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A Note on the Theory of SBV Functions

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Sunto. – *In questo articolo si fornisce una dimostrazione semplice e nuova del teorema di compattezza per la classe $SBV(\Omega)$ di funzioni speciali a variazione limitata*

1. - Introduction

The space $SBV(\Omega)$ of special functions with bounded variation was introduced by De Giorgi and Ambrosio [6] as a natural extension of the space of piecewise smooth functions, to provide a weak formulation for some variational problems with free discontinuity (or problems involving both "volume" and "surface" energies as well). In particular this weak formulation was used by De Giorgi, Carriero and Leaci [7] to prove the existence of a (weak) minimizer of a functional proposed by Mumford and Shah [10] to approach an image segmentation problem.

A general theory of SBV functions was developed by Ambrosio [1-3] to find solutions to a large class of problems, via suitable compactness and lower semicontinuity results. We recall that a function u on an n-dimensional domain belongs to SBV when it has bounded variation and the singular part of the derivative (with respect to Lebesgue measure) is represented by an $\mathscr{H}^{n-1}$ summable function, where $\mathscr{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure (see paragraph 1.2 and Definition 1.3): roughly speaking, we ask that the derivative may be written as sum of a volume" part and a "surface" part.

The central point of the SBV theory is the compactness result proved in [1] (see Theorem 1.4 below): originally this result was first proved in dimension one and then extended to arbitrary dimension by a rather complicated slicing technique. Since then different approaches to SBV functions have been studied in order to obtain simpler proofs: in [4], the space SBV is characterized by an integration-by-parts formula which leads to a new proof of compactness in a special case, while in this note we establish a characterization of the singular part of the derivatives of BV functions (Proposition 2.3) which leads to a simple proof of the compactness result in its full generality.

We begin recalling some basic definitions and results.

$\mathscr{L}_n$ is the n-dimensional *Lebesgue measure* and $\mathscr{H}^k$ is the k-dimensional *Hausdorff measure*. We say that a set M in $\mathbb{R}^n$ is *rectifiable* if there exists a countable partition (M_i) of M such that $\mathscr{H}^{n-1}(M_0) = 0$ and, for every $j \geq 1$, M_i is contained in an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ of class C^1 (in GMT these sets are usually called countably $(\mathscr{H}^{n-1}, n-1)$ rectifiable but here we do not need the general definition). For $\mathscr{H}^{n-1}$ almost every point x of M we may define a *generalized normal* $\nu(x)$ so that $\nu(x)$ is a unit vector orthogonal to the tangent space of the submanifold M_i in the point x for $\mathscr{H}^{n-1}$ a.e. $x \in M \cap M_i$; clearly the generalized normal is determined $\mathscr{H}^{n-1}-$a.e. up to a change of sign (see [11, chapter 3]).

Let Ω be a bounded open subset of $\mathbb{R}^n$. By measure on Ω (or any other topological space) we always mean a measure on the σ-field of Borel subsets of Ω. When λ is a measure on Ω and E is a Borel subset of Ω, we denote by $\lambda \llcorner E$ *the restriction of* λ *to the set* E, i.e., the measure given by $(\lambda \llcorner E)(B) := \lambda(B \cap E)$, for every Borel set $B \subset \Omega$.

When μ is a positive measure on Ω, $L^p(\mu, \mathbb{R}^m)$ is the space of all $\mathbb{R}^m$-valued functions on Ω which are p-summable with respect to μ (we will write $L^p(\Omega, \mathbb{R}^m)$ when $\mu = \mathscr{L}_n \llcorner \Omega$). When f belongs to $L^1(\Omega, \mathbb{R}^m)$, we denote by f also the associated measure $f\mathscr{L}_n \llcorner \Omega$. $\mathscr{M}(\Omega, \mathbb{R}^m)$ is the space of all $\mathbb{R}^m$-valued measures on Ω with finite total variation. When $m = 1$, we omit to write $\mathbb{R}^m$.

DEFINITION 1.1 (*The Space* $BV(\Omega)$). – $BV(\Omega)$ is the space of all scalar *functions with bounded variation* on Ω, i.e., the space of all $u : \Omega \to \mathbb{R}$ whose distributional derivative Du is (represented by) a measure in $\mathscr{M}(\Omega, \mathbb{R}^n)$. For the general theory of BV functions we refer essentially to Evans and Gariepy [8, chapter 5].

REMARK 1.2 (*Decomposition of Derivatives*). – Let u be a function in $BV(\Omega)$; then, recalling Radon-Nikodym decomposition and taking into account that the measure $|Du|$ cannot charge any set which is $\mathscr{H}^{n-1}$ negligible, we may decompose Du as sum of three mutually singular measures (see for instance [1-3])

$$Du = D_a u + D_c u + D_j u \tag{1.1}$$

so that $D_a u$ (the *Lebesgue part of Du*) is the absolutely continuos part of Du with respect to Lebesgue measures, $D_j u$ (the *jump part of Du*) is a singular measure of the form $D_j u = f\mathscr{H}^{n-1}$ where f belongs to $L^1(\mathscr{H}^{n-1} \llcorner \Omega, \mathbb{R}^n)$, and $D_c u$ (the *Cantor part of Du*) is a singular measure which does not charge any set which is $\mathscr{H}^{n-1}$ σ-finite.

If ∇u denotes the density of Du with respect to Lebesgue measure, we have

$$D_a u = \nabla u \, \mathscr{L}_n, \tag{1.2}$$

and it may be proved that for $\mathscr{L}_n$ a.e. $x \in \Omega$ the vector $\nabla u(x)$ is the approximate gradient of u at x (cf. [8, section 6.1]).

The *jump set of u* is the set Su of all points $x \in \Omega$ such that u has no approximate limit at x (see [8, section 1.7] for the definition of approximate limit). We define the *precise representative* of u as the function $\tilde{u}$ which takes every $x \in \Omega \setminus Su$ in the approximate limit of u at x, and every $x \in Su$ in 0 (Su is always a Borel set, and $\tilde{u}$ a Borel function).

The set Su is rectifiable [8, section 5.9], and we denote by $\nu_u(x)$ the generalized normal to Su in the point x (for $\mathscr{H}^{n-1}$ a.e. $x \in Su$), and by $H^+(x)$ and $H^-(x)$ the half-spaces of all $y \in \mathbb{R}^n$ such that $\langle y - x, \nu_u(x)\rangle > 0$ and $\langle y - x, \nu_u(x)\rangle < 0$ respectively. Then the following result holds: for $\mathscr{H}^{n-1}$ almost every $x \in Su$, there exist the approximate limits at x of u restricted to $H^+(x)$ and $H^-(x)$, and we denote such approximate limits as $u^+(x)$ and $u^-(x)$ respectively (cf. [8, section 5.9]); this means that

$$\lim_{\rho \to 0} \rho^{-n} \int_{H^\pm(x) \cap B(x,\rho)} \left| u(y) - u^\pm(x) \right| d\mathscr{L}_n(y) \; = 0.$$

Moreover the jump part of Du may be written in term of Su, ν_u, u^+, and u^-:

$$D_j u = (u^+ - u^-)\, \nu_u \mathscr{H}^{n-1} \llcorner Su \tag{1.3}$$

(cf. [8, section 5.9]), then (1.1) becomes

$$Du = \nabla u \, \mathscr{L}_n + D_c u + (u^+ - u^-)\, \nu_u \mathscr{H}^{n-1} \llcorner Su. \tag{1.4}$$

DEFINITION 1.3 (*The Space $SBV(\Omega)$*). – We recall that the space $SBV(\Omega)$ of *special functions with bounded variation* is defined as the subspace of all $u \in BV(\Omega)$ without Cantor part, i.e., the space of all u such that $D_c u = 0$. Then, for every $u \in SBV(\Omega)$, formula (1.4) becomes

$$Du = \nabla u \,\mathscr{L}_n + (u^+ - u^-)\, \nu_u \mathscr{H}^{n-1} \llcorner Su. \tag{1.5}$$

Now we can state the compactness theorem for SBV functions (see for instance Ambrosio [1, Theorem 2.1]).

THEOREM 1.4 (*SBV Compactness Theorem*). – *Let (u_h) be a sequence of functions in $SBV(\Omega)$, and assume that*

(i) *the functions u_n are uniformly bounded in the BV norm (i.e., they are relatively compact with respect to the weak* topology of $BV(\Omega)$),*

(ii) *the approximate gradients ∇u_h are equi-integrable (i.e, they are relatively compact with respect to the weak topology of $L^1(\Omega, \mathbb{R}^n)$),*

(iii) *there exists a function $f : [0,\infty[\to [0,\infty]$ such that $f(t)/t \to \infty$ as $t \to 0$, and*

$$\int_{Su_h} f\big(|u_h^+ - u_h^-|\big)\, d\mathscr{H}^{n-1} \le C < \infty \quad \forall\, h\,. \tag{1.6}$$

Then we may extract a subsequence $(u_k) = (u_{h_k})$ which converges to some $u \in SBV(\Omega)$. Moreover the Lebesgue part and the jump part of the derivatives converge separately, i.e., $D_a u_k \to D_a u$ and $D_j u_k \to D_j u$ weakly in $\mathscr{M}(\Omega, \mathbb{R}^n)$.*

This theorem is proved in Section 2 as an immediate corollary of Proposition 2.5.

This compactness result may be applied together with suitable semicontinuity theorems to prove the existence in $SBV(\Omega)$ of the minimizers of certain integral functionals. We refer for precise statements and proofs to Ambrosio [2], [3].

2. - Proof of the SBV Compactness Theorem

We begin recalling the chain-rule formula for BV functions.

REMARK 2.1 (*Chain-rule*). – When u belongs to $BV(\Omega)$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function, then the composition function

$\phi(u)$ is a BV function too. Moreover, if ϕ is of class C^1, the following formula holds:

$$D(\phi(u)) = \phi'(\tilde{u})\,(D_a u + D_c u) + \big(\phi(u^+) - \phi(u^-)\big)\,\mathscr{H}^{n-1} \llcorner Su. \tag{2.1}$$

This formula may be easily proved by coarea formula for piecewise linear ϕ and then by approximation for all ϕ of class C^1. A similar formula holds in general for Lipschitz ϕ, but the proof is slightly more difficult (see for instance Volpert [12] and Ambrosio-Dal Maso [5]).

DEFINITION 2.2. – Now, let $f : [0,\infty[\to [0,\infty]$ be an increasing function such that

$$\lim_{t\to 0} \frac{f(t)}{t} = +\infty, \tag{2.2}$$

and define $X(f)$ as the class of all C^1 functions $\phi : \mathbb{R} \to \mathbb{R}$ with bounded derivative which satisfy

$$\big|\phi(t') - \phi(t)\big| \le f\big(|t'-t|\big) \quad \text{for all } t, t' \in \mathbb{R}. \tag{2.3}$$

Then formula (2.1) immediately yields the following inequality:

$$\sup_{\phi\in X(f)} \big\|D(\phi(u)) - \phi'(\tilde{u})\,(D_a u + D_c u)\big\| \le \int_{Su} f\big(|u^+ - u^-|\big)\, d\mathscr{H}^{n-1}. \tag{2.4}$$

This inequality characterizes in some sense the measure $D_a u + D_c u$; more precisely we have the following result:

PROPOSITION 2.3. – *Let f and $X(f)$ be given as in Definition 2.2. Let u be a function in $BV(\Omega)$, and let λ be an $\mathbb{R}^n$-valued measure on Ω such that $|\lambda|(Su) = 0$ and*

$$\sup_{\phi\in X(f)} \big\|D(\phi(u)) - \phi'(\tilde{u})\,\lambda\big\| < \infty. \tag{2.5}$$

Then $\lambda = D_a u + D_c u$, that is, $Du = \lambda + D_j u$.

PROOF. – Set $\mu := D_a u + D_c u - \lambda$. We have to prove that $\mu = 0$.

Since $|D_a u|$, $|D_c u|$ and $|\lambda|$ do not charge the set Su, we obtain that μ and $\mathscr{H}^{n-1} \llcorner Su$ are mutually singular, hence (2.1) and (2.5) yield

$$\begin{aligned} \infty > \sup_{\phi\in X(f)} & \big\|\phi'(\tilde{u})\,\mu + \big(\phi(u^+) - \phi(u^-)\big)\,\mathscr{H}^{n-1} \llcorner Su\big\| \\ & \ge \sup_{\phi\in X(f)} \big\|\phi'(\tilde{u})\,\mu\big\| = \sup_{\phi\in X(f)} \int_\Omega \big|\phi'(\tilde{u})\big|\, d|\mu|. \end{aligned} \tag{2.6}$$

Now, if we denote by σ the positive measure on $\mathbb{R}$ given by $\sigma(B) = |\mu|\big(\tilde{u}^{-1}(B)\big)$ for all Borel sets $B \subset \mathbb{R}$, inequality (2.6) yields

$$\infty > \sup_{\phi\in X(f)} \int_\Omega \big|\phi'(\tilde{u})\big|\, d|\mu| = \sup_{\phi\in X(f)} \int_{\mathbb{R}} |\phi'|\, d\sigma,$$

and by Lemma 2.4 below we obtain that σ, and μ as well, are trivial. ■

LEMMA 2.4. – *Let f and $X(f)$ be given as in Definition 2.2, and let σ be a non-trivial positive measure on $\mathbb{R}$. Then*

$$\sup_{\phi\in X(f)} \int_{\mathbb{R}} |\phi'|\, d\sigma = +\infty.$$

PROOF. – Let $M \geq 2$ be fixed. Since f satisfies condition (2.2), there exists $\varepsilon > 0$ such that $f(t) \geq Mt$ when $0 \leq t \leq \varepsilon$, and since f is increasing, $f(t) \geq M\varepsilon$ when $t \geq \varepsilon$. Hence

$$f(t) \geq Mt \wedge M\varepsilon \geq Mt \wedge 2\varepsilon \quad \text{for all } t. \tag{2.7}$$

Set $\phi(t) := \varepsilon \sin(Mt/\varepsilon)$ for every t: ϕ is an M-lipschitz function, $|\phi| \leq \varepsilon$ for every t, and then ϕ belongs to $X(f)$ by (2.7). One readily checks that $|\phi'(t)| \geq M/\sqrt{2}$ when t belongs to the set

$$A := \bigcup_{k\in\mathbf{Z}} \Big[\frac{\varepsilon}{M}\Big(-\frac{1}{4}+k\Big)\pi\,,\ \frac{\varepsilon}{M}\Big(\frac{1}{4}+k\Big)\pi\Big],$$

and then

$$\int |\phi'|\, d\sigma \geq \frac{M}{\sqrt{2}}\,\sigma(A).$$

Furthermore, if we denote by φ the translation of ϕ by $\varepsilon\pi/(2M)$, then φ belongs to $X(f)$, $|\varphi'(t)| \geq M/\sqrt{2}$ when t belongs to $\mathbb{R}\setminus A$, and

$$\int |\varphi'|\, d\sigma \geq \frac{M}{\sqrt{2}}\,\sigma(\mathbb{R}\setminus A).$$

Hence either $\int |\phi'|\, d\sigma$ or $\int |\varphi'|\, d\sigma$ have to be greater than $M\|\sigma\|$ divided by $2\sqrt{2}$, and since M can be taken arbitrarily large, the proof is complete. ■

We use Proposition 2.3 to prove the following closure result for SBV functions; Theorem 1.4 will follow as an immediate corollary.

PROPOSITION 2.5. – *Let f be given as in Definition 2.2. Let (u_h) be a sequence of SBV functions which converges to u in the weak* topology of $BV(\Omega)$, and assume that the approximate gradients ∇u_h converge to some g weakly in $L^1(\Omega, \mathbb{R}^n)$, and that*

$$\int_{Su_h} f\big(|u_h^+ - u_h^-|\big)\, d\mathscr{H}^{n-1} \le C < \infty \quad \forall\, h.$$

Then u belongs to $SBV(\Omega)$, and the Lebesgue part and the jump part of the derivatives converge separately, i.e., $D_a u_h \to D_a u$ and $D_j u_h \to D_j u$ weakly in $\mathscr{M}(\Omega, \mathbb{R}^n)$.*

PROOF. – We remark that it is enough to prove that $D_a u + D_c u = g\,\mathscr{L}_n$.

Take $\phi \in X(f)$. Then, taking into account inequality (2.4) and the fact that $u_h \in SBV$ for every h, we get

$$(2.8)\quad C \ge \int_{Su_h} f\big(|u^+ - u^-|\big)\, d\mathscr{H}^{n-1} \ge \big\| D(\phi(u_h)) - \phi'(u_h)\, \nabla u_h \mathscr{L}_n \big\|.$$

Now, with no loss in generality, we may assume that u_h converge to u almost everywhere in Ω. The functions $\phi(u_h)$ converge to $\phi(u)$ in the weak* topology of $BV(\Omega)$, and then the measures $D(\phi(u_h))$ converge to $D(\phi(u))$ in the weak* topology of measures. Since ϕ' is bounded and continuous, the functions $\phi'(u_h)$ are uniformly bounded and converge to $\phi'(u)$ a.e. Moreover the functions ∇u_h converge to g weakly in $L^1(\Omega, \mathbb{R}^n)$ by hypothesis. Hence $\phi'(u_h)\, \nabla u_h$ converge to $\phi'(u)\, g$ weakly in $L^1(\Omega, \mathbb{R}^n)$, and then

$$\big[D(\phi(u_h)) - \phi'(u_h)\, \nabla u_h \mathscr{L}_n\big] \longrightarrow \big[D(\phi(u)) - \phi'(u)\, g\, \mathscr{L}_n\big]$$

weakly* in $\mathscr{M}(\Omega, \mathbb{R}^n)$. Now (2.8) yields

$$C \ge \liminf_{h\to\infty} \big\| D(\phi(u_h)) - \phi'(u_h)\, \nabla u_h \mathscr{L}_n \big\| \ge \big\| D(\phi(u)) - \phi'(u)\, g\, \mathscr{L}_n \big\|.$$

Eventually, if we take the supremum over all $\phi \in X(f)$, and apply Proposition 2.3, we get $D_a u + D_c u = g\,\mathscr{L}_n$. ■

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REFERENCES

[1] L. AMBROSIO, *A compactness theorem for a new class of functions of bounded variation*, Boll. Un. Mat. Ital., **3-B** (1989), 857-881.

[2] L. AMBROSIO, *Existence theory for a new class of variational problems*, Arch. Rat. Mech. Anal., **111** (1990), 291-322.

[3] L. AMBROSIO, *Variational Problems in SBV*, Acta Appl. Math., **17** (1989), 1-40.

[4] L. AMBROSIO, *A new proof of the SBV compactness theorem*, Calc. Var., **3** (1995), 127-137.

[5] L. AMBROSIO - G. DAL MASO, *A general chain-rule for distributional derivatives*, Proc. Am. Math. Soc., **108** (1990), 691-702.

[6] E. DE GIORGI - L. AMBROSIO, *Un nuovo tipo di funzionale del calcolo delle variazioni*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat., **82** (1988), 199-210.

[7] E. DE GIORGI - M. CARRIERO - A. LEACI, *Existence theorem for a minimum problem with free discontinuity set*, Arch. for Rat. Mech. Anal., **108** (1989), 195-218.

[8] L.C. EVANS - R.F. GARIEPY, *Lectures notes on measure theory and fine properties of functions*, CRC Press, Ann Harbor, 1992.

[9] E. GIUSTI, *Minimal surfaces and functions of bounded variation*, Birkhäuser, Basel, 1994.

[10] D. MUMFORD - J. SHAH, *Optimal approximation by piecewise smooth functions and associated variational problems*, Comm. Pure Appl. Math., **17** (1989), 577-685.

[11] L.M. SIMON, *Lectures on Geometric Measure Theory*, .. Proc. Centre for Math. Anal. Austral. Nat. Univ., **3** (1983)

[12] A.I. VOLPERT, *Spaces BV and quasilinear equations*, Math. USSR Sb., **17** (1967), 225-267.

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