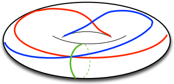
COMBINATORICS AND TOPOLOGY OF TORIC ARRANGEMENTS

II. TOPOLOGY OF ARRANGEMENTS IN THE COMPLEX TORUS



(AN INVITATION TO COMBINATORIAL ALGEBRAIC TOPOLOGY)

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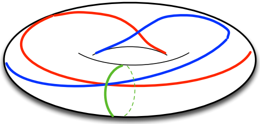
TORIC ARRANGEMENTS

Recall: a toric arrangement in the complex torus $T := (\mathbb{C}^*)^d$ is a set

$$\mathscr{A} := \{K_1, \dots, K_n\}$$

of 'hypertori' $K_i = \{z \in T \mid z^{a_i} = b_i\}$ with $a_i \in \mathbb{Z}^d \backslash 0$ and $b_i \in \mathbb{C}^*$



The *complement* of $\mathscr{A}$ is

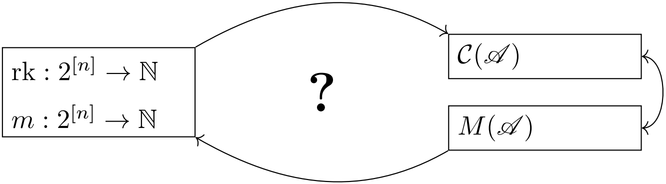
$$M(\mathscr{A}) := T \setminus \cup \mathscr{A},$$

PROBLEM: Study the topology of $M(\mathscr{A})$.

The long game

$$\text{Let } A = [a_1, \ldots, a_n] \in M_{d\times n}(\mathbb{Z})$$

(Central) hyperplane arrangement	(Centered) toric arrangement	(Centered) elliptic arrangement
$\lambda_i : \; \mathbb{C}^d \to \mathbb{C}$	$\lambda_i : \; (\mathbb{C}^*)^d \to \mathbb{C}^*$	$\lambda_i : \; \mathbb{E}^d \to \mathbb{E}$
$\underline{z} \mapsto \sum_j a_{ji} z_j$	$\underline{z} \mapsto \prod_j z_j^{a_{ji}}$	$\underline{z} \mapsto \sum_j a_{ji} z_j$
$H_i := \ker \lambda_i$	$K_i := \ker \lambda_i$	$L_i := \ker \lambda_i$
$\mathscr{A} = \{H_1, \ldots, H_n\}$	$\mathscr{A} = \{K_1, \ldots, K_n\}$	$\mathscr{A} = \{L_1, \ldots, L_n\}$
$M(\mathscr{A}) := \mathbb{C}^d \setminus \cup\mathscr{A}$	$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \cup\mathscr{A}$	$M(\mathscr{A}) := \mathbb{E}^d \setminus \cup\mathscr{A}$



$\mathrm{rk} : 2^{[n]} \to \mathbb{N}$

$m : 2^{[n]} \to \mathbb{N}$

?

$\mathcal{C}(\mathscr{A})$

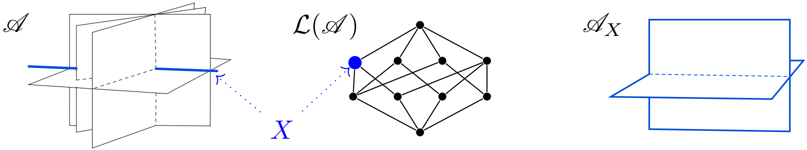
$M(\mathscr{A})$

HYPERPLANES: BRIESKORN

$\mathscr{A} := \{H_1, \dots, H_d\}$: set of (affine) hyperplanes in $\mathbb{C}^d$,

$\mathcal{C}(\mathscr{A}) = \mathcal{L}(\mathscr{A}) := \{\cap\mathscr{B} \mid \mathscr{B} \subseteq \mathscr{A}\}$: poset of intersections (reverse inclusion).

For $X \in \mathcal{L}(\mathscr{A})$: $\mathscr{A}_X = \{H_i \in \mathscr{A} \mid X \subseteq H_i\}$.



Theorem (Brieskorn 1972). The inclusions $M(\mathscr{A}) \hookrightarrow M(\mathscr{A}_X)$ induce, for every k, an isomorphism of free abelian groups

$$b : \bigoplus_{\substack{X \in \mathcal{L}(\mathscr{A}) \\ \operatorname{codim} X = k}} H^k(M(\mathscr{A}_X), \mathbb{Z}) \xrightarrow{\cong} H^k(M(\mathscr{A}), \mathbb{Z})$$

HYPERPLANES: BRIESKORN

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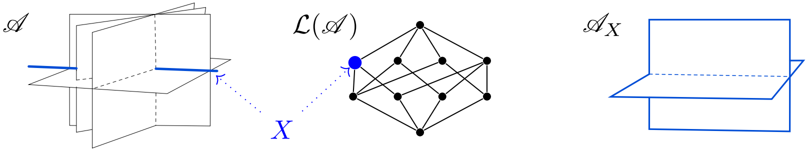


In fact: $M(\mathscr{A})$ is a *minimal space*, i.e., it has the homotopy type of a CW-complex with as many cells in dimension k as there are generators in k-th cohomology. [Dimca-Papadima '03]

HYPERPLANES: THE ORLIK-SOLOMON ALGEBRA

[Arnol'd '69, Orlik-Solomon '80]

$$H^*(M(\mathscr{A}),\mathbb{Z}) \simeq E/\mathcal{J}(\mathscr{A}), \text{ where}$$

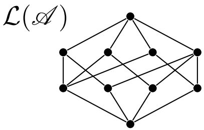
E: exterior $\mathbb{Z}$-algebra with degree-1 generators $e_1, \ldots, e_n$ (one for each H_i);

$\mathcal{J}(\mathscr{A})$: the ideal $\langle \sum_{l=1}^{k} (-1)^l e_{j_1} \cdots \widehat{e_{j_l}} \cdots e_{j_k} \mid \text{codim}(\cap_{i=1\ldots k} H_{j_i}) = k-1 \rangle$

Fully determined by $\mathcal{L}(\mathscr{A})$ (cryptomorphisms!).

For instance:

$$P(M(\mathscr{A}),t) = \sum_{X\in\mathcal{L}(\mathscr{A})} \underbrace{\mu_{\mathcal{L}(\mathscr{A})}(\hat{0},X)}_{\substack{\text{Möbius}\\ \text{function}\\ \text{of } \mathcal{L}(\mathscr{A})}} (-t)^{\overset{\text{codim}\, X}{\boxed{\text{rk}\, X}}}$$



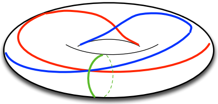
Poin$(M(\mathscr{A}),t) =$

$1 + 4t + 5t^2 + 2t^3$

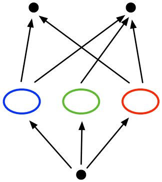
TORIC ARRANGEMENTS

Another good reason for considering $\mathcal{C}(\mathscr{A})$, the poset of *layers* (i.e. connected components of intersections of the K_i).

$\mathscr{A}$:



$\mathcal{C}(\mathscr{A})$:



Theorem [Looijenga '93, De Concini-Procesi '05]

$$\mathrm{Poin}(M(\mathscr{A}),\mathbb{Z}) = \sum_{Y\in\mathcal{C}(\mathscr{A})} \mu_{\mathcal{C}(\mathscr{A})}(Y)(-t)^{\mathrm{rk}\,Y}(1+t)^{d-\mathrm{rk}\,Y}$$

$$= (-t)^d \chi_{\mathcal{C}(\mathscr{A})}(-t(1+t))$$

TORIC ARRANGEMENTS

[De Concini – Procesi '05] compute the Poincaré polynomial and the cup product in $H^*(M(\mathscr{A}), \mathbb{C})$ when the matrix $[a_1, \ldots, a_n]$ is totally unimodular.

[d'Antonio–D. '11,'13] $\pi_1(M(\mathscr{A}))$, minimality, torsion-freeness (complexified)

[Bibby '14] $\mathbb{Q}$-cohomology algebra of unimodular abelian arrangements

[Dupont '14] Algebraic model for $\mathbb{C}$-cohomology algebra of complements of hypersurface arrangements in manifolds with hyperplane-like crossings.

[Callegaro-D. '15] Integer cohomology algebra, its dependency from $\mathcal{C}(\mathscr{A})$.

[Bergvall '16] Cohomology as repr. of Weyl group in type G_2, F_4, E_6, E_7.

Wonderful models: nonprojective [Moci'12], projective [Gaiffi-De Concini '16].

POSETS AND CATEGORIES

P - a partially ordered set

$\Delta(P)$ - the *order complex* of P

(abstract simplicial complex of totally ordered subsets)

$||P|| := |\Delta(P)|$

its geometric realization

$\mathcal{C}$ - a *s.c.w.o.l.*

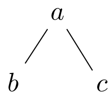
(all invertibles are endomorphisms, all endomorphisms are identities)

$\Delta\mathcal{C}$ - the nerve

(simplicial set of composable chains)

$||\mathcal{C}|| := |\Delta\mathcal{C}|$

its geometric realization



P

$$\left\{\begin{matrix} a & b & c \\ ab & ac & (\emptyset) \end{matrix}\right\}$$

ΔP

$||P||$

$\mathcal{C}$

POSETS AND CATEGORIES

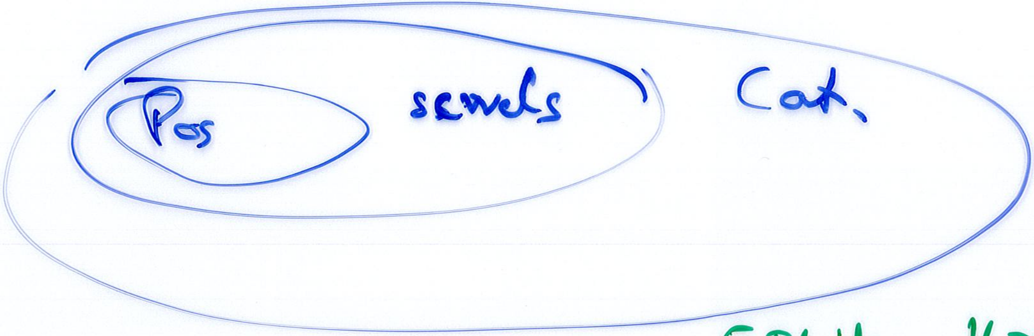
P - a partially ordered set	$\mathcal{C}$ - a *s.c.w.o.l.* (all invertibles are endomorphisms, all endomorphisms are identities)
$\Delta(P)$ - the *order complex* of P (abstract simplicial complex of totally ordered subsets)	$\Delta\mathcal{C}$ - the nerve (simplicial set of composable chains)
$\|\|P\|\| := \|\Delta(P)\|$ its geometric realization	$\|\|\mathcal{C}\|\| := \|\Delta\mathcal{C}\|$ its geometric realization

- Posets are special cases of s.c.w.o.l.s;
- Every functor $F : \mathcal{C} \to \mathcal{D}$ induces a continuous map $||F|| : ||\mathcal{C}|| \to ||\mathcal{D}||$.
- *Quillen-type theorems* relate properties of $||F||$ and F.

My favorite thing about scwols:

often - e.g., when $\mathcal{C} = \mathcal{F}(X)$, $G \curvearrowright X$ cellularly

$$\|\mathcal{C}\|/G \simeq \|\underbrace{\mathcal{C}/G}_{\text{in the cat. of scwols}}\|$$



[Bridson, Häfliger]

FACE CATEGORIES

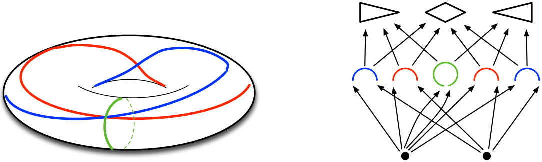
Let X be a polyhedral complex. The *face category* of X is $\mathcal{F}(X)$, with

- $\mathrm{Ob}(\mathcal{F}(X)) = \{X_\alpha, \text{ polyhedra of } X\}$.
- $\mathrm{Mor}_{\mathcal{F}(X)}(X_\alpha, X_\beta) = \{ \text{ face maps } X_\alpha \to X_\beta\}$

Theorem. There is a homeomorphism $||\mathcal{F}(X)|| \cong X$. [Kozlov / Tamaki]

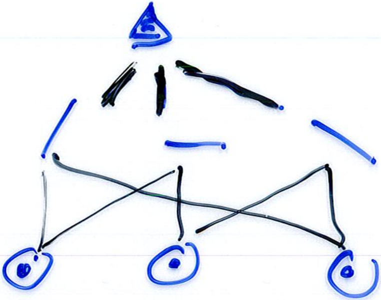
Example 1: X regular: $\mathcal{F}(X)$ is a poset, $||\mathcal{F}(X)|| = \mathrm{Bd}(X)$.

Example 2: A complexified toric arrangement ($\mathscr{A} = \{\chi_i^{-1}(b_i)\}$ with $b_i \in S^1$) induces a polyhedral cellularization of $(S^1)^d$: call $\mathcal{F}(\mathscr{A})$ its face category.

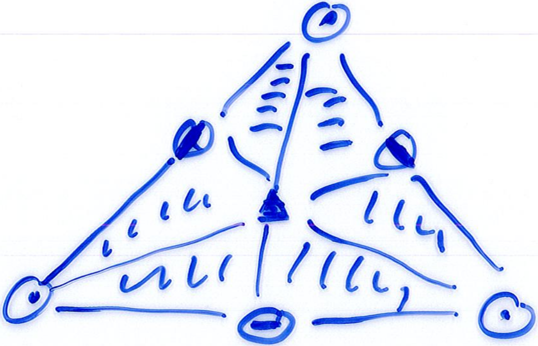


X :

$\mathcal{F}(X)$



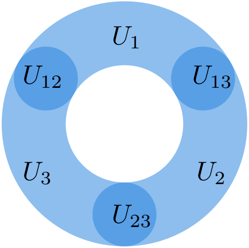
$\|\mathcal{F}(X)\|$



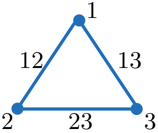
THE NERVE LEMMA

Let X be a paracompact space with a (locally) finite open cover $\mathcal{U} = \{U_i\}_I$.

For $J \subseteq I$ write $U_J := \bigcap_{i \in J} U_i$.



$$\mathcal{N}(\mathcal{U}) = \left\{ \begin{matrix} 12 & 13 & 23 \\ 1 & 2 & 3 \end{matrix} \right\}$$



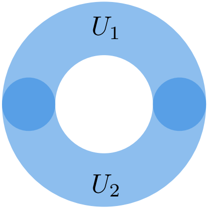
Nerve of $\mathcal{U}$: the abstract simplicial complex $\mathcal{N}(\mathcal{U}) = \{\emptyset \neq J \subseteq I \mid U_J \neq \emptyset\}$

Theorem (Weil ‘51, Borsuk ‘48). If U_J is contractible for all $J \in \mathcal{N}(\mathcal{U})$,

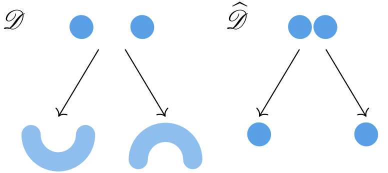
$$X \simeq |\mathcal{N}(\mathcal{U})|$$

THE GENERALIZED NERVE LEMMA

Let X be a paracompact space with a (locally) finite open cover $\mathcal{U} = \{U_i\}_I$.



$\mathscr{N}(\mathcal{U}) = \left\{ \begin{matrix} 1 & 2 \\ 12 & \end{matrix} \right\}$

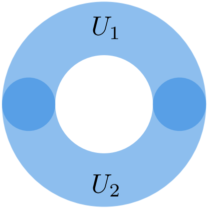


Consider the *diagram* $\mathscr{D} : \mathscr{N}(\mathcal{U}) \to \mathrm{Top}$, $\mathscr{D}(J) := U_J$ and inclusion maps.

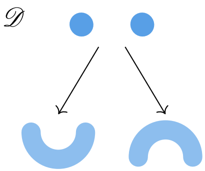
$$\begin{tikzcd}
X = \operatorname{colim}\mathscr{D} & \operatorname{hocolim}\mathscr{D} \arrow[l, "\text{G.N.L.: }\simeq"'] \arrow[r, leftrightarrow, "\simeq"'] & \operatorname{hocolim}\widehat{\mathscr{D}} \\
\biguplus_J \mathscr{D}(J) \Big/ \text{identifying along maps} \arrow[u] & \biguplus_{J_0\subseteq\dots\subseteq J_n} \Delta^{(n)} \times \mathscr{D}(J_n) \Big/ \text{glue in mapping cylinders} \arrow[u] & \simeq ||\mathscr{N} \int \widehat{\mathscr{D}}|| \\
 & & \text{Grothendieck construction}
\end{tikzcd}$$

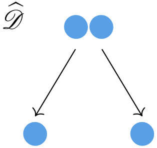
THE GENERALIZED NERVE LEMMA

Let X be a paracompact space with a (locally) finite open cover $\mathcal{U} = \{U_i\}_I$.



$$\mathscr{N}(\mathcal{U}) = \left\{\begin{matrix} 1 & 2 \\ 12 & \end{matrix}\right\}$$





Consider the *diagram* $\mathscr{D} : \mathscr{N}(\mathcal{U}) \to \mathrm{Top}$, $\mathscr{D}(J) := U_J$ and inclusion maps.

$$\begin{tikzcd}
X = \operatorname{colim}\mathscr{D} & \operatorname{hocolim}\mathscr{D} \arrow[l, "\text{G.N.L.: } \simeq"'] \arrow[r, leftrightarrow, "\simeq"'] & \operatorname{hocolim}\widehat{\mathscr{D}} \\
 & & \simeq \|\mathscr{N} \textstyle\int \widehat{\mathscr{D}}\| \\
\biguplus_{J} \mathscr{D}(J) \Big/ \text{identifying along maps} \arrow[uu] & \biguplus_{J_0 \subseteq \ldots \subseteq J_n} \Delta^{(n)} \times \mathscr{D}(J_n) \Big/ \text{glue in mapping cylinders} \arrow[uu] & \text{Grothendieck construction (...whatever.)}
\end{tikzcd}$$

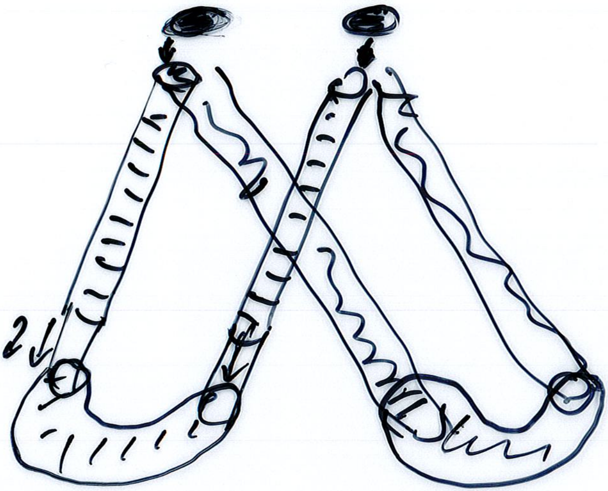
$J_0 = 12$ $\Delta^{(0)} \times \mathcal{D}(12)$

× × ● ●

$J_0 \leq J_1$

$2 \leq 12$ $\Delta^{(1)} \times \mathcal{D}(12)$

$J_0 = 2$: $\Delta^{(0)} \times \mathcal{D}(2)$



$\mathcal{N} \int \widehat{\mathcal{D}}$: the set of all (J, p) $J \in \mathcal{N}$

$p \in \widehat{\mathcal{D}}(J)$

$(J_1, p_1) \geq (J_2, p_2) \iff J_1 \geq J_2$

$p_1 \longmapsto p_2$

THE GENERALIZED NERVE LEMMA

APPLICATION: THE SALVETTI COMPLEX

Let $\mathscr{A}$ be a *complexified* arrangement of hyperplanes in $\mathbb{C}^d$ (i.e. the defining equations for the hyperplanes are real).

[Salvetti '87] There is a poset $\mathrm{Sal}(\mathscr{A})$ such that

$$||\,\mathrm{Sal}(\mathscr{A})|| \simeq M(\mathscr{A}).$$

Recall: complexified means $\alpha_i \in (\mathbb{R}^d)^*$ and $b_i \in \mathbb{R}$.

Consider the associated arrangement $\mathscr{A}^{\mathbb{R}} = \{H_i^{\mathbb{R}}\}$ in $\mathbb{R}^d$, $H_i^{\mathbb{R}} = \Re(H_i)$.

The Salvetti poset

For $z \in \mathbb{C}^d$ and all j,

$$\alpha_j(z) = \alpha_j(\Re(z)) + i\alpha_j(\Im(z)).$$

We have $z \in M(\mathscr{A})$ if and only if $\alpha_j(z) \neq 0$ for all j.

Thus, surely for very region (chamber) $C \in \mathcal{R}(\mathscr{A}^{\mathbb{R}})$ we have

$$U(C) := C + i\mathbb{R}^d \subseteq M(\mathscr{A}).$$

G.N.L. applies to the covering by $M(\mathscr{A})$-closed sets $\mathcal{U} := \left\{\overline{U(C)}\right\}_{C\in\mathcal{R}(\mathscr{A}^{\mathbb{R}})}$

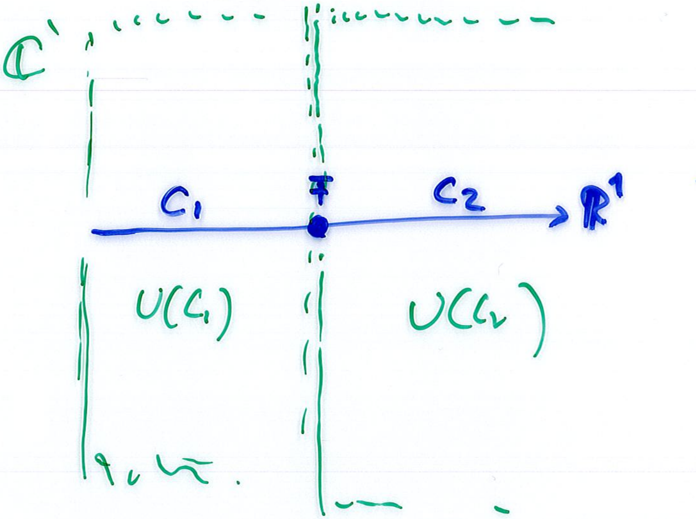
(what's important is that each $(M(\mathscr{A}), \overline{U(C)})$ is NDR-pair).

After some "massaging", $\mathscr{N}(\mathcal{U}) \int \widehat{\mathscr{D}}$ becomes

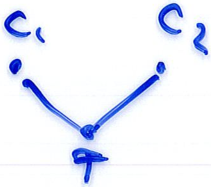
$$\mathrm{Sal}(\mathscr{A}) = \left[\begin{array}{ll} \{[F,C] \mid & F \in \mathcal{F}(\mathscr{A}^{\mathbb{R}}),\ \underbrace{C \in \mathcal{R}(\mathscr{A}^{\mathbb{R}}_{|F|})}_{\leftrightarrow C\in\mathcal{R}(\mathscr{A}^{\mathbb{R}}), C\leq F} \ \}, \\ & \\ [F,C] \geq [F',C'] & \text{if } F \leq F',\ C \subseteq C' \end{array} \right.$$

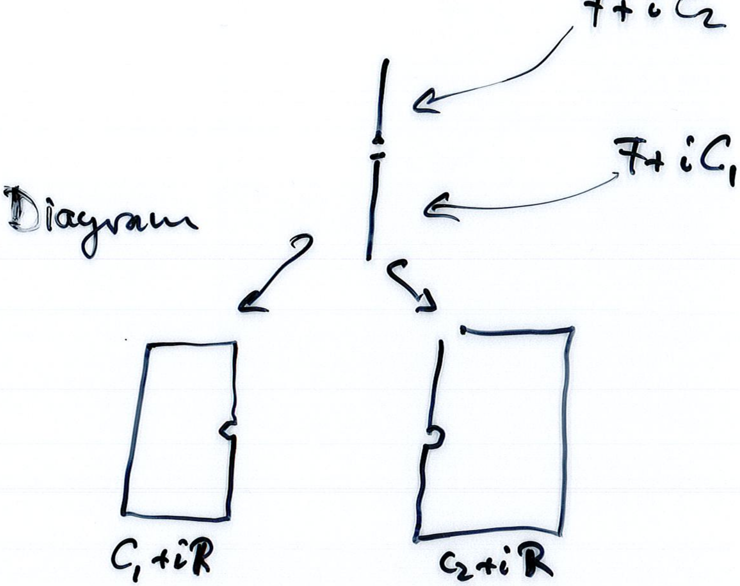
$\mathcal{A} = \{\{x=0\}\}$ in $\mathbb{C}^1$

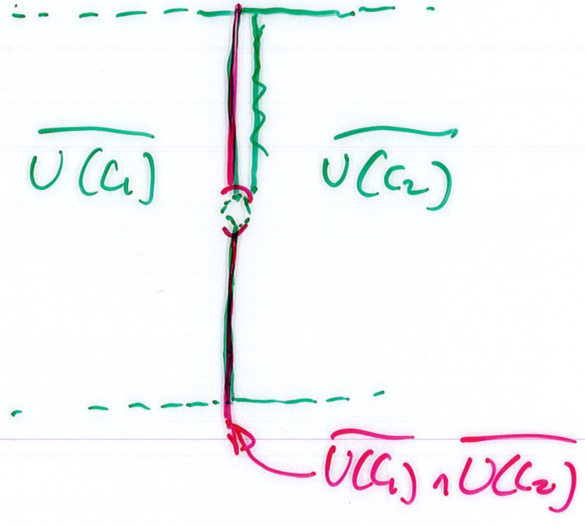
$\mathcal{A}^{\mathbb{R}} = \{\{x=0\}\}$ in $\mathbb{R}^1$



$\mathcal{F}(\mathcal{A}^{\mathbb{R}})$:



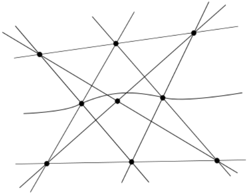




$$\begin{tikzcd}
\hat{D} & {[F, C_1]} \arrow[d] \arrow[dr] & {[F, C_2]} \arrow[dl] \arrow[d] \\
& {[C_1, \mathbb{R}]} & {[C_2, \mathbb{R}]}
\end{tikzcd}$$

SALVETTI COMPLEXES OF PSEUDOARRANGEMENTS

Notice: the definition of $\operatorname{Sal}(\mathscr{A})$ makes sense also for general pseudoarrangements (oriented matroids).



Theorem.[D.–Falk '15] The class of complexes $\|\operatorname{Sal}(\mathscr{A})\|$ where $\mathscr{A}$ is a pseudoarrangement gives rise to "new" fundamental groups. For instance, the non-pappus oriented matroid gives rise to a fundamental group that is not isomorphic to any realizable arrangement group.

THE GENERALIZED NERVE LEMMA

APPLICATION: THE SALVETTI COMPLEX

Let $\mathscr{A}$ be a *complexified* arrangement of hyperplanes in $\mathbb{C}^d$ (i.e. the defining equations for the hyperplanes are real).

[Salvetti '87] There is a poset $\mathrm{Sal}(\mathscr{A})$ such that

$$||\,\mathrm{Sal}(\mathscr{A})|| \simeq M(\mathscr{A}).$$

[Callegaro-D. '15] Let $X \in \mathcal{L}(\mathscr{A})$ with $\operatorname{codim} X = k$.

There is a map of posets $\mathrm{Sal}(\mathscr{A}) \to \mathrm{Sal}(\mathscr{A}_X)$ that induces the Brieskorn inclusion $b_X : H^k(M(\mathscr{A}_X), \mathbb{Z}) \hookrightarrow H^k(M(\mathscr{A}), \mathbb{Z})$.

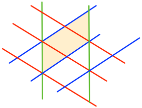
Q: "Brieskorn decomposition" in the ("wiggly") case of oriented matroids?

Salvetti Category

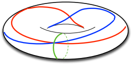
[d'Antonio-D., '11]

Any complexified toric arrangement $\mathscr{A}$ lifts to a complexified arrangement of affine hyperplanes $\mathscr{A}^{\upharpoonright}$ under the universal cover

$\mathbb{C}^d \to T, \qquad \mathscr{A}^{\upharpoonright}:$



$\xrightarrow{/\mathbb{Z}^d} \mathscr{A}:$



The group $\mathbb{Z}^d$ acts on $\mathrm{Sal}(\mathscr{A}^{\upharpoonright})$ and we can define the *Salvetti category* of $\mathscr{A}$:

$$\mathrm{Sal}(\mathscr{A}) := \mathrm{Sal}(\mathscr{A}^{\upharpoonright})/\mathbb{Z}^d$$

(quotient taken in the category of scwols).

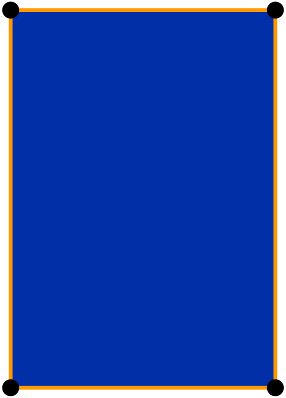
Here the realization commutes with the quotient [Babson-Kozlov '07], thus

$$||\,\mathrm{Sal}(\mathscr{A})|| \simeq M(\mathscr{A}).$$

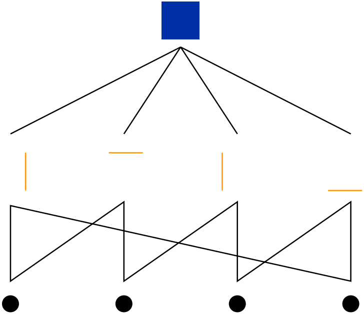
DISCRETE MORSE THEORY

[Forman, Chari, Kozlov,...; since '98]

Here is a regular CW complex



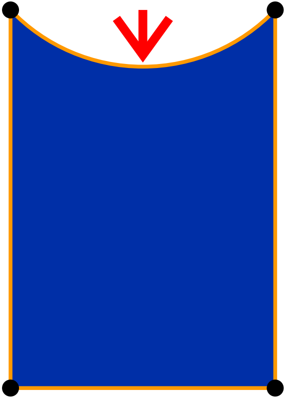
with its poset of cells:

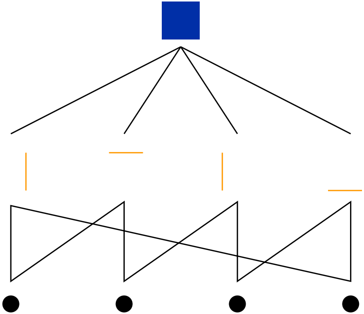


DISCRETE MORSE THEORY

[Forman, Chari, Kozlov,...; since '98]

Elementary collapses...



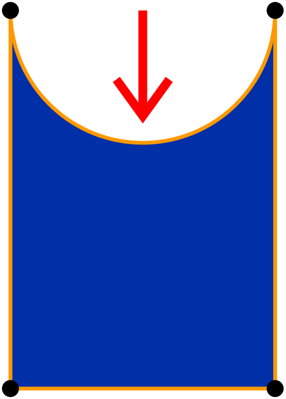


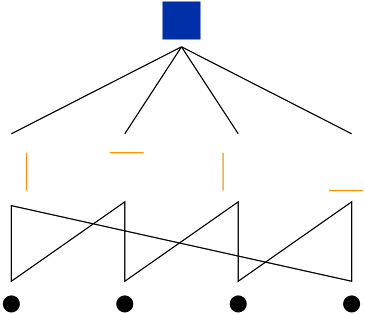
... are homotopy equivalences.

DISCRETE MORSE THEORY

[Forman, Chari, Kozlov,...; since '98]

Elementary collapses...



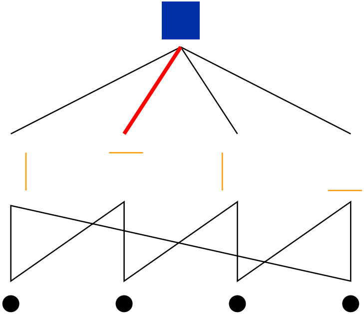


... are homotopy equivalences.

DISCRETE MORSE THEORY

[Forman, Chari, Kozlov,...; since '98]

Elementary collapses...

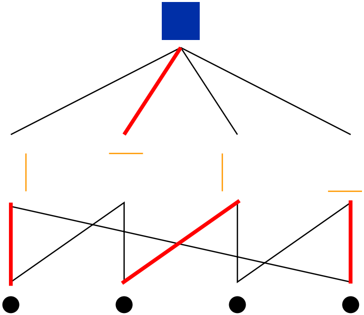


... are homotopy equivalences.

DISCRETE MORSE THEORY

[Forman, Chari, Kozlov,...; since '98]

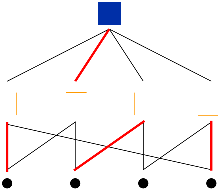
Elementary collapses...



... are homotopy equivalences.

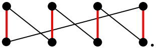
Discrete Morse Theory

The sequence of collapses is encoded in a matching of the poset of cells.



Question: Does every matchings encode such a sequence?

Answer: No. Only (and exactly) those without "cycles" like



Acyclic matchings ↔ discrete Morse functions.

DISCRETE MORSE THEORY

Main theorem of Discrete Morse Theory [Forman ‘98].

Every acyclic matching on the poset of cells of a regular CW-complex X induces a homotopy equivalence of X with a CW-complex with as many cells in every dimension as there are non-matched (“*critical*) cells of the same dimension in X.

Theorem. [d’Antonio-D. ’15] This theorem also holds for (suitably defined) acyclic matchings on face categories of polyhedral complexes.

DISCRETE MORSE THEORY

APPLICATION: MINIMALITY OF $\mathrm{Sal}(\mathscr{A})$

Let $\mathscr{A}$ be a complexified toric arrangement.

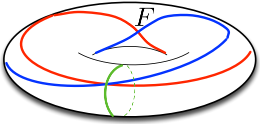
Theorem. [d'Antonio-D., '15] The space $M(\mathscr{A})$ is *minimal*, thus its cohomology groups $H^k(M(\mathscr{A}), \mathbb{Z})$ are torsion-free.

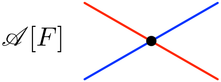
Recall: "minimal" means having the homotopy type of a CW-complex with one cell for each generator in homology.

Proof. Construction of an acyclic matching of the Salvetti category with $\mathrm{Poin}(M(\mathscr{A}), 1)$ critical cells.

THE SALVETTI CATEGORY - AGAIN

For $F \in \mathrm{Ob}(\mathcal{F}(\mathscr{A}))$ consider the hyperplane arrangement $\mathscr{A}[F]$:





[Callegaro – D. '15] $||\,\mathrm{Sal}(\mathscr{A})|| \simeq \mathrm{hocolim}\, \mathscr{D}$, where

$$\begin{array}{rccl} \mathscr{D}: & \mathcal{F}(\mathscr{A}) & \to & \mathrm{Top} \\ & F & \mapsto & ||\,\mathrm{Sal}(\mathscr{A}[F])|| \end{array}$$

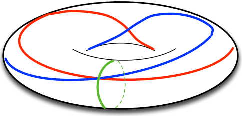
Call ${}_{\mathscr{D}}E_*^{p,q}$ the associated cohomology spectral sequence [Segal '68].

(equivalent to the Leray Spectral sequence of the canonical proj to $||\mathcal{F}(\mathscr{A})||$)

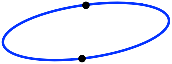
THE SALVETTI CATEGORY - ...AND AGAIN

For $Y \in \mathcal{C}(\mathscr{A})$ define $\mathscr{A}^Y = \mathscr{A} \cap Y$, the arrangement induced on Y.

$\mathscr{A}$:



$\mathscr{A}^Y = \mathscr{A} \cap Y$:



For every $Y \in \mathcal{C}(\mathscr{A})$ there is a subcategory $\Sigma_Y \hookrightarrow \mathrm{Sal}(\mathscr{A})$ with

$$Y \times M(\mathscr{A}[Y]) \simeq ||\mathcal{F}(\mathscr{A}^Y) \times \mathrm{Sal}(\mathscr{A}[Y])|| \simeq ||\Sigma_Y|| \hookrightarrow ||\,\mathrm{Sal}(\mathscr{A})||$$

and we call ${}_Y E_*^{p,q}$ the Leray spectral sequence induced by the canonical projection

$$\pi_Y : \Sigma_Y \to \mathcal{F}(\mathscr{A}^Y).$$

SPECTRAL SEQUENCES

For every $Y \in \mathcal{C}(\mathscr{A})$, the following commutative square

$$\begin{tikzcd}
M(\mathscr{A}) \simeq \; ||\,\mathrm{Sal}(\mathscr{A})|| \arrow[d, "\pi"] & ||\Sigma_Y|| \arrow[l, "\supseteq"'] \arrow[d, "\pi_Y"] \\
||\mathcal{F}(\mathscr{A})|| & ||\mathcal{F}(\mathscr{A}^Y)|| \arrow[l, "\supseteq"']
\end{tikzcd}$$

induces a morphism of spectral sequences ${}_{\mathscr{D}}E_*^{p,q} \to {}_Y E_*^{p,q}$.

Next, we examine the morphism of spectral sequences associated to the corresponding map from $\biguplus_{Y \in \mathcal{C}(\mathscr{A})} ||\Sigma_Y||$ to $||\,\mathrm{Sal}(\mathscr{A})||$.

SPECTRAL SEQUENCES

[Callegaro – D., ’15] (all cohomologies with $\mathbb{Z}$-coefficients)

$$\begin{tikzcd}
H^*(M(\mathscr{A})) \arrow[r, "\text{Hom. of rings}", "\text{Injective}"'] \arrow[d, "\text{bij.}"] & \bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^*(Y)\otimes H^*(M(\mathscr{A}[Y])) \arrow[d, "\text{bij.}"] \\
{}_{\mathscr{D}}E_2^{p,q} = \displaystyle\bigoplus_{\substack{Y\in\mathcal{C}(\mathscr{A})\\ \operatorname{rk} Y = q}} H^p(Y)\otimes H^q(M(\mathscr{A}[Y])) \arrow[r, "\text{Hom. of rings}"] & \displaystyle\bigoplus_{Y\in\mathcal{C}(\mathscr{A})} {}_Y E_2^{p,q} = \displaystyle\bigoplus_{Y\in\mathcal{C}(\mathscr{A})} H^p(Y)\otimes H^q(M(\mathscr{A}[Y]))
\end{tikzcd}$$

On Y_0-summand:

$$\omega\otimes\lambda \longmapsto \begin{pmatrix} i^*(\omega)\otimes b(\lambda) & \text{if } Y_0 \leq Y \\ 0 & \text{else.} \end{pmatrix}_Y$$

$i : Y \hookrightarrow Y_0$

“Brieskorn” inclusion

A presentation for $H^*(M(\mathscr{A}),\mathbb{Z})$

The inclusions $\phi_\bullet : \Sigma_\bullet \hookrightarrow \operatorname{Sal}(\mathscr{A})$ give rise to a commutative triangle

$$\begin{tikzcd}
\displaystyle\bigoplus_{\substack{Y'\in\mathcal{C},Y'\supseteq Y\\ \operatorname{rk} Y'=q}} H^*(Y')\otimes H^q(M(\mathscr{A}[Y'])) \arrow[d, "\sum f_{Y\supseteq Y'}"] & H^*(\|\operatorname{Sal}(\mathscr{A})\|) \arrow[l, "\oplus\phi^*_{Y'}"'] \arrow[dl, "\phi^*_Y"] \\
H^*(Y)\otimes H^q(M(\mathscr{A}[Y])) &
\end{tikzcd}$$

with $f_{Y\supseteq Y'} := \iota^*\otimes b_{Y'}$ obtained from $\iota : Y \hookrightarrow Y'$ and the Brieskorn map b.

Proof. Carrier lemma and ‘combinatorial Brieskorn’.

This defines a ‘compatibility condition’ on $\oplus_Y H^*(Y)\otimes H^*(M(\mathscr{A}[Y]))$;

the (subalgebra of) compatible elements is isomorphic to $H^*(M(\mathscr{A}),\mathbb{Z})$.

A PRESENTATION FOR $H^*(M(\mathscr{A}), \mathbb{Z})$

More succinctly, define an ‘abstract’ algebra as the direct sum

$$\bigoplus_{Y \in \mathcal{C}(\mathscr{A})} H^*(Y, \mathbb{Z}) \otimes H^{\operatorname{codim} Y}(M(\mathscr{A}[Y]), \mathbb{Z})$$

with multiplication of α, α' in the Y, resp. Y' component, as

$$(\alpha * \alpha')_{Y''} := \begin{cases} f_{Y \supseteq Y''}(\alpha) \smile f_{Y' \supseteq Y''}(\alpha') & \text{if } Y \cap Y' \supseteq Y'' \text{ and} \\ & \operatorname{rk} Y'' = \operatorname{rk} Y + \operatorname{rk} Y', \\ 0 & \text{else.} \end{cases}$$

Note: this holds in general (beyond complexified).

Question: is this completely determined by $\mathcal{C}(\mathscr{A})$?

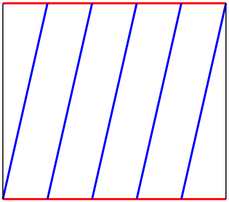
$\mathcal{C}(\mathscr{A})$ "RULES", IF A HAS A UNIMODULAR BASIS

Recall that a centered toric arrangement is defined by a $d \times n$ integer matrix $A = [\alpha_1, \ldots, \alpha_n]$.

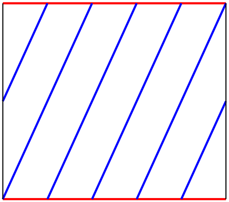
Theorem. [Callegaro-D. '15] If (S, rk, m) is an arithmetic matroid associated to a matrix A that has a maximal minor equal to 1, then the matrix A can be reconstructed from the arithmetic matroid up to sign reversal of columns. Since the poset $\mathcal{C}(\mathscr{A})$ encodes the multiplicity data, this means that, in this case, the poset in essence determines the arrangement.

An example

Consider the following two complexified toric arrangements in $T = (\mathbb{C}^*)^2$.



$\mathscr{A}_1$:



$\mathscr{A}_2$

Clearly $\mathcal{C}(\mathscr{A}_1) \simeq \mathcal{C}(\mathscr{A}_2)$.

There is an “ad hoc” ring isomorphism $H^*(M(\mathscr{A}_1, \mathbb{Z}) \to H^*(M(\mathscr{A}_2, \mathbb{Z})$;

$H^*(M(\mathscr{A}_1), \mathbb{Z})$ and $H^*(M(\mathscr{A}_2), \mathbb{Z})$ are *not* isomorphic as $H^*(T, \mathbb{Z})$-modules.

ABELIAN ARRANGEMENTS

Let $A = [a_1, \dots, a_n] \in M_{d \times n}(\mathbb{Z})$

(Central) hyperplane arrangement	(Centered) toric arrangement	(Centered) elliptic arrangement
$\lambda_i : \quad \mathbb{C}^d \to \mathbb{C}$ $\underline{z} \mapsto \sum_j a_{ji} z_j$	$\lambda_i : \quad (\mathbb{C}^*)^d \to \mathbb{C}^*$ $\underline{z} \mapsto \prod_j z_j^{a_{ji}}$	$\lambda_i : \quad \mathbb{E}^d \to \mathbb{E}$ $\underline{z} \mapsto \sum_j a_{ji} z_j$
$H_i := \ker \lambda_i$	$K_i := \ker \lambda_i$	$L_i := \ker \lambda_i$
$\mathscr{A} = \{H_1, \dots, H_n\}$	$\mathscr{A} = \{K_1, \dots, K_n\}$	$\mathscr{A} = \{L_1, \dots, L_n\}$
$M(\mathscr{A}) := \mathbb{C}^d \setminus \cup \mathscr{A}$	$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \cup \mathscr{A}$	$M(\mathscr{A}) := \mathbb{E}^d \setminus \cup \mathscr{A}$

ABELIAN ARRANGEMENTS

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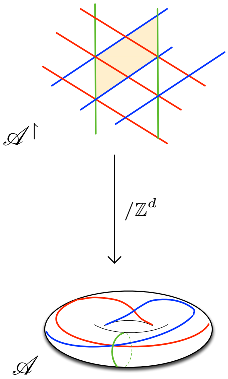
$H_i := \ker \lambda_i$	$K_i := \ker \lambda_i$	$L_i := \ker \lambda_i$
$\mathscr{A} = \{H_1, \dots, H_n\}$	$\mathscr{A} = \{K_1, \dots, K_n\}$	$\mathscr{A} = \{L_1, \dots, L_n\}$
$M(\mathscr{A}) := \mathbb{C}^d \setminus \cup\mathscr{A}$	$M(\mathscr{A}) := (\mathbb{C}^*)^d \setminus \cup\mathscr{A}$	$M(\mathscr{A}) := \mathbb{E}^d \setminus \cup\mathscr{A}$
Having a blast!	Doing pretty good.	Even Betti numbers are unknown...

TOMORROW:

Ansatz: "periodic arrangements"



$$\begin{tikzcd}
\mathcal{L}(\mathscr{A}^{\upharpoonright}) \arrow[d, "/\mathbb{Z}^d\text{(as posets)}"] & \mathcal{F}(\mathscr{A}^{\upharpoonright}) \arrow[d, "/\mathbb{Z}^d\text{ (as acyclic categories)}"] \\
\mathcal{C}(\mathscr{A}) & \mathcal{F}(\mathscr{A})
\end{tikzcd}$$

Poset of intersections

Poset (category) of polyhedral faces

Abstractly: group actions on semimatroids!