Ultrafilter Extensions of Models

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Ultracombinatorics

Introduction

Aim

Given operations $F, \ldots$ and relations $P, \ldots$ on X, we shall show that there is a natural way to extend them to operations $\tilde{F}, \ldots$ and relations $\tilde{P}, \ldots$ on $\boldsymbol{\beta} X$. Thus the model

$$\mathfrak{A} = (X, F, \ldots, P, \ldots)$$

extends to the model

$$\boldsymbol{\beta}\mathfrak{A} = (\boldsymbol{\beta} X, \tilde{F}, \ldots, \tilde{P}, \ldots).$$

The extension procedure is *canonical*: it is *unique*, it *lifts* model-theoretic interrelations, and it is the *largest* such extension.

Historical remarks

Largest compactifications were discovered independently by Čech and M. Stone (1937) for Tychonoff spaces and by Wallman (1938) for T_1 spaces.

Our ultrafilter extension procedure, applied to unary maps and relations, gives concepts described in these classical works in 30s.

Several instances of ultrafilter extensions of maps and relations of greater arities were discovered only in 60s. We isolate three areas where such instances arose.

The first area: iterated ultrapowers

Frayne, Morel, and Scott (1958) shown that finite iteration of ultrapowers gives ultrapowers by using (in our terms) ultrafilter extensions of taking n-tuples:

$$\prod_{\langle u_1,\ldots,u_n\rangle^{\sim}} \mathfrak{A} \quad \simeq \quad \prod_{u_1} \ldots \prod_{u_n} \mathfrak{A}.$$

Here $\langle\,\rangle^{\sim}$ is the ultrafilter extension of taking n-tuples.

The general construction of iterated ultrapowers, invented by Gaifman and elaborated by Kunen, has became standard in model theory and set theory.

The second area: ultrafilter extensions of semigroups

Such structures appeared as subspaces of function spaces in 60s. The first explicit construction is due to Ellis (1969):

$$u_1 \mathbin{\tilde{\cdot}} u_2 = \Big\{S \subseteq Y : \{x_1 \in X : \{x_2 \in X : x_1 \cdot x_2 \in S\} \in u_2\} \in u_1\Big\}.$$

In 70s Galvin and Glazer applied it to give an easy proof of Hindman's Finite Sums Theorem; the key idea was to use *idempotent ultrafilters*.

The method was used by Hindman, van Douwen, Blass, Strauss, and many others, and gave numerous applications in number theory, combinatorics, algebra, and dynamics. Many results have no (known) elementary proofs.

The third area: modal logic

Characterizing modal definability, van Benthem (1988) extended binary relations of frames to ultrafilters. Goldblatt (1989) and then Goranko generalized this construction to relations of arbitrary arity. Their extensions coincide with our extensions only for unary relations.

Goranko (2007, unpublished) considered extensions of operations to filters and, in another way, to ultrafilters, and proved a theorem analogous to the First Extension Theorem below.

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Our construction of ultrafilter extensions of models, together with the basic results, has appeared in 2010.

The construction and the First Extension Theorem

Topology on ultrafilters

For every $S \subseteq X$ let

$$\tilde{S} = \{u \in \boldsymbol{\beta}X : S \in u\}.$$

The sets $\tilde{S}$ form an open basis generating the standard topology on $\boldsymbol{\beta}X$. The space $\boldsymbol{\beta}X$ is:

(i) Hausdorff,
(ii) zero-dimensional (basic sets $\tilde{S}$ are clopen),
(iii) extremally disconnected (the closures of open sets are open),
(iv) contains X (identified with the set of principal ultrafilters) as an open dense subspace,
(v) compact (under a dose of AC).

It follows that the space $\boldsymbol{\beta}X$ is a compactification of the discrete space X. Moreover, $\boldsymbol{\beta}X$ is the *largest* (*Stone–Čech* or *Wallman*) compactification of X:

(vi) Any map h of X into any compact Hausdorff space Y uniquely extends to a continuous map $\tilde{h}$ of $\boldsymbol{\beta}X$ into Y.

Extending maps

Lemma. *Given an n-ary map $F : X_1 \times \ldots \times X_n \to Y$, for all ultrafilters $u_1 \in \boldsymbol{\beta} X_1, \ldots, u_n \in \boldsymbol{\beta} X_n$ let*

$$\tilde{F}(u_1, \ldots, u_n) = \Big\{ S \subseteq Y : \{x_1 \in X_1 : \ldots \{x_n \in X_n : F(x_1, \ldots, x_n) \in S\} \in u_n \ldots\} \in u_1 \Big\}.$$

Then $\tilde{F} : \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n \to \boldsymbol{\beta} Y$ and $\tilde{F} \restriction \mathrm{dom}\,(F) = F$.

Definition. $\tilde{F}$ is the *ultrafilter extension* of F.

Extending relations

Lemma. *Given an n-ary relation $P \subseteq X_1 \times \ldots \times X_n$, for all ultrafilters $u_1 \in \boldsymbol{\beta} X_1, \ldots, u_n \in \boldsymbol{\beta} X_n$ let*

$$\langle u_1, \ldots, u_n \rangle \in \tilde{P} \quad \textit{iff}$$
$$\Big\{x_1 \in X_1 : \ldots \{x_n \in X_n : \langle x_1, \ldots, x_n \rangle \in P\} \in u_n \ldots \Big\} \in u_1.$$

Then $\tilde{P} \subseteq \boldsymbol{\beta} X_1 \times \ldots \times \boldsymbol{\beta} X_n$ and $\tilde{P} \cap (X_1 \times \ldots \times X_n) = P$.

Definition. $\tilde{P}$ is the *ultrafilter extension* of P.

A redefinition:

Proposition. *Let $P \subseteq X_1 \times \ldots \times X_n$. Then*

$$\langle u_1, \ldots, u_n \rangle \in \tilde{P} \quad \textit{iff} \quad P \in \langle u_1, \ldots, u_n \rangle^{\sim}$$

for all $u_1 \in \boldsymbol{\beta} X_1, \ldots, u_n \in \boldsymbol{\beta} X_n$.

Here $\langle\,\rangle^{\sim}$ is the ultrafilter extension of taking n-tuples.

Extending models

Definition. Given a model $\mathfrak{A} = (X, F, \ldots, P, \ldots)$, the model

$$\boldsymbol{\beta}\mathfrak{A} = (\boldsymbol{\beta}X, \tilde{F}, \ldots, \tilde{P}, \ldots)$$

is the *ultrafilter extension* of the model $\mathfrak{A}$.

Lifting homomorphisms

Lemma. *Let* $h_1 : X_1 \to Y_1,\ \ldots,\ h_n : X_n \to Y_n,$ *and* $u_1 \in \boldsymbol{\beta}X_1, \ldots, u_n \in \boldsymbol{\beta}X_n$.

(i) *For any* $G : Y_1 \times \ldots \times Y_n \to Z$,

$$\tilde{G}(\tilde{h}_1(u_1), \ldots, \tilde{h}_n(u_n)) = \Big\{S \subseteq Z : \{x_1 : \ldots \{x_n : G(h_1(x_1), \ldots, h_n(x_n)) \in S\} \in u_n \ldots\} \in u_1\Big\}.$$

(ii) *For any* $P \subseteq X_1 \times \ldots \times X_n$,

$\langle \tilde{h}_1(u_1), \ldots, \tilde{h}_n(u_n)\rangle \ \in \tilde{P}$ *iff*
$\{x_1 : \ldots \{x_n : \langle h_1(x_1), \ldots, h_n(x_n)\rangle \in P\} \in u_n \ldots\} \in u_1$.

As a corollary, we get our first result:

The First Extension Theorem. *Let* $\mathfrak{A}$ *and* $\mathfrak{B}$ *be two models. If* h *is a homomorphism of* $\mathfrak{A}$ *into* $\mathfrak{B}$, *then* $\tilde{h}$ *is a homomorphism of* $\boldsymbol{\beta}\mathfrak{A}$ *into* $\boldsymbol{\beta}\mathfrak{B}$.

Topological properties of extended models and the Second Extension Theorem

Topology provides a natural language to express some properties of ultrafilter extensions. We describe topological properties of extended mapping and relations, then we isolate them in abstracto, for mapping and relations on topological spaces.

This leads to a certain class of models endowed with topologies (which is wider than the class of usual topological models). Our aim is to show that ultrafilter extensions are *largest* extensions in this class.

Extended maps, topology

Unary case. $F : X \to Y$ extends by

$$\tilde{F}(u) = \Big\{ S \subseteq Y : \{x \in X : F(x) \in S\} \in u \Big\}.$$

This gives the standard unique continuous extension of F.

Binary case. $F : X_1 \times X_2 \to Y$ extends by

$$\tilde{F}(u_1, u_2) = \Big\{ S \subseteq Y : \{x_1 \in X_1 : \{x_2 \in X_2 : F(x_1, x_2) \in S\} \in u_2\} \in u_1 \Big\}.$$

The extension can be fulfilled in two steps: first one extends left translations of F, then right translation of the partial extension. In result: all right translations of $\tilde{F}$

$$u_1 \mapsto \tilde{F}(u_1, u_2)$$

are continuous, and all its left translations

$$u_2 \mapsto \tilde{F}(u_1, u_2)$$

by *principal* ultrafilters $u_1 \in \boldsymbol{\beta} X_1$ are continuous.

General case

Lemma. *Let* $F : X_1 \times \ldots \times X_n \to Y$ *and* $1 \leq i \leq n$. *For every principal* $u_1 \in \boldsymbol{\beta} X_1, \ldots,$ $u_{i-1} \in \boldsymbol{\beta} X_{i-1}$ *and arbitrary* $u_{i+1} \in \boldsymbol{\beta} X_{i+1}, \ldots, u_n \in \boldsymbol{\beta} X_n$*, the map*

$$u \mapsto \tilde{F}(u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n)$$

of $\boldsymbol{\beta} X_i$ *into* $\boldsymbol{\beta} Y$ *is continuous. Moreover,* $\tilde{F}$ *is the only such extension of* F*.*

Remark. This description of continuiuty of extended maps cannot be improved.

We isolate this property:

Definition. Let $X_1, \dots, X_n, Y$ be topological spaces, and let $C_1 \subseteq X_1, \dots, C_n \subseteq X_n$. An n-ary map $F : X_1 \times \dots \times X_n \to Y$ is *right continuous w.r.t.* $C_1, \dots, C_n$ iff for each i, $1 \leq i \leq n$, and all $c_1 \in C_1, \dots, c_{i-1} \in C_{i-1}$ and $x_{i+1} \in X_{i+1}, \dots, x_n \in X_n$, the map

$$x \mapsto F(c_1, \dots, c_{i-1}, x, x_{i+1}, \dots, x_n)$$

of X_i into Y is continuous.

In these terms, the previous lemma states:

Lemma. *For any map F on $X_1 \times \dots \times X_n$ its ultrafilter extension $\tilde{F}$ is right continuous w.r.t. $X_1, \dots, X_n$.*

Extended relations, topology

Unary case. If $P \subseteq X$, then

$$u \in \tilde{P} \quad \text{iff} \quad P \in u.$$

Thus $\tilde{P}$ is a basic (cl)open set of $\boldsymbol{\beta}X$.

Binary case. If $P \subseteq X_1 \times X_2$, then

$$\langle u_1, u_2 \rangle \in \tilde{P} \quad \text{iff}$$
$$\Big\{x_1 \in X_1 : \{x_2 \in X_2 : \langle x_1, x_2 \rangle \in P\} \in u_2\Big\} \in u_1.$$

All sections of $\tilde{P}$

$$\tilde{P}_{u_2} = \{u_1 \in \boldsymbol{\beta}X_1 : \langle u_1, u_2 \rangle \in \tilde{P}\}$$

are clopen in $\boldsymbol{\beta}X_1$, and all its sections

$$\tilde{P}_{u_1} = \{u_2 \in \boldsymbol{\beta}X_2 : \langle u_1, u_2 \rangle \in \tilde{P}\}$$

by *principal* ultrafilters $u_1 \in \boldsymbol{\beta}X_1$ are clopen in $\boldsymbol{\beta}X_2$.

General case

Lemma. *Let* $P \subseteq X_1 \times \ldots \times X_n$ *and* $1 \leq i \leq n$. *For every principal* $u_1 \in \boldsymbol{\beta} X_1, \ldots,$ $u_{i-1} \in \boldsymbol{\beta} X_{i-1}$ *and arbitrary* $u_{i+1} \in \boldsymbol{\beta} X_{i+1}, \ldots, u_n \in \boldsymbol{\beta} X_n$, *the set*

$$\tilde{P}_{u_1,\ldots,u_{i-1},u_{i+1},\ldots,u_n} =$$
$$\{u \in \boldsymbol{\beta} X_i : \langle u_1, \ldots, u_{i-1}, u, u_{i+1}, \ldots, u_n \rangle \in \tilde{P}\}$$

of $\boldsymbol{\beta} X_i$ *is clopen.*

We isolate this property:

Definition. Let $X_1, \ldots, X_n$ be topological spaces, and let $C_1 \subseteq X_1, \ldots, C_n \subseteq X_n$. An n-ary relation $P \subseteq X_1 \times \ldots \times X_n$ is *right open w.r.t.* $C_1, \ldots, C_n$ iff for each i, $1 \leq i \leq n$, and every $c_1 \in C_1, \ldots, c_{i-1} \in C_{i-1}$ and $x_{i+1} \in X_{i+1}, \ldots, x_n \in X_n$, the subset

$$P_{c_1,\ldots,c_{i-1},x_{i+1},\ldots,x_n} =$$
$$\{x \in X_i : \langle c_1, \ldots, c_{i-1}, x, x_{i+1}, \ldots, x_n\rangle \in P\}$$

of X_i is open. Likewise for *right closed* (*right clopen*, etc.) relations.

In these terms, the previous lemma states:

Lemma. *For any relation P on $X_1 \times \ldots \times X_n$ its ultrafilter extension $\tilde{P}$ is right clopen w.r.t. $X_1, \ldots, X_n$.*

Extended models, topology

Definition. Let $\mathfrak{A} = (X, F, \ldots, P, \ldots)$ be a model equipped with a topology, and $C \subseteq X$. Then $\mathfrak{A}$ is *right open* with C its *topological center* iff all its operations are right continuous w.r.t. C and all its relations are right open w.r.t. C. Likewise for *right closed* (*right clopen*, etc.) models.

In these terms, two last lemmas state

Theorem. *For any model $\mathfrak{A}$ its ultrafilter extension $\beta\mathfrak{A}$ is right clopen with $\mathfrak{A}$ a dense topological center.*

Lifting homomorphisms, refined

The following theorem concerns rather arbitrary right open and right closed models with dense topological centers than ultrafilter extensions.

Theorem. *Let a model $\mathfrak{A}$ be right open, a model $\mathfrak{B}$ a Hausdorff right closed, and $\mathfrak{C} \subseteq \mathfrak{A}$ a dense submodel and a topological center of $\mathfrak{A}$. Let h be a continuous mapping of $\mathfrak{A}$ into $\mathfrak{B}$ such that*

(i) $h \restriction \mathfrak{C}$ *is a homomorphism, and*
(ii) $h``\mathfrak{C}$ *is a topological center of $\mathfrak{B}$.*

Then h is a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$.

Two last theorems give our main result:

The Second Extension Theorem. *Let $\mathfrak{A}$ and $\mathfrak{C}$ be two models, and let $\mathfrak{C}$ be compact Hausdorff right closed. If h be a homomorphism of $\mathfrak{A}$ into $\mathfrak{C}$ such that $h``\mathfrak{A}$ is a topological center of $\mathfrak{C}$, then $\tilde{h}$ is a homomorphism of $\boldsymbol{\beta}\mathfrak{A}$ into $\mathfrak{C}$.*

(The First Extension Theorem is a partial case when $\mathfrak{C}$ is $\boldsymbol{\beta}\mathfrak{B}$.)

Thus $\boldsymbol{\beta}\mathfrak{A}$ is the largest compactification of $\mathfrak{A}$ lifting homomorphisms. This shows that the construction provides a right generalization of the Stone–Čech (or Wallman) compactification of a discrete space to the situation when the space carries a model-theoretic structure.

Further problems

Other relationships between models

The established facts about homomorphisms hold for embeddings and other relationships between models:

Theorem. *Both Extension Theorems remain true if one replaces homomorphisms by isomorphic embeddings, as well as homotopies and isotopies.*

Problem. Characterize relationships between models that are stable under ultrafilter extensions.

Specific ultrafilters

Certain ultrafilters form natural submodels of $\boldsymbol{\beta}\mathfrak{A}$.

Theorem. *Let $\mathfrak{A}$ be a model and κ a cardinal.*

(i) *The subset $\{u \in \boldsymbol{\beta}\mathfrak{A} : u$ is κ-complete$\}$ forms a submodel of $\boldsymbol{\beta}\mathfrak{A}$.*

(ii) *The subset $\{u \in \boldsymbol{\beta}\mathfrak{A} : u$ is κ-uniform$\}$ forms a closed submodel of $\boldsymbol{\beta}\mathfrak{A}$ if the operations of $\mathfrak{A}$ are "sufficiently cancellative"* (a technical condition).

(Similarly for several other types of ultrafilters.)

Also it is possible to show that the Second Extension Theorem remains true if one replaces 'compact' by 'finally κ-compact' and $\boldsymbol{\beta}\mathfrak{A}$ by its submodel consisting of κ-complete ultrafilters.

Problem. Study the role of specific ultrafilters in ultrafilter extensions.

Connections with ultraproducts

Problem. Study an interplay of ultrafilter extensions and ultraproducts.

A simple related result:

Theorem. *If* $\tilde{F}(u_1, \ldots, u_n) = v$ *then*

$$j : \prod_v \mathfrak{A} \prec \prod_u \mathfrak{A}$$

where $u = \langle u_1, \ldots, u_n \rangle\tilde{}$ *and* j *is defined by* $j = [f \circ F]_u$ *for all* f*.*

Corollary. *If* $v \leq_{\mathrm{RK}} u$ *then*

$$\prod_v \mathfrak{A} \prec \prod_u \mathfrak{A}.$$

Formulas stable under ultrafilter extensions

Ultrafilter extensions are highly complicated objects, their equational theories quite differ from equational theories of extended models.

Problem. Characterize (atomic) formulas that are stable under ultrafilter extensions.

We have a sufficient condition:

Theorem. *Let s and t be terms such that*
(i) *the common variables of s and t appear in the same ordering, and*
(ii) *any common variable occurs in each of the terms only once.*
Then the idenitity $s = t$ is stable.

Perhaps this condition is also necessary:

Conjecture. Every identity stable under ultrafilter extensions is equivalent to an identity from Theorem.

Examples. The following identities (in the language of groupoids) are stable:

(i) $xy = (xy)z$,
(ii) $xy = xx$ (is equivalent to $xy = xz$),
(iii) $xy = (yx)z$ (is equivalent to $xy = wz$),
(iv) $(xy)z = x(yz)$ (associativity),
(v) $(xy)(zw) = x(y(zw))$,

while the next two identities are not:

(vi) $x = xx$ (idempotency),
(vii) $xy = yx$ (commutativity).

This was applied to generalize Hindman's theorem to certain non-associatice groupoids (2008).

Properties of models versus properties of their ultrafilter extensions

Ultrafilters can *encode* complex properties of the underlying models.

Basic examples.

(i) Ramsey's theorem corresponds to (arbitrary) *nonprincipal* ultrafilters.

(ii) Hindman's theorem corresponds to *idempotent nonprincipal* ultrafilters.

Problem. A translation of statements about models to statements about their ultrafilter extensions et vice versa.