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# On the differentiability of Lipschitz functions with respect to measures in the Euclidean space

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Abstract. For every finite measure $\mu$ on $\mathbb{R}^n$ we define a decomposability bundle $V(\mu,\cdot)$ related to the decompositions of $\mu$ in terms of rectifiable one-dimensional measures. We then show that every Lipschitz function on $\mathbb{R}^n$ is differentiable at $\mu$-a.e. $x$ with respect to the subspace $V(\mu,x)$, and prove that this differentiability result is optimal, in the sense that, following [4], we can construct Lipschitz functions which are not differentiable at $\mu$-a.e. $x$ in any direction which is not in $V(\mu,x)$. As a consequence we obtain a differentiability result for Lipschitz functions with respect to (measures associated to) $k$-dimensional normal currents, which we use to extend certain basic formulas involving normal currents and maps of class $C^1$ to Lipschitz maps.

Keywords: Lipschitz functions, differentiability, Rademacher theorem, normal currents.

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## 1. Introduction

The study of the differentiability properties of Lipschitz functions has a long story, and many facets. In recent years much attention has been devoted to the differentiability of Lipschitz functions on infinite dimensional Banach spaces (see the monograph by J. Lindenstrauss, D. Preiss and J. Tišer [18]) and on metric measure spaces (we just mention here the works by J. Cheeger [7], S. Keith [15] and D. Bate [6]), but at about the same time it became clear that even Lipschitz functions on $\mathbb{R}^n$ are not completely understood, and that Rademacher theorem, which states that every Lipschitz function on $\mathbb{R}^n$ is differentiable almost everywhere is not the end of story.[1]

To this regard, the first fundamental contribution is arguably the paper [24], where Preiss proved, among other things, that there exist null sets $E$ in $\mathbb{R}^2$ such that every Lipschitz function on $\mathbb{R}^2$ is differentiable at some point of $E$. Therefore Rademacher theorem is not sharp, in the sense that while the set of non-differentiability points of a Lipschitz function is always contained in a null set, the opposite inclusion does not always hold. (The construction of such sets has been variously improved in recent years, see [10], [11] for detailed references.)

Note that this differentiability result is strictly confined to *functions*, intended as real-valued maps, and indeed it was later proved by G. Alberti, M. Csörnyei

[1] As usual, the expressions "almost everywhere", "null set", "absolutely continuous / singular measure", when used without further specification, refer to the Lebesgue measure.

and D. Preiss that every null set in $\mathbb{R}^2$ is contained in the non-differentiability set of some Lipschitz map $f : \mathbb{R}^2 \to \mathbb{R}^2$. This result was announced in [2], [3] and appears in [4]; more precisely, [4] contains a complete description of the sets of non-differentiability of Lipschitz maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, and a proof of the fact that for $n = 2$ this class agrees with the class of null sets. Recently M. Csörnyei and P.W. Jones announced that the latter result holds in arbitrary dimension $n$ (cf. [13]), thus proving that every null set in $\mathbb{R}^n$ is contained in the non-differentiability set of a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$. Finally D. Preiss and G. Speight [25] completed the picture by showing that there exist null sets $E$ in $\mathbb{R}^n$ such that every Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^m$ with $m < n$ is differentiable at some point of $E$.

In this paper we approach the differentiability of Lipschitz functions from a slightly different point of view. Consider again the statement of Rademacher theorem: the "almost everywhere" there refers to the Lebesgue measure, and clearly the statement remains true if we replace the Lebesgue measure with a measure $\mu$ which is absolutely continuous, but of course it fails if $\mu$ is an arbitrary singular measure.

However in many cases it is clear how to modify the statement to make it true. For example, if $S$ is a $k$-dimensional surface of class $C^1$ contained in $\mathbb{R}^n$ and $\mu$ is the $k$-dimensional volume measure on $S$, then every Lipschitz function on $\mathbb{R}^n$ is differentiable at $\mu$-a.e. $x \in S$ in all directions in the tangent space $\mathrm{Tan}(S, x)$. Furthermore this statement is optimal in the sense that there are Lipschitz functions $f$ on $\mathbb{R}^n$ which, for every $x \in S$, are not differentiable at $x$ in any direction which is not in $\mathrm{Tan}(S, x)$ (the obvious example is the distance function $f(x) := \mathrm{dist}(x, S)$).

We aim to prove a statement of similar nature for an arbitrary finite measure $\mu$ on $\mathbb{R}^n$ (and note that all results can be immediately extended to $\sigma$-finite measures). More precisely, we want to identify for $\mu$-a.e. $x$ the largest set of directions $V(\mu, x)$ such that every Lipschitz function on $\mathbb{R}^n$ is differentiable at $\mu$-a.e. $x$ in every direction in $V(\mu, x)$.

We begin with a simple observation: let $\mu$ be a measure on $\mathbb{R}^n$ that can be decomposed as

$$\mu = \int_I \mu_t \, dt \,, \tag{1.1}$$

where $I$ is the interval $[0, 1]$ endowed with the Lebesgue measure $dt$, and each $\mu_t$ is the length measure on some rectifiable curve $E_t$ (formula (1.1) means that $\mu(E) = \int_I \mu_t(E) \, dt$ for every Borel set $E$; a precise definition of integral of a measure-valued map is given §2.3). Assume moreover that there exists a vectorfield $\tau$ on $\mathbb{R}^n$ such that for a.e. $t$ and $\mu_t$-a.e. $x \in E_t$ the vector $\tau(x)$ is tangent to $E_t$ at $x$. Then every Lipschitz function $f$ on $\mathbb{R}^n$ is differentiable at $x$ in the direction $\tau(x)$ for $\mu$-a.e. $x$.

Indeed, by applying Rademacher theorem to the Lipschitz function $f \circ \gamma_t$ where $\gamma_t$ is a parametrization of $E_t$ by arc-length, we obtain that $f$ is differentiable at the point $\gamma(s)$ in the direction $\dot\gamma(s)$ for a.e. $s$, which means that $f$ is differentiable at $x$ in the direction $\tau(x)$ for $\mu_t$-a.e. $x$ and a.e. $t$, and by formula (1.1) "for $\mu_t$-a.e. $x$ and a.e. $t$" is equivalent to "for $\mu$-a.e. $x$".



This observation suggests that the set of directions $V(\mu, x)$ we are looking for should be related to the set of all decompositions of $\mu$, or of parts of $\mu$, of the type considered in formula (1.1).

We then propose the following makeshift definition: consider all possible families of measures $\{\mu_t\}$ such that the measure $\int_I \mu_t \, dt$ is absolutely continuous w.r.t. $\mu$, and each $\mu_t$ is the restriction of the length measure to a subset $E_t$ of a rectifiable curve, and for every $x \in E_t$ let $\mathrm{Tan}(E_t, x)$ be the tangent line to this curve at $x$ (if it exists); let then $V(\mu, x)$ be the smallest linear subspace of $\mathbb{R}^n$ such that for every family $\{\mu_t\}$ as above there holds $\mathrm{Tan}(E_t, x) \subset V(\mu, x)$ for a.e. $t$. We call the map $x \mapsto V(\mu, x)$ the *decomposability bundle* of $\mu$.

As formulated, this definition would not stand a close scrutiny but hopefully it should allow the reader to understand the following theorem, which is the main result of this paper; the "correct" definition of decomposability bundle requires more preparation, and will be given in §2.6.

**1.1. Theorem.** *Let $\mu$ be a finite measure on $\mathbb{R}^n$, and let $V(\mu, \cdot)$ be the decomposability bundle of $\mu$ (see §2.6). Then the following statements hold:*

(i) *Every Lipschitz function $f$ on $\mathbb{R}^n$ is differentiable at $\mu$-a.e. $x$ with respect to the linear subspace $V(\mu, x)$, that is, there exists a linear function from $V(\mu, x)$ to $\mathbb{R}$, denoted by $d_V f(x)$, such that*

$$f(x + h) = f(x) + d_V f(x) \, h + o(|h|) \quad \textit{for } h \in V(\mu, x).$$

(ii) *The previous statement is optimal in the sense that there exists a Lipschitz function $f$ on $\mathbb{R}^n$ such that for $\mu$-a.e. $x$ and every $v \notin V(\mu, x)$ the derivative of $f$ at $x$ in the direction $v$ does not exist.*

**1.2. Remarks.** (i) Obviously the differentiability part of this theorem, namely statement (i), applies also to Lipschitz maps $f : \mathbb{R}^n \to \mathbb{R}^m$, because it applies to each component of $f$.

(ii) Theorem 4.1 below contains a stronger version of statement (ii), where the non-differentiability of $f$ in a given direction is made more precise by showing that the corresponding upper and lower directional derivatives do not agree. The possibility of a uniform quantification of the non-differentiability of $f$ (that is, of the gap between upper an lower directional derivatives) is discussed in Remarks 4.6(ii) and (iii), and Example 4.7. One may further ask to which extent the non-differentiable behaviour of $f$ could be prescribed "at many points", e.g., by requiring that the blowups of $f$ at these points includes certain nonlinear functions. This problem will be considered in [20].

(iii) Statement (ii), can be derived (with limited effort) from the characterization of the non-differentiability sets of Lipschitz maps given in [4]. More precisely, in [4] the authors define for every $k = 0, \ldots, n-1$ a class $\mathscr{N}_k$ of sets $E$ in $\mathbb{R}^n$ which are $k$-dimensional in a sense that we do not specify here, and are equipped with a suitable notion of $k$-dimensional tangent bundle $\mathrm{Tan}(E, \cdot)$; then for every such

$E$ they construct a Lipschitz map on $\mathbb{R}^n$ which, at "most" points $x \in E$,[2] is not differentiable in every direction $v \perp \mathrm{Tan}(E, x)$.

Now, using Lemma 7.3 one can show that for every measure $\mu$ on $\mathbb{R}^n$ there exist sets $E_0, \dots, E_{n-1}$ which cover $\mu$-a.e. point $x$ where $V(\mu, x) \neq \mathbb{R}^n$, and such that, for every $k$, $E_k$ belongs to $\mathscr{N}_k$, and $V(\mu, x)$ has dimension $k$ and agrees with $\mathrm{Tan}(E_k, x)$ for $\mu$-a.e. $x \in E_k$. Then one can use the non-differentiable maps associated to each $E_k$ as above to construct a function $f$ which satisfies statement (ii) in Theorem 1.1.

(iv) It turns out that the construction given in [4] can be greatly simplified when adapted to our setting, and therefore we decided to include it here with all details. The point is that in [4] the authors construct Lipschitz functions (actually maps) which are non-differentiable at every point of a given set, while here we only need Lipschitz functions which are non-differentiable $\mu$-a.e.; this means that we are allowed to discard $\mu$-null sets, which makes room for many simplifications. Moreover in our framework we can apply Baire category methods, which significantly reduces the complexity of the construction.[3]

(v) Note that the non-differentiable function $f$ in statement (ii) is a function and not a map. Thus Theorem 1.1 is not sensitive to the dimension of the codomain, unlike the results on pointwise differentiability and non-differentiability mentioned at the beginning of this introduction.

This is actually not surprising, given the following (rather elementary) statement: if $f : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map which is non-differentiable at $\mu$-a.e. point, then for (Lebesgue-) a.e. $v \in \mathbb{R}^m$ the scalar product $v \cdot f$ is a Lipschitz function which is non-differentiable at $\mu$-a.e. point. Note that this statement is no longer true if we replace both occurrences of "at $\mu$-a.e. point" with "at every point of a set $E$".

(vi) The idea that the differentiability properties of Lipschitz functions w.r.t. a general measure $\mu$ are encoded in the decompositions of $\mu$ in terms of rectifiable measures has been in the air for quite some time now; for example, it is clearly assumed as a starting point in [6], where it is extended to the context of metric measure spaces to give a characterization of Lipschitz differentiability spaces.

(vii) The notion of decomposability bundle is related to a notion of tangent space to measures introduced in [1] (see Remark 6.2(iv) for more details).

**1.3. Applications to the theory of currents.** In Section 5 we study the decomposability bundle of measures related to normal currents. More precisely, given a measure $\mu$ and a normal $k$-current $T$, we denote by $\tau$ the Radon-Nikodým derivative of $T$ w.r.t. $\mu$ (see §5.3), and show that the linear subspace of $\mathbb{R}^n$ spanned by the $k$-vector $\tau(x)$ is contained in $V(\mu, x)$ for $\mu$-a.e. $x$ (Theorem 5.10). We then use this result to give explicit formulas for the boundary of the interior product of a normal $k$-current and a Lipschitz $h$-form (Proposition 5.13) and for the push-forward of a normal $k$-current according to a Lipschitz map (Proposition 5.17).

In section 6 we give a characterization of the decomposability bundle of a measure $\mu$ in terms of 1-dimensional normal currents (Theorem 6.4); building on this result we obtain that a vectorfield $\tau$ on $\mathbb{R}^n$ can be written as the Radon-Nikodým derivative of a 1-dimensional normal current $T$ w.r.t. $\mu$ if and only if $\tau(x)$ belongs to $V(\mu, x)$ for $\mu$-a.e. $x$ (Corollary 6.5). Using this fact and a well-known result on the decomposition of 1-dimensional normal currents in terms of rectifiable currents (Theorem 5.5), we finally show that a measure $\mu$ with non-trivial decomposability bundle admits a decomposition of type (1.1) where each set $E_t$ is now 1-rectifiable and its tangent bundle is aligned with any prescribed vectorfield $\tau$ which satisfies $\tau(x) \in V(\mu, x)$ for $\mu$-a.e. $x$ (Corollary 6.6). This result is quite close in spirit to the decomposition of measures in metric spaces given in [29], Theorem 6.31.

**1.4. Computation of the decomposability bundle.** In certain cases the decomposability bundle $V(\mu, x)$ can be computed using Proposition 2.9. We just recall here that if $\mu$ is absolutely continuous w.r.t. the Lebesgue measure then $V(\mu, x) = \mathbb{R}^n$ for $\mu$-a.e. $x$, and if $\mu$ is absolutely continuous w.r.t. the restriction of the Hausdorff measure $\mathscr{H}^k$ to a $k$-dimensional surface $S$ of class $C^1$ (or a $k$-rectifiable set $S$) then $V(\mu, x) = \mathrm{Tan}(S, x)$ for $\mu$-a.e. $x$ (Proposition 2.9(iii)). On the other hand, if $\mu$ is the canonical measure associated to well-known examples of self-similar fractals such as the snowflake curve and the Sierpiński carpet, then $V(\mu, x) = \{0\}$ for $\mu$-a.e. $x$ (see Remark 2.10).

**1.5. Rademacher theorem and the dimension of $\boldsymbol{V(\mu, x)}$.** It is natural to ask for which measures $\mu$ on $\mathbb{R}^n$ Rademacher theorem holds in the usual form, that is, every Lipschitz function (or map) on $\mathbb{R}^n$ is differentiable $\mu$-a.e. Clearly the class of such measures contains all absolutely continuous measures, but does it contains any singular measure?

The answer turns out to be negative in every dimension $n$, because a singular measure $\mu$ is supported on a null set $E$, and, as mentioned above, for every null set $E$ in $\mathbb{R}^n$ there exists a Lipschitz map $f : \mathbb{R}^n \to \mathbb{R}^n$ which is non-differentiable at every point of $E$ (the case $n = 2$ is proved in [4], while the general case has been announced by Csörnyei and Jones).

On the other hand, Theorem 1.1 shows that Rademacher theorem holds for a measure $\mu$ if and only if $V(\mu, x) = \mathbb{R}^n$ for $\mu$-a.e. $x$, and therefore we conclude that if $\mu$ is a singular measure on $\mathbb{R}^n$ then $V(\mu, x) \neq \mathbb{R}^n$ for $\mu$-a.e. $x$. Using statements (i) and (iii) in Proposition 2.9 we can actually say slightly more: given a measure $\mu$ on $\mathbb{R}^n$ and denoting by $\mu_a$ and $\mu_s$ the absolutely continuous part and the singular part of $\mu$, respectively, then $V(\mu, x) = \mathbb{R}^n$ for $\mu_a$-a.e. $x$ and $V(\mu, x) \neq \mathbb{R}^n$ for $\mu_s$-a.e. $x$.

Note that for $n = 1$ it is actually easy to prove directly—that is, without using non-differentiability results—that $V(\mu, x) = \{0\}$ for $\mu$-a.e. $x$ when $\mu$ is singular: indeed $\mu$ is supported on a null set $E$, every null set in $\mathbb{R}$ is purely unrectifiable (see §2.2), and the decomposability bundle of a measure supported on a purely unrectifiable set is trivial (in any dimension, see Proposition 2.9(iv)).

For $n = 2$, the fact that $V(\mu, x) \neq \mathbb{R}^n$ for $\mu$-a.e. $x$ when $\mu$ is singular follows also from a result proved in [1] (see Remark 6.2(iv) for more details). A proof in any dimension $n$ can also be obtained as corollary of a very recent and deep result by G. De Philippis and F. Rindler [9] (it suffices to put together Corollary 1.11 and Lemma 3.1 in that paper).

[2] Here "most" is intended in a sense that, again, we do not specify.

[3] The use of Baire category methods to construct Lipschitz functions that are not differentiable on a given null set in $\mathbb{R}$ is discussed exhaustively in [26].

**1.6. Higher dimensional decompositions.** For $k = 1, \dots, n$ let $\mathscr{F}_k(\mathbb{R}^n)$ be the class of all measures $\mu$ on $\mathbb{R}^n$ which are absolutely continuous w.r.t. a measure of the form $\int_I \mu_t \, dt$ where each $\mu_t$ is the restriction of the $k$-dimensional Hausdorff measure $\mathscr{H}^k$ to a $k$-rectifiable set $E_t$. By Proposition 2.9(vi), for every such $\mu$ the decomposability bundle $V(\mu, x)$ has dimension at least $k$ at $\mu$-a.e. $x$; it is then natural to ask if the converse is true, namely that $\mu$ belongs to $\mathscr{F}_k(\mathbb{R}^n)$ when $\dim(V(\mu, x)) \geq k$ for $\mu$-a.e. $x$.

The answer is positive for $k = 1$ and $k = n$: the case $k = 1$ is trivial, while the case $k = n$ is equivalent to the statement mentioned in the previous subsection, that $\mu$ is absolutely continuous if $\dim(V(\mu, x)) = n$ for $\mu$-a.e. $x$. Recently, A. Máthé proved in [21] that the answer is negative in all other cases.

**1.7. Differentiability of Sobolev functions.** Since the continuous representatives of functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$ with $p > n$ are differentiable almost everywhere, it is natural to ask what happens to differentiability when the Lebesgue measure is replaced by a singular measure. In [4] it is shown that for every singular measure $\mu$ and every $p < +\infty$ there exists a continuous function in $W^{1,p}(\mathbb{R}^n)$ which is not differentiable in any direction at $\mu$-a.e. point; it seems therefore that Theorem 1.1 admits no significant extension to (first order) Sobolev spaces.

The rest of this paper is organized as follows: in Section 2 we give the precise definition of decomposability bundle and a few basic properties, while Sections 3 and 4 contain the proof of Theorem 1.1.

In Section 5 we study the decomposability bundle of measures associated to normal currents, and describe some applications to the theory of normal currents, while in Section 6 we give a characterization of the decomposability bundle of a measure in terms of 1-dimensional normal currents. Note that sections 5 and 6 are essentially independent of the rest of the paper (and of each other).

In order to make the structure of the main proofs more transparent, we have moved a few technical results to the appendices at the end of the paper. More precisely, Section 7 contains some statements derived from Rainwater's Lemma, while Section 8 contains two approximation results for Lipschitz functions.

## 2. Decomposability bundle

We begin this section by recalling some general definitions and notation, we then give the definition of decomposability bundle $V(\mu, \cdot)$ of a measure $\mu$ (see §2.6) and prove a few basic properties (Propositions 2.8 and 2.9).

**2.1. General notation.** Unless we specify otherwise, sets and functions on $\mathbb{R}^n$ are assumed to be Borel measurable, and measures on $\mathbb{R}^n$ are positive, finite measures on the Borel $\sigma$-algebra (the obvious exceptions being the Lebesgue and Hausdorff measures).

It is important to keep in mind that we never identify maps (and functions) which agree almost everywhere w.r.t. some measure. In other words, maps are always defined at every point, and never to be considered as equivalence classes, not even when it would appear natural to do so.



We say that a measure on $\mathbb{R}^n$ is supported on the (Borel) set $E$ if its restriction to $\mathbb{R}^n \setminus E$ vanishes (note that $E$ does not need to be closed, and hence it may not contain the support of $\mu$).

We say that a map $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at the point $x \in \mathbb{R}^n$ w.r.t. a linear subspace $V$ of $\mathbb{R}^n$ if there exists a linear map $L : V \to \mathbb{R}^m$ such that the following first-order Taylor expansion holds

$$f(x+h) = f(x) + Lh + o(|h|) \quad \text{for all } h \in V;$$

when it exists, $L$ is called the derivative of $f$ at $x$ w.r.t. $V$ and denoted by $d_V f(x)$; if $V = \mathbb{R}^n$ then $d_V f(x)$ is the usual derivative, and is simply denoted by $df(x)$.

We add below a list of frequently used notations (for the notations related to multilinear algebra and currents see §5.1):

| | |
|---|---|
| $B(x,r)$ | *closed* ball with center $x$ and radius $r$; |
| $\mathrm{dist}(x,E)$ | distance between the point $x$ and the set $E$; |
| $v \cdot w$ | scalar product of $v, w \in \mathbb{R}^n$; |
| $C(e,\alpha)$ | convex closed cone in $\mathbb{R}^n$ with axis $e$ and angle $\alpha$ (see §4.11); |
| $1_E$ | characteristic function of the set $E$, taking values 0 and 1; |
| $\mathrm{Gr}(\mathbb{R}^n)$ | set of all linear subspaces of $\mathbb{R}^n$, that is, the union of the Grassmannians $\mathrm{Gr}(\mathbb{R}^n, k)$ with $k = 0, \dots, n$. |
| $d_{\mathrm{gr}}(V,V')$ | distance between $V, V' \in \mathrm{Gr}(\mathbb{R}^n)$, defined as the maximum of $\delta(V,V')$ and $\delta(V',V)$, where $\delta(V,V')$ is the smallest number $\delta$ such that for every $v \in V$ there exists $v' \in V'$ with $\lvert v - v'\rvert \leq \delta \lvert v\rvert$; |
| $\alpha(V,V')$ | $:= \arcsin(d_{\mathrm{gr}}(V,V'))$ is the angle between $V, V' \in \mathrm{Gr}(\mathbb{R}^n)$; |
| $\langle L\,;v\rangle$ | (also written $Lv$) is the action of the linear map $L$ on the vector $v$; |
| $\lvert L\rvert$ | operator norm of the linear map $L$ (between normed spaces); |
| $D_v f(x)$ | derivative of the map $f : \mathbb{R}^n \to \mathbb{R}^m$ in the direction $v$ at the point $x$; |
| $df(x)$ | derivative of the map $f : \mathbb{R}^n \to \mathbb{R}^m$ at the point $x$, viewed as a linear map from from $\mathbb{R}^n$ to $\mathbb{R}^m$; |
| $d_V f(x)$ | derivative of the map $f : \mathbb{R}^n \to \mathbb{R}^m$ at $x$ w.r.t. the subspace $V$; |
| $\mathrm{Tan}(S,x)$ | tangent space to $S$ at the point $x$, where $S$ is a surface (submanifold) of class $C^1$ in $\mathbb{R}^n$ or a rectifiable set (see §2.2); |
| $\mathrm{Lip}(f)$ | Lipschitz constant of the map $f$; |
| $\mathscr{L}^n$ | Lebesgue measure on $\mathbb{R}^n$; |
| $\mathscr{H}^d$ | $d$-dimensional Hausdorff measure; |
| $L^p$ | stands for $L^p(\mathbb{R}^n, \mathscr{L}^n)$; for the $L^p$ space on a different measure space $(X, \mathscr{S}, \mu)$ we use the notation $L^p(\mu)$; |
| $\lVert \cdot \rVert_p$ | $L^p$-norm w.r.t. the Lebesgue measure; we use $\lVert \cdot \rVert_\infty$ also to denote the supremum norm of continuous functions; |

$\rho\mu$ measure associated to a measure $\mu$ and a function $\rho$, namely $[\rho\mu](E) := \int_E \rho\, d\mu$;
$1_E\mu$ restriction of a measure $\mu$ to a set $E$;
$f_\#\,\mu$ push-forward of a measure $\mu$ on $X$ according to a map $f : X \to X'$, that is, the measure on $X'$ given by $[f_\#\,\mu](E) := \mu(f^{-1}(E))$;
$\lambda \ll \mu$ means that the measure $\lambda$ is absolutely continuous w.r.t. $\mu$, hence $\lambda = \rho\mu$ where $\rho$ is the Radon-Nikodým derivative of $\lambda$ w.r.t. $\mu$;
$|\mu|$ total variation measure associated to a real- or vector-valued measure $\mu$; thus $\mu = \rho|\mu|$ where the Radon-Nikodým derivative $\rho$ satisfies $|\rho| = 1$ $\mu$-a.e.
$\mathbb{M}(\mu)$ $:= |\mu|(X)$, mass of a measure $\mu$ on a space $X$;
$\int_I \mu_t\, dt$ integral of the measure-valued map $t \mapsto \mu_t$ (see §2.3).

**2.2. Rectifiable and unrectifiable sets.** Given $k = 1, 2, \dots$ we say that a set $E$ contained in $\mathbb{R}^n$ is *$k$-rectifiable* if $\mathscr{H}^k(E) < +\infty$ and $E$ can be covered, except for an $\mathscr{H}^k$-null subset, by countably many images of Lipschitz maps from $\mathbb{R}^k$ to $\mathbb{R}^n$, or equivalently by countably many $k$-dimensional surfaces (submanifolds) of class $C^1$ (cf. [22], §3.10 and Proposition 3.11, or [17], Definition 5.4.1 and Lemma 5.4.2).

Fix a point $x \in \mathbb{R}^n$, and for every $r > 0$ let $\sigma_{x,r}$ be the restriction of $\mathscr{H}^k$ to the blow-up set $E_{x,r} := \frac{1}{r}(E - x)$. We say that a $k$-dimensional subspace $V$ of $\mathbb{R}^n$ is the *approximate tangent space* to $E$ at $x$ if the measures $\sigma_{x,r}$ converge to the restriction of $\mathscr{H}^k$ to $V$ in the sense of measures (that is, in the weak* topology induced by the duality with the space of continuous functions with compact support in $\mathbb{R}^n$).

If it exists, the approximate tangent space $V$ is clearly unique and is denoted by $\mathrm{Tan}(E, x)$. Moreover $\mathrm{Tan}(E, x)$ exists for $\mathscr{H}^k$-a.e. $x \in E$ (cf. [22], Proposition 3.12, or [17], Theorem 5.4.6) and is characterized up to $\mathscr{H}^k$-negligible subsets of $E$ by the property that for every $k$-dimensional surface $S$ of class $C^1$ there holds

$$\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in E \cap S. \tag{2.1}$$

Finally we say that a set $E$ in $\mathbb{R}^n$ is *purely unrectifiable* if $\mathscr{H}^1(E \cap S) = 0$ for every 1-rectifiable set $S$, or equivalently for every curve $S$ of class $C^1$.

**2.3. Integration of measures.** Let $I$ be a finite measure space and for every $t \in I$ let $\mu_t$ be a measure on $\mathbb{R}^n$, possibly real- or vector-valued, such that:

(a) the function $t \mapsto \mu_t(E)$ is measurable for every Borel set $E$ in $\mathbb{R}^n$;
(b) $\int_I \mathbb{M}(\mu_t)\, dt < +\infty$, where $dt$ denotes the measure on $I$.

Then we denote by $\int_I \mu_t\, dt$ the measure on $\mathbb{R}^n$ defined by

$$\left[\int_I \mu_t\, dt\right](E) := \int_I \mu_t(E)\, dt \quad \text{for every Borel set } E \text{ in } \mathbb{R}^n.$$

Note that assumption (a) is equivalent to say that $t \mapsto \mu_t$ is a measurable map from $I$ to the space of finite (real- or vector-valued) measures on $\mathbb{R}^n$ endowed with the weak* topology. Note that assumption (a) and the definition of mass imply that the function $t \mapsto \mathbb{M}(\mu_t)$ is measurable, thus the integral in assumption (b) is well-defined.



For the next definitions we need the following lemma (or rather, observation).

**2.4. Lemma.** *Let $\mu$ be a measure on $\mathbb{R}^n$ and let $\mathscr{G}$ be a family of Borel maps from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ which is closed under countable intersection, in the sense that for every countable family $\{V_i\} \subset \mathscr{G}$ the map $V$ defined by $V(x) := \cap_i V_i(x)$ for every $x \in \mathbb{R}^n$ belongs to $\mathscr{G}$.*

*Then $\mathscr{G}$ admits an element $V$ which is $\mu$-minimal, in the sense that every other $V' \in \mathscr{G}$ satisfies $V(x) \subset V'(x)$ for $\mu$-a.e. $x$. Moreover this $\mu$-minimal element is unique modulo equivalence $\mu$-a.e.*

**Proof.** Uniqueness follows immediately from minimality. To prove existence, set

$$\Phi(V) := \int_{\mathbb{R}^n} \dim(V(x))\, d\mu(x) \quad \text{for every } V \in \mathscr{G},$$

then take a sequence $\{V_j\}$ in $\mathscr{G}$ such that $\Phi(V_j)$ tends to the infimum of $\Phi$ over $\mathscr{G}$, and let $V$ be the intersection of all $V_j$. Thus $V$ belongs to $\mathscr{G}$ and is a minimum of $\Phi$ over $\mathscr{G}$, and we claim that it is also a $\mu$-minimal element of $\mathscr{G}$: if not, there would exist $V' \in \mathscr{G}$ such that $V''(x) := V(x) \cap V'(x)$ is strictly contained in $V(x)$ for all $x$ in some set of positive measure, thus $V''$ belongs to $\mathscr{G}$ and $\Phi(V'') < \Phi(V)$. □

**2.5. Essential span of a family of vectorfields.** Let $\mu$ be a measure on $\mathbb{R}^n$, let $\mathscr{F}$ be a family of Borel vectorfields on $\mathbb{R}^n$, and let $\mathscr{G}$ be the class of all Borel maps $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ such that for every $\tau \in \mathscr{F}$ there holds

$$\tau(x) \subset V(x) \quad \text{for } \mu\text{-a.e. } x.$$

Since $\mathscr{G}$ is closed under countable intersection, by Lemma 2.4 it admits a $\mu$-minimal element which is unique modulo equivalence $\mu$-a.e. With a slight abuse of language we call *any* of these minimal elements *$\mu$-essential span* of $\mathscr{F}$. (The abuse lies in the fact that we do *not* identify maps that agree $\mu$-a.e., and therefore the essential span is not unique.)

**2.6. Decomposability bundle.** Given a measure $\mu$ on $\mathbb{R}^n$ we denote by $\mathscr{F}_\mu$ the class of all families $\{\mu_t : t \in I\}$ where $I$ is a measured space endowed with a probability measure $dt$ and

(a) each $\mu_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E_t$;
(b) the map $t \mapsto \mu_t$ satisfies the assumptions (a) and (b) in §2.3;
(c) the measure $\int_I \mu_t\, dt$ is absolutely continuous w.r.t. $\mu$.

Then we denote by $\mathscr{G}_\mu$ the class of all Borel maps $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ such that for every $\{\mu_t : t \in I\} \in \mathscr{F}_\mu$ there holds

$$\mathrm{Tan}(E_t, x) \subset V(x) \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t \in I. \tag{2.2}$$

Since $\mathscr{G}_\mu$ is closed under countable intersection, by Lemma 2.4 it admits a $\mu$-minimal element which is unique modulo equivalence $\mu$-a.e. With a slight abuse of language and notation we call *any* of these minimal elements *decomposability bundle* of $\mu$, and denote it by $x \mapsto V(\mu, x)$.

**2.7. Remarks.** (i) This definition of the decomposability bundle differs from the one given in the Introduction in three aspects: the minimality property that characterizes $V(\mu,\cdot)$ is now precisely stated, the sets $E_t$ are 1-rectifiable sets (instead of subsets of rectifiable curves), and the "label space" $I$ in the families $\{\mu_t : t\in I\}$ is any probability space (instead of the interval $[0,1]$ equipped with the Lebesgue measure). Note that the last two modifications has been introduced for technical convenience, and do not affect the definition (see the last remark in this list).

(ii) Let $\mathscr{M}$ be the space of finite measure on $\mathbb{R}^n$ endowed with the weak* topology and the corresponding Borel $\sigma$-algebra, and let $\mathscr{R}$ be the subset of all measures $\lambda\in\mathscr{M}$ of the form $\lambda = 1_E\cdot\mathscr{H}^1$ where $E$ is a 1-rectifiable set in $\mathbb{R}^n$. Given a family $\{\mu_t : t\in I\}$ in $\mathscr{F}_\mu$, let $\Psi$ be the measure on $\mathscr{M}$ given by the push-forward of the measure $dt$ on $I$ via the map $t\mapsto\mu_t$; then $\Psi$ is a probability measure supported on $\mathscr{R}$ with finite first moment, and $\int_I \mu_t\,dt = \int_{\mathscr{M}}\lambda\,d\Psi(\lambda)$.

Thus $\mathscr{G}_\mu$ could be equivalently defined as the class of all maps $x\mapsto V(x)$ such that for every finite positive measure $\Psi$ supported on $\mathscr{R}$ with finite first moment which satisfies $\int_{\mathscr{M}}\lambda\,d\Psi(\lambda)\ll\mu$ and for $\Psi$-a.e. measure $\lambda = 1_E\cdot\mathscr{H}^1$ there holds $\mathrm{Tan}(E,x)\subset V(x)$ for $\mathscr{H}^1$-a.e. $x\in E$.

(iii) The class $\mathscr{G}_\mu$ remains the same if in the definition of $\mathscr{F}_\mu$ we add the assumption that $I$ is the interval $[0,1]$ equipped with the Lebesgue measure. This follows immediately from the previous remark and the fact that every probability measure $\Psi$ on $\mathscr{M}$ can be obtained as the push-forward of the Lebesgue measure on the interval $[0,1]$ according to a suitable Borel map $\psi : [0,1]\to\mathscr{M}$.

(iv) The class $\mathscr{G}_\mu$ remains the same if in the definition of $\mathscr{F}_\mu$ we require that $I$ is endowed with a finite measure (instead of a a probability measure), or even a $\sigma$-finite measure.

We conclude this section by giving a few properties of the decomposability bundle (Propositions 2.8 and 2.9) besides those already mentioned in §1.5.

Before stating Proposition 2.8 we must introduce an additional notion: given a measure $\mu$ on $\mathbb{R}^n$ and a family $F=\{\mu_t : t\in I\}\in\mathscr{F}_\mu$ we consider the class of all Borel maps $V:\mathbb{R}^n\to\mathrm{Gr}(\mathbb{R}^n)$ such that (2.2) holds; since this class is closed under countable intersection, by Lemma 2.4 it admits a $\mu$-minimal element which is unique modulo equivalence $\mu$-a.e.; we denote any of these minimal elements by $V(\mu,F,\cdot)$.

**2.8. Proposition.** *Let $\mu$ be a measure on $\mathbb{R}^n$. Then*

(i) *for every $F\in\mathscr{F}_\mu$ there holds $V(\mu,F,x)\subset V(\mu,x)$ for $\mu$-a.e. $x$;*

(ii) *there exists $G\in\mathscr{F}_\mu$ such that $V(\mu,G,x)=V(\mu,x)$ for $\mu$-a.e. $x$.*

**Proof.** Statement (i) is obvious, and to prove statement (ii) it suffices to find a family $G\in\mathscr{F}_\mu$ such that

$$V(\mu,x)\subset V(\mu,G,x)\quad\text{for } \mu\text{-a.e. } x. \tag{2.3}$$

For every $F\in\mathscr{F}_\mu$ we set

$$\Phi(F) := \int_{\mathbb{R}^n}\dim\big[V(\mu,F,x)\big]\,d\mu(x)\,.$$



We claim that there exists a family $G\in\mathscr{F}_\mu$ which maximizes $\Phi$ over $\mathscr{F}_\mu$, and that this family satisfies (2.3).

To prove the existence, we take a sequence of families $F_j=\{\mu_t : t\in I_j\}\in\mathscr{F}_\mu$ with $j=1,2,\dots$, which is a maximizing sequence for $\Phi$, we then take as $G$ the union of all $F_j$, and more precisely $G:=\{\mu_t : t\in I\}$ where $I$ is the (disjoint) union of the sets $I_j$, and is equipped with the probability measure defined by the property that its restriction to each $I_j$ agrees with the probability measure on $I_j$ multiplied by $2^{-j}$.

One easily checks that $G$ belongs to $\mathscr{F}_\mu$, and that for every $j$ there holds

$$V(\mu,F_j,x)\subset V(\mu,G,x)\quad\text{for } \mu\text{-a.e. } x,$$

which implies that $\Phi(F_j,x)\le\Phi(G,x)$, and therefore $G$ maximizes $\Phi$.

In turn, this implies that for every other family $F\in\mathscr{F}_\mu$ there holds

$$V(\mu,F,x)\subset V(\mu,G,x)\quad\text{for } \mu\text{-a.e. } x$$

(otherwise $\Phi(F\cup G)>\Phi(G)$, contradicting the maximality of $G$). This inclusion clearly proves that the map $x\mapsto V(\mu,G,x)$ belongs to $\mathscr{G}_\mu$, which yields (2.3). $\square$

**2.9. Proposition.** *Let $\mu$, $\mu'$ be measures on $\mathbb{R}^n$. Then the following statements hold:*

(i) [strong locality principle] *if $\mu'\ll\mu$ then $V(\mu',x)=V(\mu,x)$ for $\mu'$-a.e. $x$; more generally, if $1_E\,\mu'\ll\mu$ for some set $E$, then $V(\mu',x)=V(\mu,x)$ for $\mu'$-a.e. $x\in E$;*

(ii) *if $\mu$ is supported on a $k$-dimensional surface $S$ of class $C^1$ then $V(\mu,x)\subset$ $\mathrm{Tan}(S,x)$ for $\mu$-a.e. $x$;*

(iii) *if $\mu\ll 1_E\,\mathscr{H}^k$ where $E$ is a $k$-rectifiable set, then $V(\mu,x)=\mathrm{Tan}(E,x)$ for $\mu$-a.e. $x$; in particular if $\mu\ll\mathscr{L}^n$ then $V(\mu,x)=\mathbb{R}^n$ for $\mu$-a.e. $x$;*

(iv) *$V(\mu,x)=\{0\}$ for $\mu$-a.e. $x$ if and only if $\mu$ is supported on a purely unrectifiable set $E$.*

*Moreover, given a family of measures $\{\nu_s : s\in J\}$ as in §2.3,*

(v) *if $\int_J\nu_s\,ds\ll\mu$ then $V(\nu_s,x)\subset V(\mu,x)$ for $\nu_s$-a.e. $x$ and a.e. $s\in J$;*

(vi) *if $\mu\ll\int_J\nu_s\,ds$ and each $\nu_s$ is of the form $\nu_s=1_{E_t}\mathscr{H}^k$ where $E_s$ is a $k$-rectifiable set, then $V(\mu,x)$ has dimension at least $k$ for $\mu$-a.e. $x$.*

**2.10. Remarks.** (i) Many popular examples of self-similar fractals, including the Von Koch snowflake curve, the Cantor set, and the so-called Cantor dust (a cartesian product of Cantor sets) are purely unrectifiable, and therefore every measure $\mu$ supported on any of such sets satisfies $V(\mu,x)=\{0\}$ for $\mu$-a.e. $x$ (Proposition 2.9(iv)).

(ii) The Sierpiński carpet is a self-similar fractal that contains many segments, and therefore is not purely unrectifiable. However, the *canonical* (self-similar) probability measure $\mu$ associated to this fractal is supported on a purely unrectifiable set, and therefore $V(\mu,x)=\{0\}$ for $\mu$-a.e. $x$. The same occurs to other fractals of similar nature, such as the Sierpiński triangle and the Menger-Sierpiński sponge.

**Proof of Proposition 2.9.** Using Proposition 2.8(ii) we choose

- $G = \{\tilde\mu_t : t \in I\} \in \mathscr{F}_\mu$ such that $V(\mu, G, x) = V(\mu, x)$ for $\mu$-a.e. $x$;
- $G' = \{\tilde\mu'_t : t \in I'\} \in \mathscr{F}_{\mu'}$ such that $V(\mu', G', x) = V(\mu', x)$ for $\mu'$-a.e. $x$.

*Statement (i).* If $\mu' \ll \mu$ then one easily checks that $G'$ belongs to $\mathscr{F}_\mu$, and taking into account Proposition 2.8(i) we get

$$V(\mu', x) = V(\mu', G', x) \subset V(\mu, x) \quad \text{for } \mu'\text{-a.e. } x.$$

To prove the opposite inclusion, take a Borel set $F$ such that the restriction of $\mu$ to $F$ satisfies $1_F\mu \ll \mu'$. Then the family $G'' := \{1_F\tilde\mu_t : t \in I\}$ belongs to $\mathscr{F}_{\mu'}$ and

$$V(\mu, x) = V(\mu, G, x) = V(\mu', G'', x) \subset V(\mu', x) \quad \text{for } \mu\text{-a.e. } x \in F,$$

that is, for $\mu'$-a.e. $x$. The proof of the first part of statement (i) is thus complete.

By applying the first part of statement (i) to the measures $1_E\mu'$ and $\mu$, and then to the measures $1_E\mu'$ and $\mu'$ we obtain $V(1_E\mu', x) = V(\mu, x) = V(\mu', x)$ for $\mu'$-a.e. $x \in E$, which is the second part of statement (i).

*Statement (ii).* For every $t \in I$ let $F_t$ be a 1-rectifiable set such that $\tilde\mu_t$ is the restriction of $\mathscr{H}^1$ to $F_t$. Since $\int_I \tilde\mu_t\, dt \ll \mu$ and $\mu$ is supported on $S$ we have that

$$0 = \mu(\mathbb{R}^n \setminus S) = \int_I \tilde\mu_t(\mathbb{R}^n \setminus S)\, dt = \int_I \mathscr{H}^1(F_t \setminus S)\, dt\,,$$

which implies that, for a.e. $t \in I$, the set $F_t$ is contained in $S$ up to an $\mathscr{H}^1$-null subset. Thus $\mathrm{Tan}(F_t, x) \subset \mathrm{Tan}(S, x)$ for $\tilde\mu_t$-a.e. $x$, which implies

$$V(\mu, x) = V(\mu, G, x) \subset \mathrm{Tan}(S, x) \quad \text{for } \mu\text{-a.e. } x.$$

*Statement (iii).* Using statement (i) and the definition of $k$-rectifiable set (see §2.2) we can reduce to the case $\mu = 1_E\mathscr{H}^k$ where $E$ is a subset of a $k$-dimensional surface $S$ of class $C^1$, and we can further assume that $S$ is parametrized by a diffeomorphism $g : A \to S$ of class $C^1$, where $A$ is a bounded open set in $\mathbb{R}^k$.

Then $\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x)$ contains $V(\mu, x)$ for $\mu$-a.e. $x$ by statement (ii).

To prove the opposite inclusion we set $E' := g^{-1}(E)$ and $\mu' := 1_{E'}\mathscr{L}^k$, we fix a nontrivial vector $e \in \mathbb{R}^k$, and for every $t$ in the hyperplane $e^\perp$ we let $E'_t$ be the intersection of the set $E'$ with the line $\{x' = t + he : h \in \mathbb{R}\}$, and set $\mu'_t := 1_{E'_t}\mathscr{H}^1$. By Fubini's theorem we have that $\mu' = \int \mu'_t\, dt$ where $dt$ is the restriction of $\mathscr{H}^{k-1}$ to the hyperplane $e^\perp$.

Next we set $E_t := g(E'_t)$ and $\mu_t := 1_{E_t}\mathscr{H}^1$. Thus each $E_t$ is a 1-rectifiable set whose tangent space at $x = g(x')$ is spanned by the vector $\tau(x) := dg(x')\, e$. Moreover, taking into account that $g$ is a diffeomorphism, we get that $\int \mu_t\, dt$ and $\mu$ are absolutely continuous w.r.t. each other. Therefore $\tau(x) \in V(\mu, x)$ for $\mu_t$-a.e. $x$ and a.e. $t$, that is, for $\mu$-a.e. $x$.

Finally we let $e$ range in a basis $\mathbb{R}^k$, thus the corresponding vectors $\tau(x)$ span $\mathrm{Tan}(S, x)$ for every $x$, and we conclude that $\mathrm{Tan}(E, x) = \mathrm{Tan}(S, x)$ is contained in $V(\mu, x)$ for $\mu$-a.e. $x$.

*Statement (iv).* We prove the "if" part first. If $\mu$ is supported on a set $E$, then, arguing as in the proof of statement (ii), we obtain that for a.e. $t \in I$ the rectifiable set $F_t$ associated to $\tilde\mu_t$ is contained in $E$ up to an $\mathscr{H}^1$-null set. If in addition $E$ is purely unrectifiable we obtain that $\mathscr{H}^1(F_t) = 0$, that is, $\tilde\mu_t = 0$. Hence $V(\mu, x) = V(\mu, G, x) = \{0\}$ for $\mu$-a.e. $x$.

The "only if" part follows from Lemma 7.4; indeed alternative (ii) in this lemma is ruled out by the fact that $V(\mu, x) = \{0\}$ for $\mu$-a.e. $x$, and therefore alternative (i) holds.

*Statement (v).* We can clearly restrict to the case where measure on $J$ is a probability measure. For every $s \in J$ we choose a family $G_s \in \mathscr{F}_{\nu_s}$ according to Proposition 2.8(ii), and thanks to Remark 2.7(iii) we can assume that each $G_s$ is of the form $\{\tilde\nu_{s,t} : t \in I\}$ where $I$ is the interval $[0, 1]$ equipped with the Lebesgue measure.

It is easy to check that the family $F := \{\tilde\nu_{s,t} : (s, t) \in J \times I\}$ belongs to $\mathscr{F}_\mu$, having endowed $J \times I$ with the natural product measure. Then Proposition 2.8(i) implies that the inclusion

$$V(\mu, F, x) \subset V(\mu, x)$$

holds for $\mu$-a.e. $x$, and therefore also for $\nu_s$-a.e. $x$ and a.e. $s$ (recall that $\int_J \nu_s\, ds \ll \mu$ by assumption). On the other hand it is also easy to check that

$$V(\nu_s, x) = V(\nu_s, G_s, x) \subset V(\mu, F, x)$$

for $\nu_s$-a.e. $x$ and a.e. $s$, and statement (v) is proved.

To be precise, the proof is not correct as written, because the map $(s, t) \mapsto \tilde\nu_{s,t}$ is not necessarily Borel measurable in both variables (as required in §2.3). For a correct proof, the families $G_s$ should be chosen for every $s \in J$ in a measurable fashion, and this can be achieved by means of a suitable measurable selection theorem (we omit the details).

*Statement (vi).* By statement (i) it suffices to prove the claim when $\mu$ agrees with $\int_J \nu_s\, ds$. In this case statement (v) implies that $V(\mu, x)$ contains $V(\nu_s, x)$ for $\nu_s$-a.e. $x$ and a.e. $s$, and $V(\nu_s, x)$ has dimension $k$ by statement (iii). Thus $V(\mu, x)$ has dimension at least $k$ for $\nu_s$-a.e. $x$ and a.e. $s$, that is, for $\mu$-a.e. $x$. □



## 3. PROOF OF THEOREM 1.1(i)

We begin this section with a definition which is strictly related to the notions of *tangent space assignment* and *derivative assignment* introduced in [19], and indeed the key step in the proof of Theorem 1.1(i), namely Proposition 3.7, can be obtained from a rather general chain-rule for Lipschitz maps proved in [19], Corollary 2.24. Since the statement we need here is actually quite simple, we include a self-contained proof.

**3.1. Differentiability bundle.** Given a Lipschitz $f : \mathbb{R}^n \to \mathbb{R}$ and a point $x \in \mathbb{R}^n$, we denote by $\mathscr{D}(f, x)$ the set of all subspaces $V \in \mathrm{Gr}(\mathbb{R}^n)$ such that $f$ is differentiable at $x$ w.r.t. $V$ (cf. §2.1), and call the map $x \mapsto \mathscr{D}(f, x)$ *differentiability bundle of $f$*. We then denote by $\mathscr{D}^*(f, x)$ the set of all $V \in \mathscr{D}(f, x)$ with maximal dimension. Note that $\mathscr{D}^*(f, x)$ may contain more than one element.

Before going to Proposition 3.7, which is the core of the proof of Theorem 1.1(i), we need some measurability results related to the previous definition (Lemmas 3.5 and 3.6). To state and prove these results we need some additional notation.

**3.2. Borel multifunctions.** A *multifunction* from a set $X$ to a set $Y$ is a map that to every $x \in X$ associates a nonempty subset of $Y$. For the definition and basic results concerning (Borel) measurable multifunctions we refer to [32], Section 5.1. We just recall here that when $X$ is a topological space and $Y$ is a compact metric space, a closed-valued multifunction from $X$ to $Y$ is Borel measurable if it is Borel measurable as a map from $X$ to the space of non-empty closed subsets of $Y$, endowed with the Hausdorff distance (this case includes essentially all multifunctions considered in this paper).

**3.3. Deviation from linearity.** Given a Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, a point $x \in \mathbb{R}^n$, a linear subspace $V$ of $\mathbb{R}^n$, a linear function $\alpha : V \to \mathbb{R}$, and $\delta > 0$, we set

$$m(f, x, V, \alpha, \delta) := \sup_{h \in V,\, 0<|h|\le\delta} \frac{|f(x+h) - f(x) - \alpha h|}{|h|}$$

(the definition is completed by setting $m(f, x, V, \alpha, \delta) := 0$ when $V = \{0\}$).

Thus $m(f, x, V, \alpha, \delta)$ measures the deviation of $f$ from the linear function $\alpha$ around $x$ at the scale $\delta$. In particular $f$ is differentiable at $x$ w.r.t. $V$ with derivative $\alpha$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $m(f, x, V, \alpha, \delta) \le \varepsilon$, that is, $m(f, x, V, \alpha, \delta)$ tends to 0 as $\delta \to 0$ (note that $m$ is increasing in $\delta$).

**3.4. Lemma.** *Let be given $f$, $x$ and $V$ as above, $W$ and $W'$ linear subspaces of $V$, $\alpha$ and $\alpha'$ linear functions on $V$. Then, setting $m := m(f, x, W, \alpha, \delta)$ and $m' := m(f, x, W', \alpha', \delta)$, we have*

$$m \le m' + |\alpha' - \alpha| + (L + |\alpha'|)d\,,$$

*where $L := \mathrm{Lip}(f)$, $d := d_{\mathrm{gr}}(W, W')$ is the distance between $W$ and $W'$ (see §2.1), and the norm for linear functionals is, as usual, the operator norm.*

**Proof.** We must prove that for every $h \in W$ with $|h| \le \delta$ there holds

$$|f(x+h) - f(x) - \alpha h| \le \big[m' + |\alpha' - \alpha| + (L + |\alpha'|)d\big]\, |h|\,. \tag{3.1}$$

Let $h'$ be the orthogonal projection of $h$ on $W'$; then, taking into account the definition of $d_{\mathrm{gr}}$, we have

$$|h'| \le |h| \le \delta \quad \text{and} \quad |h - h'| \le d|h|\,. \tag{3.2}$$

Now we write $f(x+h) - f(x) - \alpha h$ as sum of the following four terms

$$\begin{aligned} I &:= f(x+h) - f(x+h')\,,\\ II &:= f(x+h') - f(x) - \alpha' h'\,,\\ III &:= \alpha' h' - \alpha' h\,,\\ IV &:= \alpha' h - \alpha h\,, \end{aligned}$$



and we obtain (3.1) by putting together the estimates

$$\begin{aligned} |I| &\le L|h - h'| \le Ld|h|\,,\\ |II| &\le m'|h'| \le m'|h|\,,\\ |III| &\le |\alpha'|\,|h' - h| \le d|\alpha'|\,|h|\,,\\ |IV| &\le |\alpha' - \alpha|\,|h|\,, \end{aligned}$$

where the first and third estimates follow from the second inequality in (3.2) and the fact that $f$ is Lipschitz, while the second one follows from the definition of $m'$ and the first inequality in (3.2). □

**3.5. Lemma.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$. Then $\mathscr{D}(f, x)$ and $\mathscr{D}^*(f, x)$ are closed, nonempty subsets of $\mathrm{Gr}(\mathbb{R}^n)$ for every $x$. Moreover $x \mapsto \mathscr{D}(f, x)$ and $x \mapsto \mathscr{D}^*(f, x)$ are Borel-measurable, closed-valued multifunctions from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$.*

**Sketch of proof.** We denote by $B$ the set of all linear functions $\alpha$ on $\mathbb{R}^n$ with $|\alpha| \le L := \mathrm{Lip}(f)$, and by $G$ the graph of the multifunction $x \mapsto \mathscr{D}(f, x)$, namely the set of all $(x, V) \in \mathbb{R}^n \times \mathrm{Gr}(\mathbb{R}^n)$ such that $V \in \mathscr{D}(f, x)$. Let then $g$ be the function on $\mathbb{R}^n \times \mathrm{Gr}(\mathbb{R}^n)$ defined by

$$g(x, V) := \inf_{\delta>0,\, \alpha\in B} m(f, x, V, \alpha, \delta)\,.$$

For every $\delta > 0$ the function $m$ is Borel measurable in the variables $x, V, \alpha$, and by Lemma 3.4 it is Lipschitz in the variables $\alpha, V$ with a Lipschitz constant independent of $\delta$. Using this fact one can easily prove that $g$ is Lipschitz in the variable $V$ and that the infimum that defines $g$ can be replaced by the infimum over a countable dense family of couples $(\delta, \alpha)$, which means that $g$ is the infimum of a countable family of Borel measurable functions, and therefore it is Borel measurable itself.

Moreover $V$ belongs to $\mathscr{D}(f, x)$ if and only if $g(x, V) = 0$ (cf. §3.3), which means that $G = g^{-1}(0)$. Since $g$ is continuous in $V$ then $\mathscr{D}(f, x)$ is closed for all $x$, and since $g$ is Borel measurable then $G$ is a Borel set. Thus $x \mapsto \mathscr{D}(f, x)$ is a closed-valued multifunction with Borel graph, which implies by a standard argument that $x \mapsto \mathscr{D}(f, x)$ is Borel measurable.

Finally, the measurability of $x \mapsto \mathscr{D}^*(f, x)$ can be easily obtained from the measurability of $x \mapsto \mathscr{D}(f, x)$ (we omit the details). □

**3.6. Lemma.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$, $E$ a Borel set in $\mathbb{R}^n$, and $x \mapsto V(x)$ a Borel map from $E$ to $\mathrm{Gr}(\mathbb{R}^n)$ such that $V(x)$ belongs to $\mathscr{D}(f, x)$ for every $x$. For every $x \in E$ we denote by $d_V f(x)$ the derivative of $f$ at $x$ w.r.t. $V(x)$, and we extend it to a linear function on $\mathbb{R}^n$ by setting $d_V f(x)\, h := 0$ for every $h \in V(x)^\perp$.*

*Then $x \mapsto d_V f(x)$ is a Borel map from $E$ to the dual of $\mathbb{R}^n$.*

**Sketch of proof.** Possibly subdividing $E$ into finitely many Borel sets, we can assume that $V(x)$ has constant dimension $d$ for all $x \in E$.

Since the map $x \mapsto V(x)$, viewed as a closed-valued multifunction from $E$ to $\mathbb{R}^n$, is Borel measurable (cf. §3.2), we can use Kuratowski and Ryll-Nardzewski's measurable selection theorem (see [32], Theorem 5.2.1) to find Borel vectorfields $e_1, \dots, e_n$ defined on $E$ so that $e_1(x), \dots, e_n(x)$ form an orthonormal basis of $\mathbb{R}^n$ for every $x \in E$ and $e_1(x), \dots, e_d(x)$ span $V(x)$.

Then for every $h > 0$ and $x \in E$ we consider the linear function $T_h(x) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\langle T_h(x) \,;\, e_i(x) \rangle := \begin{cases} \dfrac{f(x + he_i(x)) - f(x)}{h} & \text{for } i = 1, \dots, d, \\ 0 & \text{for } i = d+1, \dots, n. \end{cases}$$

Using that each $e_i$ is Borel and that $f$ is continuous, one easily verifies that $x \mapsto T_h(x)$ is a Borel map from $E$ to the dual of $\mathbb{R}^n$ for every $h > 0$. Moreover, since $f$ is differentiable w.r.t. $V(x)$ at each $x \in E$, $T_h(x)$ converges to $d_V f(x)$ as $h \to 0$. Thus $x \mapsto d_V f(x)$ is the pointwise limit of a sequence of Borel maps, and therefore it is Borel. □

**3.7. Proposition.** *Let $E$ be a 1-rectifiable set in $\mathbb{R}^n$. Then, for $\mathscr{H}^1$-a.e. $x \in E$ there holds*

$$\mathrm{Tan}(E, x) \subset V \quad \textit{for every } V \in \mathscr{D}^*(f, x).$$

**3.8. Remark.** When $E$ is a Lipschitz curve, this statement is a particular case of (the second part of) Corollary 2.24 in [19], and the general case follows quite easily. For the sake of completeness, we give below a self-contained proof.

**Proof of Proposition 3.7.** Let $E^*$ be the set of all $x \in E$ where the tangent space $\mathrm{Tan}(E, x)$ exists, and let $E'$ be the subset of all $x \in E^*$ such that $\mathrm{Tan}(E, x)$ is not contained in $V(x)$ for some $V(x) \in \mathscr{D}^*(f, x)$.

It is well-known that $E^*$ is Borel and that the map $x \mapsto \mathrm{Tan}(E, x)$ from $E^*$ to $\mathrm{Gr}(\mathbb{R}^n)$ is Borel (this is also a corollary of Lemma 6.10). Using this fact and Lemma 3.5 one easily checks that $E'$ is Borel, too, and we can use Kuratowski and Ryll-Nardzewski's measurable selection theorem to choose the subspace $V(x)$ for every $x \in E'$ so that the map $x \mapsto V(x)$ from $E'$ to $\mathrm{Gr}(\mathbb{R}^n)$ is Borel. Then also the map $x \mapsto df_V(x)$, defined in Lemma 3.6 is Borel.

We must prove that $E'$ is $\mathscr{H}^1$-null.

Assume by contradiction that it is not. Then, using Lusin's theorem and the fact that every 1-rectifiable set can be covered by countably many curves of class $C^1$ up to an $\mathscr{H}^1$-null subset, we can find a Borel set $E''$ contained in $E'$ such that

(a) $E''$ is contained in a curve $C$ of class $C^1$ and $\mathscr{H}^1(E'') > 0$;

(b) the maps $x \mapsto V(x)$ and $x \mapsto d_V f(x)$ are continuous on $E''$.

Recall now that $f$ is differentiable at every $x \in E''$ w.r.t. to $V(x)$, which means that $m(f, x, V(x), d_V f(x), \delta)$ tends to 0 as $\delta \to 0$ (see §3.3). By Egorov's theorem, we can further assume that, possibly replacing $E''$ with a suitable subset, the convergence is *uniform* w.r.t. $x \in E''$, that is, there exists a modulus of continuity $\omega : [0, +\infty) \to [0, +\infty)$ such that



(c) $m\big(f, x, V(x), d_V f(x), \delta\big) \le \omega(\delta)$ for every $\delta > 0$, $x \in E''$.

Since $\mathscr{H}^1(E'') > 0$ we can now choose a point $\bar{x} \in E''$ such that $E''$ has density 1 at $\bar{x}$ and $f$ is differentiable at $\bar{x}$ w.r.t. $T := \mathrm{Tan}(C, \bar{x}) = \mathrm{Tan}(E, \bar{x})$.

We claim that $f$ is differentiable at $\bar{x}$ w.r.t. $V(\bar{x}) \oplus T$, which implies that $V(\bar{x})$ is a proper subspace of an element of $\mathscr{D}(f, \bar{x})$ (recall that $T$ is not contained in $V(\bar{x})$ by the choice of $E'$) and therefore does not belong to $\mathscr{D}^*(f, \bar{x})$, which is the desired contradiction.

We re-write the claim as

$$f(\bar{x} + \tau + h) - f(\bar{x}) - d_T f(\bar{x})\, \tau - d_V f(\bar{x})\, h = o(|\tau| + |h|)\,, \tag{3.3}$$

for every $\tau \in T$ and $h \in V(\bar{x})$, and we let $\omega'$ be a modulus of continuity for the maps $x \mapsto V(x)$ and $x \mapsto d_V f(x)$ at the point $\bar{x}$. Since $E''$ has density 1 at $\bar{x}$, for every $\tau \in T$ we can find a point $x(\tau) \in E''$ of the form

$$x(\tau) = \bar{x} + \tau + o(|\tau|)\,. \tag{3.4}$$

Then for every $h$ and $\tau$ as above we decompose the left-hand side of (3.3) as the sum of the following three terms:

$$\begin{aligned} I &:= \big[f(\bar{x} + \tau + h) - f(x(\tau) + h)\big] + \big[f(x(\tau)) - f(\bar{x} + \tau)\big]\,, \\ II &:= f(x(\tau) + h) - f(x(\tau)) - d_V f(\bar{x})\, h\,, \\ III &:= f(\bar{x} + \tau) - f(\bar{x}) - d_T f(\bar{x})\, \tau\,, \end{aligned}$$

and (3.3) is easily obtained by putting together the following estimates

$$\begin{aligned} |I| &\le 2L\, \big|\bar{x} + \tau - x(\tau)\big| = o(|\tau|)\,, \\ |II| &\le m\big(f, x(\tau), V(\bar{x}), d_V f(\bar{x}), |h|\big)\, |h| \\ &\le \big[m\big(f, x(\tau), V(x(\tau)), d_V f(x(\tau)), |h|\big) + (1 + 2L)\, \omega'(|\tau|)\big]\, |h| \\ &\le \big[\omega(|h|) + (1 + 2L)\, \omega'(|\tau|)\big]\, |h| = o(|\tau| + |h|)\,, \\ |III| &= o(|\tau|)\,, \end{aligned}$$

where the first estimate follows from the fact that $f$ is Lipschitz with $L := \mathrm{Lip}(f)$ and (3.4); the first inequality in the second estimate follows from the definition of $m$ (cf. §3.3), the second inequality follows from Lemma 3.4, and the third one from (c); finally, the third estimate is the differentiability of $f$ at $\bar{x}$ w.r.t. $T$. □

**3.9. Corollary.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$ and let $\mu$ be a measure on $\mathbb{R}^n$ with decomposability bundle $V(\mu, \cdot)$. Then $V(\mu, x)$ belongs to $\mathscr{D}(f, x)$ for $\mu$-a.e. $x$, and more precisely*

$$V(\mu, x) \subset V \quad \textit{for every } V \in \mathscr{D}^*(f, x).$$

**Proof.** Let $E$ be the set of all $x \in \mathbb{R}^n$ such that there exists $V(x) \in \mathscr{D}^*(f, x)$ which does not contain $V(\mu, x)$. Since $x \mapsto V(\mu, x)$ is a Borel map from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ and $x \mapsto \mathscr{D}^*(f, x)$ is a Borel-measurable, close-valued multifunction from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ (Lemma 3.5), the set $E$ is Borel, and we can use Kuratowski and Ryll-Nardzewski's measurable selection theorem (see [32], Theorem 5.2.1) to choose each $V(x)$ so that the map $x \mapsto V(x)$ from $E$ to $\mathrm{Gr}(\mathbb{R}^n)$ is Borel.

We must prove that $\mu(E) = 0$.

To this end, we extend the map $x \mapsto V(x)$ by setting $V(x) := \mathbb{R}^n$ for every $x \in \mathbb{R}^n \setminus E$, and we claim that this extended map belongs to the class $\mathscr{G}_\mu$ defined in §2.6. Consider indeed an arbitrary family of measures $\{\mu_t : t \in I\}$ which belongs to the class $\mathscr{F}_\mu$ defined in §2.6. Then each $\mu_t$ is the restriction of $\mathscr{H}^1$ to a rectifiable set $E_t$, and by Proposition 3.7 for every $t \in I$ there holds

$$\mathrm{Tan}(E_t, x) \subset V(x) \quad \text{for } \mu_t\text{-a.e. } x,$$

which proves the claim.

Since $x \mapsto V(x)$ belongs to $\mathscr{G}_\mu$, by the definition of decomposability bundle we have that $V(\mu, x)$ is contained in $V(x)$ for $\mu$-a.e. $x$, and since this inclusion fails by construction for every $x \in E$, we infer that $\mu(E) = 0$. □

**Proof of statement (i) of Theorem 1.1.** By Corollary 3.9, $V(\mu, x)$ belongs to $\mathscr{D}(f, x)$ for $\mu$-a.e. $x$, which means that $f$ si differentiable at $x$ w.r.t. $V(\mu, x)$. □

## 4. Proof of Theorem 1.1(ii)

Statement (ii) of Theorem 1.1 is implied by a slightly more precise non-differentiability statement given in Theorem 4.1 below. In turn, this theorem is an immediate consequence of somewhat stronger, but also more technical results (Propositions 4.4 and 4.5) stating the residuality of certain classes of non-differentiable functions within a suitable space of Lipschitz functions.

We begin this section by stating the results mentioned above, together with the necessary definitions, while proofs are given in the second part of this section, starting with §4.8. In Remarks 4.6(ii) and (iii), and Example 4.7 we briefly discuss the quantitative form of these non-differentiability results.

Through this section $\mu$ is a measure on $\mathbb{R}^n$. Given a function $f$ on $\mathbb{R}^n$, a point $x \in \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, we consider the upper and lower (one-sided) directional derivatives

$$D_v^+ f(x) := \limsup_{h \to 0^+} \frac{f(x+hv) - f(x)}{h},$$
$$D_v^- f(x) := \liminf_{h \to 0^+} \frac{f(x+hv) - f(x)}{h}.$$

**4.1. Theorem.** *There exists a Lipschitz function $f$ on $\mathbb{R}^n$ such that, for $\mu$-a.e. $x \in \mathbb{R}^n$, $f$ is not differentiable at $x$ in any direction $v \notin V(\mu, x)$, and more precisely $D_v^+ f(x) - D_v^- f(x) > 0$.*

**4.2. The set $E$ and the space $X$.** For the rest of this section $E$ is a Borel set in $\mathbb{R}^n$ with the following property: there exist an integer $d$ with $0 < d \le n$, and continuous vectorfields $e_1, \dots, e_n$ on $\mathbb{R}^n$ such that

- $e_1(x), \dots, e_n(x)$ form an orthonormal basis of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$;
- $e_1(x), \dots, e_d(x)$ span $V(\mu, x)^\perp$ for every $x \in E$.



In particular $V(\mu, x)$ and $V(\mu, x)^\perp$ depend continuously on $x \in E$ and have dimension respectively $n - d$ and $d$ for every $x \in E$.

We then denote by $X$ the space of all Lipschitz functions $f$ on $\mathbb{R}^n$ such that

$$\left| D_{e_j(x)} f(x) \right| \le 1 \quad \text{for } \mathscr{L}^n\text{-a.e. } x \text{ and every } j = 1, \dots, n,$$

endowed with the supremum distance. It is then easy to show that $X$ is a complete metric space. Note that $X$ depends on the measure $\mu$ but also on the choice of the set $E$ and of the vectorfields $e_j$.

**4.3. Residual sets and maps of Baire class 1.** A subset of a topological space is *residual* if it contains a countable intersection of open dense sets, and by Baire Theorem a residual set in a complete metric space is dense, and in particular is not empty.

The precise definition of maps of Baire class 1 between metric spaces can be found in [14], Definition 24.1; we just recall here that every map which can be written as pointwise limit of a sequences of continuous maps is of Baire class 1,[4] and that the set of continuity points of a map of Baire class 1 is residual, see [14], Theorem 24.14.

**4.4. Proposition.** *Given a vector $v \in \mathbb{R}^n$, let $N_v$ be the set of all functions $f \in X$ such that for $\mu$-a.e. $x \in E$ there holds*

$$D_v^+ f(x) - D_v^- f(x) \ge \frac{d_v(x)}{3\sqrt{d}} \quad \textit{where} \quad d_v(x) := \mathrm{dist}(v, V(\mu, x)). \tag{4.1}$$

*Then $N_v$ is residual in $X$, and in particular it is dense.*

**4.5. Proposition.** *Let $N$ be the set of all functions $f \in X$ such that, for $\mu$-a.e. $x \in E$, inequality (4.1) holds for every $v \in \mathbb{R}^n$. Then $N$ is residual in $X$, and in particular it is dense.*

**4.6. Remarks.** (i) Proposition 4.5 and Theorem 4.1 are straightforward consequences Proposition 4.4, which is therefore the key result in the whole section. Note that in Propositions 4.4 and 4.5 the class of non-differentiable functions under consideration is proved to be residual (admittedly, in a strange-looking space), and not just nonempty.

(ii) If $V(\mu, x) = \{0\}$ for $\mu$-a.e. $x$ (which, by Proposition 2.9(iv) is equivalent to say that $\mu$ is supported on a purely unrectifiable set), then in §4.2 we can take $E = \mathbb{R}^n$ and $e_1, \dots, e_n$ equal to the standard basis of $\mathbb{R}^n$ for every $x$. Then Proposition 4.5 gives directly infinitely many Lipschitz functions $f$ which are non-differentiable at $\mu$-a.e. $x$ and in every direction $v \in \mathbb{R}^n$ with $v \ne 0$; moreover the non-differentiability of $f$ is expressed in a precise quantitative form by inequality (4.1), which becomes

$$D_v^+ f(x) - D_v^- f(x) \ge |v|/(3\sqrt{n}) \quad \text{for every } v \in \mathbb{R}^n.$$

(iii) In view of the previous remark it is natural to ask if the statement of Theorem 4.1 can be strengthened by requiring that the non-differentiability of $f$ is

[4] Note that for general metric spaces the converse is not true: there are maps of Baire class 1 which cannot be written as a pointwise limit of continuous maps.

uniform in $x$, that is, there exists an increasing function $\omega$ on $[0,+\infty)$ with $\omega(0)=0$ and $\omega(s)>0$ for $s>0$, such that for $\mu$-a.e. $x$ there holds

$$D_v^+f(x) - D_v^-f(x) \geq \omega(d_v(x)) \quad \text{for every } v \in \mathbb{R}^n.$$

Example 4.7 below—the idea of which can be traced back to [24]—shows that this is not the case. There we describe a singular measure $\mu$ on $\mathbb{R}^2$ with the following properties: $V(\mu,x)$ has dimension 1 at $\mu$-a.e. $x$, and for every Lipschitz function $f$ on $\mathbb{R}^2$ and every $\varepsilon>0$ there exists a set $E$ with $\mu(E)>0$ such that $f$ is $\varepsilon$-differentiable at $\mu$-a.e. $x\in E$, which means that there exists a linear function $L$ (depending on $x$) such that $|f(x+h)-f(x)-Lh|\leq\varepsilon|h|+o(|h|)$, and in particular

$$|D_v^\pm f(x) - Lv| \leq \varepsilon \quad \text{for every } v \in \mathbb{R}^2.$$

**4.7. Example.** Let $F$ be the union of a countable family of straight lines $\mathscr{D}$ which are dense in $\mathbb{R}^2$, in the sense that for every $x_1,x_2\in\mathbb{R}^2$ and every $\varepsilon>0$ there exists a line $R\in\mathscr{D}$ such that $\operatorname{dist}(x_i,R)\leq\varepsilon$ for $i=1,2$. Let then $\mu$ be a finite measure such that $\mu$ and the restriction of $\mathscr{H}^1$ to $F$ are absolutely continuous w.r.t. each other.

By Proposition 2.9(iii) we have that $V(\mu,x)=\operatorname{Tan}(R,x)$ for $\mu$-a.e. $x\in R$ and for every $R\in\mathscr{D}$, and in particular $V(\mu,x)$ has dimension 1 at $\mu$-a.e. $x$.

We claim that, for every $\varepsilon>0$, every Lipschitz function $f$ on $\mathbb{R}^2$ is $\varepsilon$-differentiable on a set $E$ with positive measure. To prove the claim, we set $\delta:=\varepsilon^2/(2L)$ and use the density of $\mathscr{D}$ (and the definition of Lipschitz constant) to find a line $R\in D$ and $x_1,x_2\in R$ such that

$$f(x_2)-f(x_1) > (L-\delta)|x_1-x_2|\,.$$

Set now $e:=(x_2-x_1)/|x_2-x_1|$; then the previous inequality implies that the set $E$ of all $x$ in the segment $[x_1,x_2]$ where the partial derivative $D_ef(x)$ exists and satisfies

$$D_ef(x) > L-\delta \tag{4.2}$$

is of positive measure w.r.t. $\mathscr{H}^1$, and therefore also w.r.t. $\mu$. Finally we use that $f$ is $\sqrt{2L\delta}$-differentiable at every $x$ where (4.2) holds (see for instance [8], Corollary 1), and recall that $\sqrt{2L\delta}=\varepsilon$ by the choice of $\delta$.

The rest of this section is devoted to the proofs of the results stated above, starting from Proposition 4.4. Note that since $N_{cv}=N_v$ for every $v\in\mathbb{R}^n$ and every $c>0$, it suffices to prove this statement for all $v\in\mathbb{R}^n$ with $|v|=1$.

**Attention!** From now till the end of the proof of Proposition 4.4, $v$ is a fixed vector in $\mathbb{R}^n$ with $|v|=1$.

**4.8. The maps $T^\pm_{\sigma,\sigma'}$ and $U_\sigma$.** For every $\sigma>\sigma'\geq0$ and every $f:\mathbb{R}^n\to\mathbb{R}$ we consider the functions $T^\pm_{\sigma,\sigma'}f$ and $U_\sigma f$ defined as follows for every $x\in\mathbb{R}^n$:

$$T^+_{\sigma,\sigma'}f(x) := \sup_{\sigma'<h\leq\sigma} \frac{f(x+hv)-f(x)}{h}\,,$$

$$T^-_{\sigma,\sigma'}f(x) := \inf_{\sigma'<h\leq\sigma} \frac{f(x+hv)-f(x)}{h}\,,$$



$$U_\sigma f(x) := T^+_{\sigma,0}f(x) - T^-_{\sigma,0}f(x)\,.$$

One readily checks that $T^+_{\sigma,0}f(x)$ and $T^-_{\sigma,0}f(x)$ are respectively increasing and decreasing in $\sigma$, and therefore $U_\sigma f(x)$ is increasing in $\sigma$. Moreover

$$D_v^+f(x) - D_v^-f(x) = \inf_{\sigma>0}\big(U_\sigma f(x)\big) = \inf_{m=1,2,\dots}\big(U_{1/m}f(x)\big)\,. \tag{4.3}$$

Finally we notice that $\frac{1}{h}(f(x+hv)-f(x))$ and $D_vf(x)$ (if it exists) are both smaller than $T^+_{\sigma,0}f(x)$ and larger than $T^-_{\sigma,0}f(x)$ if $h\leq\sigma$, which yields the following useful estimate:

$$U_\sigma f(x) \geq \left| D_vf(x) - \frac{f(x+hv)-f(x)}{h}\right| \quad \text{for every } 0<h\leq\sigma. \tag{4.4}$$

**4.9. Structure of the proof of Proposition 4.4.** We follow a general strategy devised by B. Kirchheim for the proof of residuality results (see [16]). In this specific case, this strategy reduces essentially to two key steps: in Lemma 4.10 we show that every $U_\sigma$ is of Baire class 1 as a map from $X$ to $L^1(\mu)$, which implies that $U_\sigma$ is continuous at residually many $f$; then in Lemma 4.15 we show that any such $f$ satisfies $U_\sigma f(x)\geq d_v(x)/(3\sqrt{d})$ for $\mu$-a.e. $x\in E$, and this inequality, together with (4.3), implies (4.1).

The proof of Lemma 4.15 is quite long, and is split in several sub-lemmas. The key step here is the construction described in Lemma 4.14, which is actually a simplification of a more refined construction given in [4].

**4.10. Lemma.** *The maps $T^\pm_{\sigma,\sigma'}$ and $U_\sigma$ take $X$ into $L^1(\mu)$ for every $\sigma,\sigma'$ as above. Moreover the maps $T^\pm_{\sigma,\sigma'}$ are continuous for $\sigma'>0$ while $T^\pm_{\sigma,0}$ and $U_\sigma$ are of Baire class 1 (as maps from $X$ to $L^1(\mu)$).*

**Proof.** The functions $T^+_{\sigma,\sigma'}f$ belong to $L^1(\mu)$ for every $\sigma>\sigma'\geq0$ and every $f\in X$ because they are bounded, and more precisely

$$\big|T^+_{\sigma,\sigma'}f(x)\big| \leq \operatorname{Lip}(f) \quad \text{for every } x\in\mathbb{R}^n. \tag{4.5}$$

Concerning the continuity of $T^+_{\sigma,\sigma'}$ for $\sigma'>0$, one readily checks that for every $f,f'\in X$ there holds

$$\big|T^+_{\sigma,\sigma'}f'(x) - T^+_{\sigma,\sigma'}f(x)\big| \leq \frac{2}{\sigma'}\|f'-f\|_\infty \quad \text{for every } x\in\mathbb{R}^n,$$

and therefore

$$\big\|T^+_{\sigma,\sigma'}f' - T^+_{\sigma,\sigma'}f\big\|_{L^1(\mu)} \leq \frac{2}{\sigma'}\mathbb{M}(\mu)\,\|f'-f\|_\infty\,.$$

To prove that $T^+_{\sigma,0}$ is of of Baire class 1 it suffices to notice that it agrees with the pointwise limit of the continuous maps $T^+_{\sigma,\sigma'}$ as $\sigma'\to0$. Indeed, it follows from the definition that, as $\sigma'$ tends to 0, $T^+_{\sigma,\sigma'}f(x)$ converges to $T^+_{\sigma,0}f(x)$ for every $f\in X$ and every $x\in\mathbb{R}^n$, and then $T^+_{\sigma,\sigma'}f$ converges to $T^+_{\sigma,0}f$ in $L^1(\mu)$ by the dominated convergence theorem (a domination is given by estimate (4.5)).

The rest of the statement can be proved in a similar way. □

**4.11. Cones and cone-null sets.** Given a unit vector $e$ in $\mathbb{R}^n$ and a real number $\alpha \in (0, \pi/2)$ we denote by $C(e, \alpha)$ the closed cone of axis $e$ and angle $\alpha$ in $\mathbb{R}^n$, that is,

$$C(e,\alpha) := \{ v \in \mathbb{R}^n : \ v \cdot e \geq \cos\alpha \cdot |v| \} .$$

Given a cone $C = C(e,\alpha)$ in $\mathbb{R}^n$, we call *$C$-curve* any set of the form $\gamma(J)$ where $J$ is a compact interval in $\mathbb{R}$ and $\gamma : J \to \mathbb{R}^n$ is a Lipschitz path such that

$$\dot\gamma(s) \in C \quad \text{for } \mathscr{L}^1\text{-a.e. } s \in J.$$

Following [4], we say that a set $E$ in $\mathbb{R}^n$ is *$C$-null* if

$$\mathscr{H}^1\big(E \cap G\big) = 0 \quad \text{for every } C\text{-curve } G.$$

The following lemma is a particular case of a result contained in [4]; we include a complete proof for the sake of completeness.

**4.12. Lemma.** *Let be given a cone $C = C(e,\alpha)$ in $\mathbb{R}^n$ and a $C$-null compact set $K$ in $\mathbb{R}^n$. Then for every $\varepsilon > 0$ there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that, for every $x \in \mathbb{R}^n$,*

(i) *$0 \leq f(x) \leq \varepsilon$;*
(ii) *$0 \leq D_e f(x) \leq 1$, and $D_e f(x) = 1$ if $x \in K$;*
(iii) *$|d_W f(x)| \leq 1/\tan\alpha$, where $W := e^\perp$, $d_W f(x)$ is the derivative of $f$ at $x$ w.r.t. $W$ (cf. §2.1), and $|d_W f(x)|$ is its operator norm.*

**Proof.** We first construct a Lipschitz function $g$ that satisfies statements (i), (ii) and (iii) with $K$ replaced by a suitable open set $A$ that contains $K$, and then we regularize $g$ by convolution to obtain $f$.

*Step 1. There exists an open set $A$ such that $A \supset K$ and*

$$\mathscr{H}^1\big(A \cap G\big) \leq \varepsilon \quad \text{for every } C\text{-curve } G. \tag{4.6}$$

More precisely, we claim that there exists $\delta > 0$ such that $\mathscr{H}^1\big(K_\delta \cap G\big) \leq \varepsilon$ for every $C$-curve $G$, where $K_\delta$ is the set of all $x$ such that $\mathrm{dist}(x,K) \leq \delta$, and then it suffices to take $A$ equal to the interior of $K_\delta$.

We argue by contradiction: if the claim does not hold, then for every $\delta > 0$ there exists a $C$-curve $G_\delta$ such that $\mathscr{H}^1\big(K_\delta \cap G_\delta\big) \geq \varepsilon$.

Let $J$ be a compact interval that contains the set $\{x \cdot e : \ x \in K\}$. We can then assume that each $G_\delta$ admits a parametrization $\gamma_\delta : J \to \mathbb{R}^n$ of the form

$$\gamma_\delta(s) = se + \eta_\delta(s) \quad \text{with } \eta_\delta(s) \in W \text{ for every } s \in J,$$

where $\eta_\delta : J \to W$ is Lipschitz and satisfies

$$|\dot\eta_\delta(s)| \leq \tan\alpha \quad \text{for a.e. } s \in J.$$

Then we set $K'_\delta := \gamma_\delta^{-1}(K_\delta) = \gamma_\delta^{-1}(K_\delta \cap G_\delta)$.

Possibly passing to a subsequence we can assume that when $\delta \to 0$ the maps $\eta_\delta$ converge uniformly to a Lipschitz map $\eta_0 : J \to W$, and that the compact sets $K'_\delta$ converge to a compact set $K'_0 \subset J$ in the Hausdorff distance. Therefore the parametrizations $\gamma_\delta$ converge to $\gamma_0$ given by $\gamma_0(s) := se + \eta_0(s)$, the set $G_0 := \gamma_0(J)$ is a $C$-curve, and $K \cap G_0$ contains $K_0 := \gamma_0(K'_0)$.



We prove next that $K_0$ has positive length, which contradicts the fact that $K$ is $C$-null. Indeed

$$\begin{aligned} \mathscr{H}^1(K_0) \geq \mathscr{L}^1(K'_0) &\geq \limsup_{\delta\to 0} \mathscr{L}^1(K'_\delta) \\ &\geq \limsup_{\delta\to 0} \big(\cos\alpha\, \mathscr{H}^1(K_\delta \cap G_\delta)\big) \geq \varepsilon \cos\alpha > 0 \end{aligned}$$

(the second inequality follows from the upper semicontinuity of the Lebesgue measure w.r.t. the Hausdorff convergence of compact sets, and the third inequality follows from the fact that $\mathscr{H}^1(\gamma_0(E)) \leq \mathscr{L}^1(E)/\cos\alpha$ for every set $E \subset J$, which in turn follows from the fact that $|\dot\eta_0(s)| \leq \tan\alpha$ for a.e. $s$).

*Step 2. Construction of $g$.*

For every $x \in \mathbb{R}^n$ we denote by $\mathscr{G}_x$ the class of all $C$-curves $G = \gamma([a,b])$ whose end-point $x_G := \gamma(b)$ is of the form $x_G = x + se$ for some $s \geq 0$, and we set

$$g(x) := \sup_{G \in \mathscr{G}_x} \big(\mathscr{H}^1(A \cap G) - |x_G - x|\big) .$$

Starting from the definition one can readily check that for every $x \in \mathbb{R}^n$ there holds:

(a) $0 \leq g(x) \leq \varepsilon$ (recall (4.6));
(b) $g(x) \leq g(x+se) \leq g(x) + s$ for every $s > 0$, and if the segment $[x, x+se]$ is contained in $A$ then $g(x+se) = g(x) + s$;
(c) $|g(x+v) - g(x)| \leq |v|/\tan\alpha$ for every $v \in W$.

Statements (b) and (c) imply that $g$ is Lipschitz and

(b') $0 \leq D_e g(x) \leq 1$ for $\mathscr{L}^n$-a.e. $x$, and $D_e g(x) = 1$ for $\mathscr{L}^n$-a.e. $x \in A$;
(c') $|d_W g(x)| \leq 1/\tan\alpha$ for $\mathscr{L}^n$-a.e. $x$.

*Step 3. Construction of $f$.*

We take $r$ so that $0 < r < \mathrm{dist}(K, \mathbb{R}^n \setminus A)$ and set $f := g * \rho$ where $\rho$ is a mollifier with support contained in the ball $B(0,r)$. Then statements (i), (ii) and (iii) follow from statements (a), (b') and (c'), respectively. □

**4.13. Lemma.** *Let be given a cone $C = C(e,\alpha)$ in $\mathbb{R}^n$, and a $C$-null compact set $K$ contained in a ball $B = B(\bar x, r)$. Then for every $\varepsilon > 0$ and every $r' > r$ there exists a smooth function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *$\|g\|_\infty \leq \varepsilon$ and the support of $g$ is contained in $B' := B(\bar x, r')$;*

*and for every $x \in B'$,*

(ii) *$-\varepsilon \leq D_e g(x) \leq 1 + \varepsilon$, and $D_e g(x) = 1$ if $x \in K$;*
(iii) *$|d_W g(x)| \leq 2/\tan\alpha$, where $W := e^\perp$.*

**Proof.** We fix $\varepsilon' > 0$ and take a smooth function $f$ that satisfies statements (i), (ii) and (iii) in Lemma 4.12 with $\varepsilon'$ in place of $\varepsilon$, and then we set

$$g := \varphi f$$

where $\varphi : \mathbb{R}^n \to [0,1]$ is a smooth cut-off function such that $\varphi = 1$ on $B$, $\varphi = 0$ on $\mathbb{R}^n \setminus B'$, and $\|d\varphi\|_\infty \leq 2/(r'-r)$.

Then $g$ is supported in $B'$ and using the properties of $f$ and $\varphi$ and the identities $D_e g = \varphi\, D_e f + f\, D_e\varphi$ and $d_W g = \varphi\, d_W f + f\, d_W\varphi$ we obtain that for every $x \in B'$ there holds

(a) $|g(x)| \le |f(x)| \le \varepsilon'$;

(b) $-\frac{2\varepsilon'}{r'-r} \le D_e g(x) \le 1 + \frac{2\varepsilon'}{r'-r}$, and $D_e g(x) = D_e f(x) = 1$ if $x \in K$;

(c) $|d_W g(x)| \le |d_W f(x)| + |f(x)|\, |d_W \varphi(x)| \le \frac{1}{\tan\alpha} + \frac{2\varepsilon'}{r'-r}$.

Thus statements (i), (ii) and (iii) follow respectively from statements (a), (b) and (c) provided that we choose $\varepsilon'$ small enough. □

**4.14. Lemma.** *Let $\varepsilon, \sigma$ be positive real numbers, $f$ a function in $X$, and $E'$ a Borel subset of $E$. Then there exist a smooth function $f'' \in X$ and a compact set $K$ contained in $E'$ such that*

(i) $\|f'' - f\|_\infty \le 2\varepsilon$;

(ii) $\mu(K) \ge \mu(E')/(4d)$;

(iii) $U_\sigma f''(x) \ge d_v(x)/(3\sqrt{d})$ *for every $x \in K$.*

**Proof.** The idea is to take a smooth function $f'$ close to $f$, and then modify it into a function $f''$ so to get $U_\sigma f''(x)$ large enough for sufficiently many $x \in E'$. This modification will be obtained by adding to $f'$ a finite number of smooth perturbations with small supremum norms and small, disjoint supports, but with large derivative in the direction $v$.

In order to simplify the notation, through this proof we write $D_j$ for the partial derivative $D_{e_j}$, where $e_j$ is any of the vectorfields that appear in the definition of the space $X$ in §4.2.

*Step 1. There exists a smooth function $f'$ on $\mathbb{R}^n$ such that*

(a) $\|f' - f\|_\infty \le \varepsilon$;

(b) $\|D_j f'\|_\infty < 1$ *for $j = 1, \dots, n$, and in particular $f'$ belongs to $X$.*

We fix a mollifier $\rho$ with compact support in $\mathbb{R}^n$, choose $s > 0$ so that $s\|f\|_\infty < \varepsilon$, and set

$$f' := (1 - s) f * \rho_t$$

where $\rho_t(x) := t^{-n}\rho(x/t)$ and $t$ has yet to be chosen.

Since $f$ is uniformly continuous, $f'$ converges uniformly to $(1 - s)f$ as $t \to 0$, then $\|f' - f\|_\infty$ converges to $s\|f\|_\infty < \varepsilon$, which implies that (a) holds if we choose $t$ small enough.

Since the vectorfield $e_j$ that defines the partial derivative $D_j$ is continuous, it is not difficult to show that $\|D_j f'\|_\infty$ converges to $(1-s)\|D_j f\|_\infty < 1$ as $t \to 0$ (recall that $\|D_j f\|_\infty \le 1$ because $f \in X$) and therefore also (b) holds if we choose $t$ small enough.

*Step 2. Construction of the set $E'_k$.*

For every $x \in E$ the vectors $e_1(x), \dots, e_d(x)$ form orthonormal basis of $V(\mu, x)^\perp$ (see §4.2); thus

$$d_v(x) := \operatorname{dist}(v, V(\mu, x)) = \Big[\sum_{k=1}^d (v \cdot e_k(x))^2\Big]^{1/2} \le \sqrt{d} \sup_{1\le k\le d} |v \cdot e_k(x)|\,,$$

and then there exists $k = 1, \dots, d$ such that $d_v(x) \le \sqrt{d}\, |v \cdot e_k(x)|$. Consequently, the set $E'$ is covered by the sets

$$E'_k := \Big\{ x \in E' \,:\, d_v(x) \le \sqrt{d}\, |v \cdot e_k(x)| \Big\}\,; \tag{4.7}$$

in particular there exists at least one value of $k$ such that

$$\mu(E'_k) \ge \frac{\mu(E')}{d}\,, \tag{4.8}$$

and for the rest of the proof $k$ is assigned this specific value.

For the next four steps we fix a point $\bar{x} \in E'_k$ and positive numbers $r, r'$ such that

$$r < \sigma/3\,, \quad r < r' \le 2r\,. \tag{4.9}$$

*Step 3. Construction of the sets $E_{\bar{x},r}$.*

Let $\alpha(\bar{x}, r)$ be the supremum of the angle between $V(\mu, x)$ and $V(\mu, \bar{x})$ as $x$ varies in $E \cap B(\bar{x}, r)$ (the angle between subspaces of $\mathbb{R}^n$ is defined in §2.1). Since $V(\mu, x)$ is continuous in $x \in E$ (cf. §4.2), we have that

$$\alpha(\bar{x}, r) \to 0 \quad \text{as } r \to 0. \tag{4.10}$$

Since $e_k(\bar{x})$ is orthogonal to $V(\mu, \bar{x})$, the angle between $e_k(\bar{x})$ and $V(\mu, x)$ is at least $\pi/2 - \alpha(\bar{x}, r)$, and therefore the cone

$$C_{\bar{x},r} := C\big(e_k(\bar{x}), \pi/2 - 2\,\alpha(\bar{x}, r)\big)$$

satisfies

$$C_{\bar{x},r} \cap V(\mu, x) = \{0\} \quad \text{for all } x \in E \cap B(\bar{x}, r).$$

Moreover, since the set $F := E'_k \cap B(\bar{x}, r)$ is contained in $E \cap B(\bar{x}, r)$, we can apply Lemma 7.5 to find a $C_{\bar{x},r}$-null set $F'$ contained in $F$ that $\mu(F') = \mu(F)$. Then we can take a compact set $K_{\bar{x},r}$ contained in $F'$ such that

$$\mu(K_{\bar{x},r}) \ge \frac{1}{2}\mu(F') = \frac{1}{2}\mu(E'_k \cap B(\bar{x}, r))\,. \tag{4.11}$$

Note that $K_{\bar{x},r}$ is $C_{\bar{x},r}$-null because it is contained in $F'$.

*Step 4. Construction of the perturbations $\bar{g}_{\bar{x},r,r'}$.*

We set

$$\varepsilon' := \min\Big\{\frac{1}{2},\, \varepsilon,\, r(r' - r),\, 1 - \|D_j f'\|_\infty \text{ with } j = 1, \dots, n.\Big\}$$

Note that $\varepsilon'$ is strictly positive because of statement (b) in Step 1 and the fact that $r' > r$. Since $K_{\bar{x},r}$ is $C_{\bar{x},r}$-null, we can use Lemma 4.13 to find a smooth function $g_{\bar{x},r,r'}$ such that

(c) $\|g_{\bar{x},r,r'}\|_\infty \le \varepsilon'$ and the support of $g_{\bar{x},r,r'}$ is contained in $B(\bar{x}, r')$;

and setting $e := e_k(\bar x)$, $W := e^\perp$, for every $x \in B(\bar x, r')$ there holds

(d) $-\varepsilon' \le D_e g_{\bar x,r,r'}(x) \le 1+\varepsilon'$ and $D_e g_{\bar x,r,r'}(x) = 1$ if $x \in K_{\bar x,r}$;

(e) $|d_W g_{\bar x,r,r'}(x)| \le 2\tan\big(2\,\alpha(\bar x,r)\big)$.

Finally we set

$$\bar g_{\bar x,r,r'} := \pm\frac{1}{2} g_{\bar x,r,r'}$$

where we convene that $\pm$ is $+$ if $D_e f'(\bar x) \le 0$ and $-$ otherwise.

*Step 5. There exists $r_0 = r_0(\bar x) > 0$ such that for $r < r_0$ there holds*

$$U_\sigma(f' + \bar g_{\bar x,r,r'})(x) \ge \frac{d_v(x)}{3\sqrt d} \quad \textit{for every } x \in K_{\bar x,r}. \tag{4.12}$$

In the following, given a quantity $m$ depending on $\bar x, r, r'$ and $x \in B(\bar x, r)$, we write $m = o(1)$ to mean that, for every $\bar x$, $m$ tends to 0 as $r \to 0$, uniformly in all remaining variables. In other words, for every $\bar x$ and every $\varepsilon > 0$ there exists $\bar r > 0$ such that $|m| \le \varepsilon$ if $r \le \bar r$.

To simplify the notation, from now on we write $g$ and $\bar g$ for $g_{\bar x,r,r'}$ and $\bar g_{\bar x,r,r'}$.

For every $x \in K_{\bar x,r} \subset B(\bar x, r)$ we take $h = h(x) > 0$ such that $x + hv$ belongs to $\partial B(\bar x, r')$. Then, taking into account that $|v| = 1$ and (4.9), we have

$$r' - r \le h \le r + r' \le 3r \le \sigma\,.$$

We can then apply estimate (4.4) to the function $f'' := f' + \bar g$; taking into account that $\bar g = \pm\frac12 g$ and $g(x+hv) = 0$ (recall that the support of $g$ is contained in $B(\bar x, r')$ by statement (c) in Step 4) we get

$$\begin{aligned} U_\sigma f''(x) &\ge \Big| D_v f''(x) - \frac{f''(x+hv) - f''(x)}{h} \Big| \\ &= \Big| D_v \bar g(x) + D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} + \frac{\bar g(x)}{h} \Big| \\ &\ge \frac12 \big|D_v g(x)\big| - \Big| D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} \Big| - \frac{|g(x)|}{2h}\,. \end{aligned} \tag{4.13}$$

Since $f'$ is of class $C^1$, we clearly have

$$\Big| D_v f'(x) - \frac{f'(x+hv) - f'(x)}{h} \Big| = o(1)\,. \tag{4.14}$$

Using statement (c) in Step 4, the inequality $r' - r < h$ given above, and the choice of $\varepsilon'$, we get

$$\frac{|g(x)|}{h} \le \frac{\varepsilon'}{r' - r} \le r = o(1)\,. \tag{4.15}$$

Finally, to estimate $|D_v g(x)|$ we decompose $v$ as $v = (v\cdot e)e + w$ with $w \in W$. Then

$$D_v g(x) = (v\cdot e)\, D_e g(x) + \langle d_W g(x)\,;\, w\rangle\,,$$



and therefore

$$\begin{aligned} |D_v g(x)| &\ge |v\cdot e|\,|D_e g(x)| - |d_W g(x)| \\ &\ge |v \cdot e| - 2\tan\big(2\,\bar\alpha(\bar x, r)\big) \\ &\ge |v\cdot e_k(x)| - |e_k(x) - e_k(\bar x)| - 2\tan\big(2\,\bar\alpha(\bar x,r)\big) \\ &\ge |v\cdot e_k(x)| - o(1) \ge d_v(x)/\sqrt d - o(1)\,, \end{aligned} \tag{4.16}$$

where the second inequality follows from statements (d) and (e) in Step 4 and the fact that $x \in K_{\bar x,r}$; for the third inequality we used that $|v| = 1$ and $e = e_k(\bar x)$; the fourth one follows from (4.10) and the fact $e_k(x)$ is continuous in $x$, and the last inequality follows from (4.7) and the fact that $x \in K_{\bar x,r} \subset E'_k$.

Putting estimates (4.13), (4.14), (4.15), (4.16) together we get

$$U_\sigma(f' + \bar g)(x) = U_\sigma f''(x) \ge \frac{d_v(x)}{2\sqrt d} - o(1)\,,$$

which clearly implies the claim in Step 5.

*Step 6. There exists $r_1 = r_1(\bar x) > 0$ such that $f' + \bar g_{\bar x,r,r'} \in X$ if $r < r_1$.*

Since $\bar g$ is supported in $B(\bar x, r')$ and $f'$ belongs to $X$ (Step 1), to prove that $f' + \bar g$ belongs to $X$ it suffices to show that

$$\big|D_j(f' + \bar g_{\bar x,r,r'})(x)\big| \le 1 \quad \textit{for every } x \in B(\bar x, r') \textit{ and } j = 1,\dots,n. \tag{4.17}$$

We begin with the case $j = k$. Recalling the identities $\bar g = \pm\frac12 g$, $e = e_k(\bar x)$, we obtain

$$\begin{aligned} D_k \bar g(x) &= D_e \bar g(x) + \langle d\bar g(x)\,;\, e_k(x) - e\rangle = \pm\frac12 D_e g(x) + o(1)\,, \\ D_k f'(x) &= D_k f'(\bar x) + o(1)\,, \end{aligned}$$

and therefore

$$\big|D_k(f' + \bar g)(x)\big| = \Big|D_k f'(\bar x) \pm \frac12 D_e g(x)\Big| + o(1)\,. \tag{4.18}$$

Recall now that $-\varepsilon' \le D_e g(x) \le 1 + \varepsilon'$ (statement (d) above), that the sign $\pm$ means $+$ when $D_k f'(\bar x) \le 0$ and $-$ otherwise, that $|D_k f'(x)| \le 1 - \varepsilon'$ and $\varepsilon' \le 1/2$ (by the choice of $\varepsilon'$). Using these facts we can easily prove that

$$\Big|D_k f'(\bar x) \pm \frac12 D_e g(x)\Big| \le 1 - \varepsilon'/2\,,$$

which, together with (4.18), clearly implies that (4.17) holds for $r$ small enough.

To prove (4.17) for $j \ne k$ is actually simpler: recall indeed that $\|D_j f'\|_\infty < 1$ (statement (b) above) and note that

$$\begin{aligned} |D_j \bar g(x)| &\le \big|\langle d\bar g(x)\,;\, e_j(\bar x)\rangle\big| + \big|\langle d\bar g(x)\,;\, e_j(x) - e_j(\bar x)\rangle\big| \\ &\le \tan\big(2\,\alpha(\bar x, r)\big) + |d\bar g(x)|\,|e_j(x) - e_j(\bar x)| = o(1)\,, \end{aligned}$$

where the second inequality follows from statement (e) in Step 4 and the fact that, by definition, $\bar g = \pm\frac12 g$.

*Step 7. Construction of the function $f''$ and the set $K$.*

We consider the family $\mathscr{G}$ of all closed balls $B(\bar x, r)$ with $\bar x \in E'_k$ and $r$ smaller than $r_1(x)$ and $r_2(x)$, so that the conclusions of Step 5 and Step 6 hold. By a standard corollary of Besicovitch covering theorem (see for example [17], Proposition 4.2.13) we can extract from $\mathscr{G}$ finitely many disjoint balls $B_i = B(\bar x_i, r_i)$ such that

$$\sum_i \mu(E'_k \cap B_i) \geq \frac{1}{2}\mu(E'_k)\,. \tag{4.19}$$

Since the balls $B_i = B(\bar x_i, r_i)$ are closed and disjoint, for every $i$ we can find $r'_i > r_i$ such that the enlarged balls $B'_i := B(\bar x_i, r'_i)$ are still disjoint. Finally, for every $i$ we set $\bar g_i := \bar g_{\bar x_i, r_i, r'_i}$, $K_i := K_{\bar x_i, r_i}$, and

$$f'' := f' + \sum_i \bar g_i\,, \quad K := \bigcup_i K_i\,.$$

We now check that $f''$ and $K$ satisfy all requirements.

The function $f''$ is smooth because so are $f'$ and $\bar g_i$, and the set $K$ is compact because so are the sets $K_i$.

Note that the supports of the functions $\bar g_i$ are disjoint (because they are contained in the balls $B'_i$), and therefore at every point $x \in \mathbb{R}^n$ the derivative of $f''$ agrees either with the derivative of $f'$ or with that of $f' + \bar g_i$ for some $i$. Therefore, since $f'$ belongs to $X$ (Step 1) and $f' + \bar g_i$ belongs to $X$ for every $i$ (Step 6), we infer that also $f''$ belongs to $X$.

Statement (i), namely that $\|f'' - f\| \leq 2\varepsilon$, follows from statements (a) in Step 1 and (c) in Step 4, and the fact that the functions $g_i$ have disjoint supports.

Statement (ii), namely that $\mu(K) \geq \mu(E')/(4d)$, follows from estimates (4.11), (4.19), and (4.8).

Consider now $x \in K_i$ for some $i$. By Step 5, $U_\sigma(f' + \bar g_i)(x) \geq d_v(x)/(3\sqrt{d})$. Moreover the proof of this estimates involves only the restriction of $f' + \bar g_i$ to the ball $B'_i$, where $f' + \bar g_i$ agrees with $f''$. Thus the same estimates holds for $U_\sigma f''(x)$ as well, which proves statement (iii). □

**4.15. Lemma.** *Take $f \in X$ and $\sigma > 0$. If $U_\sigma$ is continuous at $f$ (as a map from $X$ to $L^1(\mu)$) then*

$$U_\sigma f(x) \geq \frac{d_v(x)}{3\sqrt{d}} \quad \textit{for } \mu\textit{-a.e. } x \in E. \tag{4.20}$$

**Proof.** We assume that (4.20) fails and prove that $U_\sigma$ is not continuous at $f$. Indeed, if (4.20) does not hold, we can find a set $E'$ contained in $E$ with $\mu(E') > 0$ and $\delta > 0$ such that

$$U_\sigma f(x) \leq \frac{d_v(x)}{3\sqrt{d}} - \delta \quad \text{for every } x \in E'.$$

Then we use Lemma 4.14 to construct a sequence of smooth functions $f_h \in X$ and of compact sets $K_h$ contained in $E'$ such that $f_h \to f$ uniformly as $h \to +\infty$, and for every $h$ there holds $\mu(K_h) \geq \mu(E')/(4d)$ and

$$U_\sigma f_h(x) \geq \frac{d_v(x)}{3\sqrt{d}} \quad \text{for every } x \in K_h.$$

Thus $U_\sigma f_h$ does *not* converge to $U_\sigma f$ in the $L^1(\mu)$-norm, and more precisely

$$\big\|U_\sigma f_h - U_\sigma f\big\|_{L^1(\mu)} \geq \int_{K_h} \big|U_\sigma f_h - U_\sigma f\big|\, d\mu \geq \delta\mu(K_h) \geq \frac{\delta}{4d}\mu(E')\,. \qquad \square$$



**Proof of Proposition 4.4.** For every $\sigma > 0$, let $N_{v,\sigma}$ of all $f \in X$ which satisfy (4.20). Then each $N_{v,\sigma}$ is residual in $X$ because it contains the set of continuity points of $U_\sigma$ (Lemma 4.15), which in turn is residual because $U_\sigma$ is a map of Baire class 1 (Lemma 4.10).

To conclude we note that $N_v$ agrees with the intersection of all $N_{v,1/m}$ with $m = 1, 2, \dots$ (by (4.3)) and therefore $N_v$ is residual as well. □

**Proof of Proposition 4.5.** Let $D$ be a countable dense subset of $\mathbb{R}^n$, and let $N'$ be the intersection of all sets $N_v$ defined in Proposition 4.4 with $v \in D$. By Proposition 4.4 the sets $N_v$ are residual in $X$, and then also $N'$ is residual.

Let now be given $f \in N'$. One readily checks that for $\mu$-a.e. $x \in E$ inequality (4.1) holds for every $v \in D$, and we deduce that it actually holds for every $v \in \mathbb{R}^n$ using the fact that both sides of (4.1) are continuous in $v$ (and $D$ is dense in $\mathbb{R}^n$); notice indeed that the directional upper and lower derivatives $D_v^{\pm} f(x)$ are Lipschitz in $v$ (with the same Lipschitz constant as $f$).

We have thus proved that $f$ belongs to $N$, thus $N$ contains $N'$, and therefore is residual. □

**Proof of Theorem 4.1.** The strategy is simple: we cover $\mathbb{R}^n$ with a countable family of pairwise disjoint sets $E_i$ which satisfy the assumption in §4.2, then we use Proposition 4.5 to find functions $f_i$ which satisfy (4.1) for every $v$ and $\mu$-a.e. $x \in E_i$, and we regularize these functions out of the set $E_i$ using Proposition 8.4; finally we take as $f$ a weighted sum of these modified functions.

For every $x \in \mathbb{R}^n$ let $d(x)$ be the dimension of $V(\mu, x)^\perp$, and let $F_0$ be the set of all $x$ such that $d(x) > 0$.

*Step 1. For every (Borel) set $F$ contained in $F_0$ with $\mu(F) > 0$ there exists a compact set $E \subset F$ with $\mu(E) > 0$ which satisfies the assumption in §4.2.*

The map $x \mapsto V(\mu, x)^\perp$, viewed as a closed-valued multifunction from $E$ to $\mathbb{R}^n$, is Borel measurable, and therefore we can use Kuratowski and Ryll-Nardzewski's measurable selection theorem (see [32], Theorem 5.2.1) to choose Borel vectorfields $e_1, \dots, e_n$ on $\mathbb{R}^n$ so that

(a) $e_1(x), \dots, e_n(x)$ form an orthonormal basis of $\mathbb{R}^n$ for every $x \in \mathbb{R}^n$;

(b) $e_1(x), \dots, e_{d(x)}(x)$ span $V(\mu, x)^\perp$ for every $x \in F$.

Then we use Lusin's theorem to find a compact set $E \subset F$ with $\mu(E) > 0$ such that the restrictions of the function $d$ and the vectorfields $e_j$ to $E$ are continuous; thus $d$ is locally constant on $E$; possibly replacing $E$ with a smaller subset we can further assume that $d$ is constant on $E$ and that the restrictions of each $e_j$ to $E$

takes values in the closed ball $B_j := B(e_j(\bar x), \delta)$ for some $\bar x \in E$ and some (small) $\delta > 0$.

To conclude the proof we modify the vectorfields $e_j$ in the complement of $E$ so that they become continuous on the whole $\mathbb{R}^n$ and still satisfy assumption (a) above. This last step is achieved by first extending the restriction of each $e_j$ to $E$ to a continuous map from $\mathbb{R}^n$ to $B_j$ (using Tietze extension theorem) and then applying the Gram-Schmidt orthonormalization process to the resulting vectorfields (note that if $\delta$ is small enough these vectorfields are linearly independent at every point).

*Step 2. There exists a countable collection of pairwise disjoint compact sets $E_i$ such that each $E_i$ satisfies $\mu(E_i) > 0$ and the assumption in §4.2, and the union of all $E_i$ contains $\mu$-a.e. point.*

Let $\mathscr{G}$ be the class of all countable collections $\{E_i\}$ that satisfy all requirements except possibly the last one (the union contains $\mu$-a.e. point). The class $\mathscr{G}$ is nonempty and admits an element which is maximal with respect to inclusion. Using Step 1 it is easy to prove that this maximal element satisfies also the last requirement.

For the rest of the proof we assume that the collection $\{E_i\}$ is infinite and that $i = 1, 2, \dots$; the case of a finite collection can be treated in the same way (and is actually simpler).

*Step 3. For every $i = 1, 2, \dots$ there exists a function $g_i$ with $\mathrm{Lip}(g_i) \le 2$ which is smooth outside $E_i$ and for $\mu$-every $x \in E_i$ satisfies*

$$D_v^+ g_i(x) - D_v^- g_i(x) > 0 \quad \text{for every } v \notin V(\mu, x). \tag{4.21}$$

We use Proposition 4.5 to find a Lipschitz function $f_i$ with $\mathrm{Lip}(f_i) \le 1$ such that for $\mu$-a.e. $x \in E_i$,

$$D_v^+ f_i(x) - D_v^- f_i(x) > 0 \quad \text{for every } v \notin V(\mu, x); \tag{4.22}$$

and then we apply Proposition 8.4 to each $f_i$ to find a Lipschitz function $g_i$ with $\mathrm{Lip}(g_i) \le 2$ which agrees with $f_i$ on $E_i$, is smooth on $\mathbb{R}^n \setminus E_i$, and satisfies

$$|g_i(x) - f_i(x)| \le \big(\mathrm{dist}(x, E_i)\big)^2 \quad \text{for every } x \in \mathbb{R}^n.$$

This implies in particular that for every $x \in E_i$ and every $v \in \mathbb{R}^n$ there holds

$$g_i(x + hv) = f_i(x + hv) + O(|h|^2) \quad \text{for every } h \in \mathbb{R},$$

which yields $D_v^\pm g_i(x) = D_v^\pm f_i(x)$, and then (4.22) implies (4.21).

*Step 4. Construction of the function $f$.*

We take the functions $g_i$ as in Step 3, and note that, possibly adding a suitable constant to $g_i$, we can further assume $g_i(0) = 0$ for every $i$. Then we set

$$f(x) := \sum_{i=1}^{+\infty} \frac{g_i(x)}{2^i} \quad \text{for every } x \in \mathbb{R}^n.$$

The function $f$ is well-defined (thanks to the estimate $|g_i(x)| \le 2|x|$, which follows from $\mathrm{Lip}(g_i) \le 2$) and satisfies $\mathrm{Lip}(f) \le 2$.



We claim that for $\mu$-a.e. $x$ there holds $D_v^+ f(x) - D_v^- f(x) > 0$ for every $v \notin V(\mu, x)$. Taking into account (4.21) and the fact that the union of the sets $E_i$ contains $\mu$-a.e. $x$, it suffices to prove that for every $i$ and every $x \in E_i$ the function

$$\hat g_i := \sum_{j \ne i} \frac{g_j(x)}{2^j}$$

is differentiable at $x$.

To this end, it suffices to show that for every $\varepsilon > 0$ we can decompose $\hat g_i$ as $\hat g_i = g_i' + g_i''$ where $g_i'$ is differentiable at $x$ and $\mathrm{Lip}(g_i'') \le \varepsilon$.

Let indeed $g_i'$ be the sum of $g_j$ over all $j \ne i$ with $j \le j_0$, and $g_i''$ be the sum over all $j \ne i$ with $j > j_0$; thus $g_i'$ is a finite sum of functions which are smooth in a neighbourhood of $x$, and therefore is differentiable at $x$, while the Lipschitz constant of $g_i''$ satisfies

$$\mathrm{Lip}(g_i'') \le \sum_{j > j_0} \frac{\mathrm{Lip}(g_j)}{2^j} \le 2^{1-j_0},$$

and in particular it smaller than $\varepsilon$ for $j_0$ sufficiently large. □

## 5. Measures related to normal currents

In the main result of this section (Theorem 5.10) we establish a connection between the decomposability bundle of a measure $\mu$ and the Radon-Nikodým derivative of a normal current w.r.t. to $\mu$. Then we consider a few well-known formulas related to normal currents and smooth functions (or forms), and use the previous result to extend these formulas to the case of Lipschitz functions (or forms). More precisely, we prove formulas for the action of the boundary of a normal current on a Lipschitz form (Proposition 5.12), for the boundary of the interior product of a normal current and a Lipschitz form (Proposition 5.13), and for the push-forward of a normal current according to a Lipschitz map (Proposition 5.17).

**5.1. Notation related to currents.** We list here the notation from multilinear algebra and the theory of currents that is used in this section and in the next one:

- $\wedge_k(V)$ space of $k$-vectors in the linear space $V$;
- $\wedge^k(V)$ space of $k$-covectors on the linear space $V$;
- $\langle \alpha\,;\, v\rangle$ duality pairing of the $k$-covector $\alpha$ and the $k$-vector $v$, also written as $\langle v\,;\,\alpha\rangle$;
- $v \wedge w$ exterior product of multi-vectors (or multi-covectors);
- $v \llcorner \alpha$ interior product of the $k$-vector $v$ and the $h$-covector $\alpha$ (§5.7);
- $\langle T\,;\,\omega\rangle$ duality pairing of the $k$-current $T$ and the $k$-form $\omega$;
- $T \llcorner \omega$ interior product of the $k$-current $T$ and the $h$-form $\omega$ (§5.7);
- $d\omega$ exterior derivative of the $k$-form $\omega$;
- $d_T\omega$ exterior derivative of the $k$-form $\omega$ w.r.t. the current $T$ (§5.11);
- $\partial T$ boundary of the current $T$ (§5.2);

| | |
|---|---|
| $\mathbb{M}(T)$ | mass of the current $T$ (§5.2); |
| $[E,\tau,m]$ | current associated to a rectifiable set $E$, an orientation $\tau$, and a multiplicity $m$ (§5.4); |
| $\operatorname{span}(v)$ | span of the $k$-vector $v$ (§5.8); |
| $f^\#\omega$ | pull-back of the form $\omega$ according to the map $f$ (§5.15). |
| $f_T^\#\omega$ | restriction of $f^\#\omega$ to the tangent bundle of $T$ (§5.15). |
| $f_\# T$ | push-forward of the current $T$ according to the map $f$ (§5.16). |

**5.2. Currents and normal currents.** We recall here the basic notions and terminology of the theory of currents; elementary introductions to this theory can be found for instance in [17], [22], [30]; the most complete reference remains [12].

A *$k$-dimensional current* (or $k$-current) $T$ in $\mathbb{R}^n$ is a continuous linear functional on the space of $k$-forms on $\mathbb{R}^n$ which are smooth and compactly supported. The *boundary* of $T$ is the $(k-1)$-current $\partial T$ defined by $\langle \partial T\,;\omega\rangle := \langle T\,; d\omega\rangle$ for every smooth $(k-1)$-form $\omega$ with compact support in $\mathbb{R}^n$. The *mass* of $T$, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T\,;\omega\rangle$ over all forms $\omega$ such that $|\omega|\le 1$ everywhere.[5]

A current $T$ is called *normal* if both $T$ and $\partial T$ have finite mass.

**5.3. Representation of currents with finite mass.** By Riesz theorem a current $T$ with finite mass can be represented as a finite measure with values in the space $\wedge_k(\mathbb{R}^n)$ of $k$-vectors in $\mathbb{R}^n$, and therefore it can be written in the form $T=\tau\mu$ where $\mu$ is a finite positive measure and $\tau$ is a $k$-vectorfield in $L^1(\mu)$. In particular the action of $T$ on a form $\omega$ is given by

$$\langle T\,;\omega\rangle = \int_{\mathbb{R}^n} \langle \omega(x)\,;\tau(x)\rangle\, d\mu(x)\,,$$

and the mass $\mathbb{M}(T)$ is the mass of $T$ as a measure, that is, $\mathbb{M}(T)=\int|\tau|d\mu$.

In the following, whenever we write $T$ in the form $T=\tau\mu$ we tacitly assume that $\tau(x)\neq 0$ for $\mu$-a.e. $x$, and in this case we say that $\mu$ is a measure associated to the current $T$. Note that $\mu$ and $\tau$ are uniquely determined if we further require that $|\tau(x)|=1$ for $\mu$-a.e. $x$.

Moreover, if $T$ is a $k$-current with finite mass and $\mu$ is an arbitrary measure, we can write $T$ as $T=\tau\mu+\nu$ where $\tau$ is $k$-vectorfield in $L^1(\mu)$ (the Radon-Nikodým derivative of $T$ w.r.t. $\mu$), and $\nu$ is a measure with values in $k$-vectors which is singular w.r.t. $\mu$ (the singular part of $T$ w.r.t. $\mu$).

**5.4. Rectifiable currents.** Let $E$ be a $k$-rectifiable set. An *orientation* of $E$ is a $k$-vectorfield $\tau$ on $\mathbb{R}^n$ such that $\tau(x)$ is a *simple* $k$-vector with norm 1 that spans the approximate tangent space $\operatorname{Tan}(E,x)$ for $\mathscr{H}^k$-a.e. $x\in E$. A *multiplicity* on $E$ is any integer-valued function $m$ such that $\int_E m\, d\mathscr{H}^k<+\infty$. For every choice of $E,\tau,m$ as above we denote by $[E,\tau,m]$ the $k$-current defined by $[E,\tau,m] := m\tau 1_E\, \mathscr{H}^k$, that is,

$$\big\langle [E,\tau,m]\,;\omega\big\rangle := \int_E \langle\omega\,;\tau\rangle\, m\, d\mathscr{H}^k\,.$$

[5] We endow $\wedge_k(V)$ and $\wedge^k(V)$ with the Euclidean norms, but other norms would work as fine.



Currents of this type are called *integer-multiplicity rectifiable currents*, and in the following simply rectifiable currents.

The next statement contains a decomposition for normal 1-currents which is strictly related to a decomposition given in [31].

**5.5. Theorem.** *Let $T=\tau\mu$ be a normal 1-current with $|\tau(x)|=1$ for $\mu$-a.e. $x$. Then there exists a family of rectifiable 1-currents $\big\{T_t:=[E_t,\tau_t,1]:\ t\in I\big\}$, where $I$ is a measure space endowed with a finite measure $dt$, such that*

(i) *$T$ can be decomposed as $T=\int_I T_t\,dt$ (in the sense of §2.3) and*

$$\mathbb{M}(T)=\int_I \mathbb{M}(T_t)\,dt = \int_I \mathscr{H}^1(E_t)\,dt\,; \tag{5.1}$$

(ii) *$\tau_t(x)=\tau(x)$ for $\mathscr{H}^1$-a.e. $x\in E_t$ and for a.e. $t\in I$;*

(iii) *$\mu$ can be decomposed as $\mu=\int_I\mu_t\,dt$ (in the sense of §2.3) where each $\mu_t$ is the restriction of $\mathscr{H}^1$ to the 1-rectifiable set $E_t$.*

**Proof.** The existence of a family $\{T_t: t\in I\}$ satisfying the decomposition in statement (i) and (5.1) can be found for instance in [23], Corollary 3.3.

To prove statement (ii), we integrate the vectorfield $\tau$ against $T$, viewed as a vector measure, and using the decomposition of $T$ we obtain

$$\begin{aligned}
\mathbb{M}(T) = \int_{\mathbb{R}^n} 1\, d\mu(x) &= \int_{\mathbb{R}^n} \langle\tau(x)\,; dT(x)\rangle\\
&= \int_I \bigg[\int_{\mathbb{R}^n} \langle\tau(x)\,; dT_t(x)\rangle\bigg]\,dt\\
&= \int_I \bigg[\int_{E_t} \langle\tau(x)\,; \tau_t(x)\rangle\, d\mathscr{H}^1(x)\bigg]\,dt\\
&\le \int_I \mathscr{H}^1(E_t)\,dt = \int_I \mathbb{M}(T_t)\,dt\,,
\end{aligned}$$

where the inequality follows from the fact that $\tau(x)$ and $\tau_t(x)$ are unit vectors. Now (5.1) implies that this inequality is actually an equality, which means that the vectors $\tau(x)$ and $\tau_t(x)$ agree for $\mathscr{H}^1$-a.e. $x\in E_t$ and a.e. $t$.

Finally, the identity of scalar measures $\mu=\int_I\mu_t\,dt$ in statement (iii) is obtained by multiplying the identity of vector measures $T=\int_I T_t\,dt$ by the vectorfield $\tau$. $\square$

A consequence of Theorem 5.5 is the following.

**5.6. Proposition.** *Let $\mu$ be a positive measure and let $\tau$ be the Radon-Nikodým derivative of a 1-dimensional normal current $T$ w.r.t. $\mu$. Then*

$$\operatorname{span}(\tau(x))\subset V(\mu,x) \quad \text{for } \mu\text{-a.e. } x. \tag{5.2}$$

**Proof.** We write $T$ in the form $T=\tau'\mu'$ with $|\tau'(x)|=1$ for $\mu'$-a.e. $x$, and consider the decomposition $\mu'=\int_I\mu_t\,dt$ given in Theorem 5.5: for $\mu_t$-a.e. $x$ and a.e. $t$ we have that $\operatorname{span}(\tau'(x))$ agrees with $\operatorname{Tan}(E_t,x)$ which in turn is contained in $V(\mu',x)$ (by the definition of decomposability bundle and Remark 2.7(iv)), and this means that

$$\mathrm{span}(\tau'(x)) \subset V(\mu',x) \quad \text{for } \mu'\text{-a.e. } x. \tag{5.3}$$

Now, let $E$ be the set of all $x$ such that $\tau(x) \neq 0$. Thus $1_E\mu \ll \mu'$, and therefore Proposition 2.9(i) yields $V(\mu',x) = V(\mu,x)$ for $\mu$-a.e. $x \in E$. Moreover $\tau' = \tau/|\tau|$ $\mu$-a.e. in $E$. These facts together with (5.3) yield that $\mathrm{span}(\tau(x)) \subset V(\mu,x)$ for $\mu$-a.e. $x \in E$, and since this inclusion is trivially true for $x \notin E$, the proof of (5.2) is complete. □

In order to extend Proposition 5.6 to currents with arbitrary dimension, we need some additional notions.

**5.7. Interior product.** Let $h,k$ be integers with $0 \le h \le k$. Given a $k$-vector $v$ and an $h$-covector $\alpha$ on $V$, the *interior product* $v \llcorner \alpha$ is the $(k-h)$-vector uniquely defined by the duality pairing

$$\langle v \llcorner \alpha \,;\, \beta\rangle = \langle v \,;\, \alpha \wedge \beta\rangle \quad \text{for every } \beta \in \wedge^{k-h}(V).$$

Accordingly, given a $k$-current $T$ in $\mathbb{R}^n$ and a smooth $h$-form $\omega$ on $\mathbb{R}^n$, the *interior product* $T \llcorner \omega$ is the $(k-h)$-current defined by

$$\langle T \llcorner \omega \,;\, \sigma\rangle = \langle T \,;\, \omega \wedge \sigma\rangle \tag{5.4}$$

for every smooth $(h-k)$-form $\sigma$ with compact support on $\mathbb{R}^n$. Then the natural counterpart of the Leibniz rule for the exterior derivative of the product of forms is

$$\partial(T \llcorner \omega) = (-1)^h \big[(\partial T) \llcorner \omega - T \llcorner d\omega\big]\,. \tag{5.5}$$

Note that if $T$ has finite mass and $\omega$ is bounded and continuous then formula (5.4) still makes sense, $T \llcorner \omega$ is a current with finite mass, and given a representation $T = \tau\mu$ there holds $T \llcorner \omega = (\tau \llcorner \omega)\,\mu$. Along the same line, if $T$ is a normal current and $\omega$ is of class $C^1$, bounded and with bounded derivative, then $T \llcorner \omega$ is a normal current and formula (5.5) holds.

**5.8. Span of a $k$-vector.** Given a linear space $V$ and a $k$-vector $v$ in $V$, we denote by $\mathrm{span}(v)$ the smallest linear subspace $W$ of $V$ such that $v$ belongs to $\wedge_k(W)$.[6]

**5.9. Proposition.** *Taken $v$ and $\mathrm{span}(v)$ as above, we have that*

(i) *if $v = 0$ then $\mathrm{span}(v) = \{0\}$;*
(ii) *if $v \neq 0$ then $\mathrm{span}(v)$ has dimension at least $k$;*
(iii) *if $v$ is simple and non-trivial, that is, $v$ can be written as $v = v_1 \wedge \cdots \wedge v_k$ with $v_1, \dots, v_k$ linearly independent vectors in $V$, then $\mathrm{span}(v)$ is the subspace of $V$ spanned by $v_1, \dots, v_k$; in particular $\mathrm{span}(v)$ has dimension $k$;*
(iv) *conversely, if $\mathrm{span}(v)$ has dimension $k$ then $v$ is simple and non-trivial;*
(v) *$\mathrm{span}(v)$ consists of all vectors of the form $v \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(V)$.*

[6] If $W$ is a linear subspace of $V$ then every $k$-vector in $W$ is canonically identified with a $k$-vector in $V$ via the immersion $I : W \to V$. Assuming this identification we have that $\wedge_k(W) \cap \wedge_k(W') = \wedge_k(W \cap W')$ for every $W, W'$ subspaces of $V$, and therefore the definition of $\mathrm{span}(v)$ is well-posed.



**Proof.** Statement (i) is immediate, while statements (ii) and (iv) are consequence of the following general facts, respectively: if $\dim(W) < k$ then every $k$-vector in $W$ is null, and if $\dim(W) = k$ then every $k$-vector in $W$ is simple.

To prove statement (iii), denote by $W$ the linear subspace of $V$ generated by $v_1, \dots, v_k$. Clearly $\mathrm{span}(v)$ is contained in $W$; moreover $\mathrm{span}(v)$ has dimension at least $k$ by statement (ii) while $W$ has dimension at most $k$; therefore $\mathrm{span}(v)$ and $W$ agree.

To prove statement (v), denote by $W$ the linear subspace of $V$ consisting of all vectors $v \llcorner \alpha$ with $\alpha \in \wedge^{k-1}(V)$. The inclusion $W \subset \mathrm{span}(v)$ is immediate because $v$ is a $k$-vector in $\mathrm{span}(v)$, and therefore the interior product of $v$ by any $h$-covector on $\mathrm{span}(v)$ is a $(k-h)$-vector in $\mathrm{span}(v)$.[7]

To prove the opposite inclusion, namely that $\mathrm{span}(v) \subset W$, we introduce some additional notation. Given a basis $\{e_i : i = 1, \dots, n\}$ of $V$, we denote by $\{e_i^*\}$ the corresponding dual basis.[8] We then denote by $I(n,k)$ the set of all multi-indexes $\mathbf{i} = (i_1, \dots, i_k)$ such that $1 \le i_1 < i_2 < \cdots < i_k \le n$, and we set, as usual, $e_{\mathbf{i}} := e_{i_1} \wedge \cdots \wedge e_{i_k}$ and $e^*_{\mathbf{i}} := e^*_{i_1} \wedge \cdots \wedge e^*_{i_k}$. Thus the $k$-vectors $e_{\mathbf{i}}$ form a basis of $\wedge_k(V)$ while the $k$-covectors $e^*_{\mathbf{i}}$ form the corresponding dual basis of $\wedge^k(V)$.

Let now $k' := \dim(W)$, and choose the basis $\{e_i\}$ so that $\{e_i : i = 1, \dots, k'\}$ is a basis of $W$. Then the inclusion $\mathrm{span}(v) \subset W$ means that $v$ is a linear combination of $e_{\mathbf{i}}$ over all $\mathbf{i}$ with $i_k \le k'$, or equivalently that $\langle v \,;\, e^*_{\mathbf{j}}\rangle = 0$ for all $\mathbf{j}$ such that $j_k > k'$. Indeed we can write $e^*_{\mathbf{j}}$ as $e^*_{\mathbf{j}'} \wedge e^*_j$ with $\mathbf{j}' \in I(n, k-1)$ and $j > k'$, and then

$$\langle v \,;\, e^*_{\mathbf{j}}\rangle = \langle v \,;\, e^*_{\mathbf{j}'} \wedge e^*_j\rangle = \langle v \llcorner e^*_{\mathbf{j}'} \,;\, e^*_j\rangle = 0\,,$$

because $w := v \llcorner e^*_{\mathbf{j}'}$ belongs by definition to $W$, and $\langle w \,;\, e^*_j\rangle = 0$ for every $w \in W$ and every $j \ge k'$ by the choice of the basis $\{e_i\}$. □

We can now state and prove the main result of this section.

**5.10. Theorem.** *Let $\mu$ be a positive measure and let $\tau$ be the Radon-Nikodým derivative of a $k$-dimensional normal current $T$ w.r.t. $\mu$.*

*Then $\mathrm{span}(\tau(x))$ is contained in $V(\mu,x)$ for $\mu$-a.e. $x$. In particular $V(\mu,x)$ has dimension at least $k$ for $\mu$-a.e. $x$ such that $\tau(x) \neq 0$.*

**Proof.** For every $\alpha \in \wedge^{k-1}(\mathbb{R}^n)$, $T \llcorner \alpha$ is a normal 1-current whose Radon-Nikodým derivative w.r.t. $\mu$ is $\tau \llcorner \alpha$ (see §5.7), and therefore the vector $\tau(x) \llcorner \alpha$ belongs to $V(\mu,x)$ for $\mu$-a.e. $x$ (Proposition 5.6). In particular, taken a finite set $\{\alpha_j : j \in J\}$ that spans $\wedge^{k-1}(\mathbb{R}^n)$, for $\mu$-a.e. $x$ there holds

$$\tau(x) \llcorner \alpha_j \in V(\mu,x) \quad \text{for every } j \in J. \tag{5.6}$$

Moreover the vectors $\tau(x) \llcorner \alpha_j$ span $\{\tau(x) \llcorner \alpha : \alpha \in \wedge^{k-1}(\mathbb{R}^n)\}$, which by Proposition 5.9(v) agrees with $\mathrm{span}(\tau(x))$. This fact and (5.6) imply that $\mathrm{span}(\tau(x))$ is

[7] This is actually a consequence of the fact that the interior product commutes with the immersion $I : W \to V$, and more generally with every linear map $T : W \to V$, in the sense that $(T_\# v) \llcorner \alpha = T_\#(v \llcorner (T^\# \alpha))$ for every $v \in \wedge_k(W)$ and every $\alpha \in \wedge^h(V)$.

[8] That is, the basis of the dual $V^*$ defined by the identity $\langle e_i^* \,;\, e_j\rangle = \delta_{ij}$ for every $i,j$.

contained in $V(\mu,x)$ for $\mu$-a.e. $x$. The rest of the statement follows from Proposition 5.9(ii). $\square$

In the rest of this section we give some applications of Theorem 5.10. We begin with a simple remark.

**5.11. Exterior derivative of Lipschitz forms.** Let $\mu$ be a positive measure on $\mathbb{R}^n$ and $\omega$ a Lipschitz $h$-form on $\mathbb{R}^n$. Then the (pointwise) exterior derivative $d\omega(x)$ is defined at $\mathscr{L}^n$-a.e. $x$ but in general not at $\mu$-a.e. $x$. However, since the coefficients of $\omega$ w.r.t. any basis of $\wedge^h(\mathbb{R}^n)$ are Lipschitz functions, they are differentiable w.r.t. $V(\mu,x)$ at $\mu$-a.e. $x$, and therefore it is possible to define the exterior derivative of $\omega$ relative to $V(\mu,x)$ at $\mu$-a.e. $x$, which we denote by $d_\mu\omega(x)$.

The precise construction is the following: given a basis $\{\alpha_i\}$ of $\wedge^h(\mathbb{R}^n)$, we denote by $\omega_i$ the coefficients of $\omega$ w.r.t. this basis, so that $\omega(x)=\sum_i \omega_i(x)\,\alpha_i$ for every $x\in\mathbb{R}^n$. Then, given a point $x$ such that the functions $\omega_i$ are all differentiable at $x$ w.r.t. to $V=V(\mu,x)$, we chose a basis $\{e_j\}$ of $V$, and let $d_\mu\omega(x)$ be the $(h+1)$-covector on $V$ defined by

$$d_\mu\omega(x) := \sum_{i,j} D_{e_j}\omega_i(x)\, e_j^*\wedge\alpha_i\,.$$

Assume now that $T=\tau\mu$ is a normal $k$-current on $\mathbb{R}^n$. By Theorem 5.10, $\mathrm{span}(\tau(x))$ is contained in $V(\mu,x)$ for $\mu$-a.e. $x$, and therefore we can define the exterior derivative of $\omega$ w.r.t. $\mathrm{span}(\tau(x))$ at $\mu$-a.e. $x$, which we denote by $d_T\omega(x)$; in other words $d_T\omega(x)$ is the $(h+1)$-covector on $\mathrm{span}(\tau(x))$ given by the restriction of $d_\mu\omega(x)$.

Note that the form $d_T\omega$ is essentially independent of the specific decomposition $T=\tau\mu$, because so is the bundle $x\mapsto\mathrm{span}(\tau(x))$. Indeed, for every other decomposition $T=\tau'\mu'$ the measures $\mu$ and $\mu'$ are absolutely continuous w.r.t. each other, and $\mathrm{span}(\tau(x))=\mathrm{span}(\tau'(x))$ for $\mu$-a.e.$x$.

Now we turn our attention to the identity that defines the boundary of a $k$-current $T$, namely $\langle\partial T\,;\,\omega\rangle=\langle T\,;\,d\omega\rangle$ for every smooth $(k-1)$-form $\omega$ with compact support. If $T$ is a normal current then both terms in this identity can be represented as integrals; therefore they make sense even when $\omega$ is a form of class $C^1$ with $\omega$ and $d\omega$ bounded, and a simple approximation argument proves that they agree.

The next result shows that the same is true for Lipschitz forms, having made the necessary modifications.

**5.12. Proposition.** *Let $T=\tau\mu$ be a normal $k$-current on $\mathbb{R}^n$, and $\omega$ a bounded Lipschitz $(k-1)$-form on $\mathbb{R}^n$. Then*

$$\langle\partial T\,;\,\omega\rangle=\int_{\mathbb{R}^n}\langle d_T\omega(x)\,;\,\tau(x)\rangle d\mu(x)\,, \tag{5.7}$$

*where $d_T\omega$ is taken as in §5.11.*

Note that the duality pairing $\langle d_T\omega(x)\,;\,\tau(x)\rangle$ in (5.7) is well-defined for $\mu$-a.e. $x$ because $d_T\omega(x)$ is a $k$-covector on the span of $\tau(x)$.



**Proof.** We apply Corollary 8.3 with $V(x):=\mathrm{span}(\tau(x))$ to the coefficients of $\omega$ w.r.t. some basis of $\wedge^{k-1}(\mathbb{R}^n)$ and construct a sequence of smooth $(k-1)$-forms $\omega_j$ which are uniformly bounded, have uniformly bounded derivatives $d\omega_j$, converge to $\omega$ uniformly, and satisfy

$$\lim_{j\to+\infty} d_T\omega_j(x)=d_T\omega(x)\quad\text{for } \mu\text{-a.e. } x. \tag{5.8}$$

Then

$$\begin{aligned}\langle\partial T\,;\,\omega\rangle=\lim_{j\to\infty}\langle\partial T\,;\,\omega_j\rangle&=\lim_{j\to\infty}\langle T\,;\,d\omega_j\rangle\\&=\lim_{j\to\infty}\int_{\mathbb{R}^n}\langle d_T\omega_j(x)\,;\,\tau(x)\rangle d\mu(x)\\&=\int_{\mathbb{R}^n}\langle d_T\omega(x)\,;\,\tau(x)\rangle d\mu(x)\,,\end{aligned}$$

where the first equality follows from the fact that $\omega_j$ converge to $\omega$ uniformly, and the fourth one from (5.8) and Lebesgue's dominated convergence theorem (the domination is easily obtained using that the forms $d\omega_j$ are uniformly bounded and $\tau$ belongs to $L^1(\mu)$). $\square$

Next we consider the interior product $T\llcorner\omega$ of a $k$-current $T$ and a bounded Lipschitz $h$-form $\omega$, and prove a variant of formula (5.5) for the boundary of $T\llcorner\omega$.

**5.13. Proposition.** *Let $T=\tau\mu$ be a normal $k$-current on $\mathbb{R}^n$ and $\omega$ a bounded Lipschitz $h$-form on $\mathbb{R}^n$ with $0\le h<k$. Then $T\llcorner\omega=(\tau\llcorner\omega)\,\mu$ is a normal $(k-h)$-current with boundary*

$$\partial(T\llcorner\omega)=(-1)^h\big[(\partial T)\llcorner\omega-(\tau\llcorner d_T\omega)\,\mu\big]\,, \tag{5.9}$$

*where $d_T\omega$ is taken as in §5.11.*

Note that for $\mu$-a.e. $x$ the interior product $\tau(x)\llcorner d_T\omega(x)$ in (5.9) is a well-defined $(k-h-1)$-vector in $\mathrm{span}(\tau(x))$ (and hence a $(k-h-1)$-vector in $\mathbb{R}^n$) because $d_T\omega(x)$ is a $k$-covector on $\mathrm{span}(\tau(x))$.

**5.14. Remark.** In the special case $h=0$ Proposition 5.13 can be restated as follows: if $T=\tau\mu$ is a normal $k$-current on $\mathbb{R}^n$ and $f$ a bounded Lipschitz function on $\mathbb{R}^n$, then $fT=f\tau\mu$ is a normal $k$-current with boundary

$$\partial(fT)=f\,\partial T+(\tau\llcorner d_T f)\mu\,.$$

**Proof of Proposition 5.13.** We take a sequence of smooth forms $\omega_j$ exactly as in the proof of Proposition 5.12. Since the forms $\omega_j$ are smooth and bounded, the currents $T\llcorner\omega_j$ are normal (cf. §5.7) and it is easy to see that as $j\to+\infty$ they converge to $T\llcorner\omega$ in the mass norm. Moreover formula (5.5) yields

$$\partial(T\llcorner\omega_j)=(-1)^h\big[(\partial T)\llcorner\omega_j-T\llcorner d\omega_j\big]\,, \tag{5.10}$$

which, together with the fact that the forms $\omega_j$ and the derivatives $d\omega_j$ are uniformly bounded, implies that the masses of $\partial(T\llcorner\omega_j)$ are also uniformly bounded. Thus $\partial(T\llcorner\omega)$ has finite mass, and $T\llcorner\omega$ is a normal current.

To prove formula (5.9) we pass to the limit in (5.10), and the only delicate point is to show the convergence of $T \llcorner d\omega_j$ to $T \llcorner d_T\omega$. To this end, we use that

$$T \llcorner d\omega_j = (\tau \llcorner d\omega_j)\mu = (\tau \llcorner d_T\omega_j)\mu\,, \quad T \llcorner d_T\omega = (\tau \llcorner d_T\omega)\mu\,,$$

and that the forms $d_T\omega_j$ are uniformly bounded and converge $\mu$-a.e. to $d_T\omega$ by assumption (5.8). □

We conclude this section by proving a formula for the push-forward of a normal current according to a Lipschitz map (Proposition 5.17).

**5.15. Pull-back of forms.** Given a map $f : \mathbb{R}^n \to \mathbb{R}^m$ of class $C^1$ and a continuous $k$-form $\omega$ on $\mathbb{R}^m$, the pull-back of $\omega$ according to $f$ is the continuous $k$-form $f^\#\omega$ on $\mathbb{R}^m$ defined by

$$\big\langle (f^\#\omega)(x)\,;\, v_1 \wedge \cdots \wedge v_k \big\rangle := \big\langle \omega(f(x))\,;\, (df(x)\,v_1) \wedge \cdots \wedge (df(x)\,v_k) \big\rangle \tag{5.11}$$

for every $v_1, \dots, v_k \in \mathbb{R}^n$.

Note that when $f$ is a Lipschitz map, $(f^\#\omega)(x)$ is a well-defined $k$-covector on $\mathbb{R}^n$ only at the points $x$ where $f$ is differentiable, that is, at $\mathscr{L}^n$-a.e. $x$, but in general it is not defined at $\mu$-a.e. $x$ when $\mu$ is an arbitrary measure on $\mathbb{R}^n$. However, since $f$ is differentiable w.r.t. $V(\mu,x)$ at $\mu$-a.e. $x$, we can use formula (5.11) to define the restriction of $(f^\#\omega)(x)$ to $V(\mu,x)$; we denote this $k$-covector on $V(\mu,x)$ by $(f^\#_\mu\omega)(x)$.

Given a normal current $T = \tau\mu$ on $\mathbb{R}^n$, we use that $\mathrm{span}(\tau(x))$ is contained in $V(\mu,x)$ for $\mu$-a.e. $x$ (Theorem 5.10) to define $(f^\#_T\omega)(x)$ as the $k$-covector on $\mathrm{span}(\tau(x))$ given by the restriction of $(f^\#_\mu\omega)(x)$ to $\mathrm{span}(\tau(x))$ for $\mu$-a.e. $x$.

**5.16. Push-forward of currents.** Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ and a $k$-current $T$ in $\mathbb{R}^n$ with compact support, the *push-forward* of $T$ according to $f$ is the $k$-current $f_\#T$ in $\mathbb{R}^m$ defined by

$$\langle f_\#T\,;\,\omega\rangle := \langle T\,;\, f^\#\omega\rangle \tag{5.12}$$

for every smooth $k$-form $\omega$ on $\mathbb{R}^m$ (since $T$ has compact support, $\langle T\,;\,\sigma\rangle$ is well-defined for every smooth $k$-form $\sigma$ on $\mathbb{R}^n$, even without compact support, and in particular it is defined for $\sigma := f^\#\omega$).

If in addition $T$ has finite mass then identity (5.12) can be extended to all continuous $k$-forms $\omega$ and can be used to define $f_\#T$ when $f$ of class $C^1$.

When $f$ is Lipschitz the right-hand side of formula (5.12) does not make sense because the form $f^\#\omega$ is not defined, but the push-forward $f_\#T$ is still defined if $T$ is a normal current, although in a completely different way (see [12], §4.1.14, or [17], Lemma 7.4.3). Indeed one can prove that for every sequence of smooth maps $f_j : \mathbb{R}^n \to \mathbb{R}^m$ that are uniformly Lipschitz and converge to $f$ uniformly, the push-forwards $(f_j)_\#T$ converge in the sense of currents to the same limit, which is then taken as definition of $f_\#T$.

In the next statement we prove a modification of formula (5.12) which holds when $f$ is Lipschitz.



**5.17. Proposition.** *Let $T = \tau\mu$ be a normal $k$-current on $\mathbb{R}^n$ with compact support, let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map, and let $f_\#T$ be the push-forward of $T$ according to $f$. Then, for every continuous $k$-form $\omega$ on $\mathbb{R}^m$, there holds*

$$\langle f_\#T\,;\,\omega\rangle = \langle T\,;\, f^\#_T\omega\rangle = \int_{\mathbb{R}^n} \big\langle (f^\#_T\omega)(x)\,;\,\tau(x)\big\rangle\, d\mu(x)\,. \tag{5.13}$$

Note that the duality pairing $\big\langle (f^\#_T\omega)(x)\,;\,\tau(x)\big\rangle$ in (5.13) is well-defined for $\mu$-a.e. $x$ because $(f^\#_T\omega)(x)$ is a $k$-covector on the span of $\tau(x)$.

**Proof.** We use Corollary 8.3 to choose the approximating maps $f_j$ used to define $f_\#T$ so that for $\mu$-a.e. $x$ the linear maps $d_Tf_j(x)$ converge to $d_Tf(x)$.

Therefore, using formula (5.11) we obtain that for every smooth $k$-form $\omega$ there holds

$$f^\#_j\omega(x) \underset{j\to+\infty}{\longrightarrow} f^\#_T\omega(x) \quad \text{in } \wedge^k[\mathrm{span}(\tau(x))] \text{ for } \mu\text{-a.e. } x. \tag{5.14}$$

Hence

$$\begin{aligned}\langle f_\#T\,;\,\omega\rangle &= \lim_{j\to+\infty} \langle (f_j)_\#T\,;\,\omega\rangle\\ &= \lim_{j\to+\infty} \int_{\mathbb{R}^n} \big\langle (f^\#_j\omega)(x)\,;\,\tau(x)\big\rangle\, d\mu(x)\\ &= \int_{\mathbb{R}^n} \big\langle (f^\#_T\omega)(x)\,;\,\tau(x)\big\rangle\, d\mu(x)\,,\end{aligned}$$

where the first equality follows from the fact that $(f_j)_\#T$ converge to $f_\#T$ in the sense of currents, the second one follows from (5.12), the third one follows from (5.14) and Lebesgue's dominated convergence theorem using the domination

$$\big|\big\langle f^\#_j\omega\,;\,\tau\big\rangle\big| \le |df_j|^k|\omega|\,|\tau| \le L^k|\omega|\,|\tau|\,,$$

where $|\omega|\,|\tau|$ belongs to $L^1(\mu)$ and $L$ is the supremum of $\mathrm{Lip}(f_j)$ over all $j$.

We have thus proved identity (5.13) for every smooth $\omega$, and we extend it to every continuous $\omega$ by a standard approximation argument. □

## 6. A characterization of the decomposability bundle.

In this section we give a characterization of the decomposability bundle of a measure $\mu$ on $\mathbb{R}^n$, $n \ge 2$, in terms of normal 1-currents (Theorem 6.4), and more precisely we show that $V(\mu,x)$ agrees for $\mu$-a.e. $x$ with the space $N(\mu,x)$ defined in the next subsection. Building on this result we obtain a precise description of the vectorfields $\tau$ on $\mathbb{R}^n$ that can be obtained as the Radon-Nikodým derivative of a 1-dimensional normal current w.r.t. $\mu$ (Corollary 6.5), and a decomposition for measures with non-trivial decomposability bundle (Corollary 6.6).

Through this section $\mu$ is a fixed measure on $\mathbb{R}^n$ with $n \ge 2$.

**6.1. The auxiliary bundle $\boldsymbol{N(\mu,x)}$.** For every point $x$ in the support of $\mu$, we denote by $N(\mu,x)$ the set of all vectors $v \in \mathbb{R}^n$ for which there exists a normal

1-current $T$ in $\mathbb{R}^n$ with $\partial T = 0$ such that

$$\lim_{r\to 0} \frac{|T - v\mu|(B(x,r))}{\mu(B(x,r))} = 0 \tag{6.1}$$

(in this section we view 1-currents with finite mass on $\mathbb{R}^n$ as $\mathbb{R}^n$-valued measures; thus $|T - v\mu|$ denotes the total variation of the $\mathbb{R}^n$-valued measure $T - v\mu$).

It is sometimes convenient that $N(\mu, x)$ is defined for all $x \in \mathbb{R}^n$, and therefore we set $N(\mu, x) := \{0\}$ when $x$ does not belong to the support of $\mu$.

In the following we refer to condition (6.1) by saying that $T$ is *asymptotically equivalent* to $v\mu$ at the point $x$.

**6.2. Remarks.** (i) The set $N(\mu, x)$ is clearly a linear subspace of $\mathbb{R}^n$ and is uniquely defined at every point $x$ (by contrast, the decomposability bundle $V(\mu, x)$ is only unique up to $\mu$-negligible subsets of $x$).

(ii) If $\tau$ is the Radon-Nikodým derivative of a normal 1-current $T$ w.r.t. $\mu$, then $\tau(x)$ belongs to $N(\mu, x)$ for $\mu$-a.e. $x$. More precisely, if we write $T = \tau\mu + \nu$ with $\nu$ singular w.r.t. $\mu$, then $\tau(x) \in N(\mu, x)$ at every point $x$ where $\tau$ is $L^1(\mu)$-approximately continuous and the density of $|\nu|$ w.r.t. $\mu$ is 0.

(iii) In dimension $n = 1$, the only normal 1-current $T$ with $\partial T = 0$ is the trivial one, and therefore $N(\mu, x) = \{0\}$ for every $x$ and every $\mu$.

(iv) In dimension $n = 2$ the bundle $N(\mu, \cdot)$ is closely related to the bundle $E(\mu, \cdot)$ introduced in [1], Definition 2.1. More precisely $E(\mu, x)$ is the set of all vectors $v \in \mathbb{R}^2$ such that $v\mu$ is asymptotically equivalent at $x$ to a vector-valued measure $\lambda = \tau|\lambda|$ on $\mathbb{R}^2$ which is a (distributional) gradient,[9] which is equivalent to say that $\tau^\perp|\lambda|$ is a normal 1-current without boundary (here $v^\perp$ denotes the rotation of the vector $v$ by 90° counterclockwise), and therefore

$$N(\mu, x) = \{v^\perp : v \in E(\mu, x)\}\,.$$

If $\mu$ is a singular measure on $\mathbb{R}^2$, it was proved in [1], Theorem 3.1, that $E(\mu, x)$ has dimension at most 1 for $\mu$-a.e. $x$. Thus $N(\mu, x)$ has dimension at most 1 for $\mu$-a.e. $x$ as well, and thanks to Theorem 6.4 below we obtain that also $V(\mu, x)$ has dimension at most 1 for $\mu$-a.e. $x$ (cf. §1.5).

(v) We prove in Lemma 6.9 that the map $x \mapsto N(\mu, x)$ agrees outside a suitable $\mu$-negligible Borel set with a Borel map from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$. We actually believe that the map $x \mapsto N(\mu, x)$ itself is Borel measurable, but the only proof we could devise is rather involved, and since this result is not really needed in the following, we decided to omit it.

(vi) There are many possible variants of the definition of $N(\mu, x)$. Among these, the one given above imposes the strongest requirements on the elements of $N(\mu, x)$. Going to the opposite extreme, we may consider the set $N'(\mu, x)$ of all $v \in \mathbb{R}^n$ for which there exists a sequence of positive numbers $r_j$ that converge to 0 and a

[9] The original definition actually requires that $\lambda$ is the gradient of a $BV$ function, but the difference is irrelevant.



sequence of normal 1-currents $T_j$ such that

$$\lim_{j\to+\infty} \frac{|T_j - v\mu|(B(x,r_j))}{\mu(B(x,r_j))} = 0\,.$$

Clearly $N'(\mu, x)$ contains $N(\mu, x)$ for every $x$, and it is easy to show that this inclusion may be strict. However, it should be true that $N'(\mu, x) = V(\mu, x)$ for $\mu$-a.e. $x$, which in view of Theorem 6.4 below yields $N'(\mu, x) = N(\mu, x)$ for $\mu$-a.e. $x$ (we do not pursue this issue here).

We now give the main results of this section. The first one is a converse of the statement in Remark 6.2(ii).

**6.3. Theorem.** *Let $\tau$ be a Borel vectorfield on $\mathbb{R}^n$ which belongs to $L^1(\mu)$ and satisfies $\tau(x) \in N(\mu, x)$ for $\mu$-a.e. $x$. Then there exists a normal 1-current $T$ on $\mathbb{R}^n$ such that*

(i) *the Radon-Nikodým derivative of $T$ w.r.t. $\mu$ agrees ($\mu$-a.e.) with $\tau$, that is, $T = \tau\mu + \sigma$ where $\sigma$ is singular w.r.t. $\mu$;*

(ii) *$\partial T = 0$ and $\mathbb{M}(T) \le C\|\tau\|_{L^1(\mu)}$ where the $C$ depends only on $n$.*

**6.4. Theorem.** *There holds $V(\mu, x) = N(\mu, x)$ for $\mu$-a.e. $x$.*

Putting together Theorems 6.3 and 6.4 and Proposition 5.6 we immediately obtain the following corollary.

**6.5. Corollary.** *Let $\tau$ be a vectorfield on $\mathbb{R}^n$ which belongs to $L^1(\mu)$. Then the following statements are equivalent:*

(i) *$\tau(x) \in V(\mu, x)$ for $\mu$-a.e. $x$;*

(ii) *there exists a normal 1-current $T$ whose Radon-Nikodým derivative w.r.t. $\mu$ agrees with $\tau$, that is, $T = \tau\mu + \sigma$ where $\sigma$ is singular w.r.t. $\mu$.*

From the previous result we obtain the following decomposition for measures with non-trivial decomposability bundle (cf. [29], Theorem 6.31).

**6.6. Corollary.** *Let $\tau$ be a vectorfield on $\mathbb{R}^n$ which belongs to $L^1(\mu)$ and satisfies $\tau(x) \in V(\mu, x)$ and $\tau(x) \ne 0$ for $\mu$-a.e. $x$ (thus $V(\mu, x) \ne \{0\}$ for $\mu$-a.e. $x$). Then $\mu$ admits a decomposition $\mu = \int_I \mu_t \, dt$ in the sense of §2.3, where each $\mu_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E_t$ such that*

$$\mathrm{Tan}(E_t, x) = \mathrm{span}(\tau(x)) \quad \textit{for } \mathscr{H}^1\textit{-a.e. } x \in E_t.$$

**6.7. Remarks.** (i) In dimension $n = 1$ the statements of Theorems 6.3 and 6.4 and of Corollaries 6.5 and 6.6 are either false or irrelevant (cf. Remark 6.2(iii)).

(ii) We know from Theorem 5.10 that if $\tau$ is the Radon-Nikodým derivative of a normal $k$-current w.r.t. $\mu$, then $\mathrm{span}(\tau(x)) \subset V(\mu, x)$ for $\mu$-a.e. $x$. In the wake of Corollary 6.5 we ask now if the converse is true, that is, if every $k$-vectorfield $\tau$ in $L^1(\mu)$ that satisfies this inclusion can be obtained as the Radon-Nikodým derivative of normal $k$-current w.r.t. $\mu$.

It follows from a result by A. Máthé [21] that the answer is negative for $k = 2$ and $n = 3$ (and therefore also for every $k, n$ with $1 < k < n$). More precisely, Máthé constructs an example of a measure $\mu$ in $\mathbb{R}^3$ such that (a) $\dim(V(\mu,x)) = 2$ for $\mu$-a.e. $x$, and (b) $\mu$ cannot be decomposed in terms of measures associated to 2-dimensional rectifiable sets, that is, $\mu$ does not belong to the class $\mathscr{F}_2(\mathbb{R}^3)$ defined in §1.6. Now, property (a) implies that there exists a 2-vectorfield $\tau$ in $L^1(\mu)$ such that $\tau(x) \neq 0$ and $\operatorname{span}(\tau(x)) \subset V(\mu,x)$ for $\mu$-a.e. $x$. On the other hand every normal 2-current $T$ in $\mathbb{R}^3$ can be decomposed in terms of rectifiable 2-currents (see for instance [5]), and together with property (b) this fact implies that $T$ is singular w.r.t. to $\mu$, and in particular $\tau$ cannot be the Radon-Nikodým derivative of $T$ w.r.t. $\mu$.

The rest of this section is devoted to the proofs of Theorems 6.3 and 6.4, and Corollaries 6.5 and 6.6.

Through these proofs we use the letter $C$ to denote every constant that depends only on the dimension $n$ (the value may change at every occurrence).

**6.8. Lemma.** *Let $T$ be a normal $k$-current in $\mathbb{R}^n$, $n > k > 0$, and $B$ an open ball in $\mathbb{R}^n$ which does not intersect the support of $\partial T$. Then there exists a normal $k$-current $U$ in $\mathbb{R}^n$ such that*

(i) *the currents $U$ and $T$ agree on $B$, that is, $1_B U = 1_B T$;*
(ii) *the support of $U$ is contained in the closure $\overline{B}$ of $B$;*
(iii) *$\partial U = 0$;*
(iv) *$\mathbb{M}(U) \leq C\,|T|(\overline{B})$.*

**Proof.** First of all, we notice that it suffices to prove the statement when $B$ is the open ball with center 0 and radius 1.

We begin with an outline of the construction of $U$. We choose a point $x_0 \in B$, and construct a retraction $p$ of $\mathbb{R}^n \setminus \{x_0\}$ onto $\mathbb{R}^n \setminus B$ as follows: for $x \notin B$ we let $p(x) := x$, and for $x \in B$ we let $p(x)$ be the intersection of the sphere $\partial B$ and the half-line which starts in $x_0$ and pass through $x$. Thus

$$p(x) := x_0 + tw \quad \text{where} \quad w := \frac{x - x_0}{|x - x_0|}\,,$$

and $t > 0$ is chosen so that $|p(x)| = 1$, that is,

$$t := \sqrt{1 + (x_0 \cdot w)^2 - |x_0|^2} - x_0 \cdot w\,.$$

We then denote by $T'$ the push-forward of $T$ according to the map $p$, that is, $T' := p_\# T$. Since $p$ maps $\overline{B}$ into $\partial B$ and agrees with the identity on $\mathbb{R}^n \setminus \overline{B}$ we have that

(a) $T' = 0$ on $B$;
(b) $T' = T$ on $\mathbb{R}^n \setminus \overline{B}$;

Moreover $\partial T' = p_\#(\partial T)$, and since $\partial T$ is supported in the complement of $B$, where $p$ agrees with the identity, we have that

(c) $\partial T' = \partial T$.



Finally we set $U := T - T'$, and then statements (i), (ii) and (iii) follows immediately from (a), (b) and (c).

There are two issues with this construction: the main one is that the map $p$ is singular at $x_0$, and therefore the push-forward $p_\# T$ cannot be defined using the standard definition; the second issue is estimate (iv). Note that the same problems arise in the proof of the Polyhedral Deformation Theorem presented in [12], §4.2.9, and can be solved in the same way.

*Step 1. We can choose $x_0 \in B$ so that*

$$\int_{\mathbb{R}^n} \frac{d|T|(x)}{|x - x_0|^k} \leq C\,|T|(\overline{B})\,. \tag{6.2}$$

We actually prove that the integral of the left-hand side of (6.2) over all $x_0 \in B$ (w.r.t. the Lebesgue measure) is bounded by $C\,|T|(\overline{B})$, which implies that (6.2) holds for a set of positive measure of $x_0$. Indeed

$$\begin{aligned}\int_B \left[\int_{\overline{B}} \frac{d|T|(x)}{|x - x_0|^k}\right] dx_0 &= \int_{\overline{B}} \left[\int_{B(-x_0,1)} \frac{dy}{|y|^k}\right] d|T|(x)\\ &\leq \int_{\overline{B}} \left[\int_{B(0,2)} \frac{dy}{|y|^k}\right] d|T|(x) = C\,|T|(\overline{B})\,,\end{aligned}$$

where $dx_0$ stands (as usual) for $d\mathscr{L}^n(x_0)$, the first equality is obtained by applying Fubini's Theorem together with the change of variable $y = x - x_0$, the inequality follows from the fact that the ball $B(-x_0, 1)$ is contained in $B(0,2)$, and the last equality follows from the fact that $\int_{B(0,2)} dy/|y|^k$ is finite.

*Step 2. Construction of $T' := p_\# T$.*

The map $p$ is clearly locally Lipschitz on $\mathbb{R}^n \setminus \{-x_0\}$, and a straightforward computation shows that

$$|dp(x)| \leq u(x) \quad \text{where} \quad u(x) := \frac{C}{|x - x_0|} + 1\,. \tag{6.3}$$

Then, using estimate (6.2) and the fact that the support of $\partial T$ does not intersect $B$, we obtain that the integrals $\int_{\mathbb{R}^n} u^{-k} d|T|$ and $\int_{\mathbb{R}^n} u^{1-k} d|\partial T|$ are both finite, which allow us to define the push-forward $T' := p_\# T$ as in [12], §4.2.2. More precisely $T'$ is a normal current which satisfies the properties (a), (b) and (c) mentioned above. Moreover estimates (6.2) and (6.3) yield

$$|T'|(\overline{B}) \leq \int_{\overline{B}} |dp|^k\, d|T| \leq \int_{\overline{B}} u^{-k}\, d|T| \leq C\,|T|(\overline{B})\,. \tag{6.4}$$

*Step 3. Construction of $U$.*

As anticipated, we take $U := T - T'$. Then statements (i), (ii) and (iii) follow from statements (a), (b) and (c) above, while statement (iv) follows from estimate (6.4). □

**6.9. Lemma.** *The map $x \mapsto N(\mu, x)$ is* universally measurable *as a map from $\mathbb{R}^n$ to $\operatorname{Gr}(\mathbb{R}^n)$ (that is, measurable w.r.t. the completion of the Borel $\sigma$-algebra* *according to any finite measure on $\mathbb{R}^n$). In particular it agrees outside a suitable $\mu$-negligible Borel set $E_0$ with a Borel map.*

**Sketch of proof.** Let $K$ be the support of $\mu$, and let $G$ be the graph of the restriction of $x \mapsto N(\mu, x)$ to $K$, that is, the set of all $(x, v)$ such that $x \in K$ and $v \in N(\mu, x)$. It suffices to prove that the set $G$ is *analytic* (cf. [32], Chapter 4).

Let $N$ be the space of all normal 1-currents $T$ on $\mathbb{R}^n$ with $\mathbb{M}(T) \leq 1$ and $\partial T = 0$, endowed with the weak* topology of currents (as dual of smooth forms with compact support); thus $N$ is compact and metrizable, and in particular is a Polish space.

Now, for every $x \in K$, $v \in \mathbb{R}^n$, $T \in N$ we set

$$\psi(x, v, T) := \limsup_{r \to 0} \frac{|T - v\mu|(B(x,r))}{\mu(B(x,r))} . \tag{6.5}$$

and we remark that $v$ belongs to $N(\mu, x)$ if and only if there exists $T \in N$ such that $\psi(x, v, T) = 0$ (we have only to show that $N(\mu, x)$ does not change if we add the requirement that the current $T$ in (6.1) satisfies $\mathbb{M}(T) \leq 1$, and indeed, if need be, we simply replace $T$ by the current $U$ given by Lemma 6.8, having chosen as $B$ a ball centered at $x$ with sufficiently small radius).

It follows that $G = p(\psi^{-1}(0))$ where $p$ is the projection of $K \times \mathbb{R}^n \times N$ on $K \times \mathbb{R}^n$. Since $p$ is continuous, the analyticity of $G$ follows by the fact that $\psi^{-1}(0)$ is a Borel set, which in turn follows by the fact that $\psi$ is a Borel map.

Indeed the ratio at the right-hand side of (6.5) is left-continuous in the variable $r$, and therefore the value of $\psi$ does not change if we restrict $r$ to a fixed countable dense subset of $(0, +\infty)$. Thanks to this observation and to the fact that the ratio is Borel in the variables $x, v, T$, we obtain that $\psi$ is Borel as well. $\square$

**6.10. Lemma.** *Let $\{\sigma_t : t \in I\}$ be a family of measures on $\mathbb{R}^n$ which is Borel regular in $t$ (cf. §2.3). Assume that each $\sigma_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E_t$, and denote by $D$ the set of all $(t, x) \in I \times \mathbb{R}^n$ such that the approximate tangent line* $\mathrm{Tan}(E_t, x)$ *exists.*

*Then $D$ is a Borel set and $(t, x) \mapsto \mathrm{Tan}(E_t, x)$ is a Borel measurable map from $D$ to $\mathrm{Gr}(\mathbb{R}^n)$.*

**Sketch of proof.** We denote by $\mathscr{M}^+$ the space of all positive, locally finite measures on $\mathbb{R}^n$, and denote by $L$ the subclass of all measures given by the restriction of $\mathscr{H}^1$ to a 1-dimensional subspace of $\mathbb{R}^n$.

Given $\sigma \in \mathscr{M}^+$, a point $x \in \mathbb{R}^n$, and $r > 0$, consider the rescaled measure $\sigma_{x,r}$ given by $\sigma_{x,r}(F) := \frac{1}{r}\sigma(x + rF)$ for every Borel set $F$ in $\mathbb{R}^n$, and let $\sigma_x$ be the limit (in $\mathscr{M}^+$) of the measures $\sigma_{x,r}$ as $r \to 0$, if it exists. If $\sigma$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E$, then the approximate tangent space $\mathrm{Tan}(E, x)$ exists if and only if $\sigma_x$ exists and belongs to $L$ (cf. §2.2).

Now, since $(\sigma, x, r) \mapsto \sigma_{x,r}$ is a continuous map from $\mathscr{M}^+ \times \mathbb{R}^n \times (0, 1]$ in $\mathscr{M}^+$, it is easy to see that the set of all $(\sigma, x) \in \mathscr{M}^+ \times \mathbb{R}^n$ such that $\sigma_x$ exists is Borel, and that $(\sigma, x) \mapsto \sigma_x$ is a Borel map from this set to $\mathscr{M}^+$. Since moreover $L$ is a closed subset of $\mathscr{M}^+$, then also the set of all $(\sigma, x)$ such that $\sigma_x$ exists and belongs to $L$ is Borel.



Using these facts and recalling that $t \mapsto \sigma_t$ is Borel we conclude the proof. $\square$

The next statement is the key step in the proof of Theorem 6.3.

**6.11. Lemma.** *Let $\tau$ be a Borel vectorfield on $\mathbb{R}^n$ which belongs to $L^1(\mu)$ and satisfies $\tau(x) \in N(\mu, x)$ for $\mu$-a.e. $x$. Then there exists a normal 1-current $T$ on $\mathbb{R}^n$ such that, denoting by $\tilde{\tau}$ the Radon-Nikodým derivative of $T$ w.r.t. $\mu$,*

(i) $\|\tilde{\tau} - \tau\|_{L^1(\mu)} \leq \frac{1}{2}\|\tau\|_{L^1(\mu)}$;
(ii) $\partial T = 0$ *and* $\mathbb{M}(T) \leq C\|\tau\|_{L^1(\mu)}$.

**Proof.** We can clearly assume that $\tau$ is nontrivial, and we set

$$m := \frac{\|\tau\|_{L^1(\mu)}}{4\,\mathbb{M}(\mu)} . \tag{6.6}$$

We begin with two well-known facts: for all $x \in \mathbb{R}^n$ and all $r > 0$ except at most countably many there holds

$$\mu(\partial B(x,r)) = 0 , \tag{6.7}$$

and for $\mu$-a.e. $x$ and for $r > 0$ small enough there holds

$$\int_{B(x,r)} |\tau - \tau(x)|\, d\mu \leq m\, \mu(B(x,r)) . \tag{6.8}$$

By the definition of $N(\mu, x)$, for $\mu$-a.e. $x$ (and precisely for every $x$ in the support of $\mu$ such that $\tau(x) \in N(\mu, x)$), there exists a normal 1-current $T_x$ with $\partial T_x = 0$ such that, for $r > 0$ small enough,

$$|T_x - \tau(x)\, \mu|(B(x,r)) \leq m\, \mu((B(x,r)) . \tag{6.9}$$

Consider now the family of all closed balls $B(x, r)$ that satisfy (6.7), (6.8) and (6.9): by a standard corollary of Besicovitch covering theorem (see for example [17], Proposition 4.2.13) we can extract from this family countably many balls $B_i = B(x_i, r_i)$ which are pairwise disjoint and cover $\mu$-almost every point.

For every $i$ we set $T_i := T_{x_i}$, and use Lemma 6.8 to find a current $U_i$ with $\partial U_i = 0$ which agrees with $T_i$ in the interior of $B_i$, is supported on $B_i$, and satisfies

$$\mathbb{M}(U_i) \leq C\, |T_i|(B_i) , \tag{6.10}$$

and finally we set

$$T := \sum_i U_i .$$

We first show that $T$ is well-defined and satisfies statement (ii). Since the currents $U_i$ satisfy $\partial U_i = 0$, it suffices to show that $\sum_i \mathbb{M}(U_i) \leq C\|\tau\|_{L^1(\mu)}$. And indeed

$$\begin{aligned}\sum_i \mathbb{M}(U_i) &\leq C \sum_i |T_i|(B_i) \\ &\leq C \sum_i \big|T_i - \tau(x_i)\mu\big|(B_i) + \big|(\tau(x_i) - \tau)\mu\big|(B_i) + |\tau\mu|(B_i)\end{aligned}$$

$$\begin{aligned}&\le C\sum_i m\,\mu(B_i)+m\,\mu(B_i)+\int_{B_i}|\tau|\,d\mu\\&\le C\big(2m\,\mathbb{M}(\mu)+\|\tau\|_{L^1(\mu)}\big)=\frac{3}{2}\,C\|\tau\|_{L^1(\mu)}\,,\end{aligned}$$

where the first inequality follows from (6.10), the second one is obtained by writing the measure $T_i$ as sum of the measures $T_i-\tau(x_i)\mu$, $(\tau(x_i)-\tau)\mu$ and $\tau\mu$, the third one follows from (6.8) and (6.9), the fourth one follows from the fact that the balls $B_i$ are pairwise disjoint, and finally the fifth one follows from (6.6).

Now we prove that $T$ satisfies statement (i). Let $\tau_i$ be the Radon-Nikodým derivative of $T_i$ w.r.t. $\mu$. Since the balls $B_i$ are pairwise disjoint, in the interior of each $B_i$ the current $T$ agrees with $U_i$, which in turn agrees with $T_i$; therefore $\tilde\tau$ agrees ($\mu$-a.e.) with $\tau_i$ in the interior of each $B_i$, or equivalently on $B_i$ (because the boundary of $B_i$ is $\mu$-negligible, cf. (6.7)). Then

$$\begin{aligned}\|\tau-\tilde\tau\|_{L^1(\mu)}&=\sum_i\int_{B_i}|\tau-\tau_i|\,d\mu\\&\le\sum_i\int_{B_i}|\tau-\tau(x_i)|\,d\mu+\int_{B_i}|\tau_i-\tau(x_i)|\,d\mu\\&\le\sum_i m\,\mu(B_i)+|T_i-\tau(x_i)\,\mu|(B_i)\\&\le\sum_i 2m\,\mu(B_i)\le 2m\,\mathbb{M}(\mu)=\frac12\|\tau\|_{L^1(\mu)}\,,\end{aligned}$$

where the first equality follows by the fact the balls $B_i$ cover $\mu$-almost every point, for the second inequality we used (6.8), the third one follows from (6.9), the fourth one follows from the fact that the balls $B_i$ are disjoint, and finally the last equality follows from (6.6). □

**Proof of Theorem 6.3.** We set $\tau_0:=\tau$ and then construct currents $T_j$ and vectorfields $\tilde\tau_j$, $\tau_j$ for $j=1,2,\dots$ according to the following inductive procedure: we apply Lemma 6.11 to $\tau_{j-1}$ to obtain a normal 1-current $T_j$ such that $\partial T_j=0$ and

$$\|\tau_{j-1}-\tilde\tau_j\|_{L^1(\mu)}\le\frac12\|\tau_{j-1}\|_{L^1(\mu)}\,,\quad \mathbb{M}(T_j)\le C\|\tau_{j-1}\|_{L^1(\mu)}\,,\tag{6.11}$$

where $\tilde\tau_j$ is the Radon-Nikodým derivative of $T_j$ w.r.t. $\mu$; we then set $\tau_j:=\tau_{j-1}-\tilde\tau_j$. We finally set

$$T:=\sum_{j=1}^{\infty}T_j\,.$$

We first prove that $T$ is well-defined and satisfies statement (ii). Since the currents $T_j$ satisfy $\partial T_j=0$, it suffices to show that $\sum_j\mathbb{M}(T_j)\le C\|\tau\|_{L^1(\mu)}$. To this regard, note that the first estimates in (6.11) can be rewritten as $\|\tau_j\|_{L^1(\mu)}\le\frac12\|\tau_{j-1}\|_{L^1(\mu)}$ and therefore, recalling that $\tau_0=\tau$,

$$\|\tau_j\|_{L^1(\mu)}\le\frac{1}{2^j}\|\tau\|_{L^1(\mu)}\,.\tag{6.12}$$



Then, using the second estimates in (6.11),

$$\sum_{j=1}^{\infty}\mathbb{M}(T_j)\le\sum_{j=1}^{\infty}C\|\tau_{j-1}\|_{L^1(\mu)}\le\sum_{j=1}^{\infty}\frac{C}{2^{j-1}}\|\tau\|_{L^1(\mu)}=2C\|\tau\|_{L^1(\mu)}\,.$$

Next we show that $T$ satisfies statement (i). Since $\tilde\tau_j$ is the Radon-Nikodým derivative of $T_j$ w.r.t. $\mu$, it suffices to show that the series of all $\tilde\tau_j$ converge in $L^1(\mu)$ to $\tau$. Since $\tau_0=\tau$ and $\tilde\tau_j=\tau_{j-1}-\tau_j$ for every $j$, we have that

$$\tilde\tau_1+\dots+\tilde\tau_j=\tau-\tau_j\,,$$

and we conclude the proof by noticing that $\tau_j$ converge to 0 in $L^1(\mu)$ by (6.12). □

The next statement is the key step in the proof of Theorem 6.4.

**6.12. Lemma.** *Let $C=C(e,\alpha)$ be a closed convex cone in $\mathbb{R}^n$ (cf. §4.11) and let $\mathrm{Int}(C)$ be the interior of $C$. Let $\sigma$ be a non-trivial measure on $\mathbb{R}^n$ which can be decomposed as $\sigma=\int_I\sigma_t\,dt$ where each $\sigma_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E_t$ such that $\mathrm{Tan}(E_t,x)$ is contained in $\mathrm{Int}(C)\cup\{0\}$ for $\mathscr{H}^1$-a.e. $x\in E_t$.*

*Then there exists a normal 1-current $T$ with $\partial T=0$ whose Radon-Nikodým derivative w.r.t. $\sigma$ belongs to $C$ for $\sigma$-a.e. point and is nonzero in a set of positive $\sigma$-measure (that is, the measures $|T|$ and $\sigma$ are not mutually singular).*

**Proof.** We first construct a current $T$ that satisfies all requirements except $\partial T=0$, and at the end of the proof we explain how to modify the construction to obtain $\partial T=0$.

The idea for the construction of $T$ is quite simple: for every $t\in I$ we choose a $C$-curve $G_t$ (cf. §4.11) such that $\mathscr{H}^1(E_t\cap G_t)>0$; we then denote by $T_t$ the 1-current associated to $G_t$, and set $T:=\int T_t\,dt$. However, some care must be taken with measurability issues (for example, $G_t$ should be chosen in a Borel measurable fashion w.r.t. $t$).

Before starting with the detailed construction, we note that, possibly replacing $I$ with a suitable Borel subset, we can assume that $\mathscr{H}^1(E_t)>0$ for every $t\in I$.

We denote by $\mathscr{X}$ the class of all paths $\gamma:J\to\mathbb{R}^n$ with $J:=[-1,1]$ such that $\mathrm{Lip}(\gamma)\le1$ and $\dot\gamma(s)\in C$ for a.e. $s\in J$ (here and in the following $J$ is endowed with the Lebesgue measure, which we do not write explicitly), and we endow $\mathscr{X}$ with the supremum distance. Note that $\gamma(J)$ is a $C$-curve for every $\gamma\in\mathscr{X}$ (cf. §4.11).

*Step 1. For every $t\in I$ there exists $\gamma\in\mathscr{X}$ such that*

$$\mathscr{H}^1(E_t\cap\gamma(J))=\sigma_t(\gamma(J))>0\,.\tag{6.13}$$

Since the set $E_t$ is rectifiable and $\mathscr{H}^1(E_t)>0$, we can find a curve $G$ of class $C^1$ such that $\mathscr{H}^1(E_t\cap G)>0$. We take a point $x_0\in G$ such that $E_t\cap G$ has density 1 at $x_0$. Then $\mathrm{Tan}(G,x_0)$ agrees with $\mathrm{Tan}(E_t,x_0)$ and is contained in $\mathrm{Int}(C)\cup\{0\}$, which implies that $\mathrm{Tan}(G,x)$ is contained in $C$ for all $x$ in a suitable subarc $G'$ of $G$ that contains $x_0$, and clearly $\mathscr{H}^1(E_t\cap G')>0$. We then take as $\gamma$ a suitable parametrization of $G'$.

*Step 2. The set $F$ of all $(t,\gamma)\in I\times\mathscr{X}$ such that (6.13) holds is Borel.*
It suffices to show that $(t,\gamma)\mapsto\sigma_t(\gamma(J))$ is a Borel function on $I\times\mathscr{X}$, and this is an immediate consequence of the following facts:

- $t\mapsto\sigma_t$ is a Borel map from $I$ to the space $\mathscr{M}^+$ of finite positive Borel measures on $\mathbb{R}^n$ endowed with the weak* topology (cf. §2.3);
- $\gamma\mapsto\gamma(J)$ is a Borel map from $\mathscr{X}$ to the space $\mathscr{K}$ of compact subsets of $\mathbb{R}^n$ endowed with the Hausdorff distance;
- $(K,\sigma)\mapsto\sigma(K)$ is a Borel function on $\mathscr{K}\times\mathscr{M}^+$.

*Step 3. For every $t\in I$ we can choose $\gamma_t\in\mathscr{X}$ so that (6.13) holds and $t\mapsto\gamma_t$ agrees with a Borel map in a Borel subset $I'$ with full measure in $I$.*
The set $F$ defined in Step 2 is a Borel subset of $I\times\mathscr{X}$, and by Step 1 its projection on $I$ agrees with $I$ itself. Thus we can use the von Neumann measurable selection theorem (see [32], Theorem 5.5.2), to choose $\gamma_t\in\mathscr{X}$ for every $t\in I$ so that $(t,\gamma_t)$ belongs to $F$ (that is, $\gamma_t$ satisfies (6.13)) and the map $t\mapsto\gamma_t$ is *universally measurable*, and in particular it agrees with a Borel map in a Borel subset $I'$ with full measure in $I$.

*Step 4. Construction of the normal current $T$.*
We let $T$ be the integral (over $t\in I'$) of the 1-currents canonically associated to the paths $\gamma_t$, that is,

$$\langle T\,;\,\omega\rangle := \int_{I'}\left[\int_J \langle\omega(\gamma_t(s))\,;\,\dot\gamma_t(s)\rangle\,ds\right]dt \tag{6.14}$$

for every smooth 1-form $\omega$ on $\mathbb{R}^n$ with compact support (note that the integral in this formula is well-defined because $t\mapsto\gamma_t$ is a Borel map from $I'$ to $\mathscr{X}$ (Step 3), and then $t\mapsto\dot\gamma_t$ is a bounded Borel map from $I'$ to $L^1(J;\mathbb{R}^n)$).
A simple computation shows that

$$\langle\partial T\,;\,\varphi\rangle = \langle T\,;\,d\varphi\rangle = \int_{I'}\big[\varphi(\gamma_t(1))-\varphi(\gamma_t(-1))\big]\,dt \tag{6.15}$$

for every smooth 0-form (or function) $\varphi$ on $\mathbb{R}^n$ with compact support. It follows immediately from (6.14) and (6.15) that both $T$ and $\partial T$ have finite mass, and therefore $T$ is normal.

*Step 5. The Radon-Nikodým derivative of $T$ w.r.t. $\sigma$ takes values in $C$.*
It suffices to show that $T$, viewed as a measure, takes values in $C$. Take indeed a Borel set $E$ in $\mathbb{R}^n$: formula (6.14) yields

$$T(E) = \int_{I'}\left[\int_{\gamma_t^{-1}(E)}\dot\gamma_t(s)\,ds\right]dt\,, \tag{6.16}$$

and since $\dot\gamma_t(s)$ belongs to the cone $C$, which is closed and convex, so does $T(E)$.

*Step 6. The measures $\sigma$ and $|T|$ are not mutually singular.*
For every $t\in I'$ let $\sigma'_t$ be the restriction of $\mathscr{H}^1$ to $E_t\cap\gamma_t(J)$, or equivalently the restriction of $\sigma_t$ to $\gamma_t(J)$, and set $\sigma' := \int_{I'}\sigma'_t\,dt$.



Note that the measure $\sigma'$ is nontrivial because of the choice of $\gamma_t$, and therefore we can prove the claim by showing that $\sigma'\le\sigma$ and $\cos\alpha\,\sigma'\le|T|$. The first inequality is immediate. Concerning the second one, for every Borel set $E$ in $\mathbb{R}^n$ we have that

$$\begin{aligned}|T|(E)\ge T(E)\cdot e &= \int_{I'}\left[\int_{\gamma_t^{-1}(E)}\dot\gamma_t(s)\cdot e\,ds\right]dt\\ &\ge \int_{I'}\left[\int_{\gamma_t^{-1}(E)}\cos\alpha\,|\dot\gamma_t(s)|\,ds\right]dt\\ &\ge \cos\alpha\int_{I'}\mathscr{H}^1(\gamma_t(J)\cap E)\,dt \ge \cos\alpha\,\sigma'(E)\,,\end{aligned}$$

where the equality follows from (6.16), the second inequality follows from the fact that $\dot\gamma_t(s)$ belongs to $C=C(e,\alpha)$, the third one from the area formula, and the last one from the definition of $\sigma'$.

*Step 7. How to modify the construction of $T$ to obtain $\partial T=0$.*
We choose an open ball $B$ such that $\sigma(B)>0$. Then, possibly replacing $\sigma$ with its restriction to $B$, we can assume that $\sigma$ is supported in $B$, which means the set $E_t$ is contained in $B$ up to an $\mathscr{H}^1$-negligible subset for (almost) every $t$.
We then proceed with the construction of $T$ shown above, with the only difference that $\mathscr{X}$ is now the class of all paths $\gamma$ from $J=[-1,1]$ to the closure of $B$ such that the endpoints $\gamma(\pm1)$ belong to $\partial B$, $\mathrm{Lip}(\gamma)\le r/\cos\alpha$ where $r$ is the radius of $B$, and $\dot\gamma(s)\in C$ for a.e. $s\in J$, as before. The only modification in the proof occurs in step 1, where the path $\gamma$ must be suitably extended so that the endpoints belongs to $\partial B$.
We thus obtain a current $T$ that satisfies the same properties as before, and in addition its boundary is supported on $\partial B$ (see (6.15)). Finally we apply Lemma 6.8 to the current $T$ and the ball $B$, and obtain a current $U$ without boundary that agrees with $T$ in $B$. Using this property and the fact that $\sigma$ is supported in $B$ we easily conclude that Radon-Nikodým derivative of $U$ w.r.t. $\mu$ agrees ($\mu$-a.e.) with that of $T$. Finally we replace $T$ by $U$. □

**Proof of Theorem 6.4.** We first prove that $N(\mu,x)\subset V(\mu,x)$ for $\mu$-a.e. $x$.
We argue by contradiction, and assume that this inclusion does not hold. Then, using the Kuratowski and Ryll-Nardzewski's measurable selection theorem (see [32], Theorem 5.2.1), we can find a bounded Borel vectorfield $\tau$ on $\mathbb{R}^n$ such that $\tau(x)\in N(\mu,x)\setminus V(\mu,x)$ for every $x$ in a set of positive $\mu$-measure (here we need Lemma 6.9).
Then Theorem 6.3 yields a normal 1-current $T$ whose Radon-Nikodým derivative w.r.t. $\mu$ agrees ($\mu$-a.e.) with $\tau$, and Proposition 5.6 implies that $\tau(x)\in V(\mu,x)$ for $\mu$-a.e. $x$, in contradiction with the choice of $\tau$.

We now prove that $V(\mu,x)\subset N(\mu,x)$ for $\mu$-a.e. $x$.
First of all, we use Lemma 6.9 to modify the map $x\mapsto N(\mu,x)$ in a $\mu$-negligible set and make it Borel measurable.

By the definition of $V(\mu,\cdot)$ it suffices to show that the map $x \mapsto N(\mu,x)$ belongs to the class $\mathscr{G}_\mu$ (see §2.6). In other words, given a measure $\mu'$ of the form $\mu' = \int_I \mu_t\,dt$ such that $\mu' \ll \mu$ and each $\mu_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $E_t$, we must show that

$$\mathrm{Tan}(E_t,x) \subset N(\mu,x) \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t \in I. \tag{6.17}$$

We now argue by contradiction, and assume that (6.17) does not hold.

*Step 1. There exist a cone $C = C(e,\alpha)$ and a non-trivial measure $\sigma$ such that*

(a) *$C$ and $\sigma$ satisfy the assumptions in Lemma 6.12;*

(b) *$\sigma \ll \mu' \ll \mu$;*

(c) *$N(\mu,x) \cap C = \{0\}$ for $\sigma$-a.e. $x$.*

Let $\mu''$ be the measure on $I \times \mathbb{R}^n$ given by $\mu'' := \int_I (\delta_t \times \mu_t)\,dt$ where $\delta_t$ is the Dirac mass at $t$, and let $F$ be the set of all $(t,x) \in I \times \mathbb{R}^n$ such that $\mathrm{Tan}(E_t,x)$ exists and is not contained in $N(\mu,x)$ (note that $F$ is Borel by Lemma 6.10). Then the assumption that (6.17) does not hold can be restated by saying that $\mu''(F) > 0$.

Now, let $\mathscr{F}$ be a family of cones $C = C(e,\alpha)$ where $e$ ranges in a given countable dense subset of the unit sphere in $\mathbb{R}^n$, and $\alpha$ ranges in a given countable dense subset of $(0,\pi/2)$; for every $C \in \mathscr{F}$ let $F_C$ be the subset of $(t,x) \in F$ such that $\mathrm{Tan}(E_t,x)$ and $N(\mu,x)$ are separated by $C$, that is, $\mathrm{Tan}(E_t,x) \subset \mathrm{Int}(C) \cup \{0\}$ and $N(\mu,x) \cap C = \{0\}$ (the set $F_C$ is Borel because the set $F$ is Borel and the maps $(t,x) \mapsto \mathrm{Tan}(E_t,x)$ and $x \mapsto N(\mu,x)$ are Borel).

Then the sets $F_C$ with $C \in \mathscr{F}$ form a countable cover of $F$, and since $\mu''(F) > 0$ there exists at least one $C \in \mathscr{F}$ such that $\mu''(F_C) > 0$.

We then take $\sigma$ equal to the push-forward according to $p$ of the restriction of $\mu''$ to the set $F_C$, where $p$ is the projection of $I \times \mathbb{R}^n$ on $\mathbb{R}^n$. Note that $\sigma = \int_I \sigma_t\,dt$ where $\sigma_t$ is the restriction of $\mu_t$ to the set of all $x$ such that $(t,x) \in F_C$.

*Step 2. Completion of the proof.*

By applying Lemma 6.12 to the cone $C$ and the measure $\sigma$ constructed in Step 1 we obtain a normal 1-current $T$ with $\partial T = 0$ whose Radon-Nikodým derivative w.r.t. $\sigma$ belongs to $C$ $\sigma$-a.e., and is nonzero on a set of positive $\sigma$-measure.

Since $\sigma \ll \mu$ (statement (b) above) we deduce that also the Radon-Nikodým derivative of $T$ w.r.t. $\mu$, which we denote by $\tau$, belongs to $C$ $\sigma$-a.e. and is nonzero on a set of positive $\sigma$-measure.

Moreover we have that $\tau(x) \in N(\mu,x)$ for $\mu$-a.e. $x$ (cf. Remark 6.2(ii)) and therefore also for $\sigma$-a.e. $x$. Therefore $N(\mu,x) \cap C \neq \{0\}$ for a set of positive $\sigma$-measure of $x$, in contradiction with statement (c) above. □

**Proof of Corollary 6.5.** The implication (ii) ⇒ (i) is an immediate consequence of Proposition 5.6, while the implication (i) ⇒ (ii) follows from Theorems 6.3 and 6.4. □

**Proof of Corollary 6.6.** By Corollary 6.5 there exists a normal 1-current of the form $T = \tau\mu + \sigma$ where $\sigma$ is singular w.r.t. $\mu$. We then set $\tilde\mu := \mu + |\sigma|$ and write $T$ in the form $T = \tilde\tau\tilde\mu$.



Since $\mu$ and $|\sigma|$ are mutually singular, there exists a Borel set $E$ such that $\mu$ is supported on $E$ and $|\sigma|$ is supported on $\mathbb{R}^n \setminus E$, which means that $\mu$ is the restriction of $\tilde\mu$ to $E$. Accordingly, $\tau(x) = \tilde\tau(x)$ for $\tilde\mu$-a.e. $x \in E$, and modifying $\tilde\tau$ in a $\tilde\mu$-null set we can assume that $\tau(x) = \tilde\tau(x)$ for every $x \in E$.

By Theorem 5.5 the measure $\tilde\mu$ can be decomposed as $\tilde\mu = \int_I \tilde\mu_t\,dt$ where each $\tilde\mu_t$ is the restriction of $\mathscr{H}^1$ to a 1-rectifiable set $\widetilde E_t$ such that

$$\mathrm{Tan}(\widetilde E_t,x) = \mathrm{span}(\tilde\tau(x)) \quad \text{for } \mathscr{H}^1\text{-a.e. } x \in \widetilde E_t.$$

Therefore $\mu = \int_I \mu_t\,dt$ where each $\mu_t$ is the restriction of $\tilde\mu_t$ to $E$, that is, the restriction of $\mathscr{H}^1$ to the 1-rectifiable set $E_t := \widetilde E_t \cap E$, and clearly for $\mathscr{H}^1$-a.e. $x \in E_t$ the tangent space $\mathrm{Tan}(E_t,x)$ agrees with $\mathrm{Tan}(\widetilde E_t,x)$, which is spanned by $\tilde\tau(x) = \tau(x)$. □

## 7. Appendix: Rainwater's lemma and applications

In this appendix we give two technical results used in the previous sections (Lemmas 7.4 and 7.5), which are derived from Rainwater's lemma.

**7.1. Rainwater's Lemma.** (See [27] or [28], Lemma 9.4.3). *Let $X$ be a compact metric space, $\mathscr{F}$ a family of probability measures on $X$ which is convex and weak* compact, and $\mu$ a measure on $X$ which is singular with respect to every $\lambda \in \mathscr{F}$. Then $\mu$ is supported on a Borel set $E$ which is $\lambda$-null for every $\lambda \in \mathscr{F}$.*

For our purposes we need the following variant of Rainwater's lemma:

**7.2. Corollary.** *Let $X$ be a compact metric space and $\mathscr{F}$ a weak* compact family of probability measures on $X$. Then for every measure $\mu$ on $X$ one of the following (mutually incompatible) alternatives holds:*

(i) *$\mu$ is supported on a Borel set $E$ which is $\lambda$-null for every $\lambda \in \mathscr{F}$;*

(ii) *there exists a probability measure $\sigma$ supported on $\mathscr{F}$ and a Borel set $E$ such that the measure*

$$\int_{\lambda\in\mathscr{F}} (1_E\,\lambda)\,d\sigma(\lambda)$$

*(intended as in §2.3) is nontrivial and absolutely continuous w.r.t. $\mu$.*

**Proof.** We denote by $\mathscr{P}(\mathscr{F})$ the space of probability measures on the compact space $\mathscr{F}$, and for every $\sigma \in \mathscr{P}(\mathscr{F})$ we denote by $[\sigma]$ the corresponding average of the elements of $\mathscr{F}$, that is, the measure on $X$ given by

$$[\sigma] := \int_{\lambda\in\mathscr{F}} \lambda\,d\sigma(\lambda)\,.$$

We claim that the class $\mathscr{F}'$ of all $[\sigma]$ with $\sigma \in \mathscr{P}(\mathscr{F})$ is convex and compact (w.r.t. the weak* topology of measures on $X$). Convexity is indeed obvious, and compactness follows from the compactness of the space $\mathscr{P}(\mathscr{F})$ (endowed with the weak* topology of measures on $\mathscr{F}$) and the continuity of the map $\sigma \mapsto [\sigma]$, which in turn follows from the identity $\langle[\sigma]\,;\varphi\rangle = \langle\sigma\,;\hat\varphi\rangle$ where $\varphi$ is any continuous function on $X$ and $\hat\varphi$ is the continuous function on $\mathscr{F}$ defined by $\hat\varphi(\lambda) := \langle\lambda\,;\varphi\rangle$.

There are now two possibilities: either $\mu$ is singular with respect to all measures in $\mathscr{F}'$ or not.

In the first case Rainwater's Lemma (Lemma 7.1) implies that $\mu$ is supported on a set $E$ which is null w.r.t. all measures in $\mathscr{F}'$, and therefore also w.r.t. all measures in $\mathscr{F}$ (because $\mathscr{F}$ is contained in $\mathscr{F}'$). Thus (i) holds.

In the second case there exists $\sigma \in \mathscr{P}(\mathscr{F})$ such that $\mu$ is not singular with respect to $[\sigma]$, and therefore by the Lebesgue-Radon-Nikodým theorem there exists a set $E$ such that the restriction of $[\sigma]$ to $E$ is nontrivial and absolutely continuous w.r.t. $\mu$. Thus (ii) holds with such $\sigma$ and $E$. □

**7.3. Lemma.** *Let $C = C(e, \alpha)$ be a cone in $\mathbb{R}^n$ with axis $e$ and angle $\alpha$ (see §4.11). Then, for every measure $\mu$ on $\mathbb{R}^n$, one of the following (mutually incompatible) alternatives holds:*

(i) *$\mu$ is supported on a Borel set $E$ which is $C$-null (see §4.11);*

(ii) *there exists a nontrivial measure of the form $\mu' = \int_I \mu_t \, dt$ where $\mu'$ is absolutely continuous w.r.t. $\mu$, each $\mu_t$ is the restriction of $\mathscr{H}^1$ to some 1-rectifiable set $E_t$, and*

$$\mathrm{Tan}(E_t, x) \subset [C \cup (-C)] \quad \text{for } \mu_t\text{-a.e. } x \text{ and a.e. } t \in I.$$

**Proof.** The idea is to apply Corollary 7.2 to the measure $\mu$ and a sequence of suitably chosen families $\mathscr{F}_k$ of probability measures.

*Step 1. Construction of the families $\mathscr{F}_k$.*

For every $k = 1, 2, \dots$, we define the following objects:

- $\mathscr{G}_k$ set of all paths $\gamma$ from $[0,1]$ to the closed ball $B_k := B(0,k)$ such that $\mathrm{Lip}(\gamma) \le 1$ and $\dot\gamma(s) \cdot e \ge \cos\alpha$ for $\mathscr{L}^1$-a.e. $s \in [0,1]$;
- $G_\gamma$ $:= \gamma([0,1])$, image of the path $\gamma \in \mathscr{G}_k$;
- $\mu_\gamma$ restriction of $\mathscr{H}^1$ to the curve $G_\gamma$;
- $\lambda_\gamma$ push-forward according $\gamma$ of the Lebesgue measure on $[0,1]$;
- $\mathscr{F}_k$ set of all $\lambda_\gamma$ with $\gamma \in \mathscr{G}_k$.

One easily checks that each $G_\gamma$ is a $C$-curve (see §4.11) contained in $B_k$, and $\lambda_\gamma$ is a probability measure supported on $G_\gamma$ such that

$$\mu_\gamma \le \lambda_\gamma \le \frac{1}{\cos\alpha} \mu_\gamma \, . \tag{7.1}$$

In particular $\mathscr{F}_k$ is a subset of the space $\mathscr{P}(B_k)$ of probability measures on $B_k$.

*Step 2. Each $\mathscr{F}_k$ is a weak* compact subset of $\mathscr{P}(B_k)$.*

This is a consequence of the following statements:

(a) the space $\mathscr{G}_k$ endowed with the supremum distance is compact;

(b) $\mathscr{F}_k$ is the image of $\mathscr{G}_k$ according to the map $\gamma \mapsto \lambda_\gamma$, which is continuous as a map from $\mathscr{G}_k$ to $\mathscr{P}(B_k)$ endowed with the weak* topology.

Statement (a) follows from the well-known compactness of the class of all paths $\gamma : [0,1] \to B_k$ with $\mathrm{Lip}(\gamma) \le 1$ and the fact that we can re-write the constraint $\dot\gamma(s) \cdot e \ge \cos\alpha$ in the form

$$(\gamma(s') - \gamma(s)) \cdot e \ge \cos\alpha \, (s' - s) \quad \text{for every } s, s' \text{ with } 0 \le s \le s' \le 1,$$

which is clearly closed with respect to uniform convergence. To prove statement (b) we observe that for every $\gamma \in \mathscr{G}_k$ and every continuous test function $\varphi : B_k \to \mathbb{R}$ there holds

$$\langle \lambda_\gamma \, ; \varphi \rangle = \int_{B_k} \varphi \, d\lambda_\gamma = \int_{[0,1]} \varphi(\gamma(s)) \, d\mathscr{L}^1(s) \, ,$$

and therefore the function $\gamma \mapsto \langle \lambda_\gamma \, ; \varphi \rangle$ is continuous on $\mathscr{G}_k$.

*Step 3. Completion of the proof.*

Thanks to Step 2, for every $k = 1, 2, \dots$ we can apply Corollary 7.2 to the family $\mathscr{F}_k$ and to the measure $\mu_k$ given by the restriction of $\mu$ to $B_k$. There are then two possibilities: either there exists $k$ such that statement (ii) of Corollary 7.2 holds, or statement (i) of Corollary 7.2 holds for every $k$.

In the first case there exists a probability measure $\sigma$ on the space $\mathscr{G}_k$ and a Borel set $E$ such that the measure

$$\int_{\mathscr{G}_k} (1_E \, \lambda_\gamma) \, d\sigma(\gamma)$$

is nontrivial and absolutely continuous w.r.t. $\mu_k$, and therefore also w.r.t. $\mu$. Then, using (7.1) we obtain that also the measure

$$\mu' := \int_{\mathscr{G}_k} (1_E \, \mu_\gamma) \, d\sigma(\gamma)$$

is nontrivial and absolutely continuous w.r.t. $\mu$, and since each measure $1_E \, \mu_\gamma$ is the restriction of $\mathscr{H}^1$ to a subset of the $C$-curve $G_\gamma$, we have that $\mu'$ satisfies all the requirements in statement (ii), which therefore holds true.

In the second case we obtain that for every $k$ the measure $\mu_k$ is supported on a set $E_k$ contained in $B_k$ which is null w.r.t. all measures in $\mathscr{F}_k$, and using the first inequality in (7.1) we obtain that

$$\mathscr{H}^1(E_k \cap G_\gamma) = 0 \quad \text{for every } \gamma \in \mathscr{G}_k. \tag{7.2}$$

Now we notice that intersection of every $C$-curve $G$ with the ball $B_k$ is contained in a curve $G_\gamma$ with $\gamma \in \mathscr{G}_k$ and therefore (7.2) implies $\mathscr{H}^1(E_k \cap G) = 0$. We have thus proved that $E_k$ is $C$-null.

We then let $E$ be the union of all $E_k$ and observe that $E$ is $C$-null, too, and $\mu$ is supported on $E$. Thus (i) holds. □

**7.4. Lemma.** *For every measure $\mu$ on $\mathbb{R}^n$ one of the following (mutually incompatible) alternatives holds:*

(i) *$\mu$ is supported on a purely unrectifiable set $E$ (see §2.2);*

(ii) *there exists a nontrivial measure of the form $\mu' = \int_I \mu_t \, dt$ where $\mu'$ is absolutely continuous w.r.t. $\mu$ and each $\mu_t$ is the restriction of $\mathscr{H}^1$ to some 1-rectifiable set $E_t$.*

**Proof.** We choose finitely many cones $C_i$, the interiors of which cover $\mathbb{R}^n \setminus \{0\}$, and then apply Lemma 7.3 to $\mu$ and to each $C_i$. There are now two possibilities: either there exists $i$ such that statement (ii) of Lemma 7.3 holds, or statement (i) of Lemma 7.3 holds for every $i$.

In the first case we immediately obtain that statement (ii) holds. In the second case, for every $i$ there exists a set $E_i$ which supports $\mu$ and is $C_i$-null. We then let $E$ be the intersection of all $E_i$ and claim that $E$ satisfies the requirements in statement (i), which therefore holds true.

It is indeed obvious that $E$ supports $\mu$. Concerning the unrectifiability of $E$, note that since the interiors of the cones $C_i$ cover $\mathbb{R}^n \setminus \{0\}$, we can cover every curve $G$ of class $C^1$ in $\mathbb{R}^n$ by countably many sub-arcs $G_j$, each one contained in a $C_i$-curve for some $i$. Therefore $\mathscr{H}^1(E \cap G_j) = 0$ because $E$ is $C_i$-null. Hence $\mathscr{H}^1(E \cap G) = 0$, and we have proved that $E$ is purely unrectifiable. $\square$

**7.5. Lemma.** *Let be given a Borel set $F$ in $\mathbb{R}^n$, a cone $C = C(e, \alpha)$ in $\mathbb{R}^n$, and a measure $\mu$ such that*

$$V(\mu, x) \cap C = \{0\} \quad \text{for } \mu\text{-a.e. } x \in F.$$

*Then there exists a $C$-null Borel set $F'$ contained in $F$ such that $\mu(F \setminus F') = 0$.*

**Proof.** Let $\tilde\mu$ be the restriction of $\mu$ to the set $F$; thus $V(\tilde\mu, x) = V(\mu, x)$ for $\tilde\mu$-a.e. $x$ by Proposition 2.9(i), and in particular

$$V(\tilde\mu, x) \cap C = \{0\} \quad \text{for } \tilde\mu\text{-a.e. } x. \tag{7.3}$$

We must prove that $\tilde\mu$ is supported on a $C$-null set. To this end we apply Lemma 7.3 (to the measure $\tilde\mu$ and the cone $C$) and show that, of the two alternatives given in that statement, only (i) is viable. Indeed the definition of the decomposability bundle in §2.6 and (7.3) imply that for every family $\{\mu_t : t \in I\}$ in $\mathscr{F}_{\tilde\mu}$ there holds $\mathrm{Tan}(E_t, x) \cap C = \{0\}$ for $\mu_t$-a.e. $x$ and a.e. $t$, and this contradicts alternative (ii). $\square$

## 8. Appendix: approximation of Lipschitz functions

In this appendix we prove two approximation results for Lipschitz functions used in the previous sections, namely Corollary 8.3 (obtained as a consequence of Proposition 8.1) and Proposition 8.4.

**8.1. Proposition.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$, $\mu$ a measure on $\mathbb{R}^n$, and $V : \mathbb{R}^n \to \mathrm{Gr}(\mathbb{R}^n)$ a Borel map such that $V(x) \in \mathscr{D}(f, x)$ for $\mu$-a.e. $x$ (see §3.1). Then for every $\varepsilon > 0$ there exist a compact set $K$ in $\mathbb{R}^n$ and a function $g : \mathbb{R}^n \to \mathbb{R}$ of class $C^1$ such that:*

(i) $\mu(\mathbb{R}^n \setminus K) \le \varepsilon$;
(ii) $\|g - f\|_\infty \le \varepsilon$;
(iii) $\mathrm{Lip}(g) \le \mathrm{Lip}(f) + \varepsilon$;
(iv) $|d_V g(x) - d_V f(x)| \le \varepsilon$ *for every* $x \in K$.



**8.2. Remark.** In the special case where $V(x)$ does not depend on $x$ we can simply take $g := f * \rho$ with a suitable asymmetric mollifier $\rho$. For example, when $n = 2$ and $V$ is the line $\mathbb{R} \times \{0\}$, it suffices to take as $\rho$ the characteristic function of the rectangle $[-r, r] \times [-r^2, r^2]$, renormalized so to have integral equal to 1, and $r$ sufficiently small. This is the idea behind Step 5 in the proof below.

**Proof of Proposition 8.1.** We set $L := \mathrm{Lip}(f)$ and denote by $E$ be the set of all $x \in \mathbb{R}^n$ such that $V(x) \in \mathscr{D}(f, x)$. Then it follows from Lemma 3.5 that $E$ is a Borel set.

For every $x$ in $E$ we extend the linear function $d_V f(x)$ to a linear function $\alpha(x)$ on $\mathbb{R}^n$ by setting $\alpha(x)\, h := 0$ for every $h \in V(x)^\perp$; thus $|\alpha(x)| = |d_V f(x)| \le L$. Note that the map $x \mapsto \alpha(x)$ is a Borel measurable map from $E$ to the dual of $\mathbb{R}^n$ by Lemma 3.6.

The rest of the proof is divided in several steps.

*Step 1. There exist $\delta > 0$ and finitely many pairwise disjoint compact sets $K_i$ with the following properties:*

(a) $\mu(\mathbb{R}^n \setminus K) \le \varepsilon$ *where $K$ is the union of all $K_i$ (thus statement (i) holds);*

*and for every $i$,*

(b) $d_{\mathrm{gr}}(V(x), V(x')) \le \varepsilon / L$ *for every* $x, x' \in K_i$;
(c) $|\alpha(x) - \alpha(x')| \le \varepsilon$ *for every* $x, x' \in K_i$;
(d) $m(f, x, V(x), \alpha(x), \delta) \le \varepsilon$ *for every* $x \in K_i$ *(see §3.3).*

For every $x \in E$ the function $f$ is differentiable w.r.t. $V(x)$ with derivative $\alpha(x)$, and therefore there exists $\delta > 0$, depending on $x$, such that the estimate in (d) holds (cf. §3.3). Since moreover $\mu(\mathbb{R}^n \setminus E) = 0$, we can find a subset $E'$ of $E$ such that $\mu(\mathbb{R}^n \setminus E') \le \varepsilon/2$ and the estimate in (d) holds with the same $\delta$ for all $x \in E'$. This value of $\delta$ is the one we choose.

Next we partition $E'$ into finite a number $N$ of Borel sets $E_i$ such that the oscillations of the maps $x \mapsto V(x)$ and $x \mapsto \alpha(x)$ on each $E_i$ are less that $\varepsilon/L$ and $\varepsilon$, respectively. Finally for every $i$ we take a compact set $K_i$ contained in $E_i$ such that $\mu(E_i \setminus K_i) \le \varepsilon/(2N)$. It is now easy to check that statements (a–d) hold.

*Step 2. For every $i$ we choose $x_i \in K_i$ and set $V_i := V(x_i)$ and $\alpha_i := \alpha(x_i)$. Then for every $x \in K_i$ there holds $m(f, x, V_i, \alpha_i, \delta) \le 4\varepsilon$.*

We obtain this estimate by applying Lemma 3.4 together with the estimates in statements (b), (c) and (d) and the fact that $|\alpha(x)| \le L$.

*Step 3. Given $h \in \mathbb{R}^n$ and an index $i$, we write $h = h' + h''$ with $h' \in V_i$ and $h'' \in V_i^\perp$. If $|h'| \le \delta$ then for every $x \in K_i$ there holds*

$$|f(x + h) - f(x) - \alpha_i h'| \le 4\varepsilon |h'| + L|h''|\,. \tag{8.1}$$

The estimate in Step 2 yields $|f(x + h') - f(x) - \alpha_i h'| \le 4\varepsilon |h'|$, and using that $|f(x + h) - f(x + h')| \le L|h''|$ we obtain (8.1).

*Step 4. Let $\rho$ be a positive function on $\mathbb{R}^n$ with integral 1 and support contained in the ball $B(0, r)$. Then $f * \rho$ is a function of class $C^1$ that satisfies*

(e) $\|f - f * \rho\|_\infty \le Lr$;

(f) $\|d(f*\rho)\|_\infty \le L$.

Statement (e) is obtained by a simple computation taking into account that $f$ has Lipschitz constant $L$ and that the support of $\rho$ is contained in $B(0,r)$.

The distributional derivative $d(f*\rho) = df*\rho$, being the convolution of an $L^\infty$ and an $L^1$ function, is bounded and continuous, which means that $f*\rho$ is of class $C^1$ and has bounded derivative. Moreover $\|d(f*\rho)\|_\infty \le \|df\|_\infty\|\rho\|_1 = L$, and statement (f) is proved.

*Step 5. For every $i$ and every $r>0$ there exists a positive function $\rho_i$ with integral 1 and support contained in $B(0,r)$ such that $f_i := f*\rho_i$ satisfies the following property: for every $x\in K_i$ the restriction of the linear function $df_i(x)-\alpha_i$ to the subspace $V_i$ has norm at most $M\varepsilon$, where the constant $M$ depends only on $n$.*

We assume that $k := \dim(V_i) > 0$, otherwise there is nothing to prove. We then take $r'>0$ and denote by $B'$ the ball with center 0 and radius $r'$ contained in $V_i$, and by $B''$ the ball with center 0 and radius $r'' := \varepsilon r'/L$ contained in $V_i^\perp$. We then identify $\mathbb{R}^n$ with the product $V_i\times V_i^\perp$ and set

$$\rho_i := c\,1_{B'\times B''} \quad \text{with} \quad c := \frac{1}{\mathscr{L}^k(B')\,\mathscr{L}^{n-k}(B'')}\,.$$

We claim that if $r'\le\delta/2$ then $f_i := f*\rho_i$ satisfies

$$\big|f_i(x+h)-f_i(x)-\alpha_i h\big| \le M\varepsilon|h| \tag{8.2}$$

for every $x\in K_i$, every $h\in V_i$ with $|h|\le r'$, and a suitable $M$. This inequality shows that $f_i$ has the property required in Step 5.

We fix $x$ and $h$ as above. A simple computation yields

$$f_i(x+h)-f_i(x)-\alpha_i h = \int_{\mathbb{R}^n} e(z)\,(\rho_i(h-z)-\rho_i(-z))\,d\mathscr{L}^n(z)\,, \tag{8.3}$$

where $e(z) := f(x+z)-f(x)-\alpha_i z$ for every $z\in\mathbb{R}^n$. We observe now that estimate (8.1) yields

$$|e(z)| \le 4\varepsilon|z'| + L|z''| \tag{8.4}$$

for every $z\in\mathbb{R}^n$ such that $|z'|\le\delta$, where $z'$ and $z''$ come from the decomposition $z = z'+z''$ with $z'\in V_i$ and $z''\in V_i^\perp$ (cf. Step 3). Therefore, in order to use (8.4) to estimate the integral in (8.3), we must check that $|z'|\le\delta$ for every $z$ such that $\rho_i(h-z)-\rho_i(-z)\ne 0$. Indeed, taking into account the definition of $\rho_i$ and the fact that $h$ belongs to $V_i$, we obtain

$$\rho_i(h-z)-\rho_i(-z) = \begin{cases} \pm c & \text{if } z'\in(B'+h)\triangle B' \text{ and } z''\in B'', \\ 0 & \text{otherwise,}\end{cases} \tag{8.5}$$

and therefore if $\rho_i(h-z)-\rho_i(-z)\ne 0$ then $z'$ belongs to the symmetric difference $(B'+h)\triangle B'$; in particular $|z'|\le r'+|h|\le 2r'\le\delta$, as required.



Then, denoting by $c_k$ the volume of the unit ball in $\mathbb{R}^k$, we obtain

$$\begin{aligned} \big|f_i(x+h) &- f_i(x)-\alpha_i h\big| \\ &\le \int_{\mathbb{R}^n} \big[4\varepsilon|z'|+L|z''|\big]\,\big|\rho_i(h-z)-\rho_i(-z)\big|\,d\mathscr{L}^n(z) \\ &\le \big[8\varepsilon r'+Lr''\big]\int_{\mathbb{R}^n} |\rho_i(h-z)-\rho_i(-z)|\,d\mathscr{L}^n(z) \\ &\le 9\varepsilon r'\,\frac{\mathscr{L}^k\big((B'+h)\triangle B'\big)}{\mathscr{L}^k(B')} \le \frac{18\,c_{k-1}}{c_k}\varepsilon|h|\,, \end{aligned}$$

where the first inequality follows from (8.3) and (8.4), for the second we use that $|z'|\le 2r'$ and $|z''|\le r''$ whenever $\rho_i(h-z)-\rho_i(-z)\ne 0$ (cf. (8.5)), for the third one we use that $r'' = \varepsilon r'/L$, formula (8.5), and the definition of $c$, and finally for the fourth inequality we use that the volume of $B'$ is $c_k(r')^k$ and the volume of $(B'+h)\triangle B'$ is at most $2c_{k-1}(r')^{k-1}|h|$.

We have thus proved (8.2) with $M$ equal to the maximum of $18\,c_{k-1}/c_k$ over all $k=1,\dots,n$.

*Step 6. Take $M$ and $f_i$ as in Step 5. Then for every $x\in K_i$ there holds*

$$|d_V f_i(x) - d_V f(x)| \le (M+3)\varepsilon\,. \tag{8.6}$$

Taking into account §3.3 and the fact that $d_V f(x)$ agrees with $\alpha(x)$ on $V(x)$ we rewrite claim (8.6) as

$$m(df_i(x),0,V(x),\alpha(x),1) \le (M+3)\varepsilon\,, \tag{8.7}$$

and the estimate in Step 5 as

$$m(df_i(x),0,V_i,\alpha_i,1) \le M\varepsilon\,. \tag{8.8}$$

We then derive (8.7) from (8.8) by applying Lemma 3.4 together with the following estimates: $d_{\mathrm{gr}}(V(x),V_i)\le\varepsilon/L$ (statement (b)), $|\alpha(x)-\alpha_i|\le\varepsilon$ (statement (c)), and $|df_i(x)|$, $|\alpha_i|\le L$.

*Step 7. Construction of the function $g$.*

Since the sets $K_i$ are compact and pairwise disjoint we can find smooth functions $\sigma_i:\mathbb{R}^n\to[0,1]$ such that $\sum_i\sigma_i(x)=1$ for every $x\in\mathbb{R}^n$ (thus $\{\sigma_i\}$ is a smooth partition of unity of $\mathbb{R}^n$) and each $\sigma_i$ is constant outside some compact set and takes value 1 on $K_i$. Thus the derivatives $d\sigma_i$ have compact support and therefore are bounded, and

$$m := \max\Big\{1;\,\sum_i\|d\sigma_i\|_\infty\Big\} < +\infty\,.$$

Now we take $f_i = f*\rho_i$ as in Step 5, where $\rho_i$ supported in the ball $B(0,r)$ with $r := \varepsilon/(mL)$, and set

$$g := \sum_i \sigma_i f_i\,.$$

The function $g$ is clearly of class $C^1$. We prove next that $g$ satisfies statements (ii), (iii) and (iv).

Note that statement (e) and the choice of $r$ and $m$ yield

$$|f_i(x) - f(x)| \le Lr = \frac{\varepsilon}{m} \le \varepsilon \quad \text{for every } x \in \mathbb{R}^d, \tag{8.9}$$

and since the number $g(x)$ is a convex combination of the numbers $f_i(x)$, it satisfies $|g(x) - f(x)| \le \varepsilon$ as well, which proves statement (ii).

Given $x \in K$, take $i$ such that $x \in K_i$, and note that $g = f_i$ on the neighbourhood of $K_i$ where $\sigma_i = 1$; hence (8.6) becomes $|d_V g(x) - d_V f(x)| \le (M+3)\varepsilon$, which is the inequality in statement (iv) with $(M+3)\varepsilon$ instead of $\varepsilon$...

It remains to prove statement (iii). By deriving the identity $\sum_i \sigma_i(x) = 1$ we obtain that $\sum_i d\sigma_i(x) = 0$, and then

$$dg(x) = \sum_i \sigma_i(x)\, df_i(x) + \sum_i (f_i(x) - f(x))\, d\sigma_i(x)\,.$$

Using this identity together with the estimates $|df_i(x)| \le L$ (see statement (f)) and $|f_i(x) - f(x)| \le \varepsilon/m$ (see (8.9)), and the fact that $\sum_i |d\sigma_i(x)| \le m$ by the choice of $m$, we finally obtain that $|dg(x)| \le L+\varepsilon$ for every $x$, which concludes the proof. □

**8.3. Corollary.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$, $\mu$ a measure on $\mathbb{R}^n$, and $x \mapsto V(x)$ a Borel map from $\mathbb{R}^n$ to $\mathrm{Gr}(\mathbb{R}^n)$ such that $V(x) \in \mathscr{D}(f,x)$ for $\mu$-a.e. $x$.*

*Then there exists a sequence of smooth functions $f_j : \mathbb{R}^n \to \mathbb{R}$ such that the following statements hold (as $j \to +\infty$):*

(i) *the functions $f_j$ converge to $f$ uniformly;*
(ii) $\mathrm{Lip}(f_j)$ *converge to* $\mathrm{Lip}(f)$*;*
(iii) $d_V f_j(x) \to d_V f(x)$ *for $\mu$-a.e. $x$, where convergence is intended in the sense of the operator norm for linear functions on $V$.*

**Proof.** We first construct a sequence of approximating functions $f_n$ of class $C^1$ that satisfy requirements (i), (ii) and (iii) using Proposition 8.1, and then regularize these functions by convolution. □

**8.4. Proposition.** *Let $f$ be a Lipschitz function on $\mathbb{R}^n$, $K$ a compact set in $\mathbb{R}^n$, and $\phi$ an increasing, strictly positive function on $(0,+\infty)$. Then for every $\varepsilon > 0$ there exists a Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

(i) *$g$ agrees with $f$ on $K$ and is smooth in $\mathbb{R}^n \setminus K$;*
(ii) $|g(x) - f(x)| \le \phi(\mathrm{dist}(x,K))$ *for every $x \in \mathbb{R}^n$;*
(iii) $\mathrm{Lip}(g) \le \mathrm{Lip}(f) + \varepsilon$.

**Proof.** We let $L := \mathrm{Lip}(f)$ and for every $k = 1, 2, \dots$ we set

$$A_k := \Big\{ x \in \mathbb{R}^n : \ \frac{1}{k+1} < \mathrm{dist}(x,K) < \frac{1}{k-1} \Big\}$$

(here we adopt the convention $1/0 = +\infty$).

Then $\{A_k\}$ is an open cover of the open set $A := \mathbb{R}^n \setminus K$, and we take smooth functions $\sigma_k : A \to [0,1]$ which form a partition of unity of $A$ subject to this cover (that is, the support of each $\sigma_k$ is contained in $A_k$ and $\sum_k \sigma_k(x) = 1$ for every $x \in A$). Note that each $\sigma_k$ has compact support in $A$.



Next we choose a decreasing sequence of positive real numbers $r_k$ such that for every $k$ there holds

$$L\, \|d\sigma_k\|_\infty r_k \le 2^{-k}\varepsilon \quad \text{and} \quad Lr_k \le \phi\Big(\frac{1}{k+1}\Big)\,, \tag{8.10}$$

and a sequence of positive smooth mollifiers $\rho_k$ with support contained in the ball $B(0, r_k)$. Finally we set

$$g := f + \sum_{k=1}^{+\infty} \sigma_k (f * \rho_k - f)\,. \tag{8.11}$$

To prove statement (i) note first that $g$ agrees with $f$ on $K$ because $\sigma_k(x) = 0$ for every $x \in K$ and every $k$ (because $x$ does not belong to $A_k$). To see that $g$ is well-defined and smooth on the open set $A$ we rewrite it as

$$g = \sum_{k=1}^{+\infty} \sigma_k (f * \rho_k)\,,$$

and note that the functions in the sum are smooth, and the sum is locally finite (more precisely, $\sigma_k$ vanish on $A_h$ for all $k$ except $k = h-1, h, h+1$).

Let us prove statement (ii). Since the support of $\rho_k$ is contained in the ball $B(0, r_k)$ and $f$ has Lipschitz constant $L$, a simple computation shows that for every $x \in \mathbb{R}^n$ there holds

$$|f * \rho_k(x) - f(x)| \le Lr_k\,. \tag{8.12}$$

Therefore, given $x \in A$ and denoting by $k(x)$ the smallest $k$ such that $x \in A_k$, we have

$$\begin{aligned} |g(x) - f(x)| &\le \sum_{k \ge k(x)} \sigma_k(x)\, |f * \rho_k(x) - f(x)| \\ &\le \sum_{k \ge k(x)} \sigma_k(x)\, Lr_k \\ &\le Lr_{k(x)} \le \phi\Big(\frac{1}{k(x)+1}\Big) \le \phi(\mathrm{dist}(x,K)) \end{aligned}$$

(for the first inequality we use that $\sigma_k(x) = 0$ for $k < k(x)$ because $x \notin A_k$; the second inequality follows from (8.12); the third one follows from the fact that the sum of all $\sigma_k(x)$ is 1 and $r_k(x) \ge r_k$ for every $k \ge k(x)$; the fourth one follows from the second inequality in (8.10), the fifth one from the fact that $x$ belongs to $A_{k(x)}$ and from the definition of the sets $A_k$).

We conclude the proof by showing that $g$ is Lipschitz and satisfies statement (iii). For every $h = 1, 2, \dots$ set

$$g_h := f + \sum_{k=1}^{h} \sigma_k (f * \rho_k - f)\,.$$

Since the functions $g_h$ are Lipschitz and converge pointwise to $g$ as $h \to +\infty$, it suffices to show that $\mathrm{Lip}(g_h) \le L + \varepsilon$ for every $h$, or equivalently that the distributional derivatives $dg_h$ satisfies

$$\|dg_h\|_\infty \le L + \varepsilon\,. \tag{8.13}$$

Let $h$ be fixed for the rest of the proof. We can write $g_h$ as

$$g_h = \sum_{k=0}^{h} \sigma_k f_k\,,$$

where $\sigma_0 := 1 - (\sigma_1 + \cdots + \sigma_h)$, $f_0 := f$, and $f_k := f * \rho_k$ for $0 < k \le h$.

Since $\sigma_0 + \cdots + \sigma_h = 1$ we have that $d\sigma_0 + \cdots + d\sigma_h = 0$, and then

$$dg_h = \sum_{k=0}^{h} \sigma_k\, df_k + \sum_{k=1}^{h} (f_k - f)\, d\sigma_k\,. \tag{8.14}$$

Observe now that $df_k = df * \rho_k$ where $df$ is the distributional derivative of $f$, and then $\|df_k\|_\infty \le \|df\|_\infty \|\rho_k\|_1 \le L$; hence the first sum in line (8.14) is a (pointwise) convex combinations of functions with $L^\infty$-norm at most $L$, and therefore its $L^\infty$-norm is at most $L$ as well. Thus it remains to show that the $L^\infty$-norm of the second sum in line (8.14) is at most $\varepsilon$, and indeed

$$\Big\| \sum_{k=1}^{h} (f_k - f)\, d\sigma_k \Big\|_\infty \le \sum_{k=1}^{h} \|f_k - f\|_\infty \, \|d\sigma_k\|_\infty \le \sum_{k=1}^{h} L r_k \|d\sigma_k\|_\infty \le \varepsilon\,,$$

where the second inequality follows from the fact that $f_k = f * \rho_k$ and (8.12), and the last inequality follows from the first inequality in (8.10). $\square$

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