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On some geometric properties of currents and Frobenius theorem

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Abstract. In this note we announce some results, due to appear in [2], [3], on the structure of integral and normal currents, and their relation to Frobenius theorem. In particular we show that an integral current cannot be tangent to a distribution of planes which is nowhere involutive (Theorem 3.6), and that a normal current which is tangent to an involutive distribution of planes can be locally foliated in terms of integral currents (Theorem 4.3). This statement gives a partial answer to a question raised by Frank Morgan in [1].

Keywords: non-involutive distributions, Frobenius theorem, Sobolev surfaces, integral currents, normal currents, foliations, decomposition of normal currents.

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1. Introduction

Consider a distribution of k-dimensional planes in $\mathbb{R}^n$, namely a map that associates to each point $x \in \mathbb{R}^n$ a k-dimensional subspace $V(x)$ of $\mathbb{R}^n$, and assume that V is spanned by vectorfields $v_1, \dots, v_k$ of class C^1. We say that V is *involutive* at a point $x \in \mathbb{R}^n$ if the commutators of the vectorfields $v_1, \dots, v_k$, evaluated at x, belong to $V(x)$ (see §2.1). Moreover, given a k-dimensional surface S in $\mathbb{R}^n$, we say that S is *tangent* to V if the tangent space $\mathrm{Tan}(S, x)$ agrees with $V(x)$ for every $x \in S$.

In this context, the first part of Frobenius theorem states that, if S is tangent to V, then V must be involutive at every point of S. Or, in a slightly weaker form, that if V is nowhere involutive then there exist no tangent surfaces (cf. [8], Theorem 14.5).

The classical version of this theorem requires that the surface S is at least of class C^1, and it is then natural to ask if similar statements hold for weaker notions of surface. To this regard, we mention that a positive answer for *Sobolev surfaces*, that is, Sobolev images of open subsets of $\mathbb{R}^{2h}$, has been given in [9], Theorem 1.2, when V is the distribution of $2h$-planes in $\mathbb{R}^{n=2h+1}$ corresponding to the horizontal distribution in the sub-Riemannian Heisenberg group $\mathbb{H}^h$.

In section 3 we give a positive answer for *integral currents*,[1] and more precisely we show that given an integral k-dimensional current T which is tangent to V, then V must be involutive on the support of T (Theorem 3.6). Note that the assumption that T is integral is crucial, and indeed the analogous statement for rectifiable sets does not hold, cf. Remark 3.7(a).

It turns out that Theorem 3.6 is an immediate consequence of the following geometric property of the boundary of integral currents, which is actually the heart of

[1] The basic definitions and terminology concerning currents are recalled in Section 2.

the matter: if T is an integral k-dimensional current tangent to a continuous distribution of k-planes V, then ∂T is tangent to V as well (see §2.5 for the definition of tangency, and Theorems 3.1 and 3.2).

In Section 4 we turn to the other part of Frobenius theorem, which states that if V is everywhere involutive, then $\mathbb{R}^n$ can be locally foliated with k-dimensional surfaces which are tangent to V. In Theorem 4.3 we prove the following generalization: if V is everywhere involutive and T is a k-dimensional *normal current* tangent to V, then T can be locally foliated by a family k-dimensional integral currents tangent to V (the definition of foliation, or mass decomposition, of a current is given in §4.1). Conversely, if T can be foliated then V must be involutive at every point in the support of T.

The first part of Theorem 4.3 gives a partial positive answer to a question raised by Frank Morgan in [1], namely if every normal current admits a foliation in terms of integral currents (other positive results were given in [6], [10], [12], see Remark 4.4). On the other hand, the second part shows that a normal current which is tangent to a nowhere involutive distribution of planes admits no foliation of a certain type: this result was first stated in [13], but in a form which is not correct (see Remark 4.4(c) for more details).

2. Notation

In this section we briefly recall some notation and basic definitions. For rectifiable sets and currents we essentially follow [7]. As usual, $\mathscr{H}^k$ stands for the k-dimensional Hausdorff measure and $\mathscr{L}^n$ for the Lebesgue measure on $\mathbb{R}^n$.

In the following we fix an open set Ω in $\mathbb{R}^n$.

2.1. The vectorfield v and the distribution of planes V. In the following we consider $v_1, \dots, v_k$ continuous vectorfields on Ω with $0 < k < n$, and the simple k-vectorfield

$$v := v_1 \wedge \cdots \wedge v_k \,.$$

Moreover we assume that v is unitary, that is, $|v(x)| = 1$ for every $x \in \Omega$, and we denote by V the distribution of k-planes spanned by v, that is,

$$V(x) := \mathrm{span}(v(x)) := \mathrm{span}\big\{v_1(x), \dots, v_k(x)\big\} \quad \text{for every } x \in \Omega.$$

With a slight abuse of language, we say that V (or v) is of class C^1 to mean that $v_1, \dots, v_k$ are of class C^1, and if this is the case we say that V is *involutive* at a point $x \in \Omega$ if

$$[v_i, v_j](x) \in V(x) \quad \text{for every } 1 \le i, j \le k,$$

where $[v_i, v_j]$ is the Lie bracket, or commutator, of v_i and v_j.[2]

[2] That is, the vectorfield defined by

$$[v_i, v_j](x) := \big\langle \nabla v_j(x); v_i(x) \big\rangle - \big\langle \nabla v_i(x); v_j(x) \big\rangle = \frac{\partial v_j}{\partial v_i}(x) - \frac{\partial v_i}{\partial v_j}(x) \,,$$

where $\langle \; ; \; \rangle$ denotes the usual pairing of matrices and vectors.

2.2. Rectifiable sets, orientation. A set Σ in Ω is *rectifiable* of dimension k, or k-rectifiable, if it has finite $\mathscr{H}^k$ measure and can be covered, except for an $\mathscr{H}^k$-null subset, by countably many surfaces of dimension k and class C^1.[3]

Then at $\mathscr{H}^k$-a.e. $x \in \Sigma$ there exists an *approximate tangent space* $\mathrm{Tan}(\Sigma, x)$, which is characterized (for $\mathscr{H}^k$-a.e. $x \in \Sigma$) by the following property: for every k-surface S of class C^1 there holds

$$\mathrm{Tan}(\Sigma, x) = \mathrm{Tan}(S, x) \quad \text{for } \mathscr{H}^k\text{-a.e. } x \in \Sigma \cap S.$$

An *orientation* of Σ is a simple k-vectorfield τ defined on Σ such that $\tau(x)$ spans $\mathrm{Tan}(\Sigma, x)$ and has norm 1 for $\mathscr{H}^k$-a.e. $x \in \Sigma$.

2.3. Currents, boundary, mass, normal currents. A k-dimensional current, or k-current, in Ω is a (continuous) linear functional on the space of smooth k-forms with compact support on Ω. The *boundary* of a k-current T is the $(k-1)$-current ∂T defined by $\langle \partial T; \omega \rangle := \langle T; \mathrm{d}\omega \rangle$, where $\mathrm{d}\omega$ is the exterior differential of the form ω.

The *mass* of T, denoted by $\mathbb{M}(T)$, is the supremum of $\langle T; \omega \rangle$ over all forms ω such that $|\omega(x)| \leq 1$ for every x. A current T with finite mass can be represented as a vector measure, that is, there exist a positive finite measure μ on Ω and a map τ from Ω to the set of k-vectors with norm 1, called *orientation*, such that

$$\langle T; \omega \rangle := \int_\Omega \big\langle \tau(x); \omega(x) \big\rangle \, d\mu(x) \, ,$$

where $\langle \ ; \ \rangle$ is the usual pairing of k-vectors and k-covectors. In this case we simply write $T = \tau\mu$. Note that the mass of T is $\mathbb{M}(T) = \mu(\Omega) = \|\mu\|$.

A current T is called *normal* if both T and ∂T have finite mass.

2.4. Rectifiable and integral currents. A k-current T is called *rectifiable* (with integral multiplicity) if there exist a k-rectifiable set Σ, an orientation τ of Σ, and a *positive*, integer-valued multiplicity $\theta \in L^1(\Sigma, \mathscr{H}^k)$ such that

$$\langle T; \omega \rangle := \int_\Sigma \big\langle \tau(x); \omega(x) \big\rangle \, \theta(x) \, d\mathscr{H}^k(x) \, .$$

In this case we write $T = [\![\Sigma, \tau, \theta]\!]$. Note that the mass of T agrees with the k-dimensional measure of Σ counted with multiplicity, that is, $\mathbb{M}(T) = \int_\Sigma \theta \, d\mathscr{H}^k$; in particular $\mathbb{M}(T)$ is finite.

A current T is called *integral* if both T and ∂T are rectifiable; in particular every integral current is normal.

2.5. Notions of tangency. Take v and V as in §2.1 We say that an h-rectifiable set Σ with $h \leq k$ is tangent to V if the tangent space $\mathrm{Tan}(\Sigma, x)$ is contained in $V(x)$ for $\mathscr{H}^h$-a.e. $x \in \Sigma$.

Accordingly, a rectifiable h-current $T = [\![\Sigma, \tau, \theta]\!]$ is tangent to V if the supporting rectifiable set Σ is so. More generally, an h-current with finite mass $T = \tau\mu$ is tangent to V if the span of the h-vector $\tau(x)$ is contained in $V(x)$ for μ-a.e. x.[4]

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[3] Through this paper sets, maps and vectorfields are always (at least) Borel measurable.

[4] The span of a h-vector w in $\mathbb{R}^n$ is defined as the smallest subspace W of $\mathbb{R}^n$ such that w is also a h-vector in W. If w is a simple vector we recover the usual definition.

Moreover we say that a rectifiable k-current $T = [\![\Sigma, \tau, \theta]\!]$ is *oriented* by v if $\tau(x) = v(x)$ for $\mathscr{H}^k$-a.e. $x \in \Sigma$, and more generally, a k-current with finite mass $T = \tau\mu$ is oriented by v if $\tau(x) = v(x)$ for μ-a.e. x.

Remark 2.6. If $T = \tau\mu$ is a k-current with finite mass, then T is tangent to V if and only if $\tau(x) = \pm v(x)$ for μ-a.e. x (recall that v is unitary). In particular if T is oriented by v then it is also tangent to V, but clearly the converse does not hold.

3. Geometric structure of the boundary

Through this section, v and V are taken as in §2.1.

The next two statements are the main results in this section, and establish a natural (and apparently obvious) relation between the tangent space of a current T and the tangent space of the boundary ∂T, namely that, under suitable assumptions, the former contains the latter.

Theorem 3.1. (See [2].) *If T is an integral k-current oriented by v, then the boundary ∂T is tangent to V.*

Theorem 3.2. (See [3].) *If V is of class C^1 and T is an integral k-current tangent to V, then ∂T is tangent to V.*

Remark 3.3. (a) Theorem 3.2 can be viewed as the "non-oriented version" of Theorem 3.1, and under the assumption that V is of class C^1 it is actually a stronger statement (cf. Remark 2.6).

(b) Theorem 3.1 can be proved in a slightly stronger form, and under slightly weaker assumptions on the current T (see [2] for more details); the key step of the proof consists in taking the blow-up of T at "almost every point of the boundary", and here the assumption that T is integral (or slightly less) plays an essential role.

(c) Theorem 3.2 can be proved under much weaker assumptions on the current T, including the case where T is normal and ∂T is singular with respect to T.[5] The proof is completely different from that of Theorem 3.1, and relies heavily on the fact that V is of class C^1 (see [3]). Note that this regularity assumption on V can perhaps be weakened, but cannot be entirely dropped: indeed in [2] we construct a continuous distribution V of 2-planes in $\mathbb{R}^3$ and an integral 2-current T such that T is tangent to V but ∂T is not.[6]

(d) In [3] we also show that if V is everywhere involutive then Theorem 3.2 holds for every normal k-current T. This is no longer true if V is not everywhere involutive, the counterexample being any current on Ω of the form $T := v\mu$ where $\mu := \rho\mathscr{L}^n$ and ρ is a function of class C^1 whose support is compact and contained in the (open) set of all points where v is not involutive.

The relation between the geometric property of the boundary of T proved in Theorem 3.2 and Frobenius theorem is made clear in the following statement.

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[5] Here both T and ∂T are viewed as (vector-valued) measures.

[6] This current is actually (supported on) the graph of a continuous Sobolev function.

Proposition 3.4. *Assume that V is of class C^1, and let T be a normal k-current tangent to V such that ∂T is also tangent to V. Then V is involutive at every point of the support of T.*[7]

This result is an immediate consequence of the following lemma:

Lemma 3.5. (See [2].) *If V is of class C^1 and is not involutive at a point $x_0 \in \Omega$, then there exists a $(k-1)$-form α of class C^1 on Ω such that*

(i) *for every $x \in \mathbb{R}^n$ the restriction of $\alpha(x)$ to $V(x)$ is zero;*[8]

(ii) $\langle v(x_0); \mathrm{d}\alpha(x_0)\rangle \neq 0$.

Proof of Proposition 3.4. We write $T = \tau\mu$, and we assume by contradiction that there exists a point x_0 in the support of μ where v is not involutive.

We take α as in Lemma 3.5. Then for every smooth function φ with compact support on Ω there holds

$$\begin{aligned} 0 = \langle \partial T; \varphi\alpha\rangle = \langle T; \mathrm{d}(\varphi\alpha)\rangle &= \langle T; \mathrm{d}\varphi \wedge \alpha\rangle + \langle T; \varphi\, \mathrm{d}\alpha\rangle \\ &= \langle T; \varphi\, \mathrm{d}\alpha\rangle = \int_\Omega \langle \tau; \mathrm{d}\alpha\rangle\, \varphi\, d\mu \end{aligned}$$

(the first equality follows from the fact that ∂T is tangent to V and property (i) in Lemma 3.5; the third one from the identity $\mathrm{d}(\varphi\alpha) = \mathrm{d}\varphi \wedge \alpha + \varphi\, \mathrm{d}\alpha$; the fourth one by the fact that T is tangent to V and the restriction of the k-form $\mathrm{d}\varphi \wedge \alpha$ to V is null, again by property (i) in Lemma 3.5).

Since φ is arbitrary we infer that $\langle \tau; \mathrm{d}\alpha\rangle = 0$ μ-a.e., and since $\tau = \pm v$ (because T is tangent to V, cf. Remark 2.6) we obtain that $\langle v; \mathrm{d}\alpha\rangle = 0$ μ-a.e.

On the other hand, property (ii) in Lemma 3.5 implies that $\langle v; \mathrm{d}\alpha\rangle \neq 0$ in a neighbourhood of x_0. Since x_0 is in the support of μ, this neighbourhood has positive μ measure, and we have a contradiction. □

Using Theorem 3.2 and Proposition 3.4 we immediately obtain the following:

Theorem 3.6. *Assume that V is of class C^1 and that T is an integral k-current tangent to V. Then V is involutive at every point in the support of T.*

Remark 3.7. (a) The analogue of Theorem 3.6 for rectifiable sets does not hold. Indeed in [2] we show that for every distribution V, even a nowhere involutive one, it is possible to find a k-dimensional surface S of class C^1 whose tangency set

$$\Sigma := \big\{x \in S : \mathrm{Tan}(S, x) = V(x)\big\}$$

has positive $\mathscr{H}^k$-measure; in particular Σ is a non-trivial k-rectifiable set tangent to V. (This result was first proved in a slightly less general form in [4], Theorem 1.4.)

(b) Using Theorem 3.6 we can partly recover (and even extend) the Frobenius theorem for Sobolev surfaces proved in [9], Theorem 1.2. To be precise, by Sobolev surface we mean a k-rectifiable set Σ of the form $\Sigma = f(U)$ where U is an open set

[7] By support of a current with finite mass $T = \tau\mu$ we mean the support of the measure μ, that is, the smallest closed set F such that $\mu(\mathbb{R}^n \setminus F) = 0$. If T is rectifiable, that is $T = [\![\Sigma, \tau, \theta]\!]$, then the support of T turns out to be the closure of the set of all points $x \in \Sigma$ where the k-dimensional density of Σ is 1.

[8] That is, $\langle w; \alpha(x)\rangle = 0$ for every $(k-1)$-vector w whose span is contained in $V(x)$.

in $\mathbb{R}^k$ and $f : U \to \Omega$ is a continuous map of class $W^{1,p}$ with $p > k$, and we can show the following (see [2]): if V is of class C^1 and Σ is a Sobolev surface tangent to V, then V is involutive at $\mathscr{H}^k$-a.e. point of Σ.

4. Foliations of normal currents

We begin this section by giving the definition or foliation of a current, and then we show that for a normal current which is tangent to a distribution of planes V of class C^1 the existence of a foliation is strictly related to the involutivity of V (Theorem 4.3).

4.1. Foliations of currents. Let $T = \tau\mu$ be a k-current with finite mass in Ω, and let $\{R_t\}$ be a family of rectifiable k-currents in Ω, where t varies in some index space I endowed with a measure dt.[9] We say that $\{R_t\}$ is a *mass decomposition*, or *foliation*, of T if

(i) $\langle T; \omega\rangle = \int_I \langle R_t; \omega\rangle\, dt$ for every k-form ω on Ω of class C_c^∞;

(ii) $\mathbb{M}(T) = \int_I \mathbb{M}(R_t)\, dt$.

If T is a normal current, we may also consider the following additional conditions:

(iii) $\int_I \mathbb{M}(\partial R_t)\, dt < +\infty$;

(iv) $\mathbb{M}(\partial T) = \int_I \mathbb{M}(\partial R_t)\, dt$.

Remark 4.2. (a) Condition (i) is often written in compact form: $T = \int_I R_t\, dt$.

(b) If T is oriented by a continuous k-vectorfield v, then condition (ii) implies that R_t is oriented by v for a.e. $t \in I$.[10] This explains the term “foliation”.

(c) Conditions (i) and (iii) imply that $\partial T = \int_I \partial R_t\, dt$. Condition (iv) is stronger than (iii), and implies that the family $\{\partial R_t\}$ is a foliation of ∂T.

(d) A current of finite mass $T = \tau\mu$ may admit no foliation. For example this happens if μ is a Dirac mass, or more generally a measure supported on a set E which is purely k-unrectifiable.[11] Or if μ is the restriction of $\mathscr{H}^k$ to a k-surface S but τ does not span the tangent bundle of S.[12]

While the question of the existence of foliations for currents with finite mass is not particularly interesting, the same question for normal currents is quite relevant, and was first formulated by Frank Morgan (see [1], Problem 3.8). The next result answers this question for normal currents which are tangent to a distribution of planes of class C^1. If no regularity assumption is made on the tangent bundle of the currents there are a few partial results (see Remark 4.4) and the question is not completely settled.

Theorem 4.3. (See [2].) *Let V be a distribution of k-planes of class C^1 on Ω.*

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[9] We also assume that the function $t \mapsto \mathbb{M}(R_t)$ and $t \mapsto \langle R_t; \omega\rangle$ are Borel measurable for every k-form ω on Ω of class C_c^∞ (or, equivalently, of class C_0).

[10] Conversely, if $\int_I \mathbb{M}(R_t)\, dt$ is finite, (i) holds, and R_t is oriented by v for a.e. t, then (ii) holds.

[11] That is, $\mathscr{H}^k(E \cap \Sigma) = 0$ for every k-rectifiable set Σ.

[12] The point is that for currents with finite mass the measure μ can be chosen independently of the orientation τ. This is not the case with normal currents, and indeed none of these examples is a normal current.

(i) *If V is everywhere involutive, then every point of Ω admits a neighbourhood U such that every normal k-current in U tangent to V admits a foliation $\{R_t\}$ satisfying conditions (i), (ii), (iv) in §4.1.*

(ii) *Conversely, if T is a normal k-current in Ω which is tangent to V and admits a foliation $\{R_t\}$ satisfying conditions (i), (ii) in §4.1 and such that the currents R_t are integral, then V is involutive at every point in the support of T.*

Remark 4.4. (a) Statement (ii) is an immediate consequence of Theorem 3.6.

(b) Statement (ii) shows that the answer to Morgan's question is negative whenever $1 < k < n$, at least if we require that the currents in the foliation are integral (and not just rectifiable); the example is given by any normal current T tangent to distribution of k-planes which is nowhere involutive, for example the normal k-current T given in Remark 3.3(d).

(c) A negative answer to Morgan's question was first given by M. Zworski, who proposed the following variant of statement (ii) above (see [13], Theorem 2): if v is a nowhere involutive k-vectorfield and T is a current of the form $T = v\mathscr{L}^n$, then T admits no foliation. However this statement is not correct, because it does not require that the currents in the foliation are integral, and for $k = n-1$ it contradicts the fact that every normal current admits a foliation, see remark (e) below.

(d) Every normal 1-current in Ω admits a foliation satisfying conditions (i), (ii), (iii) in §4.1; this is essentially a consequence of the decomposition result by S. Smirnov [12] (see also [10]), even though it is not explicitly stated there. This result does not hold if we require that condition (iv) holds.

(e) Consider a normal $(n-1)$-current T in Ω. A consequence of the coarea formula for BV functions is that if T is a boundary then it admits a foliation satisfying conditions (i), (ii), (iv) in §4.1 (see [5], Theorem 4.5.9(13)). Such a foliation exists also if the boundary of T is rectifiable, as proved in [13], Theorem 1, using an idea from [6]. By modifying the argument in [6] we prove in [2] that that every normal $(n-1)$-current admits a foliation.[13]

(f) The existence of foliations for normal currents of dimension $d = 1$ or $d = n-1$ mentioned in items (d) and (e) above has no counterpart for $2 \leq d \leq n-2$. Indeed, for any $n \geq 4$, Andrea Schioppa constructed in [11] a normal current T of codimension 2 in $\mathbb{R}^n$ whose support is purely 2-unrectifiable. Clearly such T admits no foliation, and more precisely it cannot even be decomposed as $T = \int_I R_t\, dt$ with the only assumption that $\int_I \mathbb{M}(R_t)\, dt$ is finite.

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[13] In this case the currents in the foliation are no better than rectifiable.

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