in $\mathbb{R}^4$

Sia V l'insieme delle soluzioni di

$$S = \begin{cases} x + 2y + z = 0 \\ -x - y + 3t = 0 \end{cases}$$

Sia $W = \operatorname{span}\left(\begin{pmatrix}2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}3\\-2\\-2\\0\end{pmatrix}\right)$

calcolare $\dim(V \cap W)$ e $\dim(V+W)$.

<u>Soluzione</u>:

$$\begin{pmatrix}x\\y\\z\\t\end{pmatrix} \in V \iff \begin{pmatrix}1 & 2 & 1 & 0\\ -1 & -1 & 0 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

Mosse di riga

$$\begin{pmatrix}1 & 2 & 1 & 0\\ -1 & -1 & 0 & 3\end{pmatrix} \xrightarrow[r_2+r_1]{} \begin{pmatrix}1 & 2 & 1 & 0\\ 0 & 1 & 1 & 3\end{pmatrix}$$

$$\xrightarrow[r_1-2r_2]{} \begin{pmatrix}1 & 0 & -1 & -6\\ 0 & 1 & 1 & 3\end{pmatrix}$$

$$S \equiv \begin{pmatrix}1 & 0 & -1 & -6\\ 0 & 1 & 1 & 3\end{pmatrix}\begin{pmatrix}x\\y\\z\\t\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

z, t liberi.

$$\begin{cases} y = -z - 3t \\ x = z + 6t \end{cases}$$

soluzione generica $\begin{pmatrix}z+6t\\-z-3t\\z\\t\end{pmatrix} \in V$

$$= \begin{pmatrix}z\\-z\\z\\0\end{pmatrix} + \begin{pmatrix}6t\\-3t\\0\\t\end{pmatrix} = z\begin{pmatrix}1\\-1\\1\\0\end{pmatrix} + t\begin{pmatrix}6\\-3\\0\\1\end{pmatrix} \in \operatorname{span}\left(\begin{pmatrix}1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}6\\-3\\0\\1\end{pmatrix}\right)$$

$= V$

Quindi

$$V+W = \text{span}\begin{pmatrix} 1 & 6 & 2 & 3 \\ -1 & -3 & 0 & -2 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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Mosse di colonna non alterano lo span

$$\xrightarrow[C_2-6C_1]{} \begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 3 & 0 & -2 \\ 1 & -6 & 1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[C_3-2C_1]{} \begin{pmatrix} 1 & 0 & 0 & 3 \\ -1 & 3 & 2 & -2 \\ 1 & -6 & -1 & -2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow[C_3-\frac{2}{3}C_2]{} \begin{pmatrix} 1 & 0 & 0 & 3 \\ -1 & 3 & 0 & -2 \\ 1 & -6 & 3 & -2 \\ 0 & 1 & 1/3 & 0 \end{pmatrix} \xrightarrow[C_4-3C_1]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 1 \\ 1 & -6 & 3 & -5 \\ 0 & 1 & 1/3 & 0 \end{pmatrix}$$

$$\xrightarrow[C_4-\frac{1}{3}C_2]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & -6 & 3 & -3 \\ 0 & 1 & 1/3 & -1/3 \end{pmatrix} \xrightarrow[C_4-C_3]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 3 & 0 & 0 \\ 1 & -6 & 3 & 0 \\ 0 & 1 & 1/3 & 0 \end{pmatrix}$$

dim $V+W$ = Rango colonne $= 3$.

Resta da calcolare $\dim(V \cap W)$. Con Grassmann:

$$\begin{aligned} \dim(V+W) &= \dim(V \times W) - \dim(V \cap W) \\ &= \dim V + \dim W - \dim(V \cap W) \\ 3 &= 2 + 2 - \dim(V \cap W) \end{aligned}$$

Quindi $\dim(V \cap W) = 1$.

~~Domanda aggiuntiva: trovare una base di $V \cap W$.~~

$w \in W \Rightarrow w = a(2,0,1,1) + b(3,-2,-2,0)$

~~generico elemento di W.~~

~~Impongo che stia in V. Sostituisco nel sistema che definisce V e ottengo~~

$w = (2a+3b, -2b, a-2b, a) \in V \Leftrightarrow$