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Density of Morse Functions on a Complex Space

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1. Introduction

Theorem 1 fills a gap, pointed out by Fischer in [2], in the proof of a theorem of Andreotti and Grauert in [1].

Let X be a reduced, Hausdorff, q-convex complex space, with countable basis. $\mathscr{E}(X)$ denotes the set of real valued, infinitely differentiable functions on X. Let K be the compact set of X and φ the continuous function on X, strongly q-convex on $X-K$, respectively, as stated in the definition of [1] and [2]; we can suppose that $\varphi \in \mathscr{E}(X)$. We shall prove the following:

Theorem 1. *There exist a compact set $K^* \subset X$ and $\varphi^* \in \mathscr{E}(X)$, such that:*

1. φ^* is strongly q-convex in $X-K^*$ and $B_c = \{x \in X\,;\ \varphi^*(x) < c\} \subset\subset X$ for every $c \in \mathbb{R}$.

2. The set $\{$local minima of φ^* on $X\}$ is discrete in X.

Actually we shall prove a density property of the functions satisfying property 2 of Theorem 1. We note that *(iii)* in the definition of the *Espaces fortement* (p,q)-*convexes-concaves* in [5] is, by means of Theorem 1, needless.

2.

Let $\{V_n^0\}_{n\in\mathbb{N}}$, $\{V_n^1\}_{n\in\mathbb{N}}$, $\{V_n^2\}_{n\in\mathbb{N}}$, be locally finite (hence countable) open coverings of X such that

1. For all i, j, $V_j^i \subset\subset X$.
2. $\{V_n^{i+1}\}_{n\in\mathbb{N}}$ is a shrinking of $\{V_n^i\}_{n\in\mathbb{N}}$, $i=0,1$.
3. For every j, there is an analytic isomorphism $h_j : V_j^0 \to A_j^0$ of V_j^0 onto an analytic set A_j^0 of a domain $D_j^0 \subset\subset \mathbb{C}^{N(j)}$.
4. For every j, there are domains $D_j^2 \subset\subset D_j^1 \subset\subset D_j^0$ such that

$$D_j^i \cap A_j^0 = h_j(V_j^i) \equiv A_j^i \qquad i=1,2.$$

5. For every j, there is $\Phi_j \in \mathscr{E}(D_j^0)$ such that $\Phi_j | A_j^0 = \varphi \circ h_j^{-1}$; if $V_j^0 \cap K = \emptyset$ then Φ_j is strongly q-convex.

Set, for all j, $D_j^1 = D_j$, $V_j^1 = V_j$, $A_j^1 = A_j$. We choose on every $\mathscr{E}(D_j)$ a metric d_j that induces the usual Fréchet space topology on it. Furthermore we consider on $E = \prod_j \mathscr{E}(D_j)$ the topology determined by the following basis of neighbourhoods of $0=(0,0,\dots)$: for every $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{N}}$, $\varepsilon_j\in\mathbb{R}$, $\varepsilon_j>0$, $U(\varepsilon,0)=\{f=(f_0,f_1,\dots)\in E;\ d_j(f_j,0)<\varepsilon_j, j\in\mathbb{N}\}$. We define: $S=\{f\in E;$ for all i,j $f_i|h_i(V_i\cap V_j)=f_j|h_j(V_i\cap V_j)\}$. S is not empty. For every $f\in E$ and for every ε, set: $U'(\varepsilon,f)=U(\varepsilon,f)\cap S$. For every ε, set $\frac{\varepsilon}{t}=\left(\frac{\varepsilon_0}{t},\frac{\varepsilon_1}{t},\dots\right)$, $t\in\mathbb{R}-\{0\}$.

Proposition 1. *S is a Baire space.*

Proof. Let $\{C_n\}_{n\in\mathbb{N}}$ be a countable family of dense open sets in S. It is enough to prove that, for every $f\in S$, for every $\varepsilon=(\varepsilon_0,\varepsilon_1,\dots)$, $U'(\varepsilon,f)\cap(\cap C_n)\neq\emptyset$. Let $\{r_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers $r_j>0$, $r_n\to 0$. Set $f^0=f$, $\varepsilon^0=(\varepsilon_0^0,\ \varepsilon_1^0,\ \varepsilon_2^0,\dots)=\varepsilon$. $U'\left(\frac{\varepsilon^0}{4},f^0\right)\cap C_0$ is open and $\neq\emptyset$. Hence there exist $f^1\in U'\left(\frac{\varepsilon^0}{4},f^0\right)\cap C_0$ and $\varepsilon^1=(\varepsilon_0^1,\varepsilon_1^1,\dots)$ such that: $U'(\varepsilon^1,f^1)\subset U'(\varepsilon^0,f^0)\cap C_0$.

We can choose ε^1 satisfying moreover: $\varepsilon_j^1<\min\left(r_0,\frac{\varepsilon_j^0}{4}\right)$ for all j. $U'\left(\frac{\varepsilon^1}{4},f^1\right)\cap C_1$ is open and $\neq\emptyset$. Hence there exist $f^2\in U'\left(\frac{\varepsilon^1}{4},f^1\right)\cap C_1$ and $\varepsilon^2=(\varepsilon_0^2,\varepsilon_1^2,\dots)$ such that: $U'(\varepsilon^2,f^2)\subset U'(\varepsilon^1,f^1)\cap C_1$.

We can choose ε^2 satisfying moreover: $\varepsilon_j^2<\min\left(r_1,\frac{\varepsilon_j^1}{4}\right)$ for all j. We thus construct inductively a countable family $\{U'(\varepsilon^n,f^n)\}_{n\in\mathbb{N}}$ such that:

(i) $\varepsilon_j^n<\min\left(r_{n-1},\frac{\varepsilon_j^{n-1}}{4}\right)$ for all j.

(ii) $d_j(f_j^{n+1},f_j^n)<\frac{\varepsilon_j^n}{4}$ for all j.

(iii) $U'(\varepsilon^n,f^n)\subset U'(\varepsilon^{n-1},f^{n-1})\cap C_{n-1}$.

Hence: $\cap U'(\varepsilon^n,f^n)\subset U'(\varepsilon,f)\cap(\cap C_n)$; so, if we prove that $\cap U'(\varepsilon^n,f^n)\neq\emptyset$ we are done. Now, for all j, $\{f_j^n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{E}(D_j)$. Since it is a Fréchet space $\{f_j^n\}$ has a limit F_j and clearly $F=(F_0,F_1,\dots)\in S$.

Moreover, using the inequalities (i) and (ii) it is easy to prove that $F\in U'(\varepsilon^n,f^n)$, for every n. This completes our proof.

Let $\mathscr{N}(D_i)\subset\mathscr{E}(D_i)$ the set: $\{f\in\mathscr{E}(D_i);\ \operatorname{supp} f\subset\subset D_i\}$.

Proposition 2. *There is a continuous map $s\colon\mathscr{N}(D_i)\to S$, such that $p_i\circ s=\mathrm{Id}$, where p_i is the projection on the i-factor of S.*

Proof. Let Φ be an element of $\mathscr{N}(D_i)$. We will determine the corresponding $f=(f_0,f_1,\dots)\in S$. If $V_i\cap V_j=\emptyset$, we put $f_j=0$. Let $V_i\cap V_j\neq\emptyset$. We can consider Φ as an element of $\mathscr{N}(D_i^0)$ and $\overline{h_j(V_i\cap V_j)}$ in D_j^0 is compact. For every $x\in\overline{h_j(V_i\cap V_j)}$ there are an open neighbourhood $U_x\subset\subset D_j^0$ of x such that $U_x\cap A_j^0=U_x\cap h_j(V_i^0\cap V_j^0)$ and a holomorphic map with image in $\mathbb{C}^{N(i)}$, $g=(g_1,\dots,g_{N(i)})$, defined on a neighbourhood of U_x in D_j^0, such that:

$$g|h_j(V_i^0\cap V_j^0)\cap U_x=h_i\circ h_j^{-1}|h_j(V_i^0\cap V_j^0)\cap U_x.$$

Let us select a finite covering $\{U_1, \ldots, U_k\}$ of $\overline{h_j(V_i \cap V_j)}$ and let $g^t = (g_1^t, \ldots, g_{N(i)}^t)$, $t = 1, \ldots, k$, be the corresponding holomorphic maps. Let $\{\chi_t\}_{t=0,1,\ldots,k}$ be an infinitely differentiable partition of unity subordinate to $\{U_0, U_1, \ldots, U_k\}$, where $U_0 = D_j^0 - \overline{h_j(V_i \cap V_j)}$. For every $t = 1, \ldots, k$ the function: $\Phi_t(y) = \chi_t(y)\Phi(g^t(y))$ for $y \in U_t$, $\Phi_t(y) = 0$ for $y \in D_j^0 - U_t$, is infinitely differentiable. For $y \in D_j$ we put: $f_j(y) = \sum_{1 \leq t \leq k} \Phi_t(y)$. Repeating this for every j such that $V_i \cap V_j \neq \emptyset$ (this is a finite set) we obtain an element of S; clearly this correspondence is continuous and verifies the statement of the proposition.

We fix now $A = A_j$ ($D = D_j$, $\mathbb{C}^{N(j)}$). Let $W^1 \supset W^2 \supset \ldots \supset W^r$ where $W^1 = A$ and $W^{i+1} = \text{Sing. } W^i$. The following properties hold:

1. The family of W^i is locally finite.
2. W^i is a complex space in D, for all i.
3. $S^i = W^i - W^{i+1}$ is a complex manifold, for all i.

Moreover, if we call stratum (of complex dimension $n_{i,j}$) every connected component S_j^i of S^i we have:

4. ∂S_j^i is a union of strata of dimension $< n_{i,j}$.
5. The family of all strata is locally finite.

We observe that for our purpose it is enough to use the natural stratification of A without refering to Whitney's theory.

Definition 1. We say that $S_j^p < S_i^q$ if $S_j^p \subset \partial S_i^q$. Now, let S_j^p, S_i^q be strata such that $S_j^p < S_i^q$. Then, we define: $T(S_j^p, S_i^q) = \{H \in G(n_{q,i}, N)$; there are $x \in S_j^p$ and $\{y_n\}_{n \in \mathbb{N}} \subset S_i^q$ such that: $y_n \to x$ and $T(y_n, S_i^q) \to H$ in $G(n_{q,i}, N)\}$, where: $G(n_{q,i}, N)$ is the Grassman manifold of complex $n_{q,i}$-planes through the origin in $\mathbb{C}^N$; $T(y_n, S_i^q)$ is the tangent vector space to S_i^q in y_n.

Definition 2. We say that $\Phi \in \mathscr{E}(D)$ is a Morse function on A if:

1. $\Phi | S_j^p$ has no degenerate critical points for all p, j such that: $n_{p,j} > 0$.
2. For all i, j, p, q (such that: $S_j^p < S_i^q$), for all $H \in T(S_j^p, S_i^q)$, the linear form $d\Phi(x)$ is not null on H.

Note. If Φ is a Morse function on A then the set of local minima of $\Phi | A$ is discrete in A; indeed a non degenerate critical point is isolated; moreover 2. of Definition 2 says that there are no sequences of critical points of $\Phi | S_i^q$ converging to a point of an $S_j^p < S_i^q$.

Proposition 3. *For every $\Phi \in \mathscr{E}(D)$ and for every $\varepsilon > 0$, there exist $\Psi \in \mathscr{N}(D)$ such that: (i) $d(\Psi, 0) < \varepsilon$. (ii) $\Phi + \Psi$ is a Morse function on A^2.*

Proof. We identify $\mathbb{C}^N$ with $\mathbb{R}^{2N}$:

$$(z_1, z_2, \ldots) = (x_1 + iy_1, x_2 + iy_2, \ldots) \to (x_1, y_1, x_2, y_2, \ldots).$$

Now, we consider: $h: D \to \mathbb{R}^{2N+1}$ where $h(x) = (h_1(x), h_2(x), \ldots) = (\Phi(x), x_1, y_1, \ldots)$. Let $p \in \mathbb{R}^{2N+1}$. We define $L_p: D \to \mathbb{R}$ by $L_p(s) = \|p - h(s)\|^2$, where $\|\cdot\|$ is the Euclidean norm.

For all q, j, the set of $p \in \mathbb{R}^{2N+1}$ such that $L_p | S_j^q$ has degenerate critical points is of measure 0 in $\mathbb{R}^{2N+1}$ ([3]). Since the S_j^q are a countable set, the set of $p \in \mathbb{R}^{2N+1}$ such that $L_p|S_j^q$ has degenerate critical points on at least one S_j^q is of measure 0 in $\mathbb{R}^{2N+1}$. We call this set B. Let $S^n=S_j^n$ and $S^r=S_i^r$ be such that $S^r<S^n$. We can suppose that $\dim_{\mathbb{C}} S^n=n$. $\overline{S^n}$ and $\overline{S^r}$ (in D) are complex analytic subspaces of D ([4] Chapter IV, Theorem 1). Calling W the set: $\{(x,H);\, x\in S^n,\ H\in \boldsymbol{T}(x,S^n)\}$, $\bar{W}$ in $\overline{S^n}\times \boldsymbol{G}(n,N)$ is an analytic subspace of $\overline{S^n}\times \boldsymbol{G}(n,N)$ ([4] Chapter IV, Proposition 4'). Moreover $\bar{W}$ is irreducible of $\dim_{\mathbb{C}}=n$ (W is a connected set of regular points and $\dim_{\mathbb{C}} W=n$). If $(x,H)\in \bar{W}\cap S^r\times \boldsymbol{G}(n,N)$, then $H\in \boldsymbol{T}(S^r,S^n)$.

$\Theta=\{(x,H)\in\bar{W};\ x\in\overline{S^r}\}$ is an analytic subspace of $\bar{W}$ and $\dim_{\mathbb{C}}\Theta\leqq n-1$. For every $(x,H)\in\overline{S^r}\times \boldsymbol{G}(n,N)$ we call $H_h=J(x)H$, where $J(x)$ is the Jacobian matrix of h in x (we consider H as a real $2n$-plane). In $\overline{S^r}\times \boldsymbol{G}(n,N)\times\mathbb{R}^{2N+1}$ let Θ' be the set $\Theta'=\{(x,H,y);(x,H)\in\Theta,\ y\in\{(H_h)^{\perp}+h(x)\}\}$, where $(H_h)^{\perp}$ is the orthogonal space to H_h in $\mathbb{R}^{2N+1}$ with respect to the usual scalar product. By means of the restriction to Θ' of the projection $\overline{S^r}\times \boldsymbol{G}(n,N)\times\mathbb{R}^{2N+1}\to\overline{S^r}\times \boldsymbol{G}(n,N)$ we obtain an infinitely differentiable fiber bundle on every stratum of any stratification of Θ. The fiber is a real $(2N+1-2n)$-plane. Then Θ' is (as set) a countable union of differentiable manifolds of $\dim_{\mathbb{R}}\leqq 2N-1$. Hence, by Sard's theorem, the image of Θ' in $\mathbb{R}^{2N+1}$ with respect to the projection $\overline{S^r}\times \boldsymbol{G}(n,N)\times\mathbb{R}^{2N+1}\to\mathbb{R}^{2N+1}$ has measure 0 in $\mathbb{R}^{2N+1}$. Repeating it for every couple of strata of A like S^n and S^r (this couple set is countable), the union of images of Θ''s in $\mathbb{R}^{2N+1}$ is of measure 0 in $\mathbb{R}^{2N+1}$. We call this set C. Now, $Q=\mathbb{R}^{2N+1}-(B\cup C)$ is a dense set in $\mathbb{R}^{2N+1}$ and for every $p\in Q$ we have: 1. $L_p|S_j^q$ has no degenerate critical points for every j,q. 2. L_p has the property 2 of Definition 2. Let U an open set, $D^2\subset\subset U\subset\subset D$. Let $\chi\in\mathcal{N}(D)$ be such that $\chi|\bar{U}=1$. Let $c\in\mathbb{R}$ be and choose $p=(-c+\varepsilon_1,\varepsilon_2,\dots,\varepsilon_{2N+1})\in Q$. ε_i can be arbitrarily small. Set: $g=\dfrac{(L_p-c^2)}{2c}$; g has property 1 and property 2 of Definition 2. A short computation shows that

$$g=\Phi+\sum_{1\leqq i\leqq 2N+1}\frac{h_i^2}{2c}-\sum_{1\leqq i\leqq 2N+1}\frac{\varepsilon_i h_i}{c}+\sum_{1\leqq i\leqq 2N+1}\frac{\varepsilon_i^2}{2c}-\varepsilon_1\,.$$

If ε_i are small and c is large we have: for $\Psi=\chi(g-\Phi)$, $d(\Psi,0)<\varepsilon$; indeed for $f\in\mathcal{E}(D)$, if $\lambda_m\to 0$ then $d(\lambda_m f,0)\to 0$. This concludes our proof.

Proposition 4. *For every i, $M_i=\{f\in S;\ f_i$ is a Morse function on $A_i^2\}$ is an open and dense set in S.*

Proof. It is clear that M_i is an open set. Density follows by means of Proposition 2 and Proposition 3.

Now, we have proved Theorem 1. We put: $K^*=\cup\bar{V}_i^0$, where $V_i^0\cap K\neq\emptyset$. Proposition 1 and Proposition 4 say that $M=\cap M_i$ is a dense set in S. It is enough to choose $\varepsilon=\{\varepsilon_n\}_{n\in\mathbb{N}}$, so small such that a function $\Phi^*=(\Phi_0^*,\Phi_1^*,\dots)\in M\cap U'(\varepsilon,\Phi)$, where $\Phi=(\Phi_0,\Phi_1,\dots)$ has the following properties: (i) Φ_i^* is strongly q-convex on D_i^2(if Φ_i had same property). (ii) Noting φ^* the function on X that we obtain by pulling back to X, we have $B_c\subset\subset X$, for every $c\in\mathbb{R}$. Every f on X that we obtain by pulling back to X an element $F\in M$ has the property 2 of Theorem 1. (with respect to local maxima too).

References

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