Topological and Nonstandard Extensions*

Mauro Di Nasso
Dipart. di Matematica "L. Tonelli"
Università di Pisa, Italy.
dinasso@dm.unipi.it

Marco Forti
Dipart. di Matem. Applicata "U. Dini"
Università di Pisa, Italy.
forti@dma.unipi.it

Abstract

We introduce a notion of *topological extension* of a given set X. The resulting class of topological spaces includes the Stone-Čech compactification βX of the discrete space X, as well as all nonstandard models of X in the sense of nonstandard analysis (when endowed with a "natural" topology). In this context, we give a simple characterization of nonstandard extensions in purely topological terms, and we establish connections with special classes of ultrafilters whose existence is independent of ZFC.

Introduction

The problem of extending the sum and product operations on the natural numbers $\mathbb{N}$ to the Stone-Čech compactification $\beta\mathbb{N}$ was considered already in the late fifties, when a study of the nonstandard models of arithmetics began. By a number of different straightforward arguments, it was soon shown that

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there are no such extensions which are *continuous*. Later on in the sixties, in the early days of nonstandard analysis, the similar question was raised as to whether $\beta\mathbb{N}$ could be naturally given a structure of nonstandard model, thus yielding "canonical" extensions for all n-place operations. The answer was again in the negative, in that any nonstandard model containing $\beta\mathbb{N}$ realizes each non-principal ultrafilter on $\mathbb{N}$ *infinitely many* times (see e.g. the discussion contained in A. Robinson's paper [28]).

The connections between nonstandard extensions and ultrafilters have been repeatedly considered in the literature, starting from the seminal paper [24] by W.A.J. Luxemburg. Of particular relevance is also the work by C. Puritz [26], where an investigation of the model-theoretic properties of special classes of ultrafilters is started (see also [13]).

In the seventies, an interest arose on $\beta\mathbb{N}$ viewed as an algebraic object with compact topology. Starting from the proof of the famous Finite Sums Theorem by F. Galvin and S. Glazer, N. Hindman started a systematic study of the compact right-topological semigroups $\langle\beta(\mathbb{N}\times\mathbb{N}),+\rangle$ and $\langle\beta(\mathbb{N}\times\mathbb{N}),\cdot\rangle$. The resulting theory yielded a number of remarkable results and applications in Ramsey Theory (see the recent book [19]).

In this paper we introduce a notion of *topological extension* that naturally accomodates both Stone-Čech compactifications of discrete spaces and nonstandard models, within a general unified framework. A main feature shared by compactifications and completions in topology and by nonstandard models of analysis is the existence of a "canonical" extension ${}^*f : {}^*X \to {}^*X$ for each function $f : X \to X$. Given an arbitrary set X, we consider here a topological extension of X as a sort of "topological completion" ${}^*X$, where the "$*$" operator provides a *distinguished continuous extension* of each function $f : X \to X$. Moreover the "$*$" operator can be extended also to subsets of X as the *closure* operator.

Topological extensions which are *Hausdorff spaces* have been introduced and studied in [6]. Every such extension ${}^*X$ can be identified with an "invariant" subspace of the Stone-Čech compactification βX of the discrete space X (i.e. a subspace closed under all extended functions $\overline{f}$). In this way, the nonstandard elements of ${}^*X$ may be viewed as nonprincipal ultrafilters over X. More important, the continuous extension ${}^*f$ of the function f is uniquely determined by the topology of βX, and agrees with the usual function $\overline{f}$ on ultrafilters, namely ${}^*f(\alpha) = \overline{f}\,(\alpha) = \{f^{-1}\,(A) : A \in \alpha\}$.

The existence of such subspaces of $\beta\mathbb{N}$ that preserve the arithmetic properties (and in fact are nonstandard models of $\mathbb{N}$) is equivalent to the existence of a special class of ultrafilters, appropriately named *Hausdorff* here. Hausdorff ultrafilters are characterized by a property, labelled (C) in [14] and [8], which has been rarely considered in the literature. The only known examples involve *selective* ultrafilters, whose existence is consistent with, but independent of Zermelo-Fraenkel set theory ZFC. Actually, extensions corresponding to *selective* ultrafilters are of interest for their own sake. They provide models that are minimal and canonical under various respects (see e.g. [7] and Subsection 6.4 below).

As the Hausdorff case reveals to be too restricted, we only require here that topological extensions be T_1 spaces (i.e. that points are closed). As a consequence we lose *uniqueness* of continuous extensions, and so we postulate that the functions ${}^*f$ are chosen in such a way that compositions and identities are preserved. (I.e. ${}^*(f \circ g) = {}^*f \circ {}^*g$, and if f is the identity on a subset A, then ${}^*f$ is the identity on its closure $\overline{A}$.) This choice increases dramatically the range of possible models. It allows for embracing at once all possible nonstandard models together with many more general structures. In particular we endow all nonstandard extensions with a natural "Star" topology. The resulting topological extensions are characterized by two simple and natural additional properties, called *analiticity* and *coherence*. The former property isolates a fundamental feature of nonstandard extensions, which marks the difference with respect to the commonly considered continuous extensions: "if $f(x) \neq g(x)$ for all $x \in X$, then ${}^*f(\xi) \neq {}^*g(\xi)$ for all $\xi \in {}^*X$". The latter property is a sort of "amalgamation", useful for extending multivariate functions: "for all $\eta, \zeta \in {}^*X$, there are $\xi \in {}^*X$ and functions $f, g : X \to X$ such that ${}^*f(\xi) = \eta$ and ${}^*g(\xi) = \zeta$".

In our opinion, this topological approach to nonstandard models should be considered on a par with several other approches that avoid a direct use of logic, like, e.g., the interesting and elementary approach of [22], or the arithmetical approach of [4].

The paper is organized as follows. In Section 1, we introduce the notion of topological extension and we give the first important "preservation properties". In Section 2, we study the canonical map from a topological extension of X into the Stone-Čech compactification βX of the discrete space X. In Section 3 we define the "Star" topology of any nonstandard extension. In Section 4 we consider "small" (*principal*) extensions, and we give necessary and sufficient conditions for being ultrapower extensions. In Section 5, we characterize all elementary (nonstandard) extensions. Section 6 contains, *in-*

ter alia, algebraic, topological, and set-theoretic characterizations of *proper* and *simple* extensions; a sketchy study of weak compactness properties of topological extensions; a review of the set-theoretic problems originated by simple and Hausdorff extensions. We conclude with some open questions and suggestions for further research.

In general, we refer to [16] for all the topological notions and facts used in this paper, and to [12] for definitions and facts concerning ultrapowers, ultrafilters, and nonstandard models. General references for nonstandard analysis could be [22, 1], and the recent [17].

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1 Topological extensions

The main common feature, shared by *compactifications* and *completions* in topology, and by *nonstandard extensions* of analysis, is the existence of a canonical extension ${}^*f : {}^*X \to {}^*X$ of any (standard) function $f : X \to X$. We use this property to define the notion of *topological extension* of a set X.

Definition 1.1 Let X be a *dense* subspace of the T_1 topological space ${}^*X$. We say that ${}^*X$ is a *topological extension* of X if to every function $f : X \to X$ is associated a distinguished *continuous extension* ${}^*f : {}^*X \to {}^*X$ in such a way that compositions and local identities are preserved, i.e.

(c) ${}^*g \circ {}^*f = {}^*(g \circ f)$ for all $f, g : X \to X$, and

(i) if $f(x) = x$ for all $x \in A \subseteq X$, then ${}^*f(\xi) = \xi$ for all $\xi \in \overline{A}$.

Notice that a finite set X cannot have *nontrivial* topological extensions, because finite sets are closed in T_1 spaces. Hence we may restrict ourselves to consider only infinite sets X. For convenience we shall assume in the sequel that $\mathbb{N} \subseteq X$. In particular we have an extension of the *characteristic function*

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

of any subset A of X. Following the common use in nonstandard analysis, we shall call *standard* the points of X and *nonstandard* those of ${}^*X \setminus X$.

Notice that, if a topological extension ${}^*X$ of X is Hausdorff, then ${}^*f$ is the *unique* continuous extension of f, for X is dense. In this case (c) and (i) are automatically satisfied, and our definition would have been much simpler. However considering only Hausdorff spaces would have turned out too restrictive. In fact, as shown in [6], they reduce to a suitable class of subspaces of βX (see Section 2 below). In particular, it follows from Theorem 6.5 that a nonstandard Hausdorff extension cannot be $(2^{\aleph_0})^+$-enlarging.[1]

Topological extensions satisfy various natural "preservation properties". We begin by stating the following lemma.

Lemma 1.2 *Let ${}^*X$ be a topological extension of X. Then extensions of functions preserve finite ranges, i.e. ${}^*f({}^*X) = f(X)$ whenever $f(X)$ is finite. In particular*

(i) if $c_x : X \to X$ is the constant function with value $x \in X$, then its extension ${}^*c_x$ is the constant function with value x on ${}^*X$;

(ii) if $\chi_A : X \to X$ is the characteristic function of the subset $A \subseteq X$, then its extension ${}^*\chi_A = \chi_{{}^*A}$ is the characteristic function of a set ${}^*A$, which is the closure $\overline{A}$ of A in ${}^*X$.

Moreover $\overline{A}$ is (cl)open for all $A \subseteq X$, and the closure operator $A \mapsto \overline{A}$ is an isomorphism of the boolean algebra $\mathcal{P}(X)$ onto the field $\mathcal{CO}({}^*X)$ of all clopen subsets of ${}^*X$, whose inverse map is $C \mapsto C \cap X$. (It is in fact a complete isomorphism if $\mathcal{CO}({}^*X)$ is viewed as the complete boolean algebra $\mathcal{RO}({}^*X)$ of the regular open subsets of ${}^*X$.)

Proof. The inclusion ${}^*f({}^*X) \subseteq \overline{{}^*f(X)}$ holds for all continuous functions, since X is dense in ${}^*X$. Hence ${}^*f(X) = f(X) = \overline{f(X)}$ whenever $f(X)$ is finite.

In particular, extensions of constant or characteristic functions are constant or characteristic functions, respectively. Moreover ${}^*\chi_A(\overline{A}) \subseteq \{1\}$ and ${}^*\chi_A(\overline{X \setminus A}) \subseteq \{0\}$. Therefore $\overline{A}$ and $\overline{X \setminus A}$ form a clopen partition of ${}^*X$.

The closure operator commutes with binary unions and with complements of *clopen* sets. Moreover different clopen subsets of ${}^*X$ cannot have the same intersection with X, which is dense in ${}^*X$. It follows that $\overline{U \cap X}$ is the interior of the closure of the open subset $U \subseteq {}^*X$. Thus $\mathcal{CO}({}^*X) = \mathcal{RO}({}^*X)$, and the last statement is completely proven.

□

[1] The enlarging property is commonly used in nonstandard analysis. For instance, κ^+-enlargments are used in the study of topological spaces of character κ (see e.g. [23]).

Notice that neither of the properties (i) and (c) has been used in proving the above lemma. Many more preservation properties depend on these assumptions. We list below a few of the most important and natural ones, concerning *restrictions, ranges,* and *injectivity.*

Lemma 1.3 *Let* $^*X$ *be a topological extension of* X*. Then, for all* $A \subseteq X$ *and all* $f, g : X \to X$*:*

(i) *if* $f(x) = g(x)$ *for all* $x \in A$*, then* $^*f(\xi) = {}^*g(\xi)$ *for all* $\xi \in \overline{A}$*;*

(ii) $^*f(\overline{A}) = \overline{f(A)}$ *(in particular* $^*f$ *is surjective iff* f *is surjective);*

(iii) *if* $f : X \to X$ *is injective on* A*, then* $^*f$ *is injective on* $\overline{A}$*.*

Proof. In order to prove (i), assume that for all $x \in A$ $f(x) = g(x)$, and pick a function h satisfying $h(X) = A$ and $h(x) = x$ for all $x \in A$. Then $f \circ h = g \circ h$, hence $^*f \circ {}^*h = {}^*g \circ {}^*h$, by (c). Moreover $^*h$ is the identity on $\overline{A}$, by (i). Hence $^*f$ and $^*g$ must agree on $\overline{A}$.

The inclusion $^*f(\overline{A}) \subseteq \overline{f(A)}$ holds for all continuous functions. Therefore (ii) and (iii) follow from (c) and (i), because (the restriction of) a function is injective (resp. surjective) if and only if it has a left (resp. right) inverse. $\square$

For the benefit of the reader we recall below some basic *separation properties*, which are used in the sequel (see [16]). Let S be a topological space.

- S is T_0 if any two *different points* $x, y \in S$ are *separated by an open set* O (i.e. $\emptyset \neq O \cap \{x, y\} \neq \{x, y\}$).
- S is T_1 if all *points* of S are *closed*.
- S is *Hausdorff* (T_2) if any two *different points have disjoint neighborhoods*.
- S is *regular* (T_3) if S is T_1 and the *closed* neighborhoods are a *neighborhood basis* of any point.
- S is 0-*dimensional* if S is T_1 and the *clopen* sets are a *basis* of the topology.

It is quite natural to consider on $^*X$ the S-*topology*, i.e. the topology generated by the (clopen) sets $^*A = \overline{A}$ for $A \subseteq X$.[2] The S-topology is obviously coarser than or equal to the original topology of $^*X$, and we have

[2] The S-topology (for *Standard* topology) is a classical notion of nonstandard analysis, already considered in [27].

Theorem 1.4 *Let ${}^*X$ be a topological extension of X. Then*
1. *The S-topology of ${}^*X$ is either* 0*-dimensional or not* T_0.
2. *${}^*X$ is Hausdorff if and only if the S-topology is T_1, hence* 0*-dimensional.*
3. *${}^*X$ is regular if and only if the S-topology is the topology of ${}^*X$ (and so ${}^*X$ is* 0*-dimensional).*

Proof.
1. The S-topology has a clopen basis by definition. In this topology the closure of a point ξ is $M_\xi = \bigcap_{\xi \in \overline{A}} \overline{A}$. If $M_\xi = \{\xi\}$ for all $\xi \in {}^*X$, then the S-topology is T_1, hence 0-dimensional. Otherwise let $\eta \neq \xi$ be in M_ξ. Then η belongs to the same clopen sets as ξ, and the S-topology is not T_0. In fact, given $A \subseteq X$, $\xi \in \overline{A}$ implies $\eta \in \overline{A}$, by the choice of η. Similarly $\xi \notin \overline{A}$ implies $\xi \in \overline{X \setminus A}$, hence $\eta \in \overline{X \setminus A}$ and $\eta \notin \overline{A}$.

2. By point 1, the S-topology is Hausdorff (in fact 0-dimensional) whenever it is T_1. Therefore also the topology of ${}^*X$ is Hausdorff, being finer than the S-topology. For the converse, let U, V be disjoint neighborhoods of the points $\xi, \eta \in {}^*X$, and put $A = U \cap X$, $B = V \cap X$. Then $\xi \in \overline{A}$, $\eta \in \overline{B}$, and $\overline{B} \cap \overline{A} = \emptyset$. Therefore $\eta \notin M_\xi$, and the S-topology is T_1.

3. The closure of an open subset $U \subseteq {}^*X$ is the clopen set $\overline{U \cap X}$. Therefore any closed neighborhood of $\xi \in {}^*X$ includes a clopen one. Since the clopen sets are a basis of the S-topology, ${}^*X$ can be regular if and only if its original topology is the S-topology (and so the latter is T_1, hence 0-dimensional).

□

2 The canonical map into βX

Any Hausdorff extension of X is canonically embeddable into the Stone-Čech compactification[3] βX of X, and the corresponding subspaces can be completely characterized (see Theorem 1.5 of [6], and [25] for the case of nonstandard models).

Here we define a *canonical map* $\upsilon : {}^*X \to \beta X$, for any topological extension ${}^*X$ of X. Namely

[3] For various definitions and properties of the Stone-Čech compactification see [16]. Here we only recall that if X is a discrete space, then βX can be identified with the set of all ultrafilters over X, endowed with the topology having as basis $\{\mathcal{O}_A \mid A \in \mathcal{P}(X)\}$, where $\mathcal{O}_A$ is the set of all ultrafilters containing A. (The embedding $e : X \to \beta X$ is given by the principal ultrafilters.)

- for $\xi \in {}^*X$ let $\mathcal{C}_\xi$ be the family of all clopen subsets containing ξ, which is a maximal filter in $\mathcal{CO}({}^*X)$;

- let $\mathcal{U}_\xi = \{A \subseteq X \mid \xi \in {}^*A\}$ be the ultrafilter over X corresponding to $\mathcal{C}_\xi$ in the isomorphism between $\mathcal{CO}({}^*X)$ and $\mathcal{P}(X)$;

- let $\upsilon(\xi)$ be the point determined by $\mathcal{U}_\xi$ in the Stone-Čech compactification of the discrete space X.[4]

Theorem 2.1 *Let ${}^*X$ be a topological extension of X. Then the canonical map $\upsilon : {}^*X \to \beta X$ is the unique continuous map extending the canonical embedding $e : X \to \beta X$.*

For any $f : X \to X$ let $\overline{f}$ be the unique continuous extension[5] of f to βX; then the following diagram commutes:

$$\begin{tikzcd}
\beta X \arrow[rrr, "\overline{f}"] & & & \beta X \\
& X \arrow[ul, "e"'] \arrow[r, "f"] \arrow[dl, "\imath"] & X \arrow[ur, "e"] \arrow[dr, "\imath"'] & \\
{}^*X \arrow[uu, "\upsilon"] \arrow[rrr, "{}^*f"] & & & {}^*X \arrow[uu, "\upsilon"']
\end{tikzcd}$$

Moreover υ is injective if and only if ${}^*X$ is Hausdorff, and it is a homeomorphism if and only if ${}^*X$ is regular. Finally υ is surjective if and only if the S-topology is quasi-compact (i.e. every filter in $\mathcal{CO}({}^*X)$ has nonempty intersection).

Proof. For all $x \in X, \mathcal{U}_x$ is the principal ultrafilter generated by x, hence υ induces the canonical embedding of X into βX. If $\mathcal{O}_A$ is a basic open set of βX, then $\upsilon^{-1}(\mathcal{O}_A) = \overline{A}$, hence υ is continuous w.r.t. the S-topology, and *a fortiori* w.r.t. the (not coarser) topology of ${}^*X$. On the other hand, let a continuous map $\varphi : {}^*X \to \beta X$ be given. Since $\mathcal{O}_A$ is clopen, also $\varphi^{-1}(\mathcal{O}_A)$ is clopen and, by Lemma 1.2, it is the closure $\overline{B}$ of some $B \subseteq X$. If φ is the identity on X, then $\overline{B} \cap X = A$, hence $B = A$ by the same lemma. Therefore all points of $M_\xi = \bigcap \mathcal{C}_\xi$ are mapped by φ onto $\upsilon(\xi)$, and $\upsilon = \varphi$.

By point (ii) of Lemma 1.3, we have $\xi \in \overline{A} \iff {}^*f(\xi) \in \overline{f(A)}$, for all $\xi \in {}^*X$, or equivalently $A \in \mathcal{U}_\xi \iff f(A) \in \mathcal{U}_{{}^*f(\xi)}$. It follows that $\overline{f} \circ \upsilon = \upsilon \circ {}^*f$, which is the only nontrivial commutation in the diagram.

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[4] A detailed study of the canonical map υ and its properties in the context of the S-topology of arbitrary nonstandard models can be found in [25].

[5] In terms of ultrafilters, $\overline{f}$ can be defined by putting $A \in \overline{f}(\mathcal{U}) \Leftrightarrow f^{-1}(A) \in \mathcal{U}$.

The map υ is injective if and only if the S-topology is T_1, and this fact is equivalent to ${}^*X$ being Hausdorff, by Theorem 1.4. Moreover in this case υ is a homeomorphism w.r.t. the S-topology, which is the same as the topology of ${}^*X$ if and only if the latter is regular (hence 0-dimensional).

Finally, the map υ is surjective if and only if every maximal filter in the field of sets $\mathcal{CO}({}^*X)$ has nonempty intersection. This is equivalent to every proper filter in $\mathcal{CO}({}^*X)$ having nonempty intersection, which in turn is equivalent to every proper filter of closed sets in the S-topology having nonempty intersection, i.e. to quasi-compactness.

□

Notice that the map υ provides a bijection between the basic open sets $\mathcal{O}_A$ of βX and the clopen subsets $\overline{A}$ of ${}^*X$. Therefore υ is open if and only if ${}^*X$ has the S-topology.

Call *invariant* a subspace S of ${}^*X$ (or of βX) if it is closed under *functions, i.e. if $\xi \in S$ implies ${}^*f(\xi) \in S$ for all $f : X \to X$.

It is easily seen that any invariant subspace S of ${}^*X$ is itself a topological extension of X, and it is mapped by υ onto an invariant subspace of βX.

It follows that we can so characterize all Hausdorff extensions of X:

Corollary 2.2 (see Theorem 1.5 of [6]) *Every invariant subspace $Y \subseteq \beta X$ is a Hausdorff (actually 0-dimensional) extension of X with the S-topology.*

Conversely, ${}^*X$ is a Hausdorff (regular) extension of X if and only if the map υ provides a continuous bijection (a homeomorphism) of ${}^*X$ onto an invariant subspace Y of βX.

Proof. If ${}^*X$ is homeomorphic to a subspace of βX, then it is 0-dimensional, hence it has the S-topology, by Theorem 1.4. Conversely, if ${}^*X$ has the S-topology, then υ is injective. Moreover, for all $A \subseteq X$, $\upsilon(A) = \mathcal{O}_A \cap \upsilon({}^*X)$, hence υ is a homeomorphism between ${}^*X$ and its image.

If ${}^*X$ is Hausdorff but not regular, then υ is injective and continuous, but not open.

□

Notice that υ can always be turned into a homeomorphism whenever ${}^*X$ is Hausdorff. Namely one can either endow Y with a finer topology, whose open sets are the images of all open subsets of ${}^*X$, or else one can take ${}^*X$ with the (coarser) S-topology. In both cases (the restrictions to Y of) all those continuous functions $f : \beta X \to \beta X$ which map X into X remain obviously continuous. Thus all Hausdorff extensions use substantially the same "function-extending mechanism", namely that arising from the Stone-Čech compactification. Therefore, in some sense, the choice of the topology is immaterial (provided that it is *not coarser* than the S-topology).

3 The Star topology

Let us now give a closer look at the topology of topological extensions. We have already seen that this topology is 0-dimensional if and only if it coincides with the S-topology. It is worth mentioning that the latter is a sort of "Zariski topology", having as closed sets all sets of *zeroes of systems of *functions*

$$Z(f_i \mid i \in I) = \{\xi \in {}^*X \mid {}^*f_i(\xi) = 0 \quad \forall i \in I\}.$$

In fact any such *finite* system describes the set $\overline{A}$ for suitable $A \subseteq X$.

However, when considering extensions having a topology strictly finer than the S-topology, and so necessarily not regular, we have to take into account more general closed sets. As every topological extension ${}^*X$ is a T_1 space, we know that all sets of the form

$$E(f, \eta) = \{\xi \in {}^*X \mid {}^*f(\xi) = \eta\} \;\text{ for } f : X \to X, \; \eta \in {}^*X$$

are closed in ${}^*X$. Since this family may not be stable under finite unions, we have to consider the sets

$$E(\vec{f}, \vec{\eta}) = E(f_1, \ldots, f_n; \eta_1, \ldots, \eta_n) = \bigcup_{1 \le i \le n} E(f_i, \eta_i)$$

for all n-tuples of functions $f_i : X \to X$, and of points $\eta_i \in {}^*X$. The (arbitrary) intersections of these sets form a family which is stable under finite unions, by distributivity

$$\bigcup_{i \in I} \bigcap_{j \in J} E_{ij} = \bigcap_{f : I \to J} \bigcup_{i \in I} E_{i f(i)}.$$

Hence the $E(\vec{f}, \vec{\eta})$s can be taken as a basis of the *closed* sets of a topology, which is clearly the *coarsest T_1 topology*[6] on ${}^*X$ *that makes all functions ${}^*f$ continuous.* Since X is dense in ${}^*X$ w.r.t. this topology, we have:

[6] In general, the family of all topologies on a space S that make continuous a given set of functions $\mathcal{F} \subseteq S^S$ is closed under arbitrary intersections. Therefore, given a topological extension ${}^*X$ of X, there is always a *least T_1 topology* that leaves all functions ${}^*f$ continuous.

Theorem 3.1 *Let ${}^*X$ be a topological extension. Then the sets $E(\vec{f},\vec{\eta})$ are a basis of the closed sets of the coarsest topology on ${}^*X$ that makes ${}^*X$, with the given choice of the functions ${}^*f$, a topological extension of X.*

$\square$

Since this topology is intrinsically connected with the choice of the distinguished continuous extensions ${}^*f$, we call it the *Star topology* of ${}^*X$. We say that ${}^*X$ is a *star-extension* if it has the Star topology.

Whenever the interest focuses on the "nonstandard behaviour" of the topological extension ${}^*X$, one can obviously assume w.l.o.g. to have already a star-extension. Moreover this assumption has also several advantages from the topological point of view: e.g. no strictly finer topology can be *quasi-compact*, or make the canonical map υ *open*. Therefore we may restrict ourself to star-extensions whenever convenient.

Clearly the *Star topology* agrees with the S-topology when the latter is T_1, or equivalently when ${}^*X$ is Hausdorff. In the general context, the Star topology can be considered even closer than the S-topology to the spirit of the Zariski topology in algebraic geometry. Actually the Zariski closed subsets of an algebraic variety can also be defined as the counterimages of the finite subsets under algebraic functions.

The main tool of the so called *nonstandard methods* is the study of extensions which preserve those properties of the standard structure which are being considered. The *Transfer* (*Leibniz*) *Principle* states that all properties that are expressible in a(*n in*)sufficiently expressive language are preserved by passing to the nonstandard models. We shall see that our notion of topological extension is general enough to naturally embody all nonstandard models in the sense of nonstandard analysis.

More precisely, we construct a *star-extension* starting from an arbitrary *elementary extension* ${}^*X$ of X, provided that *every function* $f : X \to X$ *has a corresponding symbol in the language* (which we denote again by f, for sake of simplicity). Let ${}^*f$ be the interpretation of f in ${}^*X$. We proceed in the natural way and define the sets $E(\vec{f},\vec{\eta})$ as above. Then we have

Theorem 3.2 *Let ${}^*X$ be an elementary extension of X. The sets $E(\vec{f},\vec{\eta})$ are a basis of the closed sets of a topology τ on ${}^*X$. When endowed with τ, the space ${}^*X$ becomes a star-extension of X.*

Proof. First of all the topology τ is T_1, since $\{\xi\} = E(\imath,\xi)$, where $\imath$ is the identity on X. Secondly, the property (c) of Definition 1.1 holds in ${}^*X$, by elementary equivalence, and so $({}^*g)^{-1}(E(\vec{f},\vec{\eta})) = E(\vec{f}\circ g,\vec{\eta})$ is closed for all $g,\vec{f}$. Therefore all functions ${}^*f$ are continuous w.r.t. τ, which is indeed the coarsest T_1 topology with this property.

In order to prove that X is dense, assume that $X \subseteq E(\vec{f},\vec{\eta})$. The function ${}^*f_i$ maps the point $x \in X$ to the point $f_i(x) \in X$. Hence we can consider only those components $\eta_i = y_i$ of $\vec{\eta}$ which belong to X, and we have that

$$\forall x \in X \ (f_1(x) = y_1 \vee \ldots \vee f_n(x) = y_n).$$

Again by elementarity, we have that

$$\forall \xi \in {}^*X \ ({}^*f_1(\xi) = y_1 \vee \ldots \vee {}^*f_n(\xi) = y_n),$$

and so ${}^*X$ is included in $E(\vec{f},\vec{\eta})$.

We are thus left with property (i) of Definition 1.1, which is apparently not first order, and so not immediately transferable. However, we can express the left member of (i) by means of the characteristic function χ of the subset $A \subseteq X$, and obtain

$$\forall x \in X \ (\chi(x) = 1 \Rightarrow f(x) = x).$$

From this, by transfer we get

$$\forall \xi \in {}^*X \ ({}^*\chi(\xi) = 1 \Rightarrow {}^*f(\xi) = \xi).$$

Now $E(\chi, 1)$ is a closed set including A, and so the last implication yields the right member of (i).

□

We shall see in Section 5 below that every elementary extension in a language including all unary functions is in fact a *nonstandard model*, i.e. a *complete* elementary extension w.r.t. all n-ary functions and relations. Many interesting properties can be derived by transfer, so as to further specify those topological extensions that can be generated by nonstandard models. We shall consider this topic in the following sections.

4 Principal and analytic extensions

In this section we deal with important classes of topological extensions, which are strictly connected with the ultrapower construction.

Recall that two functions $f, g : X \to X$ are equivalent modulo $\mathcal{U}$, where $\mathcal{U}$ is an arbitrary ultrafilter over X, if they agree on some $U \in \mathcal{U}$. The ultrapower $X^X/\mathcal{U}$ is the set of the equivalence classes modulo $\mathcal{U}$ of all functions $f : X \to X$. We refer to [12] for basic facts about ultrapowers. In the sequel, in dealing with ultrapowers, we shall adhere to the following notation:

- $[f] \in X^X/\mathcal{U}$ is the equivalence class of the function $f : X \to X$;
- $\overline{g} : X^X/\mathcal{U} \to X^X/\mathcal{U}$ is the *interpretation* of the function g in the ultrapower, i.e. $\overline{g}([f]) = [g \circ f]$ for all $f, g : X \to X$;
- $A^X/\mathcal{U} \subseteq X^X/\mathcal{U}$ is the *interpretation* of $A \subseteq X$ in the ultrapower.

The subsets $A^X/\mathcal{U}$, for $A \subseteq X$, are a *clopen basis* of a topology, which is precisely the *S-topology* of the ultrapower $X^X/\mathcal{U}$, viewed as a nonstandard (elementary) extension of X. Similarly, the *Star topology* on $X^X/\mathcal{U}$ can be defined as the coarsest topology where all sets $E(g,h) = \{[f] \mid [g \circ f] = [h]\}$ are closed.

Let ${}^*X$ be a topological extension of X. For $\alpha \in {}^*X$ put

$$ {}^*X_\alpha = \{{}^*f(\alpha) \mid f : X \to X\}. $$

- ${}^*X_\alpha$ is *invariant*, and actually the *least* invariant subspace of ${}^*X$ *containing* α. We call it the *principal subspace* of ${}^*X$ *generated by* α.
- We say that ${}^*X$ is *principal* if it is equal to ${}^*X_\alpha$ for some $\alpha \in {}^*X$, and we call any such α a *generator* of ${}^*X$.

The connection with ultrapowers is given by the following lemma.

Lemma 4.1 *Let ${}^*X$ be a topological extension of X. For $\alpha \in {}^*X$ let $\mathcal{U}_\alpha$ be the ultrafilter over X associated to the image $\upsilon(\alpha)$ of α in βX. Then*

(i) there exists a unique map $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X$ such that $\psi_\alpha([f]) = {}^*f(\alpha)$ for all $f \in X^X$;

(ii) the range of ψ_α is ${}^*X_\alpha$, and ψ_α is continuous and open w.r.t. the S-topologies of both $X^X/\mathcal{U}_\alpha$ and ${}^*X$;

(iii) ψ_α is the unique map $\psi : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ satisfying ${}^*g \circ \psi = \psi \circ \overline{g}$ for all $g \in X^X$;

(iv) ψ_α is injective if and only if ${}^*X_\alpha$ satisfies the following property:
$\forall f, g : X \to X \ (\forall x \in X. f(x) \neq g(x) \implies \forall \xi \in {}^*X_\alpha. {}^*f(\xi) \neq {}^*g(\xi))$.

Proof. Define $\pi_\alpha : X^X \to {}^*X$ by $\pi_\alpha(f) = {}^*f(\alpha)$. Then the range of π_α is ${}^*X_\alpha$, and $\pi_\alpha(f) = \pi_\alpha(g)$ whenever f and g agree on some $A \subseteq X$ s.t. $\alpha \in \overline{A}$, i.e. on some $A \in \mathcal{U}_\alpha$, by point (ii) of Lemma 1.3. Therefore π_α induces a unique $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$, and

$$\psi_\alpha([f]) \in {}^*A \iff \exists U \in \mathcal{U}_\alpha \,.\, f(U) \subseteq A \iff [f] \in A^X/\mathcal{U}_\alpha.$$

Therefore ψ_α induces a bijection between a basis of the S-topology of $X^X/\mathcal{U}_\alpha$ and one of the S-topology of ${}^*X$, and both points (i) and (ii) are true.

We have, for all $f, g : X \to X$,

$${}^*g(\psi_\alpha([f])) = {}^*g({}^*f(\alpha)) = {}^*(g \circ f)(\alpha) = \psi_\alpha([g \circ f]) = \psi_\alpha(\overline{g}([f])).$$

Hence ψ_α satisfies the required condition. On the other hand, for any such ψ, one has $\psi([f]) = {}^*f(\psi(\imath))$, where $\imath$ is the identity of X. Put $\psi(\imath) = {}^*h(\alpha)$. We claim that $\upsilon({}^*h(\alpha)) = \overline{h}(\mathcal{U}_\alpha) = \mathcal{U}_\alpha$, whence $[h] = [\imath]$, by a well known property of ultrafilters. Assume the contrary and take $A \in \mathcal{U}_\alpha \setminus \overline{h}(\mathcal{U}_\alpha)$. Let f, g be respectively the constant 1 and the characteristic function of A: then $1 = {}^*f({}^*h(\alpha)) = \psi([f]) = \psi([g]) = {}^*g({}^*h(\alpha)) = 0$, contradiction.

In order to prove (iv), assume first that ψ_α is 1-1 and that ${}^*f({}^*h(\alpha)) = {}^*g({}^*h(\alpha))$. Then $[f \circ h] = [g \circ h]$, hence f and g agree on $h(A)$ for some $A \in \mathcal{U}_\alpha$, and so they are not disjoint.

Conversely, if $[f] \neq [g]$, we can assume that $f(x) \neq g(x)$ for all $x \in X$, and the condition implies that ${}^*f(\alpha) \neq {}^*g(\alpha)$, i.e. ψ_α is 1-1.

$\square$

Injectivity of the canonical maps is an essential tool in using topological extensions as *nonstandard models*. According to point (iv) above, it depends on the property that "disjoint functions have disjoint extensions":

(a) for all $f, g : X \to X$:

$$f(x) \neq g(x) \text{ for all } x \in X \quad \Longrightarrow \quad {}^*f(\xi) \neq {}^*g(\xi) \text{ for all } \xi \in {}^*X.$$

This property has a clear "analytic" flavour, and in fact it can be considered as the most characteristic feature of *nonstandard* extensions when compared with the usual *continuous* extensions of functions, where equality can be reached only at limit points. Another characteristic feature of nonstandard models of analysis is that "standard functions behave like germs":

(e) for all $f, g : X \to X$ and all $\xi \in {}^*X$

$${}^*f(\xi) = {}^*g(\xi) \iff \exists A \subseteq X.\ \xi \in \overline{A}\ \ \&\ \forall x \in A\ \ f(x) = g(x).$$

It is apparent that (e) expresses a sort of "preservation of equalizers", i.e.

$$\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} = \overline{\{x \in X \mid f(x) = g(x)\}}.$$

It turns out that the apparently weaker assumption (a), which corresponds to the particular case of *empty* equalizers, is equivalent to (e). Namely:

Lemma 4.2 *Properties* (a) *and* (e) *are equivalent in any topological extension* ${}^*X$ *of* X*, and both hold if and only if all maps* ψ_α *are injective.*

Proof Since (a) is a particular case of (e), and it is equivalent to injectivity of all ψ_αs, by point (*iv*) of Lemma 4.1, we have only to prove that (a) implies (e). Now the inclusion

$$\overline{\{x \in X \mid f(x) = g(x)\}} \subseteq \{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\}$$

is clearly equivalent to point (*ii*) of Lemma 1.3. Therefore we are left with the inclusion

$$\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} \subseteq \overline{\{x \in X \mid f(x) = g(x)\}}.$$

Put $A = \{x \in X \mid f(x) = g(x)\}$, and let the functions f', g' agree with f, g outside A, and take the values $0, 1$ on A, respectively. Since f', g' are disjoint on X, also ${}^*f'$ and ${}^*g'$ are disjoint on ${}^*X$, by (a). But f, f' and g, g' agree on $X \setminus A$, hence ${}^*f' = {}^*f$ and ${}^*g' = {}^*g$ outside $\overline{A}$, again by point (*ii*) of Lemma 1.3. Therefore $\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\} \subseteq \overline{A}$.

□

It seems now natural to give the following definition:

Definition 4.3 A topological extension ${}^*X$ of X is *analytic* if any of the properties (a) and (e) is satisfied.

Every star-extension obtained by topologizing an elementary extension as we did in Section 3, is analytic according to the above definition, since all instances of (a) or (e) are apparently instances of transfer. We shall see in Section 5 that only one more property is needed in order to characterize all nonstandard extensions.

When both the ultrapower $X^X/\mathcal{U}_\alpha$ and the principal subspace ${}^*X_\alpha$ are endowed with the respective Star topologies, then ψ_α is a homeomorphism whenever it is one-one. Therefore we have the following characterization of all principal analytic star-extensions:

Corollary 4.4 *Let ${}^*X$ be a topological extension of X. Then ${}^*X$ is isomorphic to an ultrapower $X^X/\mathcal{U}$ if and only if it is principal and analytic. In this case, if α is a generator of ${}^*X$, then the canonical map ψ_α is a homeomorphism w.r.t. the Star topologies of both spaces.* □

We can also give an algebraic characterization of all *Hausdorff principal extensions* of X. To this aim it seems appropriate to call *Hausdorff* an ultrafilter $\mathcal{U}$ satisfying the following property:

(H) for all $f, g : X \to X$ $\ (\,\overline{f}(\mathcal{U}) = \overline{g}(\mathcal{U}) \iff \{\, x \in X \mid f(x) = g(x)\} \in \mathcal{U}\,)$.[7]

The property (H) is a mere translation in terms of ultrafilters of the condition (e), which characterizes analytic extensions. Hence we have

Corollary 4.5 *A principal extension ${}^*X = {}^*X_\alpha$ of X is Hausdorff and analytic if and only if the ultrafilter $\mathcal{U}_\alpha$ is Hausdorff (and so ${}^*X$ is isomorphic to the ultrapower $X^X/\mathcal{U}_\alpha$). Moreover any topological extension where all ultrafilters $\mathcal{U}_\alpha$ are Hausdorff is analytic.* □

We conclude this section by giving an interpretation of the various kinds of extensions introduced above in terms of the natural generalization of the Rudin-Keisler and Puritz preorderings.

Remark 4.6 Recall that the *Rudin-Keisler preordering of ultrafilters* $\leq_{RK}$ is defined by putting, for $\mathcal{U}, \mathcal{V}$ ultrafilters on X,

- $\mathcal{U} \leq_{RK} \mathcal{V}$ if there exists $f : X \to X$ s.t. $A \in \mathcal{U} \Leftrightarrow f^{-1}(A) \in \mathcal{V}$, i.e. $\overline{f}(\mathcal{V}) = \mathcal{U}$.

A similar *natural preorder* $\leq_*$ can be defined on any topological extension of X as follows[8]

- $\xi \leq_* \eta$ if there exists $f : X \to X$ s.t. $\xi = {}^*f(\eta)$.

It is immediately seen that the canonical map υ is order-preserving, i.e.

- $\xi \leq_* \eta$ implies $\upsilon(\xi) \leq_{RK} \upsilon(\eta)$.

[7] The property (H) is labelled (C) in [14], where various connected properties of ultrafilters are considered.

[8] In the context of nonstandard models, the ordering $\leq_*$ has been introduced by Puritz, and its properties are studied in [26] and [25]. Also in axiomatic nonstandard set theory the relation $\leq_*$ has been extensively studied under the name *relative standardness*, beginning with Gordon's paper [18].

It is worth mentioning that the basic property (i) of topological extensions has a suggestive interpretation in terms of the preordering $\leq_*$:

- $\leq_*$ is *reflexive* if and only if the extension of the identity of X is the identity of ${}^*X$.

Characterizations in terms of the natural preordering $\leq_*$can be given for all kinds of subspaces isolated above, namely:

- ${}^*X$ is *principal* if the preorder $\leq_*$ is *dominated*, i.e. there exists $\xi \in {}^*X$ such that $\eta \leq_* \xi$ for all $\eta \in {}^*X$.
- $S \subseteq {}^*X$ is *invariant* if it is *downward closed* w.r.t. $\leq_*$, i.e. $\xi \leq_* \zeta \in S$ implies $\xi \in S$.

Once defined the corresponding notion of *dominated subspace* of βX w.r.t. the Rudin-Keisler preordering $\leq_{RK}$, it is easily seen that if a topological extension ${}^*X$ of X is principal, then the corresponding subspace $\upsilon({}^*X) \subseteq \beta X$ is dominated in the Rudin-Keisler preordering (but the reverse implication can fail, see Example 6.3).

5 Nonstandard topological extensions

The interest in *analytic* extensions lies in the fact that combining their characteristic property (a) with the sole *simple* condition (f), which we give below, already yields the strongest *Transfer Principle for all first order properties*, i.e. it provides complete elementary extensions. Recall that a *complete elementary extension* is one that is elementary w.r.t. the *complete* first-order language of X, i.e. when all n-ary relations on X have a corresponding symbol in the language.

Definition 5.1 A topological extension ${}^*X$ is *coherent* if any two elements of ${}^*X$ belong to some principal subspace, i.e.

(f) for all $\eta, \zeta \in {}^*X$ there exists $\xi \in {}^*X$ such that ${}^*X_\eta \cup {}^*X_\zeta \subseteq {}^*X_\xi$.

Coherence is apparently an *amalgamation* property, but it can also be expressed as a *filtration* property, since ${}^*X$ is *coherent* if and only if the preorder $\leq_*$ is *filtered*, i.e.

(f) for all $\eta, \zeta \in {}^*X$ there exists $\xi \in {}^*X$ such that $\eta, \zeta \leq_* \xi$.

Before proceeding, let us remark that all the properties we have isolated in the previous sections, like (i), (c), (a), (e), *etc.*, are particular (and in general *perspicuous*) cases of the Transfer Principle. This seems *prima facie* not to apply to the condition (f). But this impression is misleading. In fact a *strong uniform version* of that property can be obtained by transfer as well, namely

- *there exist functions* $p, q : X \to X$ *such that for all* $\xi, \eta \in {}^*X$ *there is a unique* $\zeta \in {}^*X$ *s.t.* ${}^*p(\zeta) = \xi$, ${}^*q(\zeta) = \eta$.

Such functions p, q can be easily obtained by composing any bijection between X and $X \times X$ with the natural projections.

So, if we want a reasonable Transfer Principle in our topological extensions, then we have to assume both properties (a) and (f). On the other hand, in order to show that Transfer holds in full form in all coherent analytic extensions, we have to take care of extending n-ary functions and relations. The *ratio* of considering only unary functions lies in the following fact:

Lemma 5.2 *Let* ${}^*X$ *be a coherent topological extension of* X. *Then there is at most one way of assigning an extension* ${}^*\varphi$ *to every function* $\varphi : X^n \to X$ *in such a way that all compositions are preserved, i.e. for all* $m, n \geq 1$, *all* $\varphi : X^n \to X$, *and all* $\psi_1, \ldots, \psi_n : X^m \to X$,

$$ {}^*\varphi \circ ({}^*\psi_1, \ldots, {}^*\psi_n) = {}^*(\varphi \circ (\psi_1, \ldots, \psi_n)). $$

Proof. We can easily generalize the property (f) and prove by induction on n that

- *For all* $\xi_1, \ldots, \xi_n \in {}^*X$ *there exist* $p_1, \ldots, p_n : X \to X$ *and* $\zeta \in {}^*X$ *such that* ${}^*p_i(\zeta) = \xi_i$ *for* $i = 1, \ldots, n$.

Whenever $\xi_1, \ldots, \xi_n$, and ζ satisfy the above conditions, the extension of any n-ary function φ must satisfy the equality

$$ {}^*\varphi(\xi_1, \ldots, \xi_n) = {}^*(\varphi \circ (p_1, \ldots, p_n))(\zeta). $$

Therefore the extensions of *unary* functions completely determine those of all n-ary functions.

□

To be sure, it might happen that the extensions, as given above, were *overdetermined*, since each n-tuple $\xi_1, \ldots, \xi_n \in {}^*X$ can be obtained from different ζs, by using different p_is. Surprisingly enough, it turns out that all coherent analytic extensions have already the required property of "substitution of equals":

Lemma 5.3 *The following property holds in any coherent analytic extension:*

- Let $p_1, \dots, p_n, q_1, \dots, q_n : X \to X$ and $\xi, \eta \in {}^*X$ satisfy ${}^*p_i(\xi) = {}^*q_i(\eta)$ for $i = 1, \dots, n$. Then, for all $\varphi : X^n \to X$
$$ {}^*(\varphi \circ (p_1, \dots, p_n))(\xi) = {}^*(\varphi \circ (q_1, \dots, q_n))(\eta). $$

More precisely, a topological extension ${}^*X$ of X is an analytic extension if and only if it satisfies all instances of the above property with $\xi = \eta$.

Proof. Let ${}^*X$ be a coherent analytic extension of X. Taking a common upper bound of ξ and η, we can assume w.l.o.g. that $\xi = \eta$. Then, applying (e), we get

$$\{\zeta \in {}^*X \mid {}^*(\varphi \circ (p_1, \dots, p_n))(\zeta) = {}^*(\varphi \circ (q_1, \dots, q_n))(\zeta)\} =$$
$$\overline{\{x \in X \mid (\varphi \circ (p_1, \dots, p_n))(x) = (\varphi \circ (q_1, \dots, q_n))(x)\}} \supseteq$$
$$\bigcap_{1 \le i \le n} \overline{\{x \in X \mid p_i(x) = q_i(x)\}} \;=\; \bigcap_{1 \le i \le n} \{\zeta \in {}^*X \mid {}^*p_i(\zeta) = {}^*q_i(\zeta)\}.$$

We have in fact proved the "only if" part of the second assertion. In order to prove the "if" part, we use only the case where $n = 2$, $p_1 = f$, $p_2 = q_1 = q_2 = g$.

Let $\chi : X \times X \to X$ be the characteristic function of the diagonal. Then $\chi \circ (f, g)$ is the characteristic function of $A = \{x \in X \mid f(x) = g(x)\}$, and $\chi \circ (g, g)$ is the constant 1. Hence their extensions are the characteristic function of $\overline{A}$ and the constant 1, respectively. The above property yields that these functions agree on $\{\xi \in {}^*X \mid {}^*f(\xi) = {}^*g(\xi)\}$, which is therefore included in $\overline{A}$. In particular, by taking $A = \emptyset$, we obtain (a), and so ${}^*X$ is an analytic extension.

□

An *unambiguous* and *unique* definition of the extension ${}^*\varphi$ of any function $\varphi : X^n \to X$ is now possible by putting

$$ {}^*\varphi(\xi_1, \dots, \xi_n) = {}^*(\varphi \circ (p_1, \dots, p_n))(\zeta), \text{ where } {}^*p_i(\zeta) = \xi_i \text{ for } i = 1, \dots, n. $$

The above lemmata ensure that such p_is and ζ always exist and that the result is independent of their choice. Moreover all compositions are preserved. Hence we have proved

Theorem 5.4 *In any coherent analytic extension ${}^*X$ of X, all n-ary functions on X can be uniquely extended to ${}^*X$, preserving compositions.*

□

***Caveat*:** For all $n > 1$ there are functions of n variables whose extensions *cannot be continuous w.r.t. the product topology* (if the extension is nontrivial).

In fact, let χ be the characteristic function of the diagonal in X^n. Any point $\xi \in {}^*X \setminus X$ is a cluster point of X, hence the extension ${}^*\chi$ takes on both values 0 and 1 in the Cartesian power U^n of any neighborhood U of ξ. Therefore ${}^*\chi$ cannot be continuous at $(\xi, \ldots, \xi)$ w.r.t. the product topology.

Notice that if $X \times X$ is identified with X by means of some "projections" p_1, p_2, then ${}^*\chi$ becomes the characteristic function of the diagonal Δ in ${}^*X \times {}^*X$. Therefore Δ is clopen. But then the topology induced by ${}^*X$ on ${}^*X \times {}^*X$ is *strictly finer* then the product topology (otherwise ${}^*X$ would be discrete, contradicting the density of X). This fact might shed new light on the differences between the naive ideas which are encompassed by the notion of topological extension. Compare the topological notion of *compactification*, where the fact that, e.g., $\beta\mathbb{N} \times \beta\mathbb{N}$ is quite different from $\beta(\mathbb{N} \times \mathbb{N})$ seems almost obvious, with the logical notion of *elementary (nonstandard)* extension, where ${}^*\mathbb{N} \times {}^*\mathbb{N}$ has to be naturally identified with ${}^*(\mathbb{N} \times \mathbb{N})$.

Using the characteristic functions in n variables one can assign an extension ${}^*R$ to any n-ary relation R on X. In this way, given a first order structure $\mathfrak{X} = \langle X; R, \ldots; c, \ldots \rangle$ and a coherent analytic extension ${}^*X$ of its universe, one obtains a structure ${}^*\mathfrak{X} = \langle {}^*X; {}^*R, \ldots; c, \ldots \rangle$ having $\mathfrak{X}$ as substructure. It turns out that $\mathfrak{X}$ is always an *elementary substructure* of ${}^*\mathfrak{X}$.

In order to prove this fact, one could proceed as usual and show, by induction on the complexity of the formula Φ, that

$$\forall x_1, \ldots, x_n \in X.\ {}^*\mathfrak{X} \models \Phi[x_1, \ldots, x_n] \iff \mathfrak{X} \models \Phi[x_1, \ldots, x_n].$$

We do not enter in the details of such a logical proof. We prefer instead to present a different and simpler argument, based on the results of Section 4. Namely we show that any coherent analytic extension can be construed as a *direct limit of a directed system of ultrapowers.*

It follows immediately from Lemmata 4.2 and 5.3 that the canonical map $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X_\alpha$ turns into a *model-theoretic isomorphism of extensions of* $\mathfrak{X}$, provided it is injective. Hence any principal subspace ${}^*X_\alpha$ of an analytic extension ${}^*X$ can be viewed as an ultrapower extension of $\mathfrak{X}$. Moreover, for each $\beta \in {}^*X_\alpha$, the inclusion of the corresponding principal submodel ${}^*X_\beta$ into ${}^*X_\alpha$ corresponds to an elementary embedding

$$\jmath_{\beta\alpha} = \psi_\alpha^{-1} \circ \psi_\beta : X^X/\mathcal{U}_\beta \to X^X/\mathcal{U}_\alpha.$$

Therefore we obtain the system of ultrapowers $\langle X^X/\mathcal{U}_\alpha \mid \alpha \in {}^*X\rangle$ together with the system of elementary embeddings $\langle \jmath_{\beta\alpha} \mid \beta \leq_* \alpha\rangle$. If ${}^*X$ is coherent, then this system is directed, and its direct limit is isomorphic to the union ${}^*X = \bigcup_{\alpha\in{}^*X} {}^*X_\alpha$. It follows that the structure ${}^*\mathfrak{X}$, being isomorphic to a direct limit of ultrapowers of $\mathfrak{X}$, is itself a complete elementary extension of $\mathfrak{X}$.

It is now evident that the definitions have been appropriately chosen so as to obtain

Theorem 5.5 *If ${}^*X$ is a coherent analytic extension of X, then the structure ${}^*\mathfrak{X}$ is a complete elementary extension of $\mathfrak{X}$.* □

If ${}^*X$ is a *coherent analytic extension* of X, then we can consider it as a nonstandard model according to Theorem 5.5 and then apply Theorem 3.2 so as to turn it into a *nonstandard star-extension.* Then clearly this operation results in nothing but a (possible) weakening of the topology:

Corollary 5.6 *Any coherent analytic extension of X is a nonstandard star-extension, possibly endowed with a finer topology.* □

Recalling the correspondence between Hausdorff extensions of X and invariant subspaces of βX, we obtain the following algebraic characterization:

Corollary 5.7 *A subspace S of βX is a complete elementary extension of X if and only if S is filtered and downward closed w.r.t. the Rudin-Keisler preordering, and all points of S correspond to Hausdorff ultrafilters.* □

6 Final remarks

6.1 Some examples

We present here a few basic examples of topological extensions. In partuicular we show that the characteristic properties (a) and (f) are independent.

Example 6.1 Let α be any point of $\beta X \setminus X$ and put $Y = (\beta X)_\alpha$. Then Y is a principal Hausdorff extension of X. Hence it trivially satisfies (f), whereas it satisfies (a) if and only if α corresponds to a Hausdorff ultrafilter (see Corollary 4.5).

Example 6.2 Let ξ, η be two Rudin-Keisler incomparable points of $\beta X \setminus X$. Then $Z = (\beta X)_\xi \cup (\beta X)_\eta$ is an invariant subspace which is clearly not coherent. Moreover Z is analytic if and only if both ξ and η correspond to Hausdorff ultrafilters. Therefore we can either falsify both (f) and (a), or maintain true (a) alone.

Example 6.3 Let $\mathcal{U}$ be a nonprincipal ultrafilter on X, and let ${}^*X$ be the star-extension given by the ultrapower $X^X/\mathcal{U}$. Let W be the union of two copies of ${}^*X$ where only the standard parts are identified. Then W is a non-Hausdorff analytic extension that is clearly not coherent, whereas its canonical image in βX is principal. Moreover, if $\mathcal{U}$ is a Hausdorff ultrafilter, then all ultrafilters $\mathcal{U}_\alpha$ of W are Hausdorff. So even this additional hypothesis does not yield coherence.

6.2 Proper extensions

In order to make an effective use of nonstandard models, it is always assumed by nonstandard analysts that *all infinite sets* are indeed *extended*, i.e. $A = {}^*A = \overline{A}$ if and only if A is finite. We never used this assumption, up to now. Let us call *proper* the topological extensions satisfying this property. We give below topological, algebraic, and set-theoretic characterizations of proper extensions. In particular we show that *nontrivial improper extensions* require *uncountable measurable cardinals*.[9]

Let us recall some basic definitions:

- the topological space S is *Weierstraß* if all continuous functions $f : S \to \mathbb{R}$ are *bounded*, or equivalently if all such f have *compact ranges*.
- The *additivity* of the nonprincipal ultrafilter $\mathcal{U}$ is the least size of a family of sets not in $\mathcal{U}$ whose union is in $\mathcal{U}$. The additivity is always a *measurable* cardinal (possibly $\aleph_0$).
- Ultrafilters of additivity $\aleph_0$ are called *countably incomplete*. Recall that sets of size less than the additivity of $\mathcal{U}$ have only *trivial ultrapowers* mod $\mathcal{U}$.

Theorem 6.4 *Let ${}^*X$ be a topological extension of X. Then the following properties are equivalent*

[9] A cardinal κ is measurable if it carries a nonprincipal κ-complete ultrafilter. Notice that, according to this definition, we include $\aleph_0$ among the measurable cardinals.

(*i*) *${}^*X$ is proper, i.e. $A = {}^*A \iff A$ is finite;*

(*ii*) ${}^*\mathbb{N} \neq \mathbb{N}$;

(*iii*) *${}^*A \neq A$ for some $A \subseteq X$ of size less than κ, the least uncountable measurable cardinal (if any);*

(*iv*) *there is $\alpha \in {}^*X$ s.t. the ultrafilter $\mathcal{U}_\alpha$ is countably incomplete.* (*In fact any nonstandard $\alpha \in {}^*X$ has this property.*)

(*v*) *${}^*X$ is Weierstraß.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are trivial.
$(iii) \Rightarrow (iv) \Rightarrow (i)$. Notice that the property $A = {}^*A$ depends only on the size of A, since bijective functions have bijective extensions. Therefore either the ultrafilters $\mathcal{U}_\alpha$ are all countably incomplete, and then $A = {}^*A \Leftrightarrow A$ is finite, or $|A| < \kappa \Rightarrow A = {}^*A$.

We are left with the equivalence $(i) \Leftrightarrow (v)$.

Assume ${}^*X$ proper and let $\varphi : {}^*X \to \mathbb{R}$ be continuous. We show that the range of φ is closed and bounded in $\mathbb{R}$. Assume the contrary. Then, by density of X, there exists a sequence x_n in X such that either $\varphi(x_n) > n$ if φ is unbounded, or else $|\varphi(x_n) - r| < 1/n$, where $r \in \mathbb{R}$ is a cluster point of $\varphi({}^*X)$ not in $\varphi({}^*X)$. Pick a point α in the boundary of $\{x_n \mid n \in \mathbb{N}\}$, which exists since ${}^*X$ is proper. Then clearly α is mapped to r in the latter case, and cannot have an image at all in the former, contradiction.

Conversely, assume ${}^*X$ not proper. Let $\{x_n \mid n \in \mathbb{N}\} \subseteq X$ be a countable subset that is closed in ${}^*X$. Define the functions $\varphi, \psi : {}^*X \to \mathbb{R}$ by

$$\varphi(x_n) = n, \;\; \psi(x_n) = 1 - 1/n, \;\text{ and }\; \varphi(\xi) = \psi(\xi) = 0, \text{ otherwise.}$$

Clearly both φ and ψ are continuous, and φ is unbounded, while the range of ψ is not closed.

□

6.3 Weak compactness and saturation

Let us recall a "weak compactness" property, which is commonly considered only for Hausdorff spaces:

- a topological space S is *Bolzano* if every infinite subset of S has a *cluster point*.

It is readily checked that a space S is Bolzano if and only if every countable open cover has a finite subcover, and so S is *countably compact* if and only if it is Bolzano and Hausdorff.

We recall also two general model-theoretic properties, which are basic in current applications of nonstandard methods, and are strictly connected with weak compactness properties of the S- and Star topologies.

Let ${}^*X$ be a topological extension of X, and let κ be a cardinal. Then

- ${}^*X$ is a *κ-enlargement* if every family of less than κ *standard* (= *clopen*) subsets of ${}^*X$ with empty intersection has a *finite subfamily* with empty intersection.
- ${}^*X$ is *κ-saturated* if every family of less than κ *internal* subsets of ${}^*X$ with empty intersection has a *finite subfamily* with empty intersection.

The identification *"standard = clopen"* is quite obvious, once we have posed ${}^*A = \overline{A}$ for all $A \subseteq X$. On the other hand, the notion of "internal" seems not to have a similar straightforward introduction in our general topological context.[10] However we have no need of this general notion. We only stipulate the minimal assumption that every *basic closed set* $E(\vec{f}, \vec{\eta})$ of the Star topology is *internal*.

It seems to us a very interesting evenience that what are perhaps the most important qualities of topological extensions, namely to be *Hausdorff, analytic,* and *(quasi-)compact*, cannot be achieved all together, but only pairwise. This fact is a consequence of the next theorem.

Theorem 6.5 *Let ${}^*X$ be a topological extension of X. Then*
1. If ${}^*X$ is $(2^{|X|})^+$-saturated, then the star topology of ${}^*X$ is Bolzano. Hence every set X has Bolzano analytic star-extensions.
2. The canonical map υ is surjective if and only if ${}^*X$ is a $(2^{|X|})^+$-enlargement. Hence no $(2^{\aleph_0})^+$-enlargement can be simultaneously analytic and Hausdorff. In particular there exists no compact analytic extension.

Proof. 1. Let ${}^*X$ be $(2^{|X|})^+$-saturated. Let $A = \{\alpha_n \mid n \in \mathbb{N}\}$ be a countable subset of ${}^*X$ such that no α_n is a cluster point of A. We claim that A is not closed. The closure of A is the intersection of a family of basic

[10] As one referee suggests, one could define the internal sets as the fibers of ${}^*P$, where P is any subset of $X \times X$. This definition provides the usual notion in the case of nonstandard (= coherent analytic) extensions. In general it calls for a suitable coding of pairs, by means of "projections" $p, q : X \to X$ (see Section 5).

closed sets $E(\vec{f},\vec{\eta})$. We can assume w.l.o.g. that if $E(f_i,\eta_i)$ appears in this family, then $\eta_i \in {}^*f_i(A)$. Otherwise $E(f_i,\eta_i)$ is disjoint from A, and so the union of the remaining $E(f_j,\eta_j)$ already contains $\overline{A}$. Therefore we have to intersect at most $2^{|X|}$ such sets in order to obtain $\overline{A}$. The same argument works for the closures of all sets $A \setminus \{\alpha_n\}$. The intersection of all these sets is disjoint from A, hence it consists only of cluster points of A. On the other hand the intersection of any finite number of them is nonempty, and so, by saturation, also the complete intersection is nonempty.

Conversely, every $(2^{|X|})^+$-saturated elementary extension of X, topologized as in Section 3, becomes a Bolzano analytic star-extension.

2. Clearly ${}^*X$ is a κ-enlargement if and only if every clopen cover of size less than κ has a finite subcover, i.e. the S-topology is quasi-κ-compact. For $\kappa = (2^{|X|})^+$ we attain the size of $\mathcal{CO}({}^*X)$, and the final part of Theorem 2.1 applies.

It is well known that in $\beta\mathbb{N}$ there are many "symmetric" elements, which correspond to ultrafilters of the form $\mathcal{U} \otimes \mathcal{U}$. Clearly any such element is mapped to $\mathcal{U}$ by both projections. Therefore, if the property (a) holds, then the canonical map v cannot be one-one, and *viceversa*.

□

Thus sufficiently saturated nonstandard extensions are Bolzano. Every Bolzano extension is necessarily *proper*, and so Weierstraß, by Theorem 6.4. So we see that Bolzano-Weierstraß together do not yield even *countable compactness*, in our context.

We conclude this subsection by proving that every nonstandard extension has plenty of non-Bolzano subextensions. In fact ultrapowers $X^X/\mathcal{U}$ provide principal analytic extensions, but their saturation cannot exceed $|X|^+$. Therefore Theorem 6.5 does not apply, and in fact we have

Theorem 6.6 *Let ${}^*X$ be a principal analytic extension of X. Then ${}^*X$ has discrete closed subsets of size $|X|$. In particular ${}^*X$ is not Bolzano.*

Proof. Let α be a generator of ${}^*X$. Fix a bijection δ between X and $X \times X$. Let $p_1, p_2 : X \to X$ be the corresponding "projections". For every $x \in X$ define the map $\sigma_x : X \to X$ by $\sigma_x(y) = \delta^{-1}(x,y)$, and put $\alpha_x = {}^*\sigma_x(\alpha)$. Then the points α_x are separated by clopen sets, and so the set $D = \{\alpha_x \mid x \in X\}$ is discrete. Suppose that $\beta = {}^*f(\alpha)$ is a cluster point of D. Then ${}^*p_2(\beta) = \alpha$, since ${}^*p_2(\alpha_x) = \alpha$ for all $x \in X$. But then ${}^*f({}^*p_2(\beta)) = \beta$, hence $f \circ p_2$ is the identity modulo $\mathcal{U}_\beta$. But this is impossible, since if $A \in \mathcal{U}_\beta$, then $\overline{A} \cap D$ is infinite, and so p_2 is not one-one on A.

$\square$

6.4 Simple extensions

We consider here an interesting class of "minimal" topological extensions of X. We say that ${}^*X$ is *simple* if all elements of ${}^*X \setminus X$ are generators, i.e. ${}^*X = {}^*X_\alpha$ for all nonstandard α. Clearly ${}^*X$ is a simple extension if and only if it has no nontrivial invariant subspaces. We give here *topological* and *algebraic* characterizations of simple extensions, improving Theorem 1.8 of [6].

Recall that X is the set of all isolated points of ${}^*X$. Hence any homeomorphism of ${}^*X$ onto itself induces a bijection of X. Therefore no topological extension can be topologically homogeneous *stricto sensu*. In our context, a more convenient notion is obtained by calling ${}^*X$ *homogeneous* if *any two nonstandard points* of ${}^*X$ are connected by a homeomorphism of ${}^*X$ onto itself.

Theorem 6.7 *Let ${}^*X$ be a topological extension of X. Then the following properties are equivalent:*

(*i*) *${}^*X$ is simple;*

(*ii*) *$\eta \leq_* \xi$ holds for all nonstandard $\xi, \eta \in {}^*X$, hence the preorder $\leq_*$ is an equivalence on ${}^*X \setminus X$;*

(*iii*) *${}^*X$ is Hausdorff and all nonprincipal ultrafilters $\mathcal{U}_\alpha$ are isomorphic;*

(*iv*) *there exists $\alpha \in {}^*X$ such that ${}^*X = {}^*X_\alpha$ and the ultrafilter $\mathcal{U}_\alpha$ is selective;*[11] (*In fact any nonstandard $\alpha \in {}^*X$ has this property.*)

(*v*) *${}^*X$ is Hausdorff and homogeneous.*

Moreover, the canonical map $\psi_\alpha : X^X/\mathcal{U}_\alpha \to {}^*X$ is a homeomorphism for all $\alpha \in {}^*X \setminus X$ if and only if ${}^*X$ is a simple star-extension. Conversely, every ultrapower of X modulo a selective ultrafilter over X, if endowed with the S-topology (which is the same as the Star topology), becomes a simple star-extension of X.

[11] Many equivalent properties can be used in defining selective (or Ramsey) ultrafilters (see, e.g. [8] or [4]). Here we need the following: *$\mathcal{U}$ is selective if and only if every $f : X \to X$ is either equivalent mod $\mathcal{U}$ to a constant or to a bijective function.*

Proof. The same argument used in proving point *(iii)* of Lemma 4.1 yields that $\upsilon(\alpha) \neq \upsilon(\xi)$ for all $\xi \in {}^*X_\alpha \setminus \{\alpha\}$. In particular the map υ is injective, when restricted to the generators of a principal subspace. Hence all simple extensions are Hausdorff. Moreover, if ${}^*X$ is simple, then for all $\alpha, \xi \in {}^*X \setminus X$, $\xi = {}^*f(\alpha)$ and $\alpha = {}^*g(\xi) = {}^*g({}^*f(\alpha))$, for suitable $f, g : X \to X$. Hence $\mathcal{U}_\alpha = \overline{g \circ f}(\mathcal{U}_\alpha)$. This implies that $[g \circ f]$ is the class of the identity, and so f is equivalent to a bijective function modulo $\mathcal{U}_\alpha$. It follows at once that ${}^*f$ is a homeomorphism that maps α to ξ, and that all ultrafilters $\mathcal{U}_\alpha$ are selective and pairwise isomorphic.
Hence $(i) \Longrightarrow (iii)\&(iv)\&(v)$.
$(i) \iff (ii)$ is obvious.
$(iii) \Longrightarrow (ii)$ because in Hausdorff extensions $\leq_{RK}$ and $\leq_*$ coincide.
$(iv) \Longrightarrow (iii)$. Assume that ${}^*X = {}^*X_\alpha$, with $\mathcal{U}_\alpha$ selective. Then $\mathcal{U}_\alpha$ is Hausdorff, and so is ${}^*X_\alpha$. Given $\xi \in {}^*X \setminus X$ pick $f : X \to X$ such that $\xi = {}^*f(\alpha)$. Then f is not equivalent to a constant, and by selectivity it is equivalent to a bijective function g. Therefore all nonprincipal ultrafilters $\mathcal{U}_\xi$ are isomorphic.
$(v) \Longrightarrow (ii)$. Any homeomorphism φ of ${}^*X$ onto itself induces a permutation f of X. Hence $\varphi = {}^*f$, for ${}^*X$ is Hausdorff. By homogeneity, for all $\xi, \eta \in {}^*X \setminus X$ there exists a bijective function $f : X \to X$ such that ${}^*f(\xi) = \eta$. Therefore all nonstandard elements are $\leq_*$-equivalent.

If ${}^*X$ is simple, then every $\alpha \in {}^*X \setminus X$ is a generator, and the ultrafilter $\mathcal{U}_\alpha$ is selective. Thus, if ${}^*f(\alpha) = {}^*g(\alpha)$, then f, g can be assumed bijective. Moreover $g \circ f^{-1}$ is the identity modulo $\mathcal{U}_\alpha$, hence $[f] = [g]$. Therefore ψ_α is both surjective and injective, and actually a homeomorphism w.r.t. the S-topologies, by Lemma 4.1. The S-topology of ${}^*X$ is the same as the Star topology, because ${}^*X$ is Hausdorff. It follows that ψ_α cannot be continuous if the topology of ${}^*X$ is strictly finer. The last assertion of the theorem is a straightforward consequence of the preceding arguments.

□

Point *(iv)* above transforms the task of constructing *simple extensions* into that of finding *selective* ultrafilters. We list below a few known facts that are relevant in this context (see e.g. [11, 12]).

- There are *no uniform countably incomplete selective ultrafilters over an uncountable set* X, and in fact the additivity of a selective ultrafilter is a *measurable* cardinal.
- Every normal ultrafilter over an uncountable measurable cardinal is selective. (But the corresponding simple extension cannot be proper.)

- If the Continuum Hypothesis CH (or Martin Axiom MA)[12] holds, then *there exist* $2^{2^{\aleph_0}}$ *selective* ultrafilters over $\mathbb{N}$.
- There are both models of ZFC with *no* selective ultrafilters, and models of ZFC with *exactly one* selective ultrafilter (up to isomorphisms) (see [10, 29]).

As a consequence we have that the following are consistent with ZFC:

- any infinite set has $2^{2^{\aleph_0}}$ *nonisomorphic proper simple analytic extensions*;
- there are *no simple extensions*;
- any infinite set has *a unique proper simple extension* (which is *analytic*).

The third possibility above seems of particular interest, because it allows for the existence of a *unique minimal* "*prime*" *nonstandard* model ${}^*X$ for any given structure $\mathfrak{X}$ with universe X. Such a ${}^*X$ would be isomorphically embedded into any nonstandard extension $\mathfrak{Y}$ of $\mathfrak{X}$, and its image would be the *unique minimal* nontrivial elementary submodel of $\mathfrak{Y}$.

The *hypernatural* and *hyperreal numbers* of all *simple* extensions share the following remarkable properties, already underlined in [6]:

- ${}^*\mathbb{N} = \{{}^*g(\alpha) \mid g : \mathbb{N} \to \mathbb{N} \text{ strictly increasing}\}$ for all nonstandard α;
- ${}^*\mathbb{N}$ is a set of *numerosities* in the sense of [4], i.e. it provides a "good" nonstandard notion of *size* of countable sets that satisfies, *inter alia*, the ancient principle that "the whole is greater than its parts";
- ${}^*\mathbb{R}$ satisfies the "Strong Cauchy Principle" of [5], i.e. every positive infinitesimal $\varepsilon \in {}^*\mathbb{R}$ is equal to ${}^*f\,(\alpha)$ for a suitable strictly decreasing function $f : \mathbb{N} \to \mathbb{R}$.

6.5 Existence of Hausdorff extensions

As shown by Theorem 4.5, Hausdorff extensions require special ultrafilters, namely those named Hausdorff in Section 4. The aim of obtaining the most general notion is the reason why we asked only for T_1-spaces, so as to comprehend all kinds of nonstandard models. Despite the apparent weakeness of the property (H), which is actually true whenever any of the involved functions is injective (or constant), Hausdorff ultrafilters on *uncountable* sets are highly problematic, connected as they are to *irregular* ultrafilters (see below). In

[12] Recall that MA is independent of ZFC and consistent with any value of $2^{\aleph_0}$.

fact, the strength of the assumption that Hausdorff analytic extensions exist is not yet completely determined. The only relevant fact to be found in the literature is that, over a *countable* set, the property (H) follows from the 3-*arrow* property of [3], which in turn is satisfied both by selective ultrafilters and by *products of pairs* of nonisomorphic selective ultrafilters (see e.g. [14, 3]). Unfortunately, while selectiveness has been deeply investigated, not much is known about the property (H) *per se*.

Recently, the authors have proved the following facts (see [15]):

(i) Let $\mathfrak{u}$ be the least size of an ultrafilter basis on $\mathbb{N}$. Then there are no regular Hausdorff ultrafilters on $\mathfrak{u}$.[13]

(ii) Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on $\mathbb{N}$. If $\mathcal{U}$ is a Hausdorff, $\mathcal{V}$ is selective, and $\mathcal{V} \not\leq_{RK} \mathcal{U}$, then the product ultrafilter $\mathcal{U} \otimes \mathcal{V}$ is Hausdorff.

It follows from (i) that even a Hausdorff extension of $\mathbb{R}$ with uniform ultrafilters $\mathcal{U}_\alpha$ would require very heavy set theoretic hypotheses.[14] On the other hand it is consistent with ZFC that the continuum is large, and that either $\mathfrak{u} = \aleph_1$ or $\mathfrak{u} = 2^{\aleph_0}$ (see [9]). In the latter case the existence of Hausdorff extensions with large uniform ultrafilters were not forbidden.

However, according to (ii), any hypothesis providing infinitely many nonisomorphic selective ultrafilters over $\mathbb{N}$, like CH or MA, provides infinitely many *invariant analytic subspaces* of $\beta\mathbb{N}$, either *principal non-simple* or *coherent non-principal.* More precisely

Theorem 6.8 *Let $\mathcal{U}_1, \ldots, \mathcal{U}_n, \ldots$ be a sequence of pairwise nonisomorphic selective ultrafilters on $\mathbb{N}$. Then any product $\mathcal{U}_{i_1} \otimes \ldots \otimes \mathcal{U}_{i_n}$ provides a principal analytic Hausdorff extension of $\mathbb{N}$. Moreover any increasing sequence of such products provides a nonprincipal analytic Hausdorff extension.*

Be it as it may, as far as we do not abide ZFC as our foundational theory, we cannot prove that Hausdorff analytic extensions exist at all. In fact, when a previous version of this paper was already submitted for publication, the authors received copy of the manuscript [2], where T. Bartoszynski and S. Shelah define a class of forcing models where there are *no Hausdorff ultrafilters.*

[13] $\mathfrak{u}$ is one of a series of cardinal invariants of the continuum considered in the literature. All what is provable in ZFC about the size of $\mathfrak{u}$ is that $\aleph_1 \leq \mathfrak{u} \leq 2^{\aleph_0}$ (see e.g. [30, 9]).

[14] E.g. it is proved in [21] that, under CH, the existence of uniform Hausdorff ultrafilters on $\mathbb{R}$ implies that of inner models with measurable cardinals. To be sure, such ultrafilters have been obtained only by assuming either a huge cardinal or Woodin's hypothesis "$\diamond$ + there exists a normal ω_1-dense ideal over ω_1" (see [20]).

6.6 Some open questions

We list below a few open questions arising from our work. They are all closely connected to the existence of ultrafilters with special properties, and thus they seem to be of independent interest.

1. Is the existence of Hausdorff analytic extensions derivable from set-theoretic hypotheses weaker than those providing selective ultrafilters? E.g. from $\mathfrak{x} = \mathfrak{c}$, where $\mathfrak{x}$ is a suitable cardinal invariant of the continuum?
2. Is the existence of countably compact analytic extensions consistent with ZFC?
3. Is it consistent with ZFC that there are nonstandard real lines ${}^*\mathbb{R}$ where all ultrafilters $\mathcal{U}_\alpha$ are uniform and the Star topology is Hausdorff?
4. Are there non-Hausdorff topological extensions ${}^*X$ where every function $f : X \to X$ has a *unique* continuous extension ${}^*f$?
5. Are there topological extensions whose topology is strictly finer than the Star topology?

References

[1] L.O. Arkeryd, N.J. Cutland, C.W. Henson (eds.) - *Nonstandard Analysis - Theory and Applications.* NATO ASI Series **C 493**, Kluwer A.P., Dordrecht 1997.

[2] T. Bartoszynski, S. Shelah - *There may be no Hausdorff ultrafilters*, manuscript (2003).

[3] J.E. Baumgartner, A.D. Taylor - Partitions theorems and ultrafilters, *Trans. Amer. Math. Soc.* **241** (1978), 283–309.

[4] V. Benci, M. Di Nasso - Numerosities of labelled sets: a new way of counting, *Adv. Math.* **173** (2003), 50–67.

[5] V. Benci, M. Di Nasso - Alpha-Theory: an elementary axiomatics for nonstandard analysis, *Expo. Math.* **21** (2003), 355–386.

[6] V. Benci, M. Di Nasso, M. Forti - Hausdorff nonstandard extensions, *Bol. Soc. Parana. Mat.* (3) **20** (2002), 9–20.

[7] M. Benedikt - Ultrafilters which extend measures, *J. Symb. Logic* **63** (1998), 638–662.

[8] A. Blass - A model-theoretic view of some special ultrafilters, in *Logic Colloquium '77* (A. MacIntyre, L. Pacholski, J. Paris, eds.). North Holland, Amsterdam 1978, 79–90.

[9] A. Blass - Combinatorial cardinal characteristics of the continuum, to appear in *Handbook of Set Theory* (M. Foreman, M. Magidor, A. Kanamori, eds.).

[10] A. Blass, S. Shelah - There may be simple $P_{\aleph_1}$- and $P_{\aleph_2}$-points and the Rudin-Keisler ordering may be downward directed, *Ann. Pure Appl. Logic* **33** (1987), 213–243.

[11] D. Booth - Ultrafilters on a countable set, *Ann. Math. Logic* **2** (1970/71), 1–24.

[12] C.C. Chang, H.J. Keisler - *Model Theory* (3rd edition). North-Holland, Amsterdam 1990.

[13] G. Cherlin, J. Hirschfeld - Ultrafilters and ultraproducts in non-standard analysis, in *Contributions to Non-standard Analysis* (W.A.J. Luxemburg, A. Robinson, eds.). North Holland, Amsterdam 1972, 261–279.

[14] M. Daguenet-Teissier - Ultrafiltres à la façon de Ramsey, *Trans. Amer. Math. Soc.* **250** (1979), 91–120.

[15] M. Di Nasso, M. Forti - *Hausdorff ultrafilters* (submitted).

[16] R. Engelking - *General Topology*. Polish S.P., Warszawa 1977.

[17] R. Goldblatt - *Lectures on the Hyperreals*. Springer, New York 1998.

[18] E. Gordon - Relatively standard elements in E. Nelson's Internal Set Theory, *Siberian Math. J.* **30** (1989), 89–95.

[19] N. Hindman, D. Strauss - *Algebra in the Stone-Čech Compactification*. W. de Gruyter, Berlin 1998.

[20] A. Kanamori - Finest partitions for ultrafilters, *J. Symb. Logic* **511** (1986), 327–356.

[21] A. Kanamori, A.D. Taylor - Separating ultrafilters on uncountable cardinals, *Israel J. Math.* **47** (1984), 131–138.

[22] H.J. Keisler - *Foundations of Infinitesimal Calculus*. Prindle, Weber and Schmidt, Boston 1976.

[23] P. Loeb - Nonstandard analysis and topology, in [1], 77–89.

[24] W.A.J. Luxemburg - A general theory of monads, in *Applications of Model Theory to Algebra, Analysis and Probability* (W.A.J. Luxemburg ed.). Holt, Rinehart and Winston, New York 1969, 18–86.

[25] S. Ng, H. Render - The Puritz order and its relationship to the Rudin-Keisler order, in *Reuniting the Antipodes - Constructive and Nonstandard Views of the Continuum* (P. Schuster, U. Berger, H. Osswald, eds.). Synthèse Library **306**, Kluwer A.P., Dordrecht 2001, 157–166.

[26] C. Puritz - Ultrafilters and standard functions in nonstandard analysis, *Proc. London Math. Soc.* **22** (1971), 706–733.

[27] A. Robinson - *Non-standard Analysis*. North Holland, Amsterdam 1966.

[28] A. Robinson - Compactification of groups and rings and nonstandard analysis, *J. Symb. Logic* **34** (1969), 576–588.

[29] S. Shelah - *Proper and Improper Forcing* (2nd edition). Springer, Berlin 1982.

[30] E.K. van Douwen - The integers and topology, in *Handbook of Set Theoretic Topology* (K. Kunen, J.E. Vaughan, eds.). North-Holland, Amsterdam 1984, 111–168.