On the cohomology of some reflection groups

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Toblach 2017

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The calculation of the *ring* structure on the cohomology $H^*(G; R)$ has traditionally been not so easy (Feshbach 2001, Swenson 2006, ...).

Outline of the presentation

Hopf ring definition

Definition

A (graded) Hopf ring over a field $\mathbb{F}$ is a graded $\mathbb{F}$-vector space A with two product $\cdot, \odot \colon A \otimes A \to A$ and a coproduct $\Delta \colon A \to A \otimes A$ that satisfy the following conditions:

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- $\forall x, y, z \in A$, if $\Delta(x) = \sum_i x_i' \otimes x_i''$, then the following equality holds:

$$x \cdot (y \otimes z) = \sum_i \left(x_i' \cdot y\right) \odot \left(x_i'' \cdot z\right)$$

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If A satisfies conditions 2 and 3, but possibly not 1, it is called an almost-Hopf ring

Almost-Hopf rings from sequences of groups (1)

Assume that we have a sequence of groups $\{G_n\}_{n=0}^{\infty}$ together with injective group homomorphisms $\mu_{k,l} \colon G_k \times G_l \hookrightarrow G_n$. Under suitable conditions we can give $A = \bigoplus_n H^* (G_n; \mathbb{F})$ the structure of an (almost-)Hopf ring.

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- Δ: $\mu_{k,l}$ induces $H^*(G_{k,l}; \mathbb{F}) \to H^*(G_k \times G_l; \mathbb{F}) \cong H^*(G_k; \mathbb{F}) \otimes H^*(G_l; \mathbb{F})$

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- $\odot$: if $\mu_{k,l}$ is injective, the corresponding map at the level of classifying spaces is homotopy equivalent to a finite covering, thus it induces a transfer map $H^*(G_k;\mathbb{F}) \otimes H^*(G_l;\mathbb{F}) \cong H^*(G_k \times G_l;\mathbb{F}) \to H^*(G_{k,l};\mathbb{F})$

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- $\cdot$: the usual $\cup$ product

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- The reflection groups of type D_n $\{W_{D_n}\}_n$ give rise to an almost-Hopf ring (not a full Hopf-ring)
- The general linear groups of a *finite* fields $\{\mathrm{Gl}_n(\mathbb{F})\}_n$ give rise to a Hopf ring

Definition of generators

Consider the *(ordered) configuration space* $\mathrm{Conf}_n(\mathbb{R}^m) = \{(x_1, \ldots, x_n) \in (\mathbb{R}^m)^n : x_i \neq x_j \forall i \neq j\}$. Let $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)$ be the quotient of $\mathrm{Conf}_n(\mathbb{R}^m)$ by the natural action of Σ_n. Let $\mathrm{Conf}_n(\mathbb{R}^\infty)$ be the direct limit $m \in \mathbb{N}$ of $\mathrm{Conf}_n(\mathbb{R}^m)$. This is contractible and has a free Σ_n-action, hence its quotient $\overline{\mathrm{Conf}}_n(\mathbb{R}^\infty)$ is a $K(\Sigma_n; 1)$.

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Consider the $(2^k l)\, m - (2^{k-1} l)$-codimensional submanifold $X_{k,l,m}$ of $\overline{\mathrm{Conf}}_{l2^k}(\mathbb{R}^m)$ consisting of the configurations of $l2^k$ points that can be partitioned in l sets consisting each of 2^k points that share their first coordinate.

$X_{k,l,m}$ is properly embedded in $\overline{\mathrm{Conf}}_n(\mathbb{R}^m)$. Take its fundamental class in locally finite homology and consider its Poincaré dual $T_{k,l,m} \in H^*\left(\overline{\mathrm{Conf}}_n(\mathbb{R}^m); \mathbb{F}_2\right)$. There esists a unique cohomology class $\gamma_{k,l} \in H^*(\Sigma_n; \mathbb{F}_2)$ that restrict to $T_{k,l,m}$ for all $m >> 0$.

Theorem (Giusti, Salvatore, Sinha)

The coproduct of $\gamma_{k,l}$ is given by the formula:

$$\Delta\left(\gamma_{k,l}\right) = \sum_{a+b=l} \gamma_{k,a} \otimes \gamma_{k,b}$$

Moreover, $\bigoplus_n H^*\left(\Sigma_n;\mathbb{F}_2\right)$ is the commutative Hopf ring generated by the elements $\gamma_{k,l}$ with the following relations:

- $\gamma_{k,l} \odot \gamma_{k,m} = \begin{pmatrix} l+m \\ l \end{pmatrix} \gamma_{k,l+m}$
- *the $\cdot$ product of generators belonging to different components is 0*

Graphical description (1)

We can describe graphically this Hopf ring by associating to $\gamma_{k,l}$ a rectangle that has width $l2^k$ and height $1 - 2^{-k}$. Taking the $\cdot$ product of two generators corresponds graphically to stacking one on top of the other the corresponding boxes, while taking the $\odot$ product corresponds to placing their rectangles next to each other horizontally.

Elements of the Hopf ring $\bigoplus_n H^* (\Sigma_n; \mathbb{F}_2)$ can be described graphically as diagrams of columns made by such rectangles.

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- For transfer product, we place the diagrams new to each other, merging two columns that are made of boxes with the same height into one, with a coefficient of 0 or 1 according to the relations from the previous slide.
- For cup product, consider all possible ways to split the diagrams into colums, then match columns of the two diagrams in all possible ways up to automorphism, and stack matched columns on top of each other.

Cohomology of W_{B_n}

Let $A_\Sigma = \bigoplus_n H^*(\Sigma_n; \mathbb{F}_2)$ and $A_B = \bigoplus_n H^*(W_{B_n}; \mathbb{F}_2)$. The projections $\pi_n \colon W_{B_n} \to \Sigma_n$ induce a Hopf ring monomorphism $A_\Sigma \to A_B$.

Theorem

A_B is generated over A_Σ by classes $\delta_n \in H^n(W_{B_n}; \mathbb{F}_2)$ with the additional relations:

- $\delta_n \odot \delta_m = \binom{n+m}{n}$
- *The product of δ_n with elements in different components is* 0

There is an analogous graphical description, by allowing a further $1 \times n$ rectangle that corresponds to δ_n.

Cohomology of W_{D_n} (1)

$A_D = \bigoplus_n H^*(W_{D_n}; \mathbb{F}_2)$ is only an almost-Hopf ring. Let $\rho\colon A_B \to A_D$ the restriction map. Let D be a column diagram representing a class in A_B. We consider three cases:

1. if D contains a rectangle corresponding to δ_{2k+1}: $\rho(D) = 0$

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3. in the other cases, we can define two classes D^+ and D^- in A_D , conjugate under the action of a reflection in $W_{B_n} \setminus W_{D_n}$, that satisfy $\rho(D) = D^+ + D^-$.

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The elements in the form D^0, D^+ and D^- above, for diagrams made of columns with pairwise different profiles, are linearly independent and their linear span $A' = \bigoplus_n A'_n$ is a sub-almost-Hopf ring of A_D.

Cohomology of W_{D-n} (2)

The products in A' satisfy some formulas similar to the B_n case, but we need to take into account the signs $0, +, -$ by means of the following rules:

- The cup product of diagrams with the same sign is again a sum of diagrams with that sign. The cup product of diagrams with opposite signs is 0.
- The transfer product of diagrams is a sum of diagrams with the same sign, determined as in the sign rule for the multiplication of real numbers.

For example $\gamma^{+}_{3,k} \odot \gamma^{-}_{3,l} = \binom{k+l}{l} \gamma^{-}_{3,k+l}$.

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$H^*\left(W_{D_{2n+1}}; \mathbb{F}_2\right) = A'_{2n+1}$, while $H^*\left(W_{D_{2n}}; \mathbb{F}_2\right)$ is a free module over A'_{2n} with basis $\{1, H_n\}$ and the ring structure is determined by the additional relation $H_n^2 + \gamma_{1,n} H_n + \delta_2 \odot \gamma^2_{1,n-1} = 0$.