Topological properties of projective maps: supports (joint work with V. Shende, to appear on "Algebraic Geometry")

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First vague question

Let $f : X \to Y$ an algebraic map of complex algebraic varieties, denote $X_y := f^{-1}(y)$

Question: How does the topology of X_y varies with y?

This is very vague, but something can be said already in this generality, depending on finiteness properties of algebraic maps.

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Ehresmann lemma

A first step: assume that X and Y are nonsingular.

A classical, result in differential topology ensures that:

If f is *proper* (=inverse image of compact is compact in the classical topology) and smooth (= Df surjective)

Then the topological (even differentiable) type of X_y is constant on connected components of Y.

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The idea behind the proof:

restrict f to a real curve in Y, lift the vector field $\frac{\partial}{\partial t}$ and follow its trajectories (properness ensures completeness of the vector field).

One may wonder if anything like that holds in positive characteristic (but what is the topological type?), we'll discuss this shortly later.

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Generic smoothness

Still assuming, for simplicity, X nonsingular and f proper

Generic smoothness Theorem: There is a dense Zariski-open set $Y_{reg} \subseteq Y$ such that:

$f_| : f^{-1}(Y_{reg}) \to Y_{reg}$ is smooth,

therefore all fibres over points of Y_{reg} (i.e. *most fibres*) have the same differentiable type.

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(H. Whitney, R. Thom, J. Mather, H. Hironaka, J.L. Verdier)

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Given *any* algebraic map $f : X \to Y$ of complex algebraic varieties:

there is a decomposition

$$Y = \coprod Y_\alpha$$

with the properties:

- The Y_α are locally closed in the Zariski topology and nonsingular.
- For every α,

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- no properness hypothesis or nonsingularity is assumed.
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Example: The universal degree d plane curve.

Let $\mathbb{P}^\vee$ the projectivization of the space of homogeneous degree d polynomials in three indeterminates (X_0, X_1, X_2), and

$$\mathscr{C} = \{(P, [X_0, X_1, X_2]) \in \mathbb{P}^\vee \times \mathbb{P}^2 : P(X_0, X_1, X_2) = 0\}.$$

Finding a stratification for this map is practically impossible for big d.

Let's draw the case $d = 3$.

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A less ambitious, but still quite interesting, question:

Understand the functions:

$P_t : Y \to \mathbb{Z}[T]$ given by $P_t(y) = \sum \dim \mathrm{H}^k(X_y, \mathbb{Q})T^k$,

or even $\chi : Y \to \mathbb{Z}$ with $\chi(y) = \sum (-1)^k \dim \mathrm{H}^k(X_y, \mathbb{Q})$.

Betti and Euler functions

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Counting functions

Assume our map is defined over a finite field $\mathbb{F}_q$

Assume that, for every r and $y \in Y_{reg}(\mathbb{F}_{q^r})$

we know the counting function $\sharp\{X_y(\mathbb{F}_{q^{rn}})\}$ for every n

What can we say about $\sharp\{X_y(\mathbb{F}_{q^{rn}})\}$ for $y \in Y$?

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$$\sum_{Z \subseteq X} n_Z \mathbf{1}_Z$$

(with coefficients in a fixed ring, which in our case is $\mathbb{Z}[T]$ for P_t and $\mathbb{Z}$ for χ) of characteristic functions of closed algebraic subvarieties.

Hence, existence of stratifications ensures that these functions are constructible

This holds true also in positive characteristic, with étale cohomology groups with coefficients in $\mathbb{Q}_\ell$ (*constructibility theorem*)

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If $Y = \coprod Y_\alpha$ is a stratification of the map, we can certainly write our functions P_t, χ as

$$\sum_\alpha n_\alpha \mathbf{1}_{\overline{Y_\alpha}}.$$

Question: Which of these strata are really necessary?

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The question becomes more sensible if we change the basis $\{\mathbf{1}_Z\}$ for constructible functions to another one, more cleverly related to the geometry of the map.

There are (at least) two ways to continue a constructible function, both due to MacPherson

1. (Euler obstruction) This is a function Eu_Z, supported on $\overline{Z}$ which is $= 1$ on the regular points of Z, but takes into account the singularities of $\overline{Z}$.
2. (Intersection cohomology) This is associated not only to a subvariety Z, but also to a locally constant sheaf $\mathscr{L}$ on an open subset Z^0. It produces a function with values in $\mathbb{Z}[T]$, which is $= rk\mathscr{L}$ on Z^0. We denote it by $IH_Z(\mathscr{L})$

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Supports

We can write

$$\chi = \sum_\alpha n_\alpha \mathrm{Eu}_{\overline{Y_\alpha}}$$

and

$$P_t = \sum_\alpha s_\alpha(T) IH_{\overline{Y_\alpha}}(\mathscr{L}_\alpha),$$

where $s_\alpha(T)$ are polynomials.

Which strata actually appear in the sum?

The strata which actually appear are called *supports* (Euler supports in the first case)

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"Unexpected smoothness"

Assume X nonsingular, and $y \in Y_\alpha$, with Y_α a codimension k stratum.

For a generic k- dimensional slice $\Sigma \subset Y$ at y, we have

$$f^{-1}(\Sigma) \text{ is nonsingular}$$

In general one cannot expect more, i.e. if $\dim \Sigma < k$, then $f^{-1}(\Sigma)$ is singular

Sometimes, though, this may not happen, then we call this *unexpected smoothness*

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"Example: the universal Weiestrass cubic."

Let $f : X \subset \mathbb{C}^2 \times \mathbb{P}^2(\mathbb{C}) \to \mathbb{C}^2 = Y$ be the family of projective curves

$$\{(a, b, [X, Y, Z]) \in \mathbb{C}^2 \times \mathbb{P}^2(\mathbb{C}), ZY^2 - X^3 - aXZ^2 - bZ^3 = 0\}. \qquad (1)$$

Let $\Delta \subset \mathbb{C}^2$ be defined by $4a^3 + 27b^2 = 0$. For $(a, b) \notin \Delta$ the fibre is a non-singular curve of genus one, while, for $(a, b) \in \Delta \setminus \{o\}$, it is a rational nodal curve. Finally $f^{-1}(o)$ is a rational curve with a cusp.

Although o is a zero dimensional stratum, the inverse image of a generic one-dimensional disc through o is nonsingular.

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We define:

$$\Delta^i(f) = \{y \in Y \text{ s.t. there is no } \mathbb{D}^{i-1} \hookrightarrow Y \text{ through } y \text{ transverse to } f\}.$$

where, given $y \in Y$, by "a $k-$dimensional disc $\mathbb{D}^k \hookrightarrow Y$ through y", we mean a germ of nonsingular $k-$dimensional subvariety passing through y.

A $k-$dimensional disc $\mathbb{D}^k \hookrightarrow Y$ through $y \in Y$ is *transverse* to f if $f^{-1}(\mathbb{D}^k)$ is nonsingular along $f^{-1}(y)$ and

$$\operatorname{codim}(f^{-1}(\mathbb{D}^k), X) = \operatorname{codim}(\mathbb{D}^k, Y).$$

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Higher discriminants of a map

Properties of the $\Delta^i(f)'$s:

- $$Y = \Delta^0(f) \supseteq \Delta^1(f) \supseteq \Delta^2(f) \supseteq \Delta^3(f) \supseteq \cdots$$
- $\Delta^1(f)$ is by definition the locus where the fibre is singular – that is, the usual discriminant.
- By generic smoothness,

$$\text{codim}\Delta^i(f) \geq i.$$

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Higher discriminants of a map

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We think about these discriminants in the following way. Moving $\delta \in \Delta(f)$ off the discriminant to $\not\delta \notin \Delta^1(f)$ changes the fibre topology: $X_\delta \not\sim X_{\not\delta}$. *But we can blur our focal point to obscure this feature*: we pass to a one dimensional disc $\mathbb{D} \ni \delta$, chosen generic and small enough to retract $f^{-1}(\mathbb{D}) =: X_{\mathbb{D}} \sim X_\delta$. A one dimensional disc cannot be perturbed off the discriminant, and indeed for δ general in $\Delta^1(f)$, a perturbation $\mathbb{D}'$ of the thickening $\mathbb{D}$ induces a homeomorphism $X_{\mathbb{D}'} \sim X_{\mathbb{D}}$. The higher discriminant $\Delta^2(f)$ is the locus which still appears to our blurred vision: where even a general perturbation of a general one parameter thickening changes the fibre topology.

Higher discriminants are supports

Theorem The supports of the map f are contained among the codimension k components of the $\Delta^k(f)$'s.

The condition is not necessary, a higher discriminant may not be a support, but ....generic points.

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