# Partition regularity of nonlinear polynomials

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# Terminology

We say that a polynomial $P(x_1, ..., x_n)$ is (injectively) partition regular on $\mathbb{N} = \{1, 2, ...\}$ if whenever the natural numbers are finitely colored there is a(n injective) monochromatic solution to the equation $P(x_1, ..., x_n) = 0$.

**Theorem (Rado)**

*Let $P(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i$ be a linear polynomial. The following conditions are equivalent:*

1. $P(x_1, ..., x_n)$ *is partition regular on* $\mathbb{N}$;
2. *there is a nonempy subset* $J$ *of* $\{1, ..., n\}$ *such that* $\sum_{j \in J} a_j = 0$.

## Hindman's Result

**Question:** Is the polynomial $x + y - zw$ injectively partition regular on $\mathbb{N}$? (P. Csikvári, K. Gyarmati and A. Sárközy)
An affirmative answer has been given (in a much more general form) by Neil Hindman in 2011 (in "Monochromatic Sums Equal to Products in $\mathbb{N}$"):

**Theorem (Hindman)**

*For every natural numbers $n, m \geq 1$, with $n + m \geq 3$, the nonlinear polynomial*

$$\sum_{i=1}^{n} x_i - \prod_{j=1}^{m} y_j$$

*is injectively partition regular.*

# Translation in terms of Ultrafilters

**Definition**

*Let* $P(x_1, ..., x_n)$ *be a polynomial, and* $\mathcal{U}$ *an ultrafilter on* $\mathbb{N}$. *Then:*

1. $\mathcal{U}$ *is a* ***$\sigma_{\mathbf{P}}$-ultrafilter*** *if and only if for every set* $A \in \mathcal{U}$ *there are* $a_1, ..., a_n \in A$ *such that* $P(a_1, .., a_n) = 0$*;*
2. $\mathcal{U}$ *is a* ***$\iota_{\mathbf{P}}$-ultrafilter*** *if and only if for every set* $A \in \mathcal{U}$ *there are mutually distinct elements* $a_1, ..., a_n \in A$ *such that* $P(a_1, .., a_n) = 0$.

## Sets of Generators of $\mathcal{U}$

Let ${}^*\mathbb{N}$ be an hyperextension of $\mathbb{N}$ satisfying the $\mathfrak{c}^+$-enlarging property.

**Definition**

*Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, its* ***set of generators*** *is*

$$G_{\mathcal{U}} = \{\alpha \in {}^*\mathbb{N} \mid \mathcal{U} = \mathfrak{U}_\alpha\},$$

*where* $\mathfrak{U}_\alpha = \{A \subseteq \mathbb{N} \mid \alpha \in {}^*A\}$.

**Question:** Given hypernatural numbers $\alpha, \beta \in {}^*\mathbb{N}$, is there a function $f : {}^*\mathbb{N}^2 \to \mathbb{N}$ such that $\mathfrak{U}_{f(\alpha,\beta)} = \mathfrak{U}_\alpha \oplus \mathfrak{U}_\beta$?

# ${}^{\bullet}\mathbb{N}$: the $\omega$-hyperextension of $\mathbb{N}$/1

**Definition**

*Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods. We call $\omega$-**hyperextension** of $\mathbb{N}$, and denote by ${}^{\bullet}\mathbb{N}$, the union of all hyperextensions $S_n(\mathbb{N})$:*

$${}^{\bullet}\mathbb{N} = \bigcup_{n \in \mathbb{N}} S_n(\mathbb{N}).$$

**Definition**

*Let $\alpha \in {}^{\bullet}\mathbb{N} \setminus \mathbb{N}$. The **height of** $\alpha$ (denoted by $h(\alpha)$) is the least natural number $n$ such that $\alpha \in S_n(\mathbb{N})$.*

## ${}^\bullet\mathbb{N}$: the $\omega$-hyperextension of $\mathbb{N}/2$

**Proposition**

*Let* $\alpha,\beta \in {}^\bullet\mathbb{N}$, $\mathcal{U}=\mathfrak{U}_\alpha$ *and* $\mathcal{V}=\mathfrak{U}_\beta$, *and suppose that* $h(\alpha)=h(\beta)=1$. *Then:*

1. *for every natural number* $n$, $\mathfrak{U}_\alpha=\mathfrak{U}_{S_n(\alpha)}$;
2. $\alpha +^* \beta \in G_{\mathcal{U}\oplus\mathcal{V}}$;
3. $\alpha \cdot^* \beta \in G_{\mathcal{U}\odot\mathcal{V}}$.

**Proposition**

*Let* $\mathcal{U}\in\beta\mathbb{N}$. *Then:*

1. $\mathcal{U}$ *is an additive idempotent ultrafilter* $\Leftrightarrow \forall\alpha,\beta\in G_{\mathcal{U}}$ $\alpha+S_{h(\alpha)}(\beta)\in G_{\mathcal{U}} \Leftrightarrow \exists\alpha,\beta\in G_{\mathcal{U}}$ $\alpha+S_{h(\alpha)}(\beta)\in G_{\mathcal{U}}$;
2. $\mathcal{U}$ *is a multiplicative idempotent ultrafilter* $\Leftrightarrow \forall\alpha,\beta\in G_{\mathcal{U}}$ $\alpha\cdot S_{h(\alpha)}(\beta)\in G_{\mathcal{U}} \Leftrightarrow \exists\alpha,\beta\in G_{\mathcal{U}}$ $\alpha\cdot S_{h(\alpha)}(\beta)\in G_{\mathcal{U}}$.

# The Polynomial Bridge Theorem

**Theorem (Polynomial Bridge Theorem)**

*Let* $P(x_1, ..., x_n)$ *be a polynomial, and* $\mathcal{U}$ *an ultrafilter on* $\beta\mathbb{N}$*. Then the following two conditions are equivalent:*

1. $\mathcal{U}$ *is a* $\iota_P$*-ultrafilter;*
2. *there are mutually distinct elements* $\alpha_1, ..., \alpha_n$ *in* $G_{\mathcal{U}}$ *such that* $P(\alpha_1, ..., \alpha_n) = 0$.

**Lemma (Reduction Lemma)**

*Let* $P(x_1, ..., x_n)$ *be a polynomial, and* $\mathcal{U}$ *a* $\iota_P$*-ultrafilter. Then there are mutually distinct elements* $\alpha_1, ..., \alpha_n \in G_{\mathcal{U}} \cap {}^*\mathbb{N}$ *such that* $P(\alpha_1, ..., \alpha_n) = 0$.

# An Example: Schur's Theorem

**Theorem (Schur)**

*The polynomial $P(x,y,z): x+y-z$ is injectively partition regular.*

**Proof:** Let $\mathcal{U}$ be an additive idempotent ultrafilter, and $\alpha \in {}^*\mathbb{N}$ a generator of $\mathcal{U}$. Then ${}^*\alpha \in \mathcal{U}$ (this holds for every ultrafilter) and $\alpha + {}^*\alpha \in \mathcal{U}$ (since $\mathcal{U}$ is an additive idempotent ultrafilter). And

$$P(\alpha, {}^*\alpha, \alpha + {}^*\alpha) = 0,$$

so we can apply the Polynomial Bridge Theorem and conclude.

# A Fundamental Lemma

**Theorem**

*If $P(x_1, ..., x_n)$ is an homogeneous injectively partition regular polynomial then there is a nonprincipal multiplicative idempotent $\iota_P$-ultrafilter.*

# $P(x,y,z,w) : x+y-zw$ is injectively partition regular

**Corollary**

*The polynomial $P(x,y,z,t) : x+y-zw$ is injectively partition regular.*

**Step 1:** Let $R(x,y,z) : x+y-z$.

**Step 2:** Let $\mathcal{U}$ be a multiplicative idempotent $\iota_R$-ultrafilter and let $\alpha,\beta,\gamma \in^*\mathbb{N}$ be generators of $\mathcal{U}$ such that $\alpha+\beta-\gamma=0$.

**Step 3:** We observe that

$$P(\alpha\cdot^*\gamma, \beta\cdot^*\gamma, \gamma, ^*\gamma) = \alpha\cdot^*\gamma + \beta\cdot^*\gamma - \gamma\cdot^*\gamma = 0,$$

and we can conclude by the Polynomial Bridge Theorem.

# Hindman's Theorem

**Theorem (Hindman)**

*For every natural numbers $n, m \geq 1$, with $n+m \geq 3$, the nonlinear polynomial $P(x_1, ..., x_n, y_1, ..., y_m) : \sum_{i=1}^{n} x_i - \prod_{j=1}^{m} y_j$ is injectively partition regular.*

**Proof:** Let $R(z_1, ..., z_{n+1}) : z_1 + .... + z_n - z_{n+1}$, $\mathcal{U}$ a multiplicative idempotent $\iota_R$-ultrafilter, $\alpha_1, ..., \alpha_n, \beta \in^* \mathbb{N}$ mutually distinct generators of $\mathcal{U}$ such that $\sum_{i=1}^{n} \alpha_i = \beta$, and $\gamma = \prod_{j=2}^{m} S_{j-1}(\beta)$. Then

$$P(\alpha_1 \cdot \gamma, ..., \alpha_n \cdot \gamma, \beta, S(\beta), ..., S_{m-1}(\beta)) = 0$$

and we can apply the Polynomial Bridge Theorem.

# Generalizing Hindman's Theorem/1

## Definition

*Let $m$ be a positive natural number, and $\{y_1, ..., y_m\}$ a set of mutually distinct variables. For every finite set $F \subseteq \{1, .., m\}$, we denote by $Q_F(y_1, ..., y_m)$ the monomial*

$$Q_F(y_1, ..., y_m) = \begin{cases} \prod_{j \in F} y_j, & \text{if } F \neq \emptyset; \\ 1, & \text{if } F = \emptyset. \end{cases}$$

E.g., if $m = 5$ and $F = \{1, 4, 5\}$ then $Q_F(y_1, ..., y_5) = y_1 \cdot y_4 \cdot y_5$.

# Generalizing Hindman's Theorem/2

**Theorem**

*Let* $n \geq 2$ *be a natural number,* $R(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i$ *an injectively partition regular polynomial, and* $m$ *a positive natural number. Then, for every* $F_1, ..., F_n \subseteq \{1, .., m\}$ *(with the request that, when* $n = 2$, $F_1 \cup F_2 \neq \emptyset$*), the polynomial*

$$P(x_1, ..., x_n, y_1, ..., y_m) = \sum_{i=1}^{n} a_i x_i Q_{F_i}(y_1, ..., y_m)$$

*is injectively partition regular.*

# A Nontrivial Example

Let us prove that $P(x_1, x_2, x_3, x_4, y_1, y_2) = 2x_1 + x_2y_1y_2 - 3x_3y_1 + x_4y_2$ is injectively partition regular.

**Step 1:** We consider $R(x, y, z, w) : 2x_1 + x_2 - 3x_3 + x_4$, and we take a multiplicative idempotent $\iota_R$−ultrafilter.

**Step 2:** We take mutually distinct $\alpha, \beta, \gamma, \delta \in G_{\mathcal{U}}$ such that $R(\alpha, \beta, \gamma, \delta) = 0$.

**Step 3:** We take $\eta \in G_{\mathcal{U}}$, and we observe that

$$P(\alpha \cdot S_1(\eta) \cdot S_2(\eta), \beta, \gamma \cdot S_2(\eta), \delta \cdot S_1(\eta), S_1(\eta), S_2(\eta)) =$$

$$= S_1(\eta) \cdot S_2(\eta)(2\alpha + \beta - 3\gamma + \delta) = 0.$$

## Definitions/1

### Definition

*A polynomial* $P(x_1, ..., x_n) : \sum_{i=1}^{k} a_i M_i(x_1, ..., x_n)$ *satisfies* ***Rado's Condition*** *if there is a nonempty subset* $J \subseteq \{1, ..., n\}$ *such that* $\sum_{j \in J} a_j = 0$.

### Definition

*Let*

$$P(x_1, ..., x_n) : \sum_{i=1}^{k} a_i M_i(x_1, ..., x_n)$$

*be a polynomial, and let* $M_1(x_1, ..., x_n), ..., M_k(x_1, ..., x_n)$ *be the distinct monic monomials of* $P(x_1, ..., x_n)$. *We say that a variable* $v$ *is* ***exclusive*** *in* $P(x_1, ..., x_n)$ *if there is an index* $i$ *such that for every* $j \leq k$, $d_{M_j}(v) \geq 1 \Leftrightarrow j = i$.

# Definitions/2

**Definition**

*Given a polynomial $P(x_1, ..., x_n)$ we denote by **NL(P)** the set of nonlinear variables in $P(x_1, ..., x_n)$:*

$$NL(P) = \{x \in \{x_1, ..., x_n\} \mid d(x) > 1\}.$$

**Definition**

*Let $P(x_1, ..., x_n) = \sum_{i=1}^{k} a_i M_i(x_1, ..., x_n)$ be a polynomial, and let $M_1(x_1, ..., x_n)$,...,$M_k(x_1, ..., x_n)$ be its monic monomials. For every index $i \leq k$ we pose*

$$l_i = \max\{d(x) - d_i(x) \mid x \in NL(P)\}.$$

# A Generalization

**Theorem**

*Let*

$$P(x_1, ..., x_n) = \sum_{i=1}^{k} a_i M_i(x_1, ..., x_n)$$

*be a polynomial, and let* $M_1(x_1, ..., x_n)$,...,$M_k(x_1, ..., x_n)$ *be the monic monomials of* $P(x_1, ..., x_n)$. *Suppose that* $k \geq 3$, *that* $P(x_1, ..., x_n)$ *satisfies Rado's Condition and that, for every index* $i \leq k$, *in the monomial* $M_i(x_1, ..., x_n)$ *there are at least* $m_i = \max\{1, l_i\}$ *linear exclusive variables.*
*Then* $P(x_1, ..., x_n)$ *is injectively partition regular.*

# An Example/1

Consider the polynomial

$$P(x_1, x_2, x_3, x_4, y) : x_1 y^2 + 2x_2 y - x_3 x_4.$$

**Step 1:** We pose $y = 1$ and consider

$$R(x_1, x_2, x_3, x_4) : x_1 + 2x_2 - x_3 x_4.$$

**Step 2:** We take a multiplicative idempotent $\iota_R$-ultrafilter $\mathcal{U}$.

## An Example/2

**Step 3:** We take $\alpha, \beta, \gamma, \delta \in G_{\mathcal{U}}$ such that $\alpha + 2\beta - \gamma\delta = 0$.

**Step 4:** We take any $\eta$ in $G_{\mathcal{U}}$ and we pose $y = S_1(\eta)$.

**Step 5:** We observe that

$$P(\alpha, \beta \cdot S_1(\eta), \gamma \cdot S_1(\eta), \delta \cdot S_1(\eta), S_1(\eta)) =$$

$$= \alpha \cdot S_1(\eta)^2 + 2\beta \cdot S_1(\eta) \cdot S_1(\eta) - \gamma \cdot S_1(\eta) \cdot \delta \cdot S_1(\eta) =$$

$$= S_1(\eta)^2(\alpha + 2\beta - \gamma\delta) = 0,$$

and we conclude by the Polynomial Bridge Theorem.

# Final Remarks

1) The request on the existence of exclusive variables is not necessary: the polynomial

$$P(x, y, z) = xy + xz - yz$$

is injectively partition regular even if it doesn't admit any exclusive variable.

2) Rado's Condition is necessary for homogeneous partition regular polynomials, but it seems to be not necessary in general: e.g., the polynomial

$$P(x_1, x_2, x_3, y_1, y_2) = x_1y_1 + x_2y_2 + x_3$$

is injectively partition regular on $\mathbb{Z}$.

3) Rado's Condition is not sufficient to ensure the partition regularity of a nonlinear polynomial: the polynomial

$$x + y - z^2$$

is not partition regular.

Thank You!