Counting occurences in almost sure limit theorems

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Abstract: Let $X, X_1, X_2 \ldots$ be a sequence of i.i.d. random variables with $X \in L^p$, $1 < p \le 2$. For $n \ge 1$, let $S_n = X_1 + \ldots + X_n$. Developing a preceding work, adressing the L^2-case only, we compare, under strictly weaker conditions than those of the central limit theorem, the deviation of the series $\sum_n w_n \mathbf{1}_{S_n/\sqrt{n} < x_n}$ with respect to $\sum_n w_n \mathbf{P}\{S_n/\sqrt{n} < x_n\}$, for suitable weights (w_n). Extensions to the case $1 < p < 2$, with suitable norming constants, and when the law of X belongs to the domain of attraction of a p-stable law, are obtained. We deduce strong versions of the a.s. central limit theorem.

1. Setting of the problem and Main results.

Let $\mathcal{X} = \{X, X_n, n \ge 1\}$ be a sequence of independent, identically distributed (*i.i.d.*) random variables defined on a probability space $(\Omega, \mathcal{B}, \mathbf{P})$, and let F denote the distribution function of X, $S_n = X_1 + \ldots + X_n$, $n \ge 1$ the partial sums of $\mathcal{X}$. Assume first that $\mathbf{E}X_1 = 0$, $\mathbf{E}X_1^2 = 1$. Let $\{x_k, k \ge 0\}$ be an arbitrary sequence of reals, and consider the events $A_k = \big\{ S_k/\sqrt{k} < x_k \big\}$. Let also a sequence of weights $w = \{w_k, k \ge 1\}$. Consider the following natural question : when the weighted deviation

$$\mathcal{D}_w(A) := \sum_{k=1}^{\infty} w_k \big(\mathbf{1}_{A_k} - \mathbf{P}(A_k)\big), \tag{1.1}$$

of the series $\sum_{k=1}^{\infty} w_k \mathbf{1}_{A_k}$ with respect to its mean $\sum_{k=1}^{\infty} w_k \mathbf{P}(A_k)$, is finite almost surely?

This question is the treated in the present work. Some partial results already exist. Put for any positive integer n, $Y_n = \sum_{2^n \le k < 2^{n+1}} \frac{1}{k}\big(\mathbf{1}_{A_k} - \mathbf{P}(A_k)\big)$. Then the series

$$\sum_{k \ge 1} c_k Y_k, \tag{1.2}$$

converges $\mathbf{P}$-almost surely, for a reasonable choice of the sequence of reals $\{c_k, k \ge 1\}$. For instance, one can take $c_k = k^{-1/2}(\log k)^{-b}$ with $b > 3/2$; so that in view of Kronecker's Lemma, (1.2) implies

$$\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \Big[\mathbf{1}_{\{S_k/\sqrt{k} < x_k\}} - \mathbf{P}\big\{S_k/\sqrt{k} < x_k\big\} \Big] \overset{a.s.}{=} 0. \tag{1.3}$$

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By using the CLT, and letting $x_k \equiv x$ in (1.3), one obtain the classical *Almost Sure Central Limit Theorem* (ASCLT) [5] : $\mathbf{P}$-almost surely, for every real number x,

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{\frac{S_k}{\sqrt{k}} \le x\}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du. \tag{1.4}$$

When (x_k) are not constant, the stronger property (1.3) does not seem connected to the CLT, although it is established under the CLT assumptions : $\mathbf{E}\, X = 0$, $\mathbf{E}\, X^2 < \infty$. In this paper, we show that (1.3) in turn, holds true under a strictly weaker assumption.
Before stating the result, we have to recall the full formulation of (1.3), and for, a useful notion ([4]) from the theory of orthogonal series. Let $(T, \mathcal{C}, \tau)$ be some probability space and consider a sequence (f_n) of elements of $L^2(\tau)$. Let $a_{j,k} = \int_T f_j(x) f_k(x) d\tau(x)$. A system of functions (f_n) such that the quadratic form defined on $l^2(\mathbf{N})$ by : $(x_n)_n \mapsto \sum_{h,k} a_{h,k} x_h x_k$ is bounded, is said *quasi-orthogonal*. Say also that a sequence $c = (c_k)_k \in \ell_2$ is *universal*, when the series $\sum c_n \psi_n$ converges almost everywhere for every orthonormal system of functions $(\psi_n)_n$. According to Schur's Theorem [6, p.56], if c is universal, then the series $\sum c_n f_n$ converges almost everywhere for any quasi-orthogonal system of functions (f_n). It follows from Rademacher-Menchov Theorem, that we can choose $c_k = k^{-1/2} (\log k)^{-b}$ with $b > 3/2$. In [3, Theorem 1.1], it is showed that

$$\textit{the sequence } (Y_n, n \ge 1) \textit{ is a quasi-orthogonal system.} \tag{1.5}$$

We refer to [3] for extensions to independent non identically distributed random variables, and to more general sequences of sets than $A_k = \{ S_k/\sqrt{k} < x_k \}$. Let $p > 1$, and consider the class $\mathcal{F}_p$ of distribution functions verifying

$$\big(F(-x) \vee (1 - F(x))\big) = \mathcal{O}\big(x^{-p}\big) \qquad x \to +\infty. \tag{$\mathcal{F}_p$}$$

We prove the following result

THEOREM 1.1. *Assume that $F \in \mathcal{F}_2$ and is a non degenerate distribution. Then (1.5) holds true. Further, for any sequence $\{x_k, k \ge 1\}$ of reals,*

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{P}\big\{ \frac{S_k}{\sqrt{k}} \le x_k \big\} = c \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{\frac{S_k}{\sqrt{k}} \le x_k\}} \overset{a.s.}{=} c.$$

We also prove results for the case $F \in \mathcal{F}_p$, $p < 2$. In this case, more is required on F. Let $\mathcal{G}_p$ of distribution functions verifying

$$x^{-p} = \mathcal{O}\big(\big(F(-x) \vee (1 - F(x)\big)\big) \qquad x \to +\infty. \tag{$\mathcal{G}_p$}$$

THEOREM 1.2. *Assume for some $1 < p < 2$, that $F \in \mathcal{F}_p \cap \mathcal{G}_p$. Let $\{x_k, k \ge 0\}$ be an arbitrary sequence of reals. Put for any positive integers k and n,*

$$A_k = \big\{ \frac{S_k}{k^{1/p}} < x_k \big\}, \qquad Z_n = \sum_{2^n \le k < 2^{n+1}} \frac{1}{k} \big(\mathbf{1}_{A_k} - \mathbf{P}(A_k).\big)$$

Then, $\{Z_n, n \ge 1\}$ is a quasi-orthogonal system.

We also prove a similar result when F belongs to the domain of attraction of a stable distribution G : there exist constants $\{a_n, n \ge 1\}$ and $\{b_n, n \ge 1\}$ such that the distribution of $a_n^{-1}S_n - b_n$ tends to G. Apart from the case $\alpha = 1$, it is known [1: p.315] that the centering constants $\{b_n, n \ge 1\}$ are unnecessary.

THEOREM 1.3. *Assume that X is centered and F belongs to the domain of attraction of a stable distribution G with exposant $p \in]1, 2]$. Let $\{x_k, k \ge 0\}$ be an arbitrary sequence of reals. Put for any positive integers k and n,*

$$A_k = \{\frac{S_k}{a_k} < x_k\}, \qquad Z_n = \sum_{2^n \le k < 2^{n+1}} \frac{1}{k}(\mathbf{1}_{A_k} - \mathbf{P}(A_k).)$$

Then, $\{Z_n, n \ge 1\}$ is a quasi-orthogonal system. In particular, $\mathbf{P}$-almost surely, for every continuity point x of G, we have

$$\lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}_{\{S_k/a_k \le x\}} = G(x).$$

2. PROOFS.

We use a notational convention : let C denote a constant depending on F only, which may change of values at each occurence. We begin with some general Lemmas. Let $\mathcal{X} = \{X, X_n, n \ge 1\}$ be a sequence *i.i.d.* random variables with basic probability space $(\Omega, \mathcal{B}, \mathbf{P})$.

LEMMA 2.1. *Assume that $\mathbf{E}|X| < \infty$ and $\mathbf{E}X = 0$. Let $b = \{b_n, n \ge 1\}$ be some non decreasing sequence of positive reals. For any integer n,*

$$\mathbf{E}\big|S_n\big| \le 2n\bigg\{b_n\mathbf{P}\{|X| > b_n\} + \int_{b_n}^{\infty} \mathbf{P}\{|X| > u\}du\bigg\} + \Big\{n\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}}\Big\}^{1/2}.$$

Proof. Write $\mathbf{E}\big|S_n\big| \le \mathbf{E}\big|\sum_{k=1}^n X_k\mathbf{1}_{\{|X_k|\le b_n\}}\big| + \mathbf{E}\big|\sum_{k=1}^n X_k\mathbf{1}_{\{|X_k|> b_n\}}\big|$. Then,

$$\mathbf{E}\Big|\sum_{k=1}^{n} X_k\mathbf{1}_{\{|X_k|>b_n\}}\Big| \le n\mathbf{E}|X|\mathbf{1}_{\{|X|>b_n\}} = n\Big[b_n\mathbf{P}\{|X| > b_n\} + \int_{b_n}^{\infty} \mathbf{P}\{|X| > u\}du\Big],$$

$$\mathbf{E}\Big|\sum_{k=1}^{n} X_k\mathbf{1}_{\{|X_k|\le b_n\}}\Big| \le \mathbf{E}\Big|\sum_{k=1}^{n}\Big(X_k\mathbf{1}_{\{|X_k|\le b_n\}} - \mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}\Big)\Big| + n\big|\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}\big|.$$

By centering $\mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}} = -\mathbf{E}X\mathbf{1}_{\{|X|>b_n\}}$. Now, by a routine symmetrization argument, letting $\varepsilon = \{\varepsilon_n, n \ge 1\}$ be a Rademacher sequence independent from the sequence $\mathcal{X}$, with corresponding expectation symbol $\mathbf{E}_\varepsilon$

$$\mathbf{E}\Big|\sum_{k=1}^{n}\Big(X_k\mathbf{1}_{\{|X_k|\le b_n\}} - \mathbf{E}X\mathbf{1}_{\{|X|\le b_n\}}\Big)\Big| \le \mathbf{E}\mathbf{E}_\varepsilon\Big|\sum_{k=1}^{n}\varepsilon_k X_k\mathbf{1}_{\{|X_k|\le b_n\}})\Big|$$

$$\le \mathbf{E}\Big\{\sum_{k=1}^{n} X_k^2\mathbf{1}_{\{|X_k|\le b_n\}}\Big\}^{1/2} \le \Big\{\mathbf{E}\sum_{k=1}^{n} X_k^2\mathbf{1}_{\{|X_k|\le b_n\}}\Big\}^{1/2} = n^{1/2}\big\{\mathbf{E}X^2\mathbf{1}_{\{|X|\le b_n\}}\big\}^{1/2}.$$

Combining both inequalities gives the claimed estimate. ■

LEMMA 2.2. *Assume for some $p > 1$ that $F \in \mathcal{F}_p$. Then, $\mathbf{E}\big|S_n\big| = \mathcal{O}(n^{1/p})$.*

Proof. Follows from Lemma 2.1, since condition $(\mathcal{F}_p)$ implies

$$\max\Big\{n^{1/p}\mathbf{P}\big\{|X| > n^{1/p}\big\}, \int_{n^{1/p}}^{\infty} \mathbf{P}\big\{|X| > u\big\}du\Big\} = \mathcal{O}(n^{1/p\,-1})$$
$$\mathbf{E}X^2\mathbf{1}_{\big\{|X|>n^{1/p}\big\}} = \mathcal{O}(n^{2/p\,-1})$$

■

LEMMA 2.3. *Assume that $F \in DA(G)$ where G is a stable distribution with index $1 < p \le 2$. Then,*

$$\mathbf{E}\big|S_n\big| = \mathcal{O}(a_n).$$

Proof. By Lemma 2.1,

$$\mathbf{E}\big|S_n\big| \le 2n\Bigg\{a_n\mathbf{P}\big\{|X| > a_n\big\} + \int_{a_n}^{\infty} \mathbf{P}\big\{|X| > u\big\}du\Bigg\} + \Big\{n\mathbf{E}X^2\mathbf{1}_{\big\{|X|\le a_n\big\}}\Big\}^{1/2}.$$

• First treat the case $1 < p < 2$. Since $F \in DA(G)$, by Theorem 1 p.312 and relation (8.6) p.313 of [1], one has that $\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}} \sim x^{2-p}L(x)$, as $x \to \infty$, where $L : \overline{\mathbf{R}}_+ \to \mathbf{R}$ is a slowly varying function; and $1 - F(x) + F(-x) \sim \frac{2-p}{p}x^{-p}L(x)$, $x \to \infty$. From [1] p. 579, also follows (for $0 < p \le 2$)

$$\frac{nL(a_n)}{a_n^p} \ \to\ c > 0. \tag{2.1}$$

Thus, immediately $n\mathbf{E}X^2\mathbf{1}_{\big\{|X|\le a_n\big\}} = \mathcal{O}(a_n^2)$, and $na_n\mathbf{P}\{|X| > a_n\} = \mathcal{O}(a_n)$. Write the last term to estimate as

$$n\int_{a_n}^{\infty} \mathbf{P}\big\{|X| > u\big\}du = n\sum_{k=0}^{\infty}\int_{a_n2^k}^{a_n2^{k+1}} \mathbf{P}\big\{|X| > u\big\}du \le n\sum_{k=0}^{\infty}\mathbf{P}\big\{|X| > a_n2^k\big\}a_n2^k.$$
$$\le Ca_n\frac{nL(a_n)}{a_n^p}\sum_{k=0}^{\infty}2^{k(1-p)}\frac{L(a_n2^k)}{L(a_n)} \le Ca_n\sum_{k=0}^{\infty}2^{k(1-p)}\frac{L(a_n2^k)}{L(a_n)}.$$

Since $L(.)$ is slowly varying, it can be represented, as $x \to \infty$ as

$$L(x) = C\big(1 + o(1)\big)\exp\Big\{\int_1^x \frac{\varepsilon(u)}{u}du\Big\},$$

where $C > 0$ and $\lim_{x\to\infty}\varepsilon(u) = 0$ (see Appendix 1 in [4]). Let $0 < \varepsilon < p - 1$. Then, for any n large enough, every k

$$\frac{L(a_n2^k)}{L(a_n)} \le C\exp\Big\{\int_{a_n}^{a_n2^k} \frac{\varepsilon(u)}{u}du\Big\} \le C\exp\big\{\varepsilon k\log 2\big\} = C2^{\varepsilon k},$$

and,

$$\sum_{k=0}^{\infty}2^{k(1-p)}\frac{L(a_n2^k)}{L(a_n)} \le C\sum_{k=0}^{\infty}2^{k(1-p+\varepsilon)} < \infty.$$

This implies that $n\int_{a_n}^\infty \mathbf{P}\big\{|X|>u\big\}du=\mathcal{O}(a_n)$ too, and finally proves the claim in that case.

• There are only minor changes for the case $p=2$. Here $U(x)=\mathbf{E}X^2\mathbf{1}_{\{|X|\le x\}}\sim L(x)$, as $x\to\infty$, where L is a slowly varying function, and $x^2\mathbf{P}\{|x|>x\}/U(x)\ \to\ 0$, as $x\to\infty$. Plainly $n\mathbf{E}X^2\mathbf{1}_{\{|X|<a_n\}}=\mathcal{O}\big(a_n^2\big)$, and $na_n\mathbf{P}\{|X|>a_n\}=\mathcal{O}\big(a_n\big)$. Let $0<\varepsilon<1$. By using again Karamata's representation of slowly varying functions, we find that $\frac{L(a_n2^j)}{L(a_n)}\le 2^{\varepsilon j}$, if n is sufficiently large, for any j.
In view of these observations and (2.1), it follows that

$$n\int_{a_n}^\infty \mathbf{P}\{|X|>u\}\le n\sum_{k=0}^\infty \mathbf{P}\{|X|>a_n2^j\}a_n2^j\le C\frac{nL(a_n)}{a_n^2}a_n\sum_{j=0}^\infty 2^{-j}\frac{L(a_n2^j)}{L(a_n)}\le Ca_n.$$

This proves the estimate in this last case.

■

We now prove a preliminary bound concerning correlations. Let $a=\{a_k,k\ge 1\}$ be some increasing unbounded sequence of positive reals. Let also $f:\mathbf{R}\to\mathbf{R}$ be bounded Lipschitz, with Lipschitz norm $\|f\|_{BL}=\|f\|_L+\|f\|_\infty<\infty$, where $\|f\|_\infty=\sup_{x\in\mathbf{R}}|f(x)|$ and

$$\parallel f\parallel_L=\ \sup\left\{\frac{|f(x)-f(y)|}{|x-y|}\ :\ x,\ y\in\mathbf{R},\ x\ne y\right\}.$$

We thus have the inequality $\big|f(x)-f(y)\big|\le 2\|f\|_{BL}\big(|x-y|\wedge 1\big)$, for $x,y\in\mathbf{R}$.
Consider the following condition linking a with $\mathcal{X}$:

there exists a constant C_0 such that, for any integers $k\ge 1$

$$\mathbf{E}\,|S_k|\le C_0a_k. \tag{$\star$}$$

The preceding Lemmas have precisely given examples for which this property is fulfilled. We now need a suitable version of the correlation inequality in [3].

PROPOSITION 2.4. *For any integers $k\le l$, for every Borel set A of $\mathbf{R}$ and every bounded Lipschitz function f, we have*

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}),f(\frac{S_l}{a_l})\Big)\Big|\le 4\|f\|_{BL}\mathbf{E}\Big(\frac{|S_k|}{a_l}\wedge 1\Big). \tag{2.2}$$

Further, assume that condition ($\star$) is satisfied. Then, for any Borel set A of $\mathbf{R}$, any bounded Lipschitz function f, and integers $k\le l$,

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}),f(\frac{S_l}{a_l})\Big)\Big|\le C\|f\|_{BL}\left(\frac{a_k}{a_l}\right). \tag{2.3}$$

Proof. Without loss of generality we can assume $H=\{\frac{S_k}{a_k}\in A\}$ to be not negligible. Let $\mathbf{E}_H$ denote the expectation with respect to the conditional probability $\mathbf{P}(\cdot|H)$, and $(X_n')_n$ an independent copy of the sequence $(X_n)_n$. Put

$$V_l=\frac{X_1'+\cdots X_k'+X_{k+1}+\cdots+X_l}{a_l}. \tag{2.4}$$

As $\mathbf{E}_H[f(V_l)] = \mathbf{E}[f(\frac{S_l}{a_l})]$, it follows that,

$$
\begin{aligned}
\Big|\mathbf{Cov}(\mathbf{1}_A(\frac{S_k}{a_k}), f(\frac{S_l}{a_l}))\Big| &= \Big|\int_H f(\frac{S_l}{a_l})d\mathbf{P} - \mathbf{P}(H)\int f(\frac{S_l}{a_l})dP\Big| = \mathbf{P}(H)\big|\mathbf{E}_H f(\frac{S_l}{a_l}) - \mathbf{E}f(\frac{S_l}{a_l})\big| \\
&= \mathbf{P}(H)\Big|\mathbf{E}_H f(\frac{S_l}{a_l}) - \mathbf{E}_H f(V_l)\Big| \leq 2\|f\|_{BL}\mathbf{P}(H)\mathbf{E}_H\Big(|\frac{S_l}{a_l} - V_l| \wedge 1\Big) \\
&= 2\|f\|_{BL}\mathbf{E}\Big(|\frac{S_l}{a_l} - V_l| \wedge 1\Big) = 2\|f\|_{BL}\mathbf{E}\Big(\frac{|S_k - S'_k|}{a_l} \wedge 1\Big) \\
&\leq 4\|f\|_{BL}\mathbf{E}\Big(\frac{|S_k|}{a_l} \wedge 1\Big),
\end{aligned}
$$

since $x \mapsto (x \wedge 1)$ is subadditive on $\mathbf{R}_+$. This establishes the first part of the Proposition. The second part is then a simple consequence of it and condition $(\star)$. ■

Introduce for any $\lambda > 0$, the *concentration function* of S_n : $Q_n(\lambda) = \sup_{x\in\mathbf{R}} \mathbf{P}(x \leq S_n \leq x+\lambda)$. According to Theorem 9 p. 49 in [7], for any *i.i.d.* sequence $\mathcal{X}$ with non degenerate distribution, there exists an absolute constant C_1 such that for any $\lambda \geq 0$, and n

$$
Q_n(\lambda) \leq C_1\frac{\lambda+1}{\sqrt{n}}. \tag{2.5}
$$

We shall now prove the following

PROPOSITION 2.5. *Let $0 < \varepsilon \leq 1$. For every Borel set A, any real x and integers $k \leq l$, we have*

$$
\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l})\Big)\Big| \leq \frac{8}{\varepsilon}\,\mathbf{E}\Big(\frac{|S_k|}{a_l} \wedge 1\Big) + 2Q_l(a_l\varepsilon). \tag{2.6}
$$

Further, assume that condition $(\star)$ is satisfied. Then, for any Borel set A of $\mathbf{R}$, *any real x, and integers $k \leq l$,*

$$
\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l})\Big)\Big| \leq C\Big\{\frac{1}{\varepsilon}\,(\frac{a_k}{a_l}) + Q_l(a_l\varepsilon)\Big\} \leq C\Big\{\frac{1}{\varepsilon}\,(\frac{a_k}{a_l}) + \frac{a_l\varepsilon+1}{\sqrt{l}}\Big\}. \tag{2.7}
$$

Proof. Let ε and x be fixed, and define the Lipschitz function f_ε as

$$
f_\varepsilon(t) = \mathbf{1}_{(-\infty,x]}(t) + g_\varepsilon(t) = \mathbf{1}_{(-\infty,x]}(t) + (1 + \frac{x-t}{\varepsilon})\mathbf{1}_{(x,x+\varepsilon)}(t).
$$

Then it is easily checked that $\|f_\varepsilon\|_{BL} = 1 + 1/\varepsilon$. Let H be the event $\{\frac{S_k}{a_k} \in A\}$; we can assume that H is not negligible. Let $\mathbf{C}$ be the conditional probability $\mathbf{P}(\cdot|H)$. Then we have

$$
\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l})\Big)\Big| = \mathbf{P}(H)\Big|\mathbf{C}(\frac{S_l}{a_l} \leq x) - \mathbf{P}(\frac{S_l}{a_l} \leq x)\Big|.
$$

But,

$$
\begin{aligned}
\mathbf{C}(\frac{S_l}{a_l} \leq x) - \mathbf{P}(\frac{S_l}{a_l} \leq x) &= \mathbf{E}^{\mathbf{C}}\big[(f_\varepsilon - g_\varepsilon)(\frac{S_l}{a_l})\big] - \mathbf{E}^{\mathbf{P}}\big[(f_\varepsilon - g_\varepsilon)(\frac{S_l}{a_l})\big] \\
&= \mathbf{E}^{\mathbf{C}}\big[(f_\varepsilon - g_\varepsilon)(\frac{S_l}{a_l})\big] - \mathbf{E}^{\mathbf{C}}\big[(f_\varepsilon - g_\varepsilon)(V_l)\big] \\
&= \mathbf{E}^{\mathbf{C}}\big[f_\varepsilon(\frac{S_l}{a_l}) - f_\varepsilon(V_l)\big] - \mathbf{E}^{\mathbf{C}}\big[g_\varepsilon(\frac{S_l}{a_l}) - g_\varepsilon(V_l)\big],
\end{aligned}
$$

where V_l is the random variable defined in (2.4). By arguing as in the proof of Proposition 2.4, we get

$$\Big|\mathbf{E}^{\mathbf{C}}[f_\varepsilon(\frac{S_l}{a_l}) - f_\varepsilon(V_l)]\Big| \le 4(1+1/\varepsilon)\frac{1}{\mathbf{P}(H)}\mathbf{E}\Big(\frac{|S_k|}{a_l} \wedge 1\Big), \tag{2.8}$$

while trivially

$$\Big|\mathbf{E}^{\mathbf{C}}[g_\varepsilon(\frac{S_l}{a_l}) - g_\varepsilon(V_l)]\Big| \le \frac{2Q_l(a_l\varepsilon)}{\mathbf{P}(H)} \tag{2.9}$$

From (2.8) and (2.9), we deduce the first claimed inequality by summing and multiplying by $\mathbf{P}(H)$. And the second inequality is easily deduced from the first, by definition of condition $(\star)$. ■

PROPOSITION 2.6. *Assume that $F \in \mathcal{F}_2$. Then, for any Borel set A of $\mathbf{R}$, any real x, and integers $k \le l$, we have*

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{\sqrt{k}}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{\sqrt{l}})\Big)\Big| \le C\big(\frac{k}{l}\big)^{1/4}.$$

Proof. We apply Proposition 2.5 with the choice $a_k = \sqrt{k}$. By Lemma 2.2, condition $(\star)$ is satisfied. Then, for every $0 < \varepsilon \le 1$

$$\sup_{A,x}|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{\sqrt{k}}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{\sqrt{l}})\Big)| \le C\Big\{\frac{1}{\varepsilon}\big(\frac{k}{l}\big)^{1/2} + \varepsilon + \frac{1}{l^{1/2}}\Big\}.$$

The proof is achieved by taking $\varepsilon = (k/l)^{1/4}$; since $\frac{1}{\varepsilon}\big(\frac{k}{l}\big)^{1/2} + \varepsilon + \frac{1}{l^{1/2}} \le 3(k/l)^{1/4}$. ■

Proof of Theorem 1.1. Combine Proposition 2.6 with Lemma 7.4.3 of [8] that we recall for convenience.

LEMMA 2.7. *Let H be an Hilbert space, and $\Phi = \{f_n, n \ge 1\} \subset H$ with correlations $a_{j,k} = \langle f_j, f_k\rangle$. In order that Φ be a quasi-orthogonal system, it is enough that $\sup_{j\ge1}\sum_{k\,:\,k\ne j}|a_{j,k}| < \infty$.*

■

For proving Theorem 1.2, we need a suitable estimate of $Q_n(\varepsilon)$. We use Esseen's estimate ([7], Theorem 3, p.43).

LEMMA 2.8. *Assume that $F \in \mathcal{G}_p$ with $1 < p < 2$. Then, there exists λ_0, such that for any $\lambda \ge \lambda_0$,*

$$Q_n(\lambda) \le Cn^{-1/2}\lambda^{p/2}.$$

Proof. Let $D(\tilde{X},\lambda) = \lambda^2\mathbf{E}\tilde{X}^2\mathbf{1}_{|\tilde{X}|<\lambda} + \mathbf{P}\{|\tilde{X}| \ge \lambda\}$ define the *censored variance* of a symmetrized version $\tilde{X}$ of X. Since $\mathcal{X}$ is an *i.i.d.* sequence, in view of Esseen's inequality, there exists an absolute constant C such that any for $\lambda > 0$, $Q_n(\lambda) \le C\big[nD(\tilde{X},\lambda)\big]^{-1/2}$. Since $X \in \mathcal{G}_p$ and $D(\tilde{X},\lambda) \ge \frac{1}{2}\mathbf{P}\{|X| \ge \lambda\}$, it follows that $D(\tilde{X},\lambda) \ge C\lambda^{-p}$ for λ is sufficiently large, say $\lambda \ge \lambda_0$. This proves our claim. ■

Corresponding to Proposition 2.6 for the case $F \in \mathcal{F}_p \cap \mathcal{G}_p$, is the following statement

PROPOSITION 2.9. *Assume that $F \in \mathcal{F}_p \cap \mathcal{G}_p$ with $1 < p < 2$, and let $a_k = k^{1/p}$. Then, there exists k_0 finite, such that for any Borel set A of $\mathbf{R}$, any real x and integers $l \geq k \geq k_0$, we have*

$$\Big|\mathbf{Cov}(\mathbf{1}_A(\frac{S_k}{k^{1/p}}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{l^{1/p}}))\Big| \leq C\big(\frac{k}{l}\big)^{\frac{1}{p+2}}.$$

Proof. We apply Proposition 2.5 with the choice $a_k = k^{1/p}$. By Lemma 2.2, condition $(\star)$ is satisfied. From estimate (2.7), we have

$$\Big|\mathbf{Cov}(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l}))\Big| \leq C\Big\{\frac{1}{\varepsilon}\big(\frac{k}{l}\big)^{1/p} + Q_l(a_l\varepsilon)\Big\}$$

Choose $\varepsilon = \big(\frac{k}{l}\big)^{\frac{2}{p(p+2)}}$. Then, $\varepsilon a_l = l^{\frac{1}{p+2}} k^{\frac{2}{p(p+2)}}$. In view of Lemma 2.8, if k is large enough, say $k \geq k_0$, then $\varepsilon a_l \geq \lambda_0$, and so

$$Q_l(a_l\varepsilon) \leq Cl^{-1/2}(a_l\varepsilon)^{p/2} \leq C\big(\frac{k}{l}\big)^{\frac{1}{p+2}}.$$

As $\frac{1}{\varepsilon}\big(\frac{k}{l}\big)^{\frac{1}{p}} = \big(\frac{k}{l}\big)^{\frac{1}{p+2}}$, this allows to conclude. ■

Proof of Theorem 1.2. Combine Proposition 2.9 with Lemma 2.7. ■

Now we pass to the

Proof of Theorem 1.3. By Lemma 2.3 and inequality (2.7) of Proposition 2.5,

$$\Big|\mathbf{Cov}\Big(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l})\Big)\Big| \leq C\Big\{\frac{1}{\varepsilon}\,(\frac{a_k}{a_l}) + Q_l(a_l\varepsilon)\Big\}$$

Choose $\varepsilon = \big(\frac{a_k}{a_l}\big)^{\frac{2}{p+2}}$. Then $a_l\varepsilon = a_l^{\frac{p}{p+2}} a_k^{\frac{2}{p+2}} (\geq a_k)$. We use the notation from the proof of Lemma 2.3 and properties of F mentionned therein. Then, $D(\tilde{X}, \lambda) \geq CL(\lambda)\lambda^{-p}$ for any $\lambda \geq \lambda_0$, where λ_0 depends on F only. And by Esseen's estimate, for $\lambda \geq \lambda_0$,

$$Q_l(\lambda) \leq C\big[lD(\tilde{X}, \lambda)\big]^{-1/2} \leq C\big(\frac{\lambda^p}{lL(\lambda)}\big)^{1/2}. \tag{2.10}$$

Choose k_0 sufficiently large to have $a_{k_0} \geq \lambda_0$. Applying (2.10) with $\lambda = a_l\varepsilon$, gives

$$Q_l(a_l\varepsilon) \leq C\frac{a_l^{\frac{p^2}{2(p+2)}} a_k^{\frac{p}{p+2}}}{l^{1/2}L(a_l\varepsilon)^{1/2}} \leq C\Big(\frac{a_k}{a_l}\Big)^{\frac{p}{p+2}}\Big(\frac{a_l^p}{lL(a_l)}\Big)^{1/2}\Big(\frac{L(a_l)}{L(a_l\varepsilon)}\Big)^{1/2}.$$

for $l \geq k \geq k_0$, where k_0 depends on F only. Let $0 < \eta < 1$. By using again Karamata's representation of slowly varying functions, we find that

$$\frac{L(a_l)}{L(a_l\varepsilon)} \leq C\exp\{\eta\log\frac{1}{\varepsilon}\} = C\exp\{\eta(\frac{p}{p+2})\log\frac{a_k}{a_l}\} = C\big(\frac{a_k}{a_l}\big)^{\eta(\frac{p}{p+2})}, \tag{2.11}$$

assuming k large enough, say $k \geq k_\eta$. By using this with relation (2.1), we obtain : there exists a constant C_η depending on F and η only, and $k_\eta < \infty$, such that for any integers $l \geq k \geq k_\eta$

$$Q_l(a_l\varepsilon) = Q_l\big(a_l^{\frac{p}{p+2}} a_k^{\frac{2}{p+2}}\big) \leq C_\eta\Big(\frac{a_k}{a_l}\Big)^{\frac{(1+\eta)p}{p+2}}. \tag{2.12}$$

By integrating this estimate into inequality (2.7) recalled at the beginning of the proof, we get

$$\Big|\mathbf{Cov}(\mathbf{1}_A(\frac{S_k}{a_k}), \mathbf{1}_{(-\infty,x]}(\frac{S_l}{a_l}))\Big| \leq C\big(\frac{a_k}{a_l}\big)^{\frac{p}{p+2}} + C_\eta\Big(\frac{a_k}{a_l}\Big)^{\frac{(1+\eta)p}{p+2}} \leq C_\eta\Big(\frac{a_k}{a_l}\Big)^{\frac{(1+\eta)p}{p+2}}. \tag{2.13}$$

One then deduce Theorem 1.3 from the combination of (2.13) with Lemma 2.7. ■

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