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Cosmological time in $(2+1)$-gravity

Riccardo Benedetti [a], Enore Guadagnini [b]

[a] *Dipartimento di Matematica, Università di Pisa, Via F. Buonarroti 2, I-56127 Pisa, Italy*
[b] *Dipartimento di Fisica, Università di Pisa, Via F. Buonarroti 2, I-56127 Pisa, Italy*

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Abstract

We consider maximal globally hyperbolic flat $(2+1)$-spacetimes with compact space S of genus $g>1$. For any spacetime M of this type, the length of time that the events have been in existence is M defines a global time, called the cosmological time CT of M, which reveals deep intrinsic properties of spacetime. In particular, the past/future asymptotic states of the cosmological time recover and decouple the linear and the translational parts of the $ISO(2,1)$-valued holonomy of the flat spacetime. The initial singularity can be interpreted as an isometric action of the fundamental group of S on a suitable real tree. The initial singularity faithfully manifests itself as a lack of smoothness of the embedding of the CT level surfaces into the spacetime M. The cosmological time determines a real analytic curve in the Teichmüller space of Riemann surfaces of genus g, which connects an interior point (associated to the linear part of the holonomy) with a point on Thurston's natural boundary (associated to the initial singularity). © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We shall be mainly concerned with maximal globally hyperbolic, matter-free spacetimes M of topological type $S\times\mathbb{R}$, where S is a compact closed oriented surface of genus $g>1$. The $(2+1)$-dimensional Einstein equation with vanishing cosmological constant actually implies that M is (Riemann) flat.

After [9] and [26], a large amount of literature has grown up about this $(2+1)$-gravity topic, regarded as a useful toy-model for the higher-dimensional case. Two main kinds of description have been experimented. A "cosmological" approach points to characterize the spacetimes in terms of some distinguished global time; for instance the *constant mean curvature* CMC time has been widely studied [3,17]. A "geometric" time-free approach

E-mail addresses: benedett@dm.unipi.it (R. Benedetti), guada@df.unipi.it (E. Guadagnini).

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eventually identifies each flat spacetime by means of its $ISO(2,1)$-valued holonomy [16,26]. With the exception of the case with toric space ($g = 1$), there is not a clear correspondence between the results obtained in these two approaches.

The aim of this paper is to show that this gap can be filled by using the canonical Cosmological Time CT, that is "the length of time that the events of M have been in existence" (see [2]). It turns out that this is a global time which reveals the fundamental properties of spacetime. It is canonically defined by means of the very basic spacetime's structures: its casual structure and the Lorentz distance. The cosmological time τ is invariant under diffeomorphisms, therefore, the $\tau = a$ level surfaces S_a provide a gauge-invariant description of space evolution in M. Both the intrinsic and extrinsic geometry of the surfaces S_a, as well as their past/future asymptotic states, are intrinsic features of spacetime. The asymptotic states are defined by the evolution of the observables associated to the length of closed geodesic curves on the surfaces S_a. Remarkably, they recover and decouple the linear and the translational parts of the holonomy. The study of the asymptotic states also leads to understand the initial singularity (we will always assume that the space is future expanding) and the way how the classical geometry degenerates, but does not completely disappear. The initial singularity can be interpreted as the isometric action of the fundamental group of S on a suitable "real tree". Differently to the case of the CMC time (for instance), the level surfaces S_a of the CT are in general only C^1-embedded into the spacetime M. This lack of smoothness takes place on a "geodesic lamination" on S_a and is a observable large scale manifestation of the intrinsic geometry of the initial singularity. Thus the initial singularity admits two complementary descriptions: one, in terms of real trees and, the second, in terms of geodesic laminations. The existence of a duality relation between real trees and laminations was already known in the context of Thurston theory of the boundary of the Teichmüller space. It is remarkable that Einstein theory of $(2+1)$-gravity sheds new light on this subject and puts duality in concrete form.

In [6] we have also used the cosmological time in order to study certain interesting families of $(2+1)$-spacetimes coupled to particles.

Our main purpose consists of elucidating the central role of the cosmological time and its asymptotic states in the description of spacetimes. The cosmological time perspective provides a new interpretation of several facts spread in the literature which are related to Thurston work. More precisely, the present article is based on, and could be considered a complement of, Mess's fundamental paper [16].

The importance of $(2+1)$-gravity has been pointed out by several authors, see for instance [8,9] and [26]; here we simply add a few comments. In general, the models which have dynamical degrees of freedom associated with the spacetime geometry are of particular interest in physics. Indeed, gravitational interactions are supposed to be described by general relativity in which the geometry of spacetime admits a nontrivial dynamical evolution. A satisfactory knowledge of all the classical and quantum aspects of general relativity is still lacking; so, toy models which provide conceptual hints in this directions are welcome. The matter-free $(2+1)$-gravity model with compact space is a remarkable example of general relativistic theory because the complete classical solution [16] has been produced. In this context, one can then find explicit answers to some open problems. In our article we have explored a few general topics which are related the problem of time in classical gravity. The resulting conceptual hints that we have obtained are:

(i) a consistent correspondence between the "dynamical" and the "static" pictures of spacetime exists, and has been explicitly produced in the model;

(ii) one can introduce a global canonical time which corresponds to the "age" of the universe, this time codifies intrinsic features of spacetime by means of the associated asymptotic states;

(iii) the asymptotic states are characterized by spectra of observables;

(iv) the geometric structure of the initial singularity gives rise to observable effects which can be detected at later times;

(v) the space slices of any global time, which plays the role of the age of the universe, are not necessarily smoothly embedded into spacetime.

2. The cosmological time function

For the basic notions of Lorentzian geometry and causality we refer, for instance, to [4,13]. Let N be any time oriented Lorentzian manifold of dimension $n+1$. The *cosmological time function*, $\tau : N \to (0, \infty]$, is defined as follows. Let $C^{-}(q)$ be the set of past-directed causal curves in N that start at $q \in N$, then

$$\tau(q) = \sup\{L(c) : c \in C^{-}(q)\},$$

where $L(c)$ denotes the Lorentzian length of the curve c:

$$L(c) = \int_c (\text{proper} - \text{time}).$$

$\tau(q)$ can be interpreted as the length of time the event q has been in existence in N. For example, if N is the standard flat Minkowski space $\mathbb{M}^{n+1}$, τ is the constant ∞-valued function, so in this case it is not very interesting. In [2] (see also [25]) one studies the properties of a manifold N with *regular* cosmological time function. Recall that τ is regular if:

(1) $\tau(q)$ is finite valued for every $q \in N$;

(2) $\tau \to 0$ along every past-directed inextensible causal curve.

The existence of a regular cosmological time function has strong consequences on the structure of N and of the constant-τ surfaces [2]. In particular when τ is regular, $\tau : N \to (0, \infty[$ is a continuous function, which is twice differentiable almost everywhere, giving a global time on N denoted by CT. Each τ level surface is a future Cauchy surface, so that N is globally hyperbolic. For each $q \in N$ there exists a future-directed time-like unit speed geodesic ray $\gamma_q : (0, \tau(q)] \to N$ such that:

$$\gamma_q(\tau(q)) = q, \qquad \tau(\gamma_q(t)) = t.$$

The union of the past asymptotic end-points of these rays can be regarded as the initial singularity of N.

The cosmological time function is not related to any specific choice of coordinates in N; it is "gauge-invariant" and so it represents an intrinsic feature of spacetime. Thus, when the cosmological time is regular, the τ-constant level surfaces and their properties have a direct physical meaning as they are observables.

We present now two basic examples of spacetime with regular cosmological time, which shall be important throughout all the paper. To fix the notations, the standard Minkowski space $\mathbb{M}^{2+1}$ is endowed with coordinates $x = (x^1, x^2, x^0)$, so that the metric is given by $ds^2 = (dx^1)^2 + (dx^2)^2 - (dx^0)^2$. $\mathbb{M}^{2+1}$ is oriented and time-oriented in the usual way.

Example 1. Consider the chronological future of the origin $0 \in \mathbb{M}^{2+1}$

$$I^+(0) = \{x \in \mathbb{M}^{2+1} : (x^1)^2 + (x^2)^2 - (x^0)^2 < 0,\ x^0 > 0\}.$$

Its cosmological time, $\tau : I^+(0) \to (0, \infty)$, is a smooth submersion; the constant-time $\{\tau = a\}$ surfaces are the (upper) hyperboloids

$$\mathbb{I}(a) = \{x \in \mathbb{M}^{2+1} : (x^1)^2 + (x^2)^2 - (x^0)^2 = -a^2,\ x^0 > 0\}.$$

Hence $\mathbb{I}(a)$ is a complete space of constant Gaussian curvature equal to $-1/a^2$, and of constant extrinsic mean curvature $1/a$. The Lorentzian length of the time-like geodesic arc connecting any $p \in I^+(0)$ with 0 equals $\tau(p)$; 0 is the initial singularity. Note that $\mathbb{I}(a)$ can be obtained from $\mathbb{I}(1)$ by means of a dilatation in $\mathbb{M}^{2+1}$ with constant factor a; shortly we write $\mathbb{I}(a) = a\mathbb{I}(1)$. We shall denote by $SO(2,1)$ the group of oriented Lorentz transformations acting on $\mathbb{M}^{2+1}$ and by $ISO(2,1)$ the Poincaré group. $SO^+(2,1)$ denotes the subgroup of $SO(2,1)$ transformations which keep $I^+(0)$ and each $\mathbb{I}(a)$ invariant. $ISO^+(2,1)$ is the corresponding subgroup of $ISO(2,1)$.

Example 2. Let us denote by $I^+(1,3)$ the chronological future in $\mathbb{M}^{2+1}$ of the line $\{x^1 = x^0 = 0\}$

$$I^+(1,3) = \{x \in \mathbb{M}^{2+1} : (x^1)^2 - (x^0)^2 < 0,\ x^0 > 0\}.$$

The Lorentzian length of the time-like geodesic arc connecting any $p = (x^1, x^2, x^0) \in I^+(1,3)$ with $q = (0, x^2, 0)$ equals the cosmological time $\tau(p)$. The level surfaces are

$$\mathbb{I}(1,3,a) = \{x \in \mathbb{M}^{2+1} : (x^1)^2 - (x^0)^2 = -a^2,\ x^0 > 0\}$$

and have constant extrinsic mean curvature equal to $(1/2a)$. Each surface $\mathbb{I}(1,3,a)$ is isometric to the flat plane $\mathbb{R}^2$. To make this manifest, it is useful to consider the following change of coordinates. Let $\Pi^{2+1} = \{(u, y, \tau) \in \mathbb{R}^{2+1} : \tau > 0\}$ be endowed with the metric $ds^2 = \tau^2 du^2 + dy^2 - d\tau^2$. Then, $x^1 = \tau sh(u)$, $x^2 = y$, $x^0 = \tau ch(u)$, is an isometry between Π^{2+1} and $I^+(1,3)$. The level set $\{\tau = a\}$ of Π^{2+1} goes isometrically onto $\mathbb{I}(1,3,a)$, so this is intrinsically flat. Note that the group of oriented isometries of Π^{2+1} is generated by the translations parallel to the planes $\{\tau = a\}$, and the rotation of angle π of the (u, y) coordinates.

We are going to show that any maximal globally hyperbolic, matter-free $(2+1)$-spacetime M, with compact space S, actually has regular cosmological time, and its initial singularity can be accurately described.

3. Flat $(2+1)$-spacetimes

A flat spacetime is, by definition, locally isometric to the Minkowski space $\mathbb{M}^{2+1}$. We assume that our maximal hyperbolic flat spacetimes are time-oriented and future expanding, and that these orientations locally agree with the usual ones on $\mathbb{M}^{2+1}$. The spacetime structures on $S \times \mathbb{R}$ are regarded up to oriented isometry homotopic to the identity.

3.1. Minkowskian suspensions

We introduce here the simplest $(2+1)$-spacetimes with compact space S of genus $g > 1$.

Recall that the upper hyperboloid $\mathbb{I}(1) \subset \mathbb{M}^{2+1}$, mentioned in the previous section, is a classical model for the hyperbolic plane $\mathbb{H}^2$ (see [7] for this and other models); the Poincaré disk is another model which can be obtained from $\mathbb{I}(1)$ by means of the stereographic projection shown in Fig. 1. We shall use the Poincaré model in Section 4.

Take any hyperbolic surface $F = \mathbb{H}^2/\Gamma$ homeomorphic to S. Γ is a subgroup of $SO^+(2,1)$ which acts freely and properly discontinuously on $\mathbb{H}^2 \cong \mathbb{I}(1)$. $\mathbb{I}(1)$ can be identified with the universal covering of S and Γ with the fundamental group $\pi_1(S)$. Γ can be thought also as a group of isometries of the spacetime $I^+(0)$ and $M(F) = I^+(0)/\Gamma$ is the required spacetime with compact space homeomorphic to S. We call it the *Minkowskian suspension* of F. This construction is well-known; sometimes $M(F)$ is also called the Lorentzian cone over F or the Löbell spacetime based on F. $I^+(0)$ can be regarded as the universal covering of $M(F)$. Let us now consider the cosmological time of $M(F)$. The CT of $I^+(0)$ naturally induces the CT of $M(F)$. Indeed, each level surface S_a of $M(F)$ has $\mathbb{I}(a)$ as universal covering; moreover, $S_1 = F$ and $S_a = aF$. In this case, the CT coincides with the CMC time and each level surface S_a smoothly embeds into $M(F)$. The initial singularity "trivially" consists of one point.

Notation. Let Y be any subset of $\mathbb{I}(1)$, we shall denote by $\widehat{Y}$ its "suspension" in $I^+(0)$ which is defined by $\widehat{Y} = \bigcup_{a\in(0,\infty[} aY$.

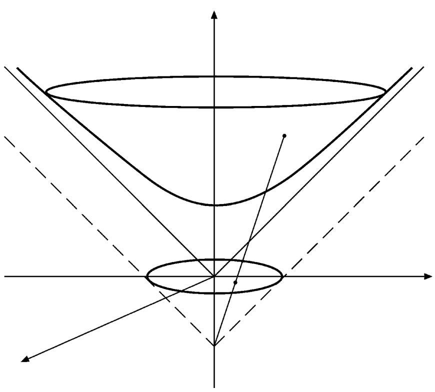

Fig. 1. The hyperbolic plane in the hyperboloid and disk models.

3.2. $(2+1)$-*spacetimes as deformed Minkowskian suspensions*

It has been shown in [16] that any maximal globally hyperbolic, future expanding flat spacetime M with compact space homeomorphic to S, as above, can be regarded as a "deformation" of some Minkowskian suspension (see also [26]). In fact M is of the form $M = U(M)/\Gamma'$, where:

(1) The domain $U(M)$ of $\mathbb{M}^{2+1}$ is a convex set

$$U(M) = \left\{x \in \mathbb{M}^{2+1};\, x^0 > f(x)\right\},$$

where $f : \{x^0 = 0\} \to [0, \infty[$ is a convex function.

(2) Γ' is a subgroup of $ISO^+(2,1)$ (also called the *holonomy* group of M) acting freely and properly discontinuously on $U(M)$. Hence $U(M)$ is the universal covering of M and Γ' is isomorphic to $\pi_1(M) \cong \pi_1(S)$.

(3) The "linear part" Γ of Γ' is a subgroup of $SO^+(2,1)$ which is isomorphic to $\pi_1(S)$ and acts freely and properly discontinuously on $\mathbb{I}(1) \cong \mathbb{H}^2$. This is a nontrivial fact which follows from a result of Goldman [12]. Each element $\gamma' \in \Gamma'$ is of the form $\gamma' = \gamma + t(\gamma)$, where $\gamma \in \Gamma$ and $t(\gamma) \in \mathbb{R}^3$ is a translation. $t : \Gamma \to \mathbb{R}^3$ is a *cocycle* representing an element of $H^1(\Gamma, \mathbb{R}^3)$. If $t_\lambda = \lambda t$, $\lambda \in \mathbb{R}^*$, then $U(M_\lambda)$ differs from $U(M)$ by: $U(M) = \lambda^{-1} U(M_\lambda)$. When λ is "small", $U(M_\lambda)$ is "close" to $I^+(0)$ (M_λ is "close" to $M(F)$, $F = \mathbb{H}^2/\Gamma$).

Note that Γ', whence $U(M)$ and t, are defined up to inner automorphism of $ISO^+(2,1)$.

3.3. *Spacetimes of simplicial type*

In this section, we shall consider the flat spacetimes that can be obtained from Minkowskian suspensions by means of particular deformations. These spacetimes will be called of *simplicial type*, the origin of this name is related to the material presented in Section 4. Spacetimes of simplicial type are important because they are "dense" in the set of all spacetimes; the shape of any spacetime and of its CT can be arbitrarily well approximated by some spacetime of simplicial type (see Proposition 4.23). So, it is enough to understand these examples in order to have a rather complete qualitative picture of our general presentation. Moreover, all the statements of this paper can easily be checked in a spacetime of simplicial type.

Start with a Minkowskian suspension $M(F)$. Assume that a *weighted multi-curve* $\mathcal{L}$ on F is given. $\mathcal{L}$ is the union of a finite number of disjoint simple closed geodesics on F, each one endowed with a strictly positive real weight. $\mathcal{L}$ governs a specific deformation of $M(F)$ producing a required flat spacetime denoted by $M(F, \mathcal{L})$. A particular spacetime deformation is associated to each component of $\mathcal{L}$ and can be obtained by means of an appropriate surgery operation in Minkowski space. As the deformations associated to the components of $\mathcal{L}$ act locally and independently from each other, we may assume for simplicity that $\mathcal{L}$ just consists of one component c, with weight r and length s.

3.3.1. Elementary deformation

In order to illustrate the deformation associated with one geodesic c with weight r, we shall now introduce a simple hyperbolic surface F_0 which can be understood as a local model for the general surface S. Let Γ_0 be an infinite cyclic subgroup of $SO^+(2,1)$ generated by an element g_0 acting on $\mathbb{I}(1)$ as an isometry of *hyperbolic type* (see, for instance, [7] for the classification of the isometries of $\mathbb{H}^2$). We can assume that g_0 is a Lorentz transformation corresponding to a boost along the x^1-direction, so that the g_0-invariant geodesic line on $\mathbb{I}(1)$ is the line $\sigma_0 = \mathbb{I}(1) \cap \{x^2 = 0\}$. The hyperbolic surface $F_0 = \mathbb{I}(1)/\Gamma_0$ is homeomorphic to the noncompact annulus $S^1 \times \mathbb{R}$ and its area is not finite. The image in F_0 of the axis of g_0 is a simple closed geodesic c of a certain length s; give it the positive weight r. So we dispose of a one-component weighted multi-curve $\mathcal{L}_0$ on F_0, as illustrated in Fig. 2.

The suspension $M(F_0) = I^+(0)/\Gamma_0$ is a flat spacetime. Let us now construct $M_0 = M(F_0, \mathcal{L}_0)$ which represents the deformation of $M(F_0)$ associated to the weighted multi-curve $\mathcal{L}_0$. We shall use the spacetimes $I^+(0)$, $I^+(1,3)$ and Π^{2+1} that we have introduced in Section 2. The universal covering $U(M_0)$ of M_0 will be the union of three domains of $\mathbb{M}^{2+1}$: $U(M_0) = A \cup B \cup C$, where $A = I^+(0) \cap \{x^2 \leqslant 0\}$, $B = I^+(1,3) \cap \{0 \leqslant x^2 \leqslant r\}$, $C = C' + r(0,1,0)$ and $C' = I^+(0) \cap \{x^2 \geqslant 0\}$. In our notations, $C' + r(0,1,0)$ denotes the set of points in $\mathbb{M}^{2+1}$ which can be obtained from C' by means of a translation of length r along the unit vector $(0,1,0)$. It is important to note that the cosmological times of the different pieces A, B and C fit well together; in fact, the CT level surfaces $\widetilde{S}_a$ of $U(M_0)$ are

$$\begin{aligned}\widetilde{S}_a = {} & \big(a\mathbb{I}(1) \cap \{x^2 < 0\}\big) \cup \big(\mathbb{I}(1,3,a) \cap \{0 \leqslant x^2 \leqslant r\}\big) \\ & \cup \big(a\mathbb{I}(1) \cap \{x^2 > 0\} + r(0,1,0)\big). \end{aligned} \tag{1}$$

As shown in Fig. 3, each surface $\widetilde{S}_a \subset \mathbb{M}^{2+1}$ can be obtained by cutting the hyperboloid $\mathbb{I}(a)$ along $a\sigma_0$ (which is the intersection of $\mathbb{I}(a)$ with the $\{x^2 = 0\}$-plane) and then by inserting a band of $\mathbb{I}(1,3,a)$ of depth r.

The surfaces $\widetilde{S}_a$ are only C^1-embedded into $U(M_0)$. The initial singularity of $U(M_0)$ is the segment $J_0 = \{x^1 = x^0 = 0\, 0 \leqslant x^2 \leqslant r\}$.

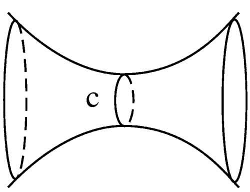

Fig. 2. The surface F_0 with the closed geodesic c.

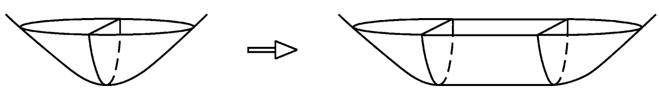

Fig. 3. Level surfaces $\widetilde{S}_a$ of $U(M_0)$.

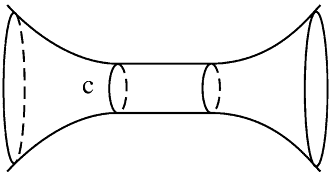

Fig. 4. Level surface of M_0.

Remark 3.1. We have the following characterization of J_0. The interior points of this segment make the subset of $\partial U(M_0)$ (boundary of the convex set $U(M_0)$) of the points with exactly two *null* supporting planes; the end-points make the subset of $\partial U(M)$ with more than two null supporting planes. Recall that a supporting plane at $x \in \partial U(M_0)$ is a plane P such that $x \in P$ and $U(M_0) \cap P = \emptyset$. P is null if it contains some null-lines.

The covering $U(M_0) \subset \mathbb{M}^{2+1}$ is flat. To get M_0, we only need to specify the action of $\pi_1 = \pi_1(F_0) \cong \mathbb{Z}$ on $U(M_0)$.

3.3.2. Action of the fundamental group

π_1 acts on A by the restriction of the action of Γ_0 on $\mathbb{I}(1)$. The domain B corresponds (via the isometry established in Section 2) to $B' = \{(u, y, \tau) \in \Pi^{2+1}; 0 \leqslant y \leqslant r\}$, so that the action of π_1 on B transported on B' is just given by the translation $(u, y, \tau) \to (u + s, y, \tau)$. Finally, if α is the translation $(x^1, x^2, x^0) \to (x^1, x^2 + r, x^0)$ on $\mathbb{M}^{2+1}$, then the action of π_1 on C is just the conjugation of Γ_0 by α.

The CT of the covering $U(M_0)$ passes to the quotient $M_0 = U(M_0)/\pi_1$; each level surface S_a is only C^1-embedded into M_0, so that it is endowed with an induced C^1-Riemannian metric. This allows anyway to define the length of curves traced on the surface S_a and the derived length-space distance. Let $\mathcal{A} = A/\pi_1$, $\mathcal{B} = B/\pi_1$ and $\mathcal{C} = C/\pi_1$. Then, $S_a \cap \mathcal{B}$ is a flat annulus of depth r and parallel geodesic boundary components of length as; $S_a \cap (\mathcal{A} \cup \mathcal{C})$ can be isometrically embedded into aF_0, and has geodesic boundary curves of length as. As shown in Fig. 4, S_a can be obtained by cutting F_0 along c and by inserting a annulus of depth r.

Remark 3.2. If g is an element of $ISO^+(2, 1)$ acting on $X \in \mathbb{M}^{2+1}$ as $g(X) = QX + w$, the transformed domain $g(U(M_0)) = Q(A \cup B \cup C) + w$ is, of course, an isometric copy of the universal covering in $\mathbb{M}^{2+1}$. The curve $\sigma = Q(\sigma_0)$ is a geodesic line of $\mathbb{I}(1)$; σ is the intersection of $\mathbb{I}(1)$ with a suitable hyperplane passing at the origin of $\mathbb{M}^{2+1}$. Let us denote by $\hat{\sigma}$ the suspension of σ; then

$$Q(B) = \bigcup_{\lambda \in [0,r]} \{\hat{\sigma} + \lambda v\},$$

where v is the unitary (in the Minkowski norm) vector tangent to $\mathbb{I}(1)$, normal to σ, and pointing towards $Q(C')$. We also denote $Q(B) = B(\sigma, v, r)$. The shape of the CT level surfaces in $g(U(M_0))$ is shown in Fig. 5. The initial singularity of $g(U(M_0))$ is given by the space-like segment $J = Q(J_0) + w$.

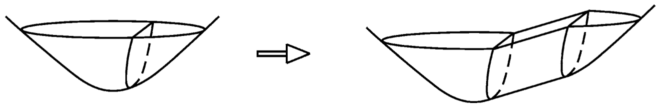

Fig. 5. Level surfaces of $g(U(M_0))$.

3.3.3. Simplicial type deformation

M_0 represents a local model of the deformation $M = M(F, \mathcal{L})$ we are interested in. In fact, there exists a neighborhood $\mathcal{W}$ of $\mathcal{B}$ in M_0 which embeds isometrically into M, respecting the cosmological time. Let us denote by $\mathcal{W}'$ the image of $\mathcal{W}$ in M. Then $M \backslash \mathcal{W}'$ embeds isometrically into the Minkowskian suspension $M(F)$, respecting again the cosmological time.

We describe now the universal covering $U(M) \subset \mathbb{M}^{2+1}$ and a cocycle $t : \Gamma \to \mathbb{R}^3$ which leads to $\Gamma' \subset ISO^+(2,1)$ such that $M = U(M)/\Gamma'$. The inverse image of $c \subset F = \mathbb{I}(1)/\Gamma$ into the covering $\mathbb{I}(1)$ is an infinite and locally finite set $\widetilde{\mathcal{L}}$ of disjoint complete geodesic lines. Given any geodesic $\sigma_0 \in \widetilde{\mathcal{L}}$, then $\widetilde{\mathcal{L}} = \{\sigma = \gamma(\sigma_0);\ \gamma \in \Gamma\}$. Let $\widehat{\mathcal{L}} \subset I^+(0)$ be the suspension of the geodesic lines of $\widetilde{\mathcal{L}}$. The set $\mathbb{I}(1) \backslash \widetilde{\mathcal{L}}$ is the union of an infinite number of connected components. Denote by R any such a component, and by $\widehat{R}$ its suspension, which is a component of $I^+(0) \backslash \widehat{\mathcal{L}}$. Every R covers a component F_R of $F \backslash \mathcal{L}$; more precisely, if Γ_R is the subgroup of Γ which keeps R invariant, then $F_R = R/\Gamma_R$.

Now, fix one base component R_0 and take in it one base point x_0. For each $\gamma \in \Gamma$, let $\gamma(x_0)$ be the point in $\mathbb{I}(1)$ which is defined by the action of γ on x_0. The geodesic arc in $\mathbb{I}(1)$ connecting x_0 with $\gamma(x_0)$ crosses a finite number of lines $\{\sigma_i\}$ belonging to $\widetilde{\mathcal{L}}$. At each crossing consider the unitary (in the norm of $\mathbb{M}^{2+1}$) vector v_i tangent to $\mathbb{I}(1)$ and normal to σ_i, pointing far from x_0. Then, the required cocycle $t(\gamma) \in \mathbb{R}^3$ is given by

$$t(\gamma) = \sum_i r v_i.$$

Note that if $\gamma_1(x_0)$ and $\gamma_2(x_0)$ belong to the same component R, then $t(\gamma_1) = t(\gamma_2)$, whence also $t(R) = t(\gamma)$ for any γ such that $\gamma(x_0) \in R$, is well defined. $U(M)$ is tiled by tiles of two types: (i) "$\widehat{R} + w$", (ii) "$B(\sigma, v, r) + w$", for some translation vector $w \in \mathbb{R}^3$. More precisely, the tiles of the first type make the open subset of $U(M)$

$$\mathcal{R} = \bigcup_R \bigl\{\widehat{R} + t(R)\bigr\}.$$

Each line $\sigma \in \widetilde{\mathcal{L}}$ is in the boundary of two regions R_σ, R'_σ and we assume that R_σ is closer to x_0 than R'_σ. Set v_σ the unitary (in the norm of $\mathbb{M}^{2+1}$) vector tangent to $\mathbb{I}(1)$ and normal to σ, pointing towards R'_σ. The two regions $\widehat{R}_\sigma + t(R_\sigma)$ and $\widehat{R}'_\sigma + t(R'_\sigma)$ are connected by the tile of the second type $B(\sigma, v_\sigma, r) + t(R_\sigma)$, so that

$$U(M) \backslash \mathcal{R} = \bigcup_{\sigma \in \widetilde{\mathcal{L}}} \bigl(B(\sigma, v_\sigma, r) + t(R_\sigma)\bigr).$$

Note that each tile has its own CT; all the cosmological times fit well together and define the CT of $U(M)$ which passes to the quotient M.

Remark 3.3. The construction of $M(F)$ and of $M(F, \mathcal{L})$ can be performed for any hyperbolic surface F, not necessarily compact nor of finite area. Similarly, by starting from *any* locally finite family of weighted geodesic lines in $\mathbb{I}(1)$, the simplicial deformation that we have just described produces a globally hyperbolic spacetime structure on $\mathbb{R}^2 \times \mathbb{R}$ with a regular cosmological time.

Remark 3.4. When F is compact, every region R (defined above) is bounded by *infinitely* many lines of $\widetilde{\mathcal{L}}$. In fact, as F is compact, every $\gamma \in \Gamma$ with $\gamma \neq 1$ is of hyperbolic type [7]. Consequently, for any $\sigma \in \widetilde{\mathcal{L}}$ and for every $\gamma \in \Gamma$ with $\gamma(\sigma) \neq \sigma$, σ and $\gamma(\sigma)$ are "ultra-parallel". This means that the hyperbolic distance satisfies $d(\sigma, \gamma(\sigma)) > 0$; moreover, σ and $\gamma(\sigma)$ have a common orthogonal geodesic line. Suppose now that a region R is bounded by finitely many lines in $\widetilde{\mathcal{L}}$. In this case, R contains a band E of infinite diameter, bounded by two half-lines contained in $\widetilde{\mathcal{L}}$. As F is compact, $\mathbb{H}^2$ is tiled by tiles of the form $\gamma(\mathcal{D})$, where $\mathcal{D}$ is a fundamental domain for Γ of finite diameter. So, one (at least) tile $\gamma(\mathcal{D})$ must be contained in E. But clearly $\gamma(\mathcal{D}) \cap \widetilde{\mathcal{L}} \neq \emptyset$ and this contradicts the fact that R is a region of $\mathbb{H}^2 \backslash \widetilde{\mathcal{L}}$.

The same conclusion holds if F is of finite area. If F is of infinite area, we can eventually have different behaviours. For instance, in the example F_0 above, $\widetilde{\mathcal{L}}_0$ just consists of one component which divides $\mathbb{H}^2$ into two regions. Other examples will be presented in Subsection 4.1.

4. The cosmological time of $(2 + 1)$-spacetimes

In this section we describe the main properties of the CT for an arbitrary spacetime M. We adopt the notations of the previous sections; in particular, M is assumed to be an expanding matter-free spacetime of topological type $S \times \mathbb{R}$ with compact surface S of genus $g > 1$. The validity of the following statements can be quite easily checked for spacetimes of simplicial type. We shall try to point out the main ideas; we postpone a commentary on the proofs, with references to the existing literature.

Proposition 4.1. *The cosmological time function,* $\tau : M \to (0, \infty[$, *is surjective and regular, so that it defines a global time on* M. *It lifts to a regular cosmological time on the covering,* $\tilde{\tau} : U(M) \to (0, \infty[$. *Each level surface* $\widetilde{S}_a$ *of* $U(M)$ *maps onto* S_a *of* M *and is its universal covering. In other words, the action of* $\pi_1(S)$ *on* $U(M)$ *restricts to a free, properly discontinuous isometric action on* $\widetilde{S}_a$ *such that* $S_a = \widetilde{S}_a / \pi_1(S)$. *Each* $\widetilde{S}_a$ (S_a) *is a future Cauchy surface.*

4.1. Initial singularity—external view

Let us give a description of the initial singularity of M as it appears "from the exterior" point of view, that is, from the Minkowski space in which the universal covering $U(M)$ is placed. In Subsection 4.5 we shall show how the initial singularity can also be characterized in terms of the observables associated with the CT asymptotic states.

Let us first give a definition.

Definition 4.2. An $\mathbb{R}$-tree (also called a *real tree*) is a metric space $(\mathcal{T}, d)$ such that for each couple of points $p \neq q \in \mathcal{T}$ there exists a *unique* arc in $\mathcal{T}$ with p and q as end-points and this arc is isometric to the interval $[0, d(p,q)] \subset \mathbb{R}$. This arc is called a *segment* of $\mathcal{T}$ and is denoted $[p,q]$.

Remark 4.3. The so-called *simplicial trees* are the simplest examples of real trees. A simplicial tree is a real tree covered by a countable family of *elementary* segments, called the "edges" of the tree, in such a way that: (a) whenever two edges intersect, then they just have one common endpoint; (b) the edge-lengths take values in a *finite* set of strictly positive numbers. Any endpoint of any edge is called a "vertex" of the tree. The distance is the natural length-space distance. Note that a simplicial tree is not necessarily locally finite; in other words, vertices of infinite "valence" may occur. In general, a real tree is more complicated than a simplicial tree because one might find, for instance, a segment containing a Cantor set made by the endpoints of other segments.

Proposition 4.4. *For any $p \in U(M) \subset \mathbb{M}^{2+1}$ there is a unique time-like geodesic arc $a(p)$ contained in $U(M)$, which starts at p and is directed in the past of p, such that the Lorentzian length of $a(p)$ equals $\tilde{\tau}(p)$. The other end-point of $a(p)$, denoted by $i(p)$, belongs to the boundary $\partial U(M)$ of $U(M)$ in $\mathbb{M}^{2+1}$. If p and q are identified in M by the action of $\pi_1(S)$, so are $a(p)$ and $a(q)$. The union of the initial points $\mathcal{T} = \{i(p);\ p \in U(M)\}$ is an $\mathbb{R}$-tree. More precisely, each segment of $\mathcal{T}$ is a rectifiable space-like curve in $\partial U(M)$ with its own length. There is a natural isometric action of the fundamental group $\pi_1(S)$ on $\mathcal{T}$. The quotient space $i(M) = \mathcal{T}/\pi_1(S)$ can be thought as the initial singularity of M.*

Remark 4.5. We have already encountered several examples of spaces of the form $X = \widetilde{X}/\pi_1(S)$ for some action of π_1 on $\widetilde{X}$: for instance, $F = \mathbb{H}^2/\Gamma$, $M = U(M)/\Gamma'$, $S_a = \widetilde{S}_a/\Gamma'$. Now, the initial singularity of spacetime also is a quotient $i(M) = \mathcal{T}/\pi_1(S)$. Instead of the bare topological quotient space, it is more interesting to consider $\widetilde{X}$ endowed with the action of π_1.

Remark 4.6. When M is of simplicial type, the corresponding real tree $\mathcal{T}$ is actually a simplicial real tree. This justifies the name we have given to these special spacetimes. In this case, the set of edges of $\mathcal{T}$ consists of the union of the space-like segments which form the initial singularity of the different tiles of the form $B(\sigma, v_\sigma, r) + t(R_\sigma)$ (see Section 3). The points of $\mathcal{T}$ can also be characterized by the properties discussed in Remark 3.1. A homeomorphic (not isometric) copy of $\mathcal{T}$ can easily be embedded into $\mathbb{I}(1)$. Select one point in each region of $\mathbb{I}(1)\backslash\widetilde{\mathcal{L}}$ and consider the set made by the union of all these points. Connect two points of this set by a geodesic segment of $\mathbb{I}(1)$ if and only if they belong to adjacent regions. In this way we get the required tree. This tree is manifestly "dual" of $\widetilde{\mathcal{L}}$; in fact, the regions of $\mathbb{I}(1)\backslash\widetilde{\mathcal{L}}$ correspond to the vertices of $\mathcal{T}$ and the lines of $\widetilde{\mathcal{L}}$ correspond to the edges of $\mathcal{T}$. We shall return on this duality in Section 4.3. Note that, as demonstrated in Remark 3.4, all the vertices of $\mathcal{T}$ are of infinite valence.

4.1.1. Examples of real trees

A hyperbolic surface $F = \mathbb{H}^2/\Gamma$ is represented in Fig. 6. F is of infinite area and is homeomorphic to a torus with one puncture; two simple closed geodesics c and a on F are depicted. The geodesic c cuts open F into a compact surface and an infinite area annulus. By using the Poincaré disk model, a fundamental domain of Γ in $\mathbb{H}^2$ is also shown in Fig. 6. This domain is delimited by four pair-wise ultra-parallel geodesic lines. The inverse images of c and a are represented on this domain. The first two terms of a sequence of partial tilings of $\mathbb{H}^2$, made by a finite number of copies of the fundamental domain, are shown in Fig. 7. The first partial tiling just contains one fundamental domain. The second is made by the union of $1 + 4 = 5$ copies of the fundamental domain. The next partial tiling of this sequence, which is not shown in the figure, contains $1 + 4 + 12 = 17$ tiles, and so on. For each partial tiling of $\mathbb{H}^2$ one can determine a corresponding partial lifting of the curves c and a. Fig. 8 shows the first two partial liftings $\tilde{c}$ of c and the structure of the associated partial dual trees. In the limit of the complete infinite tiling of $\mathbb{H}^2$, the complete lifting of c contains an infinite number of geodesics and the associated real tree is infinite. In this case, $\mathbb{H}^2 \backslash \tilde{c}$ has exactly one component with infinitely many boundary lines (the associated vertex of the dual tree has infinite valence), whereas all the remaining components have one boundary line. The first three partial liftings $\tilde{a}$ of a are shown in Fig. 9; the corresponding partial dual trees are also represented. Note that these figures are just evocative, as they are not geometrically exact.

Remark 4.7. The $\mathbb{R}$-trees and the associated $\pi_1(S)$-actions which occur in Proposition 4.4 are not arbitrary (see [20] p. 32). In fact, one can prove that the $\pi_1(S)$-action is *minimal with small edge-stabilizers*. This means that there is no nonempty strictly subtree which is invariant for this action, and that, for each segment in the tree, the subgroup of $\pi_1(S)$ which keeps the segment invariant is virtually Abelian. We shortly say that a real tree which admits such a kind of $\pi_1(S)$-action, is *geometric*.

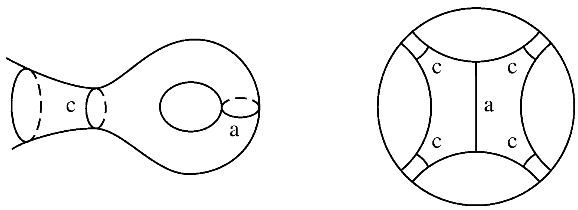

Fig. 6. Two simple curves on $F = \mathbb{H}^2/\Gamma$.

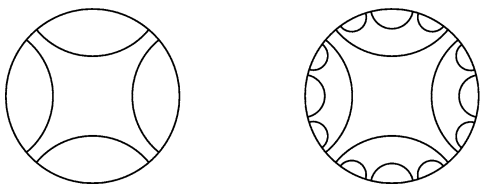

Fig. 7. Partial tilings of $\widetilde{F} = \mathbb{H}^2$.

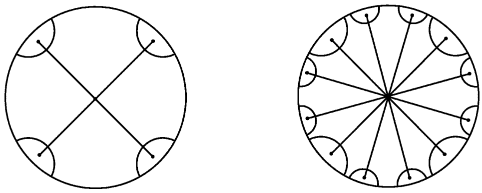

Fig. 8. Partial $\tilde{c}$ and dual trees.

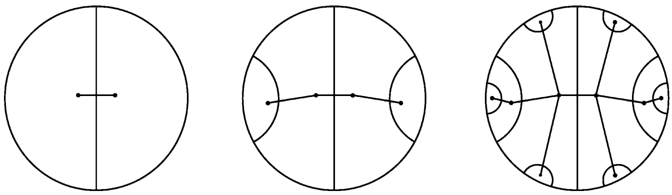

Fig. 9. Partial $\tilde{a}$ and dual trees.

4.2. Intrinsic and extrinsic geometry of the constant CT surfaces

In order to describe the geometric properties of the surfaces of constant cosmological time, it is natural to introduce the notion of geodesic lamination.

Definition 4.8. Let G be a surface endowed with a C^1-Riemannian metric. As usual, this induces a length-space distance on G and the notion of geodesic arc (line) makes sense. A *geodesic lamination* of G is a closed subset K of G, also called the support of the lamination, which is the disjoint union of complete and simple geodesics, also called the leaves of the lamination. "Complete" means that we dispose of arc-length parametrization defined on the whole real line $\mathbb{R}$; "simple" means that the geodesic has no self-normal crossing in G. In other words, each leaf is either a simple closed geodesic or a simple geodesic which is an isometric copy of $\mathbb{R}$ embedded in G. When G is compact, such a noncompact leaf is not a closed subset of G.

Remark 4.9. A finite union of disjoint simple closed geodesics is called a *multi-curve* and is the simplest example of geodesic lamination. We have already introduced multi-curves in Section 3.3. A generic geodesic lamination K can be more complicated than a multi-curve; in fact, if α is an arc embedded in G which is transverse to the leaves of K, typically $\alpha \backslash K$ is a Cantor set.

Proposition 4.10. *For every $a \in (0, \infty[$:*

(1) *$\widetilde{S}_a$ is the graph of a positive convex function defined on the plane $\{x^0 = 0\}$ in $\mathbb{M}^{2+1}$.*

(2) *$\widetilde{S}_a$ is only C^1-embedded into $U(M)$, so that it carries an induced C^1-Riemannian metric. $\widetilde{S}_a$ is geodesically complete and for each $p \neq q \in \widetilde{S}_a$, there is a unique geodesic arc connecting p and q.*

(3) *The locus $\widetilde{L}_a$ at which the embedding of $\widetilde{S}_a$ into $U(M)$ is no longer C^2 is a geodesic lamination of $\widetilde{S}_a$. $\widetilde{L}_a$ is in fact the pull-back of a geodesic lamination L_a of S_a.*

Remark 4.11. If M is the spacetime of simplicial type which corresponds to the weighted multi-curve $\mathcal{L}$ on the surface F, then L_a is just made by the boundary components of the flat annular components embedded into S_a, which are associated to the components of $\mathcal{L}$.

The content of the last remark generalizes as follows. Recall that $\pi_1(S)$ acts as Γ on each $\mathbb{I}(a)$. For every $a \in (0, \infty[$, let us consider the map

$$p_a : \widetilde{S}_a \to \mathbb{I}(a)$$

defined as follows: $p_a(x)$ is the unique point of $\mathbb{I}(a)$ such that the tangent plane to $\widetilde{S}_a$ at x is parallel to the tangent plane to $\mathbb{I}(a)$ at $p_a(x)$. This map is well-defined, surjective and $\pi_1(S)$-equivariant. By taking the union of the p_a's we get a $\pi_1(S)$-equivariant map $p : U(M) \to I^+(0)$, respecting the CT. This induces a map $p' : M \to M(F)$ respecting the CT.

Proposition 4.12.

(1) *There exists a geodesic lamination $\mathcal{F}$ on $F = \mathbb{I}(1)/\Gamma$, which lifts to a geodesic lamination $\widetilde{\mathcal{F}}$ on $\mathbb{I}(1)$, such that, for every a, one has $p_a(\widetilde{L}_a) = a\widetilde{\mathcal{F}}$ and any leaf of $\widetilde{L}_a$ is isometrically mapped onto a leaf of $a\widetilde{\mathcal{F}}$. That is, the union of $p_a(\widetilde{L}_a)$'s covers the suspension $\widehat{\mathcal{F}}$ of $\widetilde{\mathcal{F}}$.*

(2) *$\mathcal{F}$ is the disjoint union of two sublaminations*

$$\mathcal{F} = \mathcal{L} \cup \mathcal{F}',$$

where $\mathcal{L}$ is the maximal multi-curve sublamination of $\mathcal{F}$. Note that either $\mathcal{L}$ or $\mathcal{F}'$ may be empty. Then

(a) *p embeds $U(M) \backslash p^{-1}(\widehat{\mathcal{F}})$ isometrically into $I^+(0)$ respecting the CT;*

(b) *p embeds $U(M) \backslash p^{-1}(\widehat{\mathcal{L}})$ continuously into $I^+(0)$ respecting the CT.*

(3) *The set $p^{-1}(\widehat{\mathcal{L}})$ is the union of components of the type $B(\sigma, v_\sigma, r) + w$, so that $(p')^{-1}(\mathcal{L}) \cap S_a$ is the disjoint union of flat annular components of S_a, like in the case of a spacetime of simplicial type.*

We have an immediate corollary concerning the intrinsic and extrinsic geometry of the constant CT surfaces.

Corollary 4.13. *$\widetilde{W}_a = \widetilde{S}_a \backslash \widetilde{L}_a$ is an open dense set of $\widetilde{S}_a$. Each component of $\widetilde{W}_a$ is either isometric to an open set of $\mathbb{I}(a)$ or is a flat band which embeds into $\mathbb{I}(1, 3, a)$, and projects onto an annulus of S_a. Flat annuli do occur only if $\mathcal{L}$ is nonempty.*

4.3. CT duality

To sum up, two geometric structures are naturally associated to the spacetime M: the real tree $\mathcal{T}$ (the initial singularity) and the geodesic lamination $\mathcal{F}$ on $F = \mathbb{I}(1)/\Gamma$ which reflects the lack of smoothness of the embedding of S_a into M. We have already noted that for a spacetime of simplicial type these two objects are "dual" to each other. Here we want to strengthen and generalize this point.

If $\mathcal{L}$ is nonempty, we extend the lamination L_a on S_a to a lamination L'_a, by foliating the flat annular regions of S_a by closed geodesics parallel to the boundary components. As usually $\widetilde{L}'_a$ denotes the lifted lamination to $\widetilde{S}_a$. The above map p_a sends $\widetilde{L}'_a$ onto $a\widetilde{\mathcal{F}}$.

We have a natural continuous surjective map $i_a : \widetilde{S}_a \to \mathcal{T}$ which associates to x the initial point on the arc $a(x)$. So $\mathcal{T}' = \{i_a^{-1}(x)\}_{x \in \mathcal{T}}$ is a partition of $\widetilde{S}_a$ by closed subsets. $\pi_1(S)$ acts also on $\mathcal{T}'$ and, clearly, i_a induces an π_1-equivariant identification between $\mathcal{T}'$ and $\mathcal{T}$.

Proposition 4.14. *For every a, each closed set E of the partition $\mathcal{T}'$ of $\widetilde{S}_a$ is*:

(1) *either the closure of a component of $\widetilde{S}_a \backslash \widetilde{L}'_a$;*

(2) *or a leaf of the foliation of some band component of $\widetilde{L}'_a$ which projects onto a flat annular region of S_a.*

We describe how the distance d on the real tree $\mathcal{T}$ can be encoded, in dual terms, by equipping the geodesic laminations $\widetilde{\mathcal{F}}$, $\mathcal{F}$, with suitable *transverse invariant measures*.

Definition 4.15. A *measured geodesic lamination* on F is a couple $(\mathcal{F}, \mu)$, where $\mathcal{F}$ is a geodesic lamination and μ is a *transverse invariant measure* which consists of a *Borel measure* μ_J on each embedded interval $J \cong [0, 1]$ in F, transverse to the leaves of $\mathcal{F}$ such that

(1) the support of μ_J coincides with $\mathcal{F} \cap J$;

(2) if J, J' are arcs, homotopic through arcs which are transverse to the leaves of $\mathcal{F}$, keeping the endpoints either on the same leaf or in the same connected components of $F \backslash \mathcal{F}$, then $\mu_J(J) = \mu_{J'}(J')$. $(\mathcal{F}, \mu)$ lifts to $(\widetilde{\mathcal{F}}, \tilde{\mu})$ which is π_1-equivariant.

Remark 4.16. The simplest measured geodesic laminations of F are the weighted multi-curves.

Let J be an arc in $\mathbb{I}(1)$ transverse to the leaves of $\widetilde{\mathcal{F}}$. The map p_a lifts J to an arc J' in $\widetilde{S}_a$ transverse to the leaves of $\widetilde{L}'_a$. On the other hand, by means of the map i_a, the distance d on $\mathcal{T}$ lifts to a measure $\tilde{\mu}_{J'}$ on J' which finally gives us the required (π_1-equivariant) transverse measure on $\widetilde{\mathcal{F}}$.

One can invert the above construction and associate to each measured geodesic lamination $(\mathcal{F}, \mu)$ of the hyperbolic surface F a suitable geometric $\mathbb{R}$-tree.

4.3.1. From geodesic laminations to real trees

Take the measured lamination $(\mathcal{F}, \mu)$ of the surface F. $\mathcal{F}$ is in general the disjoint union of two sublaminations

$$\mathcal{F} = \mathcal{L} \cup \mathcal{F}'',$$

where $\mathcal{L}$ is the maximal weighted multi-curve sublamination of $\mathcal{F}$. Note that either $\mathcal{L}$ or $\mathcal{F}''$ may be empty. $F \backslash \mathcal{F}$ consists of a finite number of connected components, the metric completion of any such a component is isometric to a compact hyperbolic surface with geodesic polygonal boundary. If $\mathcal{L}$ is nonempty, let us consider the spacetime of simplicial type associated to $\mathcal{L}$, and let F' be the $\tau = 1$ level surface of this spacetime. Let us denote by $\mathcal{F}'$ the lamination on F' which coincides with $\mathcal{F}$ outside the flat annuli of F' and is defined as L'_1 above on these annuli. If $\mathcal{L}$ is empty, set $F' = F$. The measured lamination $(\mathcal{F}, \mu)$ "extends" to a measured lamination $(\mathcal{F}', \mu')$ on F'. The flat annular components are foliated by closed geodesics parallel to the boundary components. These annuli are endowed with a plain transverse measure of total mass equal to the corresponding annulus depth. Take the universal covering $\widetilde{F}'$ of F' with the lifted (π_1-equivariant) measured geodesic lamination $(\widetilde{\mathcal{F}}', \tilde{\mu}')$. Now define a partition of $\widetilde{F}'$ by closed subsets in the very same way we have defined above the partition $\mathcal{T}'$ of $\widetilde{S}_1$, with respect to the lamination $\widetilde{L}'_1$. Call again this partition $\mathcal{T}'$. We can give it a distance d which makes it an $\mathbb{R}$-tree. If E and E' are the closure of two components of the complement of the lamination, take two points x and x' in these closed sets such that the geodesic segment $[x, x']$ of $\widetilde{F}'$ is transverse to the leaves of the lamination. By integration, the transverse measure induces a distance on the subset of $\mathcal{T}'$ made by the closed sets intersecting $[x, x']$. In fact, by the "invariance" of the measure, this distance does not depend on the segment $[x, x']$. Finally one verifies that in this way one can actually define a distance between any two points of $\mathcal{T}'$ and that the resulting $(\mathcal{T}', d)$ is a geometric real tree.

Remark 4.17. Clearly, weighted multi-curves on the surface F dually correspond to geometric simplicial real trees; the spacetimes of simplicial type do materialize this duality.

4.4. Reconstruction of $M = U(M)/\Gamma'$

Starting from $(F = \mathbb{I}(1)/\Gamma, \mathcal{T})$ or, equivalently, from $(F = \mathbb{I}(1)/\Gamma, (\mathcal{F}, \mu))$, one can reconstruct a cocycle t, whence $M = U(M)/\Gamma'$. This generalizes what we have done for a spacetime of simplicial type in Subsection 3.3.

With the notations introduced at the end of the previous subsection, consider $(\widetilde{\mathcal{F}}', \tilde{\mu}')$ on $\widetilde{F}'$. To recover a cocycle t do as follows: fix one base point p_0^* on $\widetilde{F}'$ out from the support of the lamination. Let p_0 be its image on F'. If σ is a loop in F' based on p_0, which represents an element $[\sigma]$ of $\pi(F', p_0)$, lift it to the oriented arc σ^* in $\widetilde{F}'$ which starts at p_0^*; up to homotopy we can assume that σ^* is transverse to the leaves of the lamination. Let f be any continuous $\mathbb{R}^3$-valued function on σ^* which coincides with the unit normal to the leaves of the lamination, tangent to $\widetilde{F}'$, and oriented in agreement with σ^*. Now we can integrate f along σ^* by using the transverse measure getting a vector $t([\sigma])$. By varying $[\sigma]$, one gets such a cocycle t.

4.5. CT asymptotic states

The above discussion tells us that any spacetime $M = U(M)/\Gamma'$ is completely determined by the linear part Γ of its holonomy Γ' (or equivalently by the surface $F = \mathbb{I}(1)/\Gamma$) and by its initial singularity $i(M) = \mathcal{T}/\pi_1(S)$. The aim of this subsection is to recover these geometric objects from the "internal point of view" by "working inside the spacetime". More precisely, we would like to show that F and $\mathcal{T}$ can be interpreted as the future and past *asymptotic states* for the geometry of the CT level surfaces. To this aim we shall consider the observables defined by the lengths of the curves on the CT level surfaces. It is convenient to introduce the concept of *Marked Spectrum* associated with a metric space $(\widetilde{X}, d)$ which is endowed with an action α of the surface's fundamental group $\pi_1(S)$, so that $X = \widetilde{X}/\pi_1(S)$. Whenever we shall refer to X, we shall actually refer to the triple $(\widetilde{X}, d, \alpha)$ (see Remark 4.5).

Let us denote by $\mathcal{C}$ the set of conjugation classes of $\pi_1(S)\backslash\{1\}$ which coincide with the homotopy classes of noncontractible continuous maps $f : S^1 \to S$. Each marked spectrum is a point of the functional space $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$, endowed with the natural product topology. The Marked Spectrum s_X of X (denoted also by $s_{\widetilde{X}}$), is defined as follows: for any $c = [\gamma] \in \mathcal{C}$, $\gamma \in \pi_1(S)$, $\gamma \neq 1$,

$$s_X(c) = \inf_{p \in \widetilde{X}} d\big(p, \alpha_\gamma(p)\big).$$

The spectrum is "marked" because one takes track of the map in addition to its image.

When $X = F$ or S_a, $s_X(c)$ is just the length of a closed geodesic curve (not necessarily simple; that is, self-crossings could possibly occur in c) which minimizes the length among the curves in that homotopy class. For this reason, in such a case, s_X is called the *Marked Length Spectrum* and is denoted by l_X. When $\widetilde{X} = \mathcal{T}$, $s_{\mathcal{T}}$ can be expressed, in dual terms, as the *Marked Measure Spectrum* of the corresponding measured geodesic lamination $\mathcal{F}$ on F; usually this is denoted $I_{\mathcal{F}}$. $I_{\mathcal{F}}(c)$ is just the minimal transverse measure realized by the curves in that homotopy class. When $\mathcal{T}$ is simplicial, that is when $\mathcal{F}$ is a weighted multi-curve $\mathcal{L}$ of F, $I_{\mathcal{L}}(c)$ is easily expressed in terms of the *geometric intersection number* (this also justifies the notation): assume that all the weights are equal to 1 (that is the length of all edges of $\mathcal{T}$ is equal to 1); then it is easy to see that $I_{\mathcal{L}}(c)$ is just the minimum number of intersection points between $\mathcal{L}$ and any curve belonging to c and transverse to the components of the lamination. For arbitrary weights one just takes multiples of the contribution of each component of $\mathcal{L}$.

Remark 4.18. Instead of the whole $\mathcal{C}$, one could prefer to use the subset $\mathcal{S} \subset \mathcal{C}$ of isotopy classes of *simple* closed curve in S, and take the corresponding (restricted) marked spectra. The discussion should proceed without any substantial modification.

4.5.1. On the boundary of the Teichmüller space

It is convenient, at this stage, to recall the fundamental facts about the role that the marked spectra play in the study of the Teichmüller space and in Thurston's theory of its natural boundary. Let us denote by T_g the Teichmüller space of the hyperbolic structures on S up to isometry isotopic to the identity. It is well-known (see [7,11,23]) that the map

$$l : T_g \to (\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$$

defined by $l(F = \mathbb{H}^2/\Gamma) = l_F$, realizes a meaningful embedding of T_g onto a subset of $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$ homeomorphic to the finite-dimensional open ball B^{6g-6}. We shall identify T_g with $l(T_g)$. In fact T_g is in a natural way a real analytic submanifold of $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$.

Fix any such a hyperbolic structure $F \in T_g$ on S. Let us denote by $\mathcal{MGL}(F)$ the set of measured geodesic laminations on F. Let us denote by $\mathcal{GT}(S)$ the set of all $\pi_1(S)$-geometric $\mathbb{R}$-trees (Remark 4.7). At the end of Subsection 4.3, we have outlined a construction which associates to each $\mathcal{F} \in \mathcal{MGL}(F)$ a dual $\mathbb{R}$-tree say $\Delta(\mathcal{F}) \in \mathcal{GT}(S)$. Note that this construction did not use the fact that F was associated to a spacetime M.

Proposition 4.19. *$\Delta : \mathcal{MGL}(F) \to \mathcal{GT}(S)$ is a bijection, that is it can be naturally inverted. For each $r > 0$, $\Delta(r\mathcal{F}) = r\Delta(\mathcal{F})$; here we take either the r-multiple of the measure or the r-multiple of the distance. We can shortly say that "Δ respects the positive rays".*

Proposition 4.20. *Consider the maps, $I : \mathcal{MGL}(F) \to (\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$ and $s : \mathcal{GT}(S) \to (\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$, obtained by taking the corresponding marked spectra. Then $I = s \circ \Delta$ and is injective. The image in $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$ is a positive cone based on the origin and positive rays go onto positive rays, in a obvious sense. Moreover, T_g and the image $\mathrm{Im}(I)$ are disjoint subsets of $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$.*

Remark 4.21. These spectra represent the actual "physical" observables in our discussion. The last two propositions specify the meaning of the duality between laminations and real trees. As the spectra coincide, they reveal the same physical content.

Similarly to T_g, we identify $\mathcal{MGL}(F)$ and $\mathcal{GT}(S)$ with the image $\mathrm{Im}(I) \subset (\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$, endowed with the subspace topology.

Set $\mathcal{P}^+(\mathcal{MGL}(F)) = \mathcal{P}^+(\mathcal{GT}(S)) = \mathcal{P}^+(\mathrm{Im}(I))$ the projective quotient space, obtained by identifying to one point each positive ray in $\mathrm{Im}(I) \backslash \{0\}$. Similarly $T_g \cup \mathcal{P}^+(\mathrm{Im}(I))$ has a natural quotient topology.

Proposition 4.22. *The pair $(\overline{T}_g, \partial\overline{T}_g) = (T_g \cup \mathcal{P}^+(\mathrm{Im}(I)), \mathcal{P}^+(\mathrm{Im}(I)))$ is homeomorphic to the pair $(\overline{B}^{6g-6}, S^{6g-7})$, where $\overline{B}^{6g-6}$ is the closed ball and S^{6g-7} is its boundary sphere. The natural action on T_g of the* mapping class group *Mod_g of S extends to an action on the compactification $\overline{T}_g$. This is called the Thurston's natural compactification and $\partial\overline{T}_g$ is the* natural boundary *of the Teichmüller space.*

We can state precisely how the simplicial trees are dense, as we claimed in Section 3. Let us denote $\mathcal{ST}(S)$ the subset of $\mathcal{GT}(S)$ made by the simplicial real trees.

Proposition 4.23. *$\mathcal{ST}(S)$ is dense in $\mathcal{GT}(S)$ in the induced topology by $(\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$.*

Remark 4.24. In this remark we collect a few technical complements concerning the marked spectra and the geometric meaning of spectra convergence.

(1) The natural compactification of T_g is formally similar to the natural compactification of $\mathbb{H}^2$ in the hyperboloid model $\mathbb{I}(1)$ where S^1_∞ is obtained by adding to $\mathbb{I}(1)$ the endpoints of the rays of the future light cone.

(2) Let F_n be a sequence in T_g considered as a sequence of actions of $\pi_1(S)$ on $\mathbb{H}^2$. The meaning of the compactification is the following; up to passing to a subsequence (still denoted by F_n), one of the following situations occur: for every $c \in \mathcal{C}$,

(a) $l_{F_n}(c) \to l_{F_0}(c)$, for some $F_0 \in T_g$.

(b) There exist a geometric real tree $\mathcal{T} \in \mathcal{GT}(S)$ and a positive sequence $\epsilon_n \to 0$, such that $\epsilon_n l_{F_n}(c) \to s_{\mathcal{T}}(c)$. This is also called the Morgan–Shalen convergence of the sequence of actions. This can be reformulated in a similar, equivalent, dual way as the convergence (up to positive multiples) of a sequence of marked length spectra of hyperbolic structures on S to the marked measure spectrum of a measured geodesic lamination on a fixed *base* F_0.

(3) The convergence of marked spectra has a deep geometric content. This can be expressed in terms of the Gromov convergence. Given two metric spaces (Y, d) and (Y', d') and $\epsilon > 0$, an ϵ-relation is a set $R \subset Y \times Y'$ (i.e., a relation between the two spaces) such that:

(a) the two projections of R to Y and Y' are both surjective;

(b) if $(y, y'), (z, z') \in R$ then $|d(y, z) - d(y', z')| < \epsilon$.

Let G be a group, and $\{G \times Y_n \to Y_n\}_{n \geqslant 1}$ be a sequence of isometric actions of G on the metric spaces Y_n. We say that $(G \times Y_n \to Y_n) \to (G \times Y_0 \to Y_0)$ in the sense of Gromov, if for every compact subset $K_0 \subset Y_0$, for every $\epsilon > 0$ and for every finite subset P of G, if n is big enough, there are compact subsets $K_n \subset Y_n$ and ϵ-relations R_n between K_n and K_0 which are *P-equivariant*; this means that: if $x \in K_0$, $g \in P$, $g(x) \in K_0$, $x_n, y_n \in K_n$ and $(x, x_n), (g(x), y_n) \in R_n$, then $d_n(g(x_n), y_n) \leqslant \epsilon$.

It turns out that in case (a) above we actually have the convergence in the Gromov sense of the sequence of actions on $\mathbb{H}^2$ to an interior point of T_g. In case (b), the Morgan–Shalen convergence is equivalent to the Gromov convergence for the sequence of actions on $\epsilon_n \mathbb{H}^2$.

(4) Note that $\mathcal{GT}(S)$ is defined by using only the topology of S (its fundamental group indeed) while in order to adopt the dual view point we have to fix (in an arbitrary way) a base hyperbolic surface $F_0 \in T_g$. In fact, the dual view point can be developed by using the marked measure spectra of the measured (singular) foliations on S (instead of the measured geodesic laminations on F_0), which only depend on the differential structure of S (see [11]). On the other hand, let us consider T_g as a space of complex holomorphic structures on S (thanks to the classical Uniformization Theorem). By fixing any such structure F_0, one can realize such a spectrum as the measure spectrum of the horizontal measured foliation of a unique quadratic differential ω on F_0. These “rigidifications” (via geodesic laminations or quadratic differentials) of softer objects (the measured foliations) is reminiscent of the role of Hodge theory with respect to De Rham Cohomology.

4.5.2. CT asymptotic states as limit spectra

After this somewhat long but necessary digression, let us come back to the CT asymptotic states.

Proposition 4.25. (a) $\lim_{a \to 0} l_{S_a} = s_{\mathcal{T}}$; (b) $\lim_{a \to \infty} l_{S_a}/a = l_F$.

Remark 4.26. This means, in particular, that in a far CT future the spacetime looks like the Minkowskian suspension $M(F)$. In order to detect the dual effect of the initial singularity on the embedding of S_a into M, for large value of the cosmological time one needs to increase the accuracy in the measurement of geometric quantities. Nevertheless, this effect is, in principle, observable for any finite value a of the CT.

Proposition 4.27. *For every $a \in (0, \infty[$, l_{S_a}/a belongs to $T_g \in (\mathbb{R}_{\geqslant 0})^{\mathcal{C}}$. Hence, the cosmological time determines a curve $\gamma_M : (0, \infty[\to T_g$. This is a real analytic curve which connects $F \in T_g$ with the point on the natural boundary $[\mathcal{T}] \in \partial T_g$ (here $[.]$ denotes the projective class).*

Remark 4.28. Consider a spacetime of simplicial type. To prove Proposition 4.25 in this case, one has to note that the depth of the annular regions is constant on each S_a. When $a \to 0$, the contribution (to the length of any curve on S_a) of the part contained in the nonannular components becomes negligible, the length of the annuli boundaries goes linearly to zero, so that only the transverse crossing of the annuli becomes dominant. When $a \to \infty$, the annuli depth goes to zero because of the rescaling by $1/a$, and the length spectrum converge to the spectrum of F. The general case follows by using the density stated in Proposition 4.23. Concerning Proposition 4.27, in the special case of a spacetime of simplicial type, the curve in T_g is just given by the Fenchel–Nielsen flow obtained by "twisting" the hyperbolic surface F along the closed geodesic of the multi-curve (see [23] and also [7]).

4.6. A commentary on the proofs

The identification between cocycles of a spacetime M with measured geodesic laminations on $F = \mathbb{I}(1)/\Gamma$ is due to Mess [16]. In fact one can find other examples of such a construction of "cocycles" from measured laminations in the contest of Thurston's theory of "bending" or "earthquakes" (see for instance [10]).

Measured geodesic laminations emerged in the original Thurston's approach to the natural compactification of T_g [21–23]. See also [11] for the alternative approach by using the measured foliations (see Remark 4.24 (4)). For the claim about the quadratic differentials in Remark 4.24 (4)) see [14]. The dual approach via real trees is due to Morgan–Shalen [18,19]. This approach does apply to more general, higher-dimensional situations. The monography [20] contains a rather exhaustive introduction to this matter and we mostly refer to it (and to its bibliography) for all the details. In particular one can find in [20] a complete proof of the duality (see Proposition 4.19 and Proposition 4.20). The delicate point is just the inversion of the map Δ we have described above. The geometric interpretation of the Morgan–Shalen convergence (see Remark 4.24 (3)) is due to Paulin and Bestvina (cf. the bibliography of [20]).

It is an amazing fact that the spacetimes "materialize" this subtle duality in the way we have seen. Note also that, in the spacetime setting, the choice of the base hyperbolic surface

F (see Remark 4.24 (4)) is fixed by the linear part of the holonomy of M, that is by its future asymptotic state.

Concerning Proposition 4.27, the Fenchel–Nielsen flow generalizes to the earthquake flow (one uses again the density 4.23) with initial data $(F, \mathcal{F})$ which has real analytic orbits [15,24].

5. Complements

In this section we add a few comments about the flat spacetimes with compact space of genus $g = 1$, and about the spacetimes with negative cosmological constant. Finally we discuss a conjecture relating the CT and the CMC time.

5.1. *Toric space* $(g = 1)$

The case in which the surface S is a torus is a particular example of the so-called *Teichmüller spacetimes* which we have analysed in [5]. So we simply remind the main points. Each nonstatic spacetime determines a curve $\gamma : (0, \infty[\to T_1^*$, $\gamma(a) = (w(a), \omega(a))$, where T_1^* is the cotangent bundle of the Teichmüller space T_1 of conformal structures on the torus up to isomorphism isotopic to the identity. Let us recall that T_1 is isometric with the Poincaré disk. The cotangent vectors $\omega(a)$ at the point $w(a) \in T_1$ is a quadratic differential on a Riemann surface representing $w(a)$. It is not hard to verify that γ is just the complete orbit of the Teichmüller flow with initial data $(w(1), \omega(1))$ (see [1,5]). In particular, the projection of γ onto T_1 is a complete geodesic connecting two boundary points. These points can also be interpreted in terms of marked spectra. Let us denote by $\mathcal{H}$ and by $\mathcal{V}$ the horizontal and vertical measured foliations of $w(1)$. Then: $\lim_{a\to\infty} l_{S_a}/a = I_{\mathcal{H}}$ and $\lim_{a\to 0} l_{S_a} = I_{\mathcal{V}}$.

5.2. *Spacetime with negative cosmological constant*

The above discussion on CT for flat spacetimes (i.e., with cosmological constant $\Lambda = 0$) can be adapted to the case of negative Λ which we normalize to be $\Lambda = -1$. We denote by $\mathbb{X}^{2+1}$ the Universal anti-de-Sitter spacetime of dimension $2 + 1$. Each spacetime is now locally isometric to $\mathbb{X}^{2+1}$. The role played by $I^+(0)$ in the flat case, is played now by the diamond-shaped domain $D(2)$ (see [13] p. 132) isometric to $B^2 \times (-\pi/2, \pi/2)$ with metric, in coordinates (y^1, y^2, t), $ds^2 = (\cos^2 t)h_2 - dt^2$, where h_2 is the usual Poincaré hyperbolic metric on the open disk B^2.

5.2.1. *Anti-de-Sitter suspensions*

If $F = \mathbb{H}^2/\Gamma$ is a hyperbolic surface of genus $g > 1$, then Γ isometrically acts also on $D(2)$ and $D(F) = D(2)/\Gamma$ is the anti-de-Sitter suspension of F. Up to a translation, the function t gives the CT and it has many qualitative properties similar to the CT of the Minkowskian suspensions, but we have now both an initial and a final singularity, both reduced to one point. In a sense, $D(F)$ can be obtained by the Minkowskian suspension

$M(F)$ by a procedure of *warping and doubling*; $D(F)$ and $M(F)$ have the same initial singularity; the future asymptotic state of $M(F)$ "becomes" the level surface of the CT on $D(F)$ where the expansion ends and the collapsing begins. Also the anti-de-Sitter analogous of $I^+(1,3)$ is easy to figure out.

5.2.2. *Deforming anti-de-Sitter suspensions*

We want to generalize the above "warping and doubling" construction. Let $M = U(M)/\Gamma'$ as in the former flat-spacetime discussion, $\Gamma' = \Gamma + t(\Gamma)$. $F = \mathbb{I}(1)/\Gamma$ as usually. For $t \in (-\pi/2, 0)$, $\tau \in (0, \infty)$, set $t = -(\pi/2)e^{-\tau}$. Denote $h(a)$ the spatial metric on S_a. On the manifold $F \times (-\pi/2, 0)$ consider the metric $ds^2 = \cos^2(t)h(\tau)/\tau^2 - dt^2$, getting a spacetime $\mathcal{D}'(M)$. Similarly, take M^- and $-\mathcal{D}'(M^-)$, where $M^- = U(M^-)/-\Gamma'$, $-\Gamma' = \Gamma - t(\Gamma)$, $-\mathcal{D}'(M^-)$ is obtained from $\mathcal{D}'(M^-)$ by reversing the time and the orientation. Finally, set $\mathcal{D}(M) = \mathcal{D}'(M) \cup -\mathcal{D}'(M^-)$, by gluing the two pieces at $t = 0$. $\mathcal{D}(M)$ is locally anti-de-Sitter; up to a translation, t gives the CT. The asymptotic state for $t \to -\pi/2$ (i.e., the initial singularity) is equal to the initial singularity of M. The final singularity ($t \to \pi/2$) coincides with the initial singularity of M^-. The future asymptotic states of M and M^- "glue" at the level surface $\{t = 0\}$ of the CT where the expansion ends and the collapse begins. The orbit of $\mathcal{D}(M)$ in T_g is given by the union of two earthquake rays associated to M (pointing to the future) and to M^- (towards the past); note that the qualitative behaviour is similar to what we have remarked for $g = 1$. $\mathcal{D}(M)$ is the quotient of a domain $D(2)_M \subset \mathbb{X}^{2+1}$, which is a "deformation" of the diamond-shaped domain $D(2)$. Also in this case the spacetimes with simplicial asymptotic singularities are significant and particularly simple to be understood.

5.3. *CT versus CMC*

Assume again that the space S is of genus $g > 1$, and that the spacetimes are flat. Given any global time on a spacetime $M = U(M)/\Gamma'$, the asymptotic behaviour of the geometry of the corresponding level surfaces reflects in general a property of the specific time and not of the spacetime. On the other hand, we have seen that the asymptotic states of the cosmological time are intrinsic features of the spacetime. In this sense, we can say that a global time is "good" when it has the same asymptotic states of the CT. The CMC time, ρ say, is a widely studied global time. A natural question is whether ρ is a good global time. We conjecture that this is the case. Let us denote by W_a the $\{\rho = a\}$ level surfaces of the CMC time.

Conjecture 5.1. (a) $\lim_{a\to\infty} l_{W_a} = s_{\mathcal{T}}$; (b) $\lim_{a\to 0} l_{W_a}/a = l_F$.

There are some strong evidences supporting the conjecture; in particular by [3] we know that:

(1) ρ is a global time function with image the interval $(0, +\infty)$.
(2) If $\gamma : (0, \infty) \to T_g$ is any ρ-orbit in T_g (here T_g is intended as a space of conformal structures) then:

(i) The $\lim_{\rho\to 0}\gamma$ exists in T_g.
(ii) γ is proper, that is it goes out from any compact set of T_g, roughly it "goes to ∞".

An idea to prove the conjecture, should be to confine each W_a between two barriers made by CT-level surfaces $S_{a'}$, $S_{a''}$, in such a way that a' and a'' depend nicely on a and, when $a \to \infty$ or $a \to 0$, $S_{a'}$ and $S_{a''}$ become more and more "geometrically" close to each other. In a recent conversation, L. Andersson confirmed that this should actually work at least for a spacetime with simplicial initial singularity. A similar conjecture can be formulated in the anti-de-Sitter context.

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