dot w(t)$$

$$= \int\limits_{-1<t<t'<1} \Big[\dot K\big(v(t')-v(t)\big) \int\limits_{t<s<t'} d\dot w(s)\Big]\,dt'dt \ + \int\limits_{-1<s<1} W(s)\,d\dot w(s)$$

$$= \int\limits_{-1<s<1} \Big[W(s) + \int\limits_{\substack{s<t'<1\\ -1<t<s}} \dot K\big(v(t')-v(t)\big)\,dt'dt\Big]\,d\dot w(s)$$

$$= \int_{-1}^{1} \big(W(s)-Hv(s)\big)\,d\dot w(s)\ . \qquad \square$$

Proof of Theorem 2.14. Let $v \in Y$ be a minimizer for F° in Y. We want to show that $W \geq Hv$ everywhere in $[-1,1]$ and $W \leq Hv$ everywhere in the support of $\dot v$. Take $w \in \mathcal{A}(v) \cap BV(-1,1)$. By Proposition 2.21(b) and Lemma 2.22 we obtain

$$\int_{-1}^{1} \big(W(s)-Hv(s)\big)\,d\dot w(s) = \lim_{h\to 0^+} \frac{F^\circ(v+hw)-F^\circ(v)}{h} \geq 0\ . \tag{2.19}$$

Since every positive finite measure μ on $(-1,1)$ is the derivative of a bounded increasing function w, and every bounded and increasing function belongs to $\mathcal{A}(v)\cap BV(-1,1)$ (cf. statement (d) of Proposition 2.21), then (2.19) yields

$$\int_{-1}^{1} \big(W(s)-Hv(s)\big)\,d\mu(s) \geq 0\ .$$

Therefore $W - Hv \geq 0$ μ-almost everywhere for every positive measure μ, that is, $W - Hv \geq 0$ everywhere on $[-1,1]$.

For every Borel set B relatively compact in $(-1,1)$, the restriction of the measure $-\dot v$ to B is the derivative of some decreasing bounded function w, and w belongs to $\mathcal{A}(v)\cap BV(-1,1)$ by statement (c) of Proposition 2.21. Therefore (2.19) becomes

$$\int_B \big(Hv(s)-W(s)\big)\,d\dot v(s) \geq 0\ .$$

Since this holds for every B relatively compact in $(-1,1)$, then $W \leq Hv$ almost everywhere with respect to the measure $\dot v$, and recalling that W and Hv are continuous functions we deduce that $W \leq Hv$ everywhere in the support of $\dot v$. The first implication of Theorem 2.14 is thus proved.

Conversely, let v be a function in Y so that $W \geq Hv$ everywhere in $[-1,1]$ and $W = Hv$ everywhere in the support of the measure $\dot v$, and take an admissible variation $w \in \mathcal{A}(v)\cap BV(-1,1)$. Since $\dot w \geq -\dot v$ (statement (c) of Proposition 2.21), if we denote by $(\dot w)^-$ the negative part of the measure $\dot w$, we obtain that $(\dot w)^- \leq \dot v$. Therefore the support of $(\dot w)^-$ is included in the support of $\dot v$, and then $W-Hv=0$ almost everywhere with respect to $(\dot w)^-$. Moreover $W-Hv\geq 0$ everywhere. Hence (2.17) yields

$$\lim_{h\to 0^+}\frac{F^\circ(v+hw)-F^\circ(v)}{h}=\int_{-1}^{1}\big(W(s)-Hv(s)\big)\,d\dot w(s)\geq 0\ . \tag{2.20}$$

Since the function $h\mapsto F^\circ(v+hw)$ is convex (cf. Proposition 2.10), (2.20) implies that $F^\circ(v+hw)\geq F^\circ(v)$ for every $h>0$, and in particular

$$F^\circ(v+w)\geq F^\circ(v)\ . \tag{2.21}$$

So far we have proved that inequality (2.21) holds for every admissible variation for v which has bounded variation, and then we extend it to every admissible variation by a density argument. Hence v minimizes F° on Y. $\square$

Proof of Corollary 2.16. Let X' be the class of all increasing functions in X. By statement (a) in the proof of Theorem 2.4, the infimum of F on X is equal to the infimum on X'. Hence if u is an increasing function in X, by Theorem 2.11 we obtain that u minimizes F on X if and only if the inverse of u minimizes the conjugate functional F° on Y. Now the thesis follows from Theorem 2.14. $\square$

3. The optimal profile problem: the N-dimensional case

In this section we study the optimal profile in dimension $N>1$.

The notation is the same of Sect. 1: J, W are given in paragraph 1.2, F is defined in (1.14), and e, M, A, X^e_A are given in paragraph 1.4.

In the following we identify $\mathbb{R}^N$ with the product $M\times\mathbb{R}$ by writing every $x\in\mathbb{R}^N$ as $x=y+te$ with $y\in M$ and $t\in\mathbb{R}$; $t=\langle e,x\rangle=x_e$ is then the component of x in the direction e. In particular $A\times\mathbb{R}$ is the stripe of all $x\in\mathbb{R}^N$ of the form $x=y+te$ with $y\in A$, $t\in\mathbb{R}$.

As far as possible we use the letters x, h to denote elements of $\mathbb{R}^N$, y for elements of M, and s, t for real numbers.

Definition 3.1. *We define the one-dimensional non-negative interaction potential J^e and the double-well potential W^e as follows:*

$$\begin{aligned} J^e(s)&:=\int_{y\in M}J(y+se)\,dy\quad \textit{for every } s\in\mathbb{R},\\ W^e(v)&:=W(v)\quad \textit{for every } v\in\mathbb{R}. \end{aligned}\tag{3.1}$$

One readily checks that J^e satisfies (2.1). Eventually the one-dimensional functional F^e is defined as in (1.9) by replacing J and W with J^e and W^e respectively. We denote by γ_e the optimal profile associated with F^e (cf. Theorem 2.4).

For those functions u which vary only on the direction e, the value of F can be recovered by F^e. More precisely we have the following result:

Proposition 3.2. *If $u : \mathbb{R}^N \to \mathbb{R}$ is a function which varies only in the direction e, that is, $u(x) \equiv \bar{u}(x_e)$ for some $\bar{u}$ on $\mathbb{R}$, then*

$$F(u, B \times \mathbb{R}) = |B| \cdot F^e(\bar{u}) \quad \text{for every } B \subset M. \tag{3.2}$$

Proof. We write every $x \in B \times \mathbb{R}$ as $x = y + te$ with $y \in B$, $t \in \mathbb{R}$, and every $h \in \mathbb{R}^N$ as $h = z + se$ with $z \in M$, $s \in \mathbb{R}$. Both changes of variable preserve the measure, and then

$$\begin{aligned}
\int\limits_{\substack{x \in B \times \mathbb{R} \\ h \in \mathbb{R}^N}} J(h)\big(u(x+h) - u(x)\big)^2 dx\, dh = \\
&= \int\limits_{t,s \in \mathbb{R}} \Big[\int\limits_{y \in B,\, z \in M} J(z + se)\, dz\, dy \Big] \big(\bar{u}(t+s) - \bar{u}(t)\big)^2 dt\, ds \\
&= |B| \int\limits_{t,s \in \mathbb{R}} J^e(s) \big(\bar{u}(t+s) - \bar{u}(t)\big)^2 dt\, ds\ .
\end{aligned} \tag{3.3}$$

Similarly we obtain

$$\int\limits_{x \in B \times \mathbb{R}} W(u(x))\, dx = |B| \int\limits_{t \in \mathbb{R}} W(\bar{u}(t))\, dt = |B| \int\limits_{t \in \mathbb{R}} W^e(\bar{u}(t))\, dt\ . \tag{3.4}$$

Identities (3.3) and (3.4) together imply (3.2). □

Now we can state and prove the main result of this section:

Theorem 3.3. *Under the above stated hypotheses the minimum problem*

$$\min \Big\{ \frac{F(u, A \times \mathbb{R})}{|A|} \ :\ u \in X^e_A \Big\}\ , \tag{3.5}$$

has a solution u which varies only in the direction e. More precisely we can take $u(x) := \gamma_e(x_e)$, where γ_e is the optimal profile associated with the one-dimensional functional F^e given in Definition 3.1, and therefore the value of the minimum in (3.5) *is equal to $F^e(\gamma_e)$ (and is independent of A).*

The rest of this section is devoted to the proof of Theorem 3.3. In view of Proposition 3.2 the statement of Theorem 3.3 reduces to the following inequality: for every $u \in X^e_A$ there holds

$$F(u, A \times \mathbb{R}) \geq |A| \cdot F^e(\gamma_e)\ , \tag{3.6}$$

where F^e and γ_e are given in Definition 3.1.

The proof of this inequality is quite delicate and requires some additional notation.

Definition 3.4. *We denote by* $\mathrm{Im}\,(\gamma_e)$ *the essential image of the optimal profile* γ_e*, that is, the closure of the image of any left or right-continuous representative of* γ_e *(cf. Remark 2.17).*

We set $P := e + M = \{e + y : y \in M\}$*; given a function* u *on* $\mathbb{R}^N$*, for every* $y \in M$*,* $z \in P$*, we set*

$$u^{yz}(t) := u(y + tz) \quad \text{for every } t \in \mathbb{R}. \tag{3.7}$$

Notice that when u belongs to X_A^e then u^{yz} belongs to X for every y, z (cf. definitions (1.11) and (1.16)).

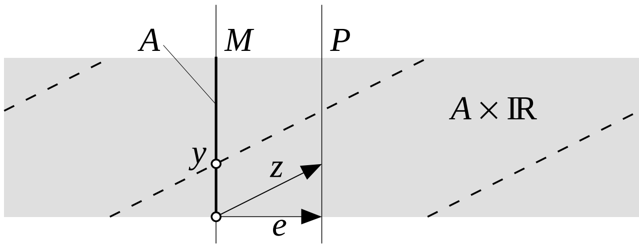

Fig.1. in grey is the stripe $A\times\mathbb{R}$. If we identify this stripe with the cylinder $\mathbb{R}^N/G$ where G is the subgroup of $\mathbb{R}^N$ generated by $\{e_1,\ldots,e_{N-1}\}$; then the dashed segments represent the straight line $R^{yz}:=\{y+tz: t\in\mathbb{R}\}$ and for every z the family $\{R^{yz}: y\in A\}$ is a fibration of the cylinder $A\times\mathbb{R}$.

Definition 3.5. *For every* $z \in P$ *we define the one-dimensional non-negative interaction potential* $\overline{J}^z$ *as follows:*

$$\overline{J}^z(s) := J(sz)\,|s|^{N-1} \quad \text{for every } s \in \mathbb{R}. \tag{3.8}$$

In the following we say that a family $\{\overline{W}^z : z \in P\}$ *of continuous non-negative functions defined on* $\mathbb{R}$ *is admissible when*

$$W(v) \geq \int_{z\in P} \overline{W}^z(v)\,dz \quad \text{for every } v \in [-1,1]\,, \tag{3.9a}$$

$$W(v) = \int_{z\in P} \overline{W}^z(v)\,dz \quad \text{for every } v \in \mathrm{Im}\,(\gamma_e). \tag{3.9b}$$

Clearly (3.9a) *yields* $\overline{W}^z(\pm 1) = 0$ *for almost every* z*; notice that we do not assume that the functions* $\overline{W}^z$ *are strictly positive out of* ± 1*, so they may be not double-well potentials.*

For every admissible family $\{\overline{W}^z : z \in P\}$ *and every* $z \in P$ *we define the one-dimensional functional* $\overline{F}^z$ *as in* (1.9)*, by replacing* J *and* W *with* $\overline{J}^z$ *and* $\overline{W}^z$ *respectively. Since* e*,* J*, and* $\overline{J}^z$ *are fixed, the functionals* $\overline{F}^z$ *depend only on the choice of the family* $\{\overline{W}^z : z \in P\}$*.*

Proposition 3.6. *For a.e. $z \in P$ the one-dimensional interaction potential $\overline{J}^z$ satisfies* (2.1)*, and for every admissible choice of $\{\overline{W}^z : z \in P\}$ there holds*

$$F^e(\gamma_e) = \int\limits_{z\in P} \overline{F}^z(\gamma_e)\, dz \ . \tag{3.10}$$

Proof. We set

$$g(z) := \int_0^\infty \overline{J}^z(s)\,(1+s)\,ds \quad \text{for every } z \in P.$$

Then $\overline{J}^z$ satisfies (2.1) if and only if $g(z)$ is finite. Thus we integrate $g(z)$ over all $z \in P$, and taking into account (3.8) and the fact that $|s| \le |sz|$ (because $|z| \ge 1$ for every $z \in P$), we obtain

$$\begin{aligned} \int\limits_{z\in P} \Big(\int_{\mathbb{R}} \overline{J}^z(s)\,(1+|s|)\,ds \Big)\, dz &\le \int\limits_{s\in\mathbb{R}, z\in P} J(sz)\,(1+|sz|)\,|s|^{N-1} ds\, dz \\ &=^{(6)} \int_{\mathbb{R}^N} J(h)\,(1+|h|)\,dh < \infty \ . \end{aligned}$$

Hence $\int g(z)\,dz$ is finite, and then $g(z)$ is finite for a.e. $z \in P$. The first part of the proposition is proved.

It remains to prove (3.10): the definitions of J^e and $\overline{J}^z$ (see (3.1) and (3.8)) and a direct computation yield

$$J^e(s) = \int\limits_{z\in P} \overline{J}^z(s)\,dz \quad \text{for every } s \in \mathbb{R}\setminus\{0\}, \tag{3.11}$$

and taking (3.9*b*) into account, we immediately obtain (3.10) [7]. □

---

[6] We apply the change of variable $h = sz$. More precisely we consider the function Φ which takes $(s,z) \in \mathbb{R}\times P$ into $h = sz \in \mathbb{R}^N$; Φ is one-to-one from $(\mathbb{R}\setminus\{0\})\times P$ to $\mathbb{R}^N \setminus M$, and has Jacobian determinant $J\Phi(s,z) = |s|^{N-1}$. Indeed if we write z as $z = \bar{z} + e$ with $\bar{z} \in M$, the differential of Φ at the point (s,z) is the linear mapping $d\Phi$ which takes $(ds, dz) \in \mathbb{R}\times M$ into $z\,ds + s\,dz = e\,ds + (\bar{z}\,ds + s\,dz)$, and then $d\Phi$ may be viewed as the linear mapping of $\mathbb{R}\times M$ into itself represented by the matrix

$$\begin{pmatrix} 1 & \bar{z} \\ 0 & sI \end{pmatrix}$$

where I is the identity on the $(N-1)$-dimensional space M. Hence $J\Phi := \big|\det(d\Phi)\big| = |s|^{N-1}$.

[7] In fact (3.10) holds even if we replace γ_e with any other function whose essential image is included in $\mathrm{Im}\,(\gamma_e)$; in the general case there holds only an inequality.

Proposition 3.7. *For every admissible choice of the family* $\{\overline{W}^z : z \in P\}$ *and every* $u : \mathbb{R}^N \to [-1,1]$ *which is A-periodic there holds*

$$F(u, A \times \mathbb{R}) \geq \int\limits_{y \in M,\, z \in P} \overline{F}^z(u^{yz})\, dz\, dy\ . \tag{3.12}$$

Proof. Let us consider the first integral in the definition of F (see (1.14)): if we apply the change of variable $h = sz$ with $z \in P$, $s \in \mathbb{R}$ (cf. the proof of Proposition 3.7) we obtain

$$\int\limits_{h \in \mathbb{R}^N} \Big(\int\limits_{x \in A \times \mathbb{R}} \big(u(x+h) - u(x)\big)^2 dx \Big)\, J(h)\, dh \ = \\ = \int\limits_{\substack{s \in \mathbb{R} \\ z \in P}} \Big(\underbrace{\int\limits_{x \in A \times \mathbb{R}} \big(u(x+sz) - u(x)\big)^2 dx}_{I(s,z)} \Big)\, J(sz)\, |s|^{N-1} ds\, dz \tag{3.13}$$

We fix $s \in \mathbb{R}$ and $z \in P$ for the moment, and we restrict our attention to the integral $I(s,z)$. We write z in the form $z = \bar{z} + e$ with $\bar{z} \in M$, and then we apply the change of variable $x = y + tz$ with $t \in \mathbb{R}$, $y \in M$ [(8)]:

$$I(s,z) = \int\limits_{t \in \mathbb{R}} \Big(\int\limits_{y \in A - t\bar{z}} \big(u(y+tz+sz) - u(y+tz)\big)^2\, dy \Big)\, dt$$

(here $A - t\bar{z}$ stands for the translation of the set A by $-t\bar{z}$). Since u is A-periodic, replacing the integration domain $A - t\bar{z}$ with A does not affect the value of the integral between brackets, and then, recalling definition (3.7),

$$I(s,z) = \int\limits_{t \in \mathbb{R},\, y \in A} \big(u^{yz}(t+s) - u^{yz}(t)\big)^2\, dy\, dt\ . \tag{3.14}$$

Now we replace the value of $I(s,z)$ in (3.13) with (3.14), and taking definition (3.8) into account we get

$$\int\limits_{\substack{x \in A \times \mathbb{R} \\ h \in \mathbb{R}^N}} J(h)\, \big(u(x+h) - u(x)\big)^2 dx\, dh \ = \\ = \int\limits_{\substack{y \in A \\ z \in P}} \Big(\int\limits_{s,t \in \mathbb{R}} \overline{J}^z(s)\, \big(u^{yz}(t+s) - u^{yz}(t)\big)^2\, ds\, dt \Big)\, dy\, dz \tag{3.15}$$

(8) More precisely we consider the function Ψ which takes $(t,y) \in \mathbb{R} \times M$ into $x = y + tz \in \mathbb{R}^N$: Ψ is one-to-one from $\mathbb{R} \times M$ to $\mathbb{R}^N$, and has Jacobian determinant equal to 1.

Reasoning in a similar way and taking (3.9a) into account, one obtains

$$\int\limits_{x\in A\times\mathbb{R}} W(u(x))\,dx \geq \int\limits_{z\in P}\Big(\int\limits_{x\in A\times\mathbb{R}} \overline{W}^z(u(x))\,dx\Big)\,dz = \int\limits_{\substack{y\in A\\ z\in P}}\Big(\int\limits_{t\in\mathbb{R}} \overline{W}^z(u^{yz}(t))\,dt\Big)\,dy\,dz\ . \tag{3.16}$$

Identities (3.15) and (3.16) together imply (3.12). □

Proof of inequality (3.6). Take any admissible family $\{\overline{W}^z : z\in P\}$ and set

$$\overline{\sigma}_z := \inf\left\{\overline{F}^z(u) :\ u\in X\right\} \quad \text{for every } z\in P.^{(9)} \tag{3.17}$$

Let $u\in X^e_A$ be fixed. We already remarked that for every $y\in M$, $z\in P$ there holds $u^{yz}\in X$ (see (3.7)), and then $\overline{F}^z(u^{yz})\geq\overline{\sigma}_z$. Hence Proposition 3.7 yields

$$F(u,A\times\mathbb{R}) = \int\limits_{y\in A, z\in P} \overline{F}^z(u^{yz})\,dy\,dz \geq \int\limits_{y\in A, z\in P} \overline{\sigma}_z\,dy\,dz = |A|\int\limits_{z\in P}\overline{\sigma}_z\,dz\ . \tag{3.28}$$

On the other hand, by Proposition 3.6 there holds

$$|A|\int\limits_{z\in P}\overline{F}^z(\gamma_e)\,dz = |A|\cdot F^e(\gamma_e)\ .$$

Therefore inequality (3.6) is proved if we choose the functions $\overline{W}^z$ so that $\overline{\sigma}_z\geq\overline{F}^z(\gamma_e)$ for (almost) every $z\in P$, that is

$$\gamma_e \text{ minimizes } \overline{F}^z \text{ in } X \text{ for every } z\in P. \tag{3.19}$$

Now the strategy is the following: we first choose $\overline{W}^z$ so that (3.19) holds, and then we show that this choice is admissible (cf. Definition 3.5). As we remarked in paragraph 2.19, the problem of finding $\overline{W}^z$ so that the minimizer of $\overline{F}^z$ on X is a given function (i.e., γ_e) may be solved using Corollary 2.16: it is enough to set

$$\overline{W}^z(v) := \big(\overline{H}^z\gamma_e^{-1}\big)(v) \quad \text{for every } v\in[-1,1]\,, \tag{3.20}$$

where $\overline{H}^z$ is defined as in (2.15) by replacing K with $\overline{K}^z := \overline{J}^z * f$ (see (2.8)) We complete the definition of $\overline{W}^z$ by setting $\overline{W}^z(v)=0$ when $v\in\mathbb{R}\setminus[-1,1]$.

By Proposition 2.13 the function $\overline{W}^z$ is continuous, non-negative, and vanishes at ±1; it remains to prove that (3.9a) and (3.9b) hold. In order to do this,

(9) Since $\overline{W}^z$ may be not strictly positive in $(-1,1)$, Theorem 2.4 does not apply and then the infimum in (3.17) may be not achieved.

we define H^e as in (2.15) by replacing K with $K^e := J^e * f$ and we integrate identity (3.20) with respect to $z \in P$: taking (3.11) into account we get

$$\int_P \overline{W}^z(v)\,dz = \int_P \big(\overline{H}^z \gamma_e^{-1}\big)(v)\,dz =^{(10)} \big(H^e \gamma_e^{-1}\big)(v) \quad \text{for every } v \in [-1,1]. \tag{3.21}$$

Since γ_e minimizes F^e on X by definition, Corollary 2.16 and Remark 2.17 yield

$$\big(H^e \gamma_e^{-1}\big)(v) \le W^e(v) \quad \text{for every } v \in [-1,1], \tag{3.22a}$$

$$\big(H^e \gamma_e^{-1}\big)(v) = W^e(v) \quad \text{for every } v \in \mathrm{Im}\,(\gamma_e). \tag{3.22b}$$

As $W^e = W$ by definition (cf. (3.1)), from (3.21) and (3.22*a*), (3.22*b*) we get (3.9*a*) and (3.9*b*). This concludes the proof of inequality (3.6). □

4. Further remarks, generalizations and open problems

In this section we add some remarks and we briefly discuss possible generalizations of the results of the previous sections and some open problems. Throughout this section we identify functions which agree almost everywhere.

4a. Uniqueness of the optimal profile

In the one-dimensional case the uniqueness of the optimal profile (up to translations) can be proved under some additional hypotheses on J and W via the Euler-Lagrange equation associated with F.

To this aim we notice that for every $u : \mathbb{R} \to \mathbb{R}$ such that $F(u)$ is finite, and every $v : \mathbb{R} \to \mathbb{R}$ bounded and compactly supported, the first variation of F at u in direction v is given by

$$\langle DF(u), v\rangle := \lim_{h\to 0} \frac{F(u+hv) - F(u)}{h} = \int_{\mathbb{R}} \big(- J * u + au + \dot{W}(u)\big)\, v\, dx\ , \tag{4.1}$$

here W is assumed of class C^1 and $a := \int J(h)\,dh$. It follows immediately that the Euler-Lagrange equation associated with F is

$$J * u - a\,u = \dot{W}(u) \quad \text{a.e. in } \mathbb{R}. \tag{4.2}$$

In particular every optimal profile solves (4.2) under the boundary condition (1.3) [(11)].

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(10) Since $\int \overline{J}^z(s)\,dz = J^e(s)$ for every $s \in \mathbb{R}$ by (3.11), then (2.8) yields $\int \dot{\overline{K}}^z(s)\,dz = \dot{K}^e(s)$, and by (2.15) for every increasing $\phi : (-1,1) \to \mathbb{R}$ there holds

$$\int_P \big(\overline{H}^z \phi(v)\big)\,dz = H^e \phi(v) \quad \text{for every } v \in [-1,1]\,.$$

(11) Replacing u with the truncated function $(u \vee 1) \wedge -1$ decreases the value of F. Hence if a function minimizes F on X, then it minimizes F also on the larger class of all locally bounded $u : \mathbb{R} \to \mathbb{R}$ which satisfy (1.3). Using this fact and (4.1) we easily deduce that every optimal profile satisfies (4.2).

Notice that both equation (4.2) and conditions (1.3) are translation invariant; therefore if u solves (4.2) with boundary condition (1.3), so does every translation of u.

As we remarked in the Introduction, every solution of (4.2) subject to (1.3) is a travelling wave solution for the one-dimensional parabolic equation

$$u_t = J * u - u - f(u) \ , \tag{4.3}$$

where f is the derivative of a potential with two wells at possibly different depth in ± 1. We refer to [6] for a detailed analysis of existence stability and regularity of such travelling waves. We just recall here the following uniqueness result (see [6], Theorem 4.1):

Theorem 4.1. *Assume that J and W satisfy the following additional hypotheses:*
(i) *W is of class C^2 and the interval $(-1,1)$ splits into three sub-intervals where W is either convex or concave.*
(ii) *J is of class C^1;*

Then the solution of equation (4.2) *subject to* (1.3) *is unique up to translations.*

This result shows that if (i) and (ii) hold, the optimal profile associated with F is unique up to translations, and every solution of (4.2) which satisfies (1.3) agrees with the optimal profile. Hence solving (4.2) subject to the boundary condition (1.3) is equivalent to minimizing F on X.

Question 4.2. *Is there any direct proof of the uniqueness result for the solutions of the minimum problem* (1.10)*?* So far the uniqueness of the optimal profile is a consequence of Theorem 4.1, which states a uniqueness result for equation (4.2), and therefore W cannot be less regular than C^1. On the other hand it would be interesting to find conditions on J and W which do not imply any additional regularity than J summable and W continuous. An answer to this question could be obtained by looking for conditions such that the conjugate functional F° is strictly convex, and every solution of (1.10) is increasing. In turn this monotonicity result could be achieved by sharpening the rearrangement result given in Theorem 5.8.

Question 4.3. *Under which hypotheses on J inequality* (5.14) *is strict whenever u is not (a.e. equal to) an increasing function (that is, u does not agree a.e. with its increasing rearrangement u^*)?*

Eventually it would be interesting to prove some uniqueness result in the N-dimensional case, that is, finding conditions so that the minimum problem (1.15) has a unique solution up to translations. This can be achieved by proving that under suitable assumptions the optimal profile γ_e associated with F^e is unique up to translations (cf. Theorem 4.1) and inequality (3.6) is strict when u cannot be written in the form $u(x) := \gamma_e(x_e)$ for a.e. $x \in \mathbb{R}^N$. A careful analysis of the proof of (3.6) shows that this refinement holds true if γ_e is the unique minimizer of $\overline{F}^z$ on X for every z in a set of positive measure in P. Notice that in this situation Theorem 4.1 is of little use: indeed it is not clear which conditions we could impose on J and W so that each potential $\overline{W}^z$ satisfies the second part of assumption (i) in Theorem 4.1.

4b. The multi-phase model

The free energy given in (1.1) may be used to describe systems which admits only two phases, or in other words, systems whose spatial inhomogeneity is described by one scalar parameter only. In order to describe a multi-phase system one may postulate a free energy of the form (1.1) where u is a *vector* density function on a domain of $\mathbb{R}^N$ taking values in $\mathbb{R}^k$, $W : \mathbb{R}^k \to [0, \infty)$ is a continuous function which vanishes at $d + 1$ affinely independent wells $\{\alpha_0, \dots, \alpha_d\}$, and J is the usual interaction potential.

The optimal profiles can be defined as in the scalar case. If $N = 1$ we proceed as in paragraph 1.3: for every $u : \mathbb{R} \to \mathbb{R}^k$, $F(u)$ is defined as in (1.9), and for every couple of indexes $i \neq j$ (with $i, j = 0, \dots, k$) we consider the minimum problem

$$\sigma^{ij} := \min \left\{ F(u) : \, u \in X^{ij} \right\} , \tag{4.4}$$

where

$$X^{ij} := \left\{ u : \mathbb{R} \to [-1, 1] : \lim_{x \to -\infty} u(x) = \alpha_i \, , \, \lim_{x \to +\infty} u(x) = \alpha_j \right\} . \tag{4.5}$$

A solution γ_{ij} of the minimum problem (4.4) is the optimal profile for the interface between the phases α_i and α_j, and σ^{ij} is the surface tension associated with the interface between the phases α_i and α_j (clearly $\sigma^{ij} = \sigma^{ji}$).

Notice that Theorem 2.4 cannot be easily extended to the multi-phase setting: in fact its proof is based on a rearrangement result (Proposition 2.3), which cannot even be stated for vector density functions. Also the statements of Theorem 2.11 and Corollary 2.16 can be hardly adapted to the vector setting.

Question 4.4. *Does the minimum problem* (4.4) *admit a solution? In case of a positive answer, is there any analog of Corollary 2.16?*

Once an existence result for problem (4.4) is proved, one may pass to the N-dimensional case: given a direction e, a rectangle A, and a couple of indexes $i \neq j$, the analog of the minimum problem (1.15) is defined in the obvious way, but as far as we know in this case Theorem 3.3 may not hold, that is, it may happen that problem (1.15) has solutions none of which varies only in the direction e, and the value of the minimum itself may depend on the choice of the rectangle A.

4c. The optimal assumptions on J

The hypothesis $J \geq 0$ cannot be removed without deep modifications in the proofs of our results. For instance, the existence of an increasing optimal profile in the one-dimensional case (Theorem 2.4) depends on the rearrangement result in Proposition 2.3, that is, on Theorem 5.8, which in turn depends on Theorem 5.2. And this theorem may not hold when J is not positive. Moreover if no increasing optimal profile exists, then the characterization of minimizers given in Corollary 2.16 does not apply, and the proof of Theorem 3.3, which is essentially based on this corollary, does not work. Yet it would be interesting to understand what happens if J takes negative values in a “small” zone (if the negative part of J is predominant with respect to the positive part, then the infimum of F on X is $-\infty$).

Question 4.5. *Under which hypothesis on the negative part of J does the one-dimensional minimum problem* (1.10) *have a solution?*

We restrict now our attention to one-dimensional functionals of the form (1.9) where the double-well potential W is taken as usual but the interaction potential J is only assumed even and non-negative. The following theorem gives the optimal assumption on J (and W) so that F is not identically equal to $+\infty$ on X.

Theorem 4.6. *Under the previous hypotheses, there exists $u \in X$ such that $F(u)$ is finite if and only if*

$$\int_0^\infty J(h)\,(h \wedge h^2)\,dh < \infty \ . \tag{4.6}$$

Proof. Assume first that there exists $u \in X$ such that $F(u)$ is finite. By inequality (2.2) we may take u increasing, and if we set $v := u^{-1}$ then $F^\circ(v)$ is finite by identity (2.11) (one can easily verify that both (2.2) and (2.11) holds true with no growth assumptions on J). From the definition of $F^\circ(v)$ we infer

$$\int\limits_{-1<t<t'<1} K\big(v(t')-v(t)\big)\,dt'dt < +\infty \ , \tag{4.7}$$

and then $K(x)$ must be finite for every $x > 0$ (recall that K is decreasing), that is,

$$\int_x^\infty J(h)\,(h-x)\,dh < +\infty \quad \text{for every } x > 0\,. \tag{4.8}$$

Moreover v is almost everywhere differentiable because it is increasing, then we may find a set of positive measure $B \subset (-1,1)$ and constants r, c such that $r > 0$, $0 < c < \infty$, and

$$v(t+s) - v(t) \le cs \quad \text{for } s \in [0,r],\ t \in B.$$

Hence, taking into account that K is decreasing,

$$\begin{aligned}\int\limits_{-1<t<t'<1} K\big(v(t')-v(t)\big)\,dt'dt &\ge \int_B \Big(\int_0^r K\big(v(t+s)-v(t)\big)\,ds\Big)\,dt \\ &\ge \int_B \Big(\int_0^r K(cs)\,ds\Big)\,dt = \frac{|B|}{c}\int_0^{cr} K(x)\,dx \ .\end{aligned}$$

We deduce that K is summable in the interval $[0, cr]$, and by the definition of K we obtain

$$\infty > \int\limits_{\substack{0<x<cr \\ x<h<\infty}} J(h)\,(h-x)\,dx\,dh \geq \int\limits_{\substack{0<h<cr \\ 0<x<h}} J(h)\,(h-x)\,dx\,dh = \frac{1}{2}\int_0^{cr} J(h)\,h^2\,dh\ . \quad (4.9)$$

Inequalities (4.8) and (4.9) yield (4.6).

Conversely, assume that (4.6) holds and set $u(x) := (x \wedge 1) \vee -1$. Then $u^{-1}(t) = t$, and a direct computation shows that $F^\circ(u^{-1})$ is finite, and so is $F(u)$. $\square$

In view of Theorem 4.6 the one-dimensional optimal profile problem (1.10) makes sense even if we replace the assumptions $J \in L^1$ and (1.8) with the more general (4.6); it may be verified that the main result of Sect. 2, namely Proposition 2.3, Theorems 2.4, 2.11 and 2.14, and Corollary 2.16, still hold true.

The proofs of Proposition 2.3 and Theorem 2.4 require no modifications at all. Theorem 2.11 (and namely identity (2.11)) can be proved by approximating J with an increasing sequence of interaction potential J_n which satisfy (2.1) and passing to the limit in the corresponding identities $F_n(u) = F_n^\circ(u^{-1})$.

In order to make the definition of the operator H in (2.15) well-posed and state Theorem 2.14, we set $\dot{K}(0)$ equal to $-\infty$ when $K(0) = +\infty$, and equal to the right derivative of K at 0 otherwise. This way Hv is a non-negative lower-semicontinuous function, vanishes at ± 1, but may be neither continuous nor bounded. To adapt the proof Theorem 2.14 requires some additional care; the main difficulty is due to the fact that Lemma 2.22 no longer holds in this case: if $\dot{w}$ is a signed measure and Hv is not bounded the right side of (2.17) may be not defined. However identity (2.17) holds for every v such that $F^\circ(v)$ is finite and every admissible variation w such that w belongs to BV and $F^\circ(v + hw)$ is finite for some $h > 0$. The rest of the proof of Theorem 2.14, and Corollary 2.16 as well, follow as in Sect. 2.

Eventually we consider the N-dimensional case. In view of Theorem 4.6, the minimum problem (1.15) makes sense even if we replace the usual assumptions $J \in L^1(\mathbb{R}^N)$ and (1.8) with

$$\int_{\mathbb{R}^N} J(h)\,\big(|h| \wedge |h|^2\big)\,dh < +\infty\ . \quad (4.10)$$

Then Theorem 3.3 can be proved as in Sect. 3, with just some minor modifications. Notice that the optimal profile γ_e associated with F^e exists because J^e satisfies (4.6).

4d. On the lower semicontinuity of F

In this subsection we briefly discuss the semicontinuity of functionals of the form (1.1).

Let us consider for simplicity the following setting: Ω is a bounded open set in $\mathbb{R}^N$, $\mathscr{E}$ is the class of all $u : \Omega \to [-1,1]$, and for every $u \in \mathscr{E}$ we set

$$F(u) := \frac{1}{4} \int_{\Omega\times\Omega} J(x'-x) \big(u(x') - u(x)\big)^2 dx'dx \; + \int_\Omega W\big(u(x)\big)\, dx \; . \tag{4.11}$$

Here W is the usual double-well potential while the interaction potential J is only assumed even and in $L^1(\mathbb{R}^N)$. Notice that J may take negative values.

If we look for a minimizer of F on $\mathscr{E}$ (subject to some constraint which makes the problem nontrivial), a natural approach is the so-called direct method. In other words one tries to endow $\mathscr{E}$ with a topology which makes F lower semicontinuous and coercive (that is, a topology such that the set $\{u \in \mathscr{E} : F(u) \leq a\}$ is compact for every $a \in \mathbb{R}$). Now $\mathscr{E}$ admits two natural topologies: the *strong topology* is induced by convergence in measure (or equivalently by the embedding of $\mathscr{E}$ into $L^p(\Omega)$ for any $1 \leq p < \infty$), and the *weak topology* is induced by convergence in the sense of distributions (or equivalently by the weak topology of $L^p(\Omega)$ for any $1 \leq p < \infty$). Both topologies are metrizable, moreover $\mathscr{E}$ is weakly compact, but not strongly. Therefore we are interested in the semicontinuity of F with respect to the weak topology.

Theorem 4.7. *For every $x \in \Omega$ we set*

$$g(x) := \frac{1}{2} \int_{x'\in\Omega} J(x'-x)\, dx' \quad , \qquad C := \operatorname*{ess\,sup}_{x\in\Omega} g(x) \; . \tag{4.12}$$

Then F is weakly lower semicontinuous on $\mathscr{E}$ if and only if the function $t \mapsto Ct^2 + W(t)$ is convex in $[-1,1]$ (that is, $\ddot{W} \geq -2C$ in $[-1,1]$ when W is of class C^2).

Proof. By replacing $\big(u(x') - u(x)\big)^2$ with $(u(x'))^2 + (u(x))^2 - 2\,u(x)\,u(x')$ in (4.11), and taking definition (4.12) into account, we obtain

$$F(u) = \underbrace{-\frac{1}{2} \int_{\Omega\times\Omega} J(x'-x)\, u(x')\, u(x)\, dx'dx}_{I_1(u)} + \underbrace{\int_\Omega \big[g(x)\,(u(x))^2 + W(u(x))\big]\, dx}_{I_2(u)} \; .$$

We claim that $I_1(u)$ is weakly continuous on $\mathscr{E}$: indeed if (u_n) converges weakly to u, then the functions $v_n(x',x) := u_n(x')\,u_n(x)$ converge to $v(x',x) := u(x')\,u(x)$ weakly* in $L^\infty(\Omega\times\Omega)$, and since $J(x'-x)$ belongs to $L^1(\Omega\times\Omega)$, then $I_1(u_n) \to I_1(u)$.

On the other hand by well-known semicontinuity theorems $I_2(u)$ is weakly lower semicontinuous on $\mathscr{E}$ if and only if the function $t \mapsto g(x)\,t^2 + W(t)$ is convex on $[-1,1]$ for almost every $x \in \Omega$, that is, if and only if $Ct^2 + W(t)$ is convex on $[-1,1]$. The proof is complete. □

Remark 4.8. We can use Theorem 4.7 to obtain existence results for some minimum problems: since $\mathscr{E}$ is weakly compact, when F is weakly lower semicontinuous it attains the minimum on $\mathscr{E}$ subject to any weakly closed constraint (e.g., subject to the volume constraint $\int_\Omega u = m\,|\Omega|$ with $m \in (-1,1)$).

Remark 4.9. One may try to modify Theorem 4.7 in order to approach the minimum problem (1.10) when J takes negative values, or when u is a vector-valued function (cf. Subsects. 4b and 4c). For example F may be taken as in (1.9) with W as usual and J even and such that $\int_0^\infty \big|J(h)\big|\,(1+h)\,dh$ is finite, $\mathscr{E}$ is the class of all functions $u : \mathbb{R} \to [-1,1]$, and we set $C := \int_0^\infty J(h)\,dh$ (cf. (4.12)). Then $\mathscr{E}$ is weak* compact in $L^\infty(\mathbb{R})$ and F is weak* lower semicontinuous on $\mathscr{E}$ if and only if the function $t \mapsto Ct^2 + W(t)$ is convex in $[-1,1]$. However the difficulty is that (1.3) and similar constraints are not weakly* closed in $\mathscr{E}$, and therefore this semicontinuity result is of little use.

5. Appendix: a rearrangement result

In this appendix we state and prove some rearrangement results we used in the proof of Theorem 2.4. The main result is Theorem 5.8. Throughout this section, J is a non-negative Borel function (we do not need any summability assumption on J).

Definition 5.1. *Let $I = [c,d]$ be a given bounded interval; for every set $A \subset I$ we define the right rearrangement of A in I as the interval $A^* = [d-|A|, d]$, where $|A|$ denotes as usual the measure of A.*

For every couple of sets A, B included in I we set

$$\Phi(A,B) := \int\limits_{A\times B} J(x'-x)\,dx'dx \ . \tag{5.1}$$

The following result holds.

Theorem 5.2. *Let $I = [c,d]$ and assume that the support of J is included in $[-r,r]$ for some positive $r \le (d-c)/2$. Then, for every couple of sets A, B such that $[d-r,d] \subset A, B \subset I$ we have*

$$\Phi(A,B) \le \Phi(A^*,B^*) \ . \tag{5.2}$$

Moreover the equality holds in (5.2) *if $A = I$ and $[d-r,d] \subset B \subset [c+r,d]$ (or conversely).*

Remark 5.3. These rearrangement problems have been widely studied in the literature; (see for instance [14], [4], [5]). In particular we refer to [14] for the *symmetric rearrangement* of sets (and functions), that is, when A^* is defined as the interval with the same measure as A and centered at the middle of I.

The reverse of inequality (5.2) was studied in [3] and [7]: if J is positive and decreasing on $\mathbb{R}^+$, then for every couple of disjoint sets A, B included in I there holds $\Phi(A,B) \ge \Phi(A^*,B_*)$ where A^* is the *right* rearrangement of A in I and B_* is the *left* rearrangement of B in I.

Remark 5.4. We point out that the condition $[d-r,d] \subset A, B$ is essential, and indeed inequality (5.2) may be false if this condition is dropped. Take for instance

$I = [0,3]$, J the characteristic function of the interval $[-1,1]$, $A = [0,3]$, and $B = [1,2]$: then $A^* = [0,3]$, $B^* = [2,3]$, but $\Phi(A,B) = 2 > 3/2 = \Phi(A^*,B^*)$.

Proof of Theorem 5.2. For every couple of sets $A, B \subset I$ we write $A \succeq B$ (resp. $A \succ B$) if $\inf A \geq \sup B$ (resp. $\inf A > \sup B$).

We first prove (5.2) when A and B are non-empty disjoint unions of finitely many closed intervals, i.e., $A = A_1 \cup A_2 \cup \ldots \cup A_{n_A}$, $B = B_1 \cup B_2 \cup \ldots \cup B_{n_B}$, and the sets A_i and B_j are closed intervals which satisfy $A_1 \succ A_2 \succ \ldots \succ A_{n_A}$ and $B_1 \succ B_2 \succ \ldots \succ B_{n_B}$.

In this case, condition $[d-r,d] \subset A, B \subset I$ yields $[d-r,d] \subset A_1, B_1 \subset I$; then $A_1 = [d-a,d]$ and $B_1 = [d-b,d]$ for suitable $a, b \geq r$.

The proof will be achieved by induction on the total number of intervals $n := n_A + n_B$. When $n = 2$ we have $A = A^* = A_1$, $B = B^* = B_1$ and then (5.2) obviously holds. Now, fix $n > 2$ and assume that (5.2) holds whenever the total number of intervals is strictly less than n: we have to prove that (5.2) holds for any couple A, B such that $n_A + n_B = n$ (we assume $n_A, n_B > 1$; the proof of the remaining cases is in fact simpler).

We set $A' := A_2 \cup \ldots \cup A_{n_A}$, $B' := B_2 \cup \ldots \cup B_{n_B}$, and

$$\delta := \big(\inf A_1 - \sup A_2\big) \wedge \big(\inf B_1 - \sup B_2\big) \ . \tag{5.3}$$

Then $\delta > 0$ and for every h such that $\delta \geq h \geq 0$, we have $A_1 \succeq A' + h$ and $B_1 \succeq B' + h$; eventually we set

$$A(h) := A_1 \cup (A' + h) \quad , \quad B(h) := B_1 \cup (B' + h) \ . \tag{5.4}$$

We claim that $\Phi\big(A(h), B(h)\big)$ is an increasing function of $h \in [0,\delta]$. In order to prove this, we write $\Phi\big(A(h), B(h)\big)$ as the sum

$$\begin{aligned} \Phi\big(A(h), B(h)\big) = \ & \underbrace{\Phi(A_1, B_1)}_{P_1(h)} + \underbrace{\Phi(A' + h, B' + h)}_{P_2(h)} + \\ & + \underbrace{\Phi(A_1, B' + h)}_{P_3(h)} + \underbrace{\Phi(A' + h, B_1)}_{P_4(h)} \ . \end{aligned}$$

P_1 and P_2 are clearly independent of h, and then it is enough to prove that P_3 and P_4 are increasing functions of h. Let us consider P_3: recalling that $A_1 = [d-a,d]$ we get

$$P_3(h) = \int_{B'+h} \Big(\int_{A_1} J(x' - x)\, dx' \Big)\, dx = \int_{B'} \Big(\int_{d-a-x-h}^{d-x-h} J(y)\, dy \Big)\, dx \ . \tag{5.5}$$

Now, since $B_1 \succeq B' + h$, for every $x \in B'$ we have $x + h \leq d - b \leq d - r$, and then $d - x - h \geq r$. Therefore, taking into account that $J(y) = 0$ for almost every $y \geq r$, (5.5) becomes

$$P_3(h) = \int_{B'} \Big(\int_{(d-a-x)-h}^{r} J(y)\, dy \Big)\, dx \ ,$$

and since J is non-negative, this is clearly an increasing function of h. Since a similar argument applies to P_4, the claim is proved. Hence

$$\Phi(A,B) = \Phi\big(A(0),B(0)\big) \le \Phi\big(A(\delta),B(\delta)\big) \ . \tag{5.6}$$

Taking (5.3) into account, it is clear that either $\inf A_1 = \sup(A_2+\delta)$ or $\inf B_1 = \sup(B_2+\delta)$, and then either $A(\delta)$ is union of $n_A - 1$ disjoint closed intervals, or $B(\delta)$ of $n_B - 1$. Hence $A(\delta)$ and $B(\delta)$ consist of a total number of closed intervals which is strictly less than n, and by the inductive hypothesis

$$\Phi\big(A(\delta),B(\delta)\big) \le \Phi\big(A(\delta)^*,B(\delta)^*\big) \ . \tag{5.7}$$

Now, since $|A(\delta)| = |A|$ and $|B(\delta)| = |B|$, we obtain that $A(\delta)^* = A^*$ and $B(\delta)^* = B^*$, and then (5.6) and (5.7) yield (5.2).

We prove inequality (5.2) when A and B are open sets by inner approximation with finite unions of closed intervals, and when A and B are Borel sets by outer approximation with open sets.

Eventually we prove that equality holds in (5.2) if $A = I$ and $[d-r,d] \subset B \subset [c+r,d]$. We write B as $I_1 \cup B_1$ where $I_1 := [d-r,d]$ and $B_1 := B \cap [c+r,d-r]$. Hence

$$\Phi(A,B) = \Phi(I,I_1) + \Phi(I,B_1) = \Phi(I,I_1) + |B_1| \int_{-r}^{r} J(y)\,dy \ .$$

Then $\Phi(A,B)$ depends only on the measure of B_1, and since this measure is preserved by the right rearrangement of B in I, then $\Phi(A,B) = \Phi(A,B^*) = \Phi(A^*,B^*)$. □

Definition 5.5. *For every Borel set $A \subset \mathbb{R}$ such that*

$$[a,\infty) \supset A \supset [b,\infty) \tag{5.8}$$

for some real numbers $a<b$, we define the right rearrangement *of A as the half-line $A^* := [c,+\infty)$, with $c := b - \big|A \cap [a,b]\big|$. For every $u \in X$ (see Definition 2.1) we define the* increasing rearrangement *of u as the function $u^* : \mathbb{R} \to [-1,1]$ such that*

$$\big\{x : \ t \le u^*(x)\big\} = \big\{x : \ t \le u(x)\big\}^* \quad \textit{for every } t \in (-1,1)\,. \tag{5.9}$$

The set A^* satisfies $[a,\infty) \supset A^* \supset [b,\infty)$ and $|A^* \cap I| = |A \cap I|$ for every interval I which includes $[a,b]$; hence the previous definition is consistent with Definition 5.1: for every interval I which includes $[a,b]$, $A^* \cap I$ is the right rearrangement in I of the set $A \cap I$. It may be immediately verified that the function u^* is well-defined and increasing.

Theorem 5.6. *Let W be a non-negative continuous function on $[-1,1]$ such that $W(\pm 1) = 0$, take $u \in X$ and let u^* be the increasing rearrangement of u. Then*

$$\int_{\mathbb{R}} W\big(u(x)\big)\,dx = \int_{\mathbb{R}} W\big(u^*(x)\big)\,dx \ . \tag{5.10}$$

Proof. Assume first that W is of class C^1 and $W = 0$ in $[-1, -1+\varepsilon]$ and $[1-\varepsilon, 1]$ for some $\varepsilon > 0$. There exist real numbers $a < b$ such that

$$u(x),\ u^*(x) \le -1+\varepsilon \text{ for every } x \le a, \text{ and } u(x),\ u^*(x) \ge 1-\varepsilon \text{ for every } x \ge b. \tag{5.11}$$

We set $I := [a, b]$, for every $t \in (-1+\varepsilon, 1-\varepsilon)$ we denote by E_t the set $\big\{x \in \mathbb{R} :\ t \le u(x)\big\}$, and by 1_t the characteristic function of E_t. Then, recalling that $\dot W(t) = 0$ in $[-1, -1+\varepsilon] \cup [1-\varepsilon, 1]$,

$$\begin{aligned}\int_{\mathbb{R}} W\big(u(x)\big) = \int_I W\big(u(x)\big)\, dx &= \int_I \Big(\int_{-1+\varepsilon}^{1-\varepsilon} \dot W(t)\, 1_t(x)\, dt \Big)\, dx \\ &= \int_{-1+\varepsilon}^{1-\varepsilon} \dot W(t)\, |E_t \cap I|\, dt\ .\end{aligned} \tag{5.12}$$

Condition (5.11) implies that the measure of $E_t \cap I$ is preserved by the right rearrangement of u for every $t \in (-1+\varepsilon, 1-\varepsilon)$, and then equality (5.12) yields (5.10).

We prove equality (5.10) in the general case by approximating W with an increasing sequence of non-negative functions of class C^1 which are 0 in a neighborhood of -1 and 1, and then applying the monotone convergence theorem. $\square$

Definition 5.7. *Let $L : \mathbb{R} \to \mathbb{R}$ be a non-negative convex function such that $L(0) = 0$. For every function $u : \mathbb{R} \to \mathbb{R}$ and every interval $I \subset \mathbb{R}$, we set*

$$G(u, I) := \int_{I \times I} J(x' - x)\, L\big(u(x') - u(x)\big)\, dx' dx. \tag{5.13}$$

The main result of this appendix reads as follows.

Theorem 5.8. *Let $u \in X$, let u^* be the increasing rearrangement of u, and let G be given as in* (5.13)*. Then*

$$G(u, \mathbb{R}) \ge G(u^*, \mathbb{R})\ . \tag{5.14}$$

In order to prove Theorem 5.8 we need the following lemma:

Lemma 5.9. *Assume that L is of class C^2, J has support included in $[-r, r]$, I is the bounded interval $[c, d]$, $2r \le d - c$, and u satisfies the following conditions:*
(i) *$|u(x)| \le 1$ for all x,*
(ii) *$u(x) = -1$ for every $x \le c + r$, $u(x) = 1$ for every $x \ge d - r$.*

Then

$$G(u, I) \ge G(u^*, I)\ . \tag{5.15}$$

Proof. First of all, we establish the following identity, which may be proved by a direct computation: for every $s, s' \in [-1,1]$

$$L(s'-s) = -\int_{-1}^{s}\Big(\int_{-1}^{s'} \ddot{L}(t'-t)\,dt'\Big)\,dt + \\ + \int_{-1}^{s'} \dot{L}(t+1)\,dt \;-\int_{-1}^{s} \dot{L}(-t-1)\,dt\ . \tag{5.16}$$

For every $t \in [-1,1]$ we denote by E_t the set $\big\{x \in I:\ t \le u(x)\big\}$, and by 1_t the characteristic function of E_t. If we take $x, x' \in I$ and set $s := u(x)$, $s' := u(x')$, identity (5.16) and condition (i) yield

$$L\big(u(x')-u(x)\big) = -\int\limits_{[-1,1]\times[-1,1]} \ddot{L}(t'-t)\,1_{t'}(x')\,1_t(x)\,dt'dt + \\ + \int\limits_{[-1,1]} \dot{L}(t+1)\,1_t(x')\,dt \;-\int\limits_{[-1,1]} \dot{L}(-t-1)\,1_t(x)\,dt\ .$$

Then, recalling Definition 5.1, $G(u,I)$ may be written as the sum of two integrals as follows:

$$G(u,I) = \int\limits_{[-1,1]\times[-1,1]} -\,\ddot{L}(t'-t)\,\Phi(E_{t'},E_t)\,dt'dt + \tag{5.17}$$

$$+\int\limits_{[-1,1]} \big(\dot{L}(t+1)-\dot{L}(-t-1)\big)\Phi(I,E_t)\,dt\ . \tag{5.18}$$

By assumption (ii) we obtain that $[d-r,d] \subset E_t \subset [c+r,d]$ for every $t \in (-1,1)$, and to replace u with its increasing rearrangement u^* means to replace every E_t with its right rearrangement E_t^* in I.

Then Theorem 5.2 applies: $\Phi(E_{t'}^*,E_t^*) \ge \Phi(E_{t'},E_t)$ for every t, t', and recalling that $-\ddot{L}$ is a non-positive function (because L is convex), we obtain that the integral in line (5.17) is decreased by the rearrangement of u. Moreover $\Phi(I,E_t^*) = \Phi(I,E_t)$ for every t, and then the integral in line (5.18) is preserved by the rearrangement of u. □

Proof of Theorem 5.8. Assume first that L is of class C^2 and J has support included in $[-r,r]$ for some positive real number r.

We prove that (5.14) holds for every u in X by approximating u with functions which satisfy the hypotheses of Lemma 5.9 on larger and larger intervals. We set $I_n := [-n,n]$ and

$$\delta_n := \sup_{-n\le x\le -n+r}\big(1+u(x)\big)\ \vee \sup_{n-r\le x\le n}\big(1-u(x)\big)\ .$$

Since u belongs to X, δ_n decreases to 0 as n tends to infinity; in particular it is smaller than 1 for n large enough, and we can set

$$u_n := \Big(\frac{u}{1-\delta_n} \wedge 1\Big) \vee -1 \ .$$

Then $(1-\delta_n)\,\big|u_n(x') - u_n(x)\big| \leq \big|u(x') - u(x)\big|$ for every x, x', and therefore

$$G(u,\mathbb{R}) \geq (1-\delta_n)^2\, G(u_n,\mathbb{R}) \geq (1-\delta_n)^2 G(u_n, I_n) \ . \tag{5.19}$$

By construction $u_n(x) = 1$ for $x \geq n-r$ and $u_n(x) = -1$ for $x \leq -n+r$; hence Lemma 5.9 yields

$$G(u_n, I_n) \geq G(u_n^*, I_n) \ . \tag{5.20}$$

Moreover u_n^* converge to u^* a.e., and then Fatou's lemma yields

$$\liminf_{n\to\infty} G(u_n^*, I_n) \geq G(u,\mathbb{R}) \ . \tag{5.21}$$

Putting together $(5.19-21)$ we get (5.14).

Finally we prove (5.14) for every non-negative convex L and every non-negative J by approximating L with an increasing sequence of non-negative convex functions L_n of class C^2 and J with an increasing sequence of non-negative summable functions J_n with compact support, and then applying the monotone convergence theorem. □

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