1 Preliminaries

Let $(N_t)_{t\geq 0}$ be a collection of random variables. The parameter t is often interpreted as time.

Definition 1.1. $(N_t)_{t\geq 0}$ is a *stochastic process.*

Definition 1.2. The stochastic process $(N_t)_{t\geq 0}$ is said to be a *counting process* if N_t represents the total numbers of "events"that have occurred up to time t.

From this definition we see that the following properties must be verified

(i) $N_t \geq 0$, $\forall\ t \geq 0$;

(ii) N_t is integer-valued, $\forall\ t \geq 0$;

(iii) If $s < t$, then $N_s \leq N_t$.

The variable $N_t - N_s$ equals the number of events that have occurred in the time interval $(s, t]$; the family of random variables $(N_t - N_s)_{0\leq s<t}$ are called the *increments* of the counting process $(N_t)_{t\geq 0}$.

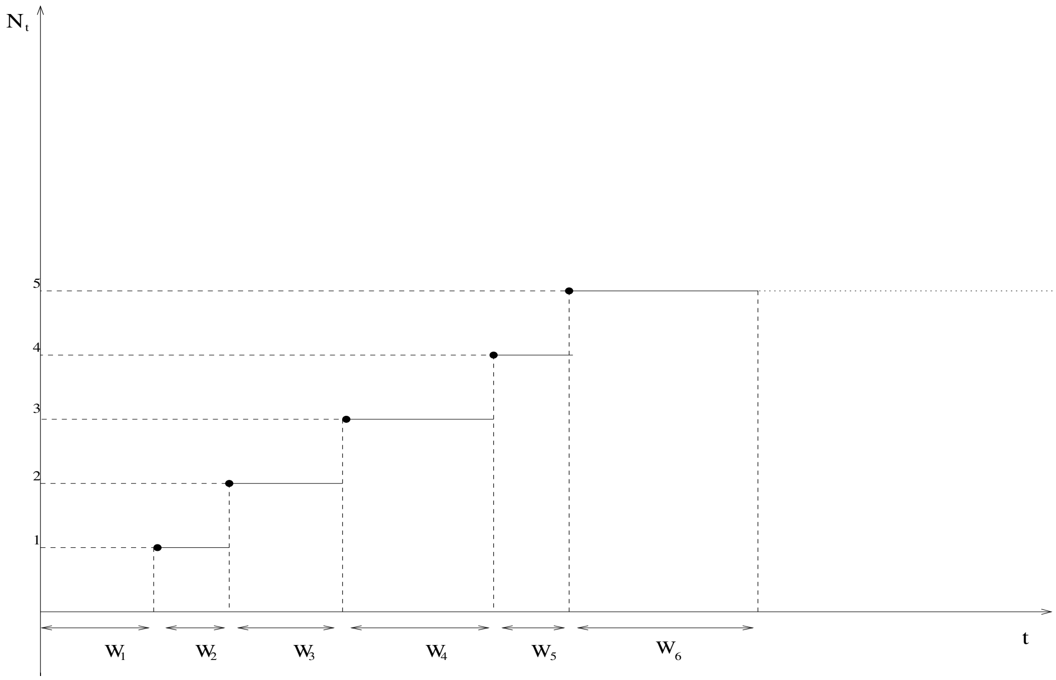
Definition 1.3. A counting process is said to possess *independent increments* if the number of events occurred in disjoint times interval are independent.

Definition 1.4. A counting process is said to possess *stationary increments* if the distribution of the number of events occurred in any time interval depends only on the length of the time interval. This means that, for all $t_1 < t_2$ and for all $s > 0$ the increment $N_{t_2+s} - N_{t_1+s}$ (i.e. the number of events occurred in the time interval $(t_1 + s, t_2 + s]$) has the same distribution as the increment $N_{t_2} - N_{t_1}$ (i.e. the number of events occurred in the time interval $(t_1, t_2]$).

2 Poisson process: first definition

Let $W_1, W_2, W_3 \ldots$ be independent random variables with law $\mathcal{E}(\lambda)$, where $\lambda > 0$ is a given number. For every $t \geq 0$ define

$$N_t = \begin{cases} 0 & \text{if} \qquad 0 \leq t < W_1 \\ 1 & \text{if} \qquad W_1 \leq t < W_1 + W_2 \\ 2 & \text{if} \qquad W_1 + W_2 \leq t < W_1 + W_2 + W_3 \\ \vdots & \end{cases}$$

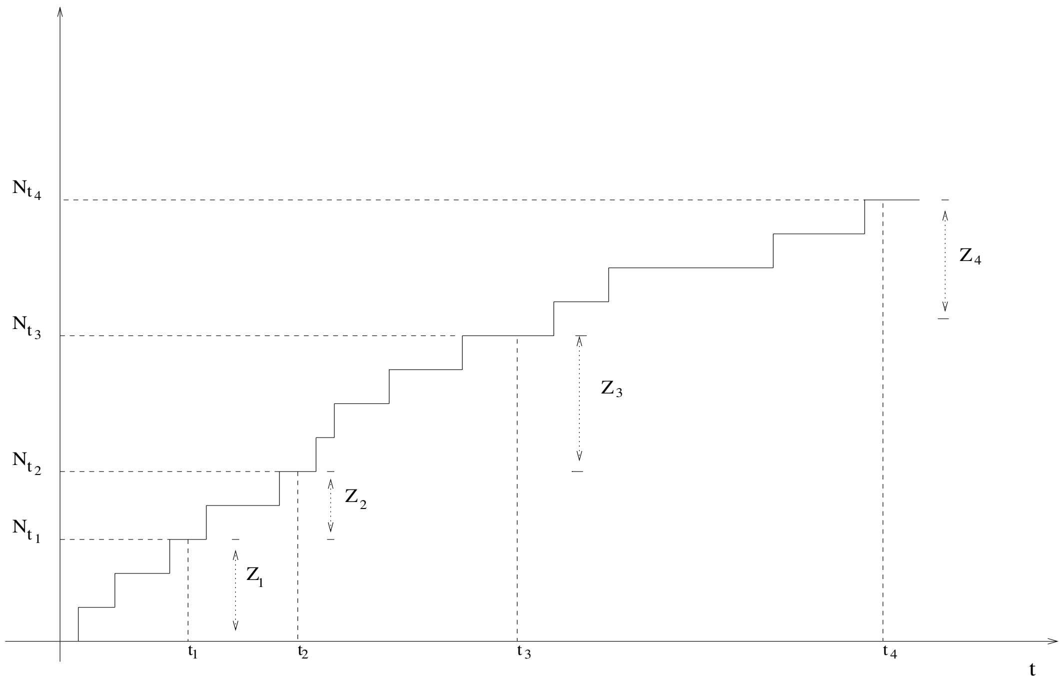


Definition 2.1. The family of random variables $(N_t)_{t\geq 0}$ is called a *Poisson process* having rate λ.

Theorem 2.2. *Let* $0 = t_0 < t_1 < t_2 < t_3 < \cdots < t_{n-1} < t_n = t$ *be a partition of* $[0, t]$. *Then the increments*

$$\begin{aligned} Z_1 &= N_{t_1} - N_{t_0} = N_{t_1} \\ Z_2 &= N_{t_2} - N_{t_1} \\ Z_3 &= N_{t_3} - N_{t_2} \\ &\vdots \\ Z_n &= N_{t_n} - N_{t_{n-1}} \end{aligned}$$

are independent and Poisson distributed with parameters $\lambda t_1, \lambda(t_2 - t_1), \ldots, \lambda(t_n - t_{n-1})$.



3 Poisson process: second definition

Definition 3.1. The counting process $(N_t)_{t\geq 0}$ is said to be a *Poisson process* having rate λ if

(i) $N_0 = 0$;

(ii) The process has independent increments;

(iii) The number of events in any interval of length t is Poisson distributed with mean (parameter) λt. This means that, for all $h, t \geq 0$

$$P\big(N_{t+h} - N_h = n\big) = \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t}, \qquad n = 0, 1, 2, \dots$$

It follows from condition (iii) that a Poisson process has stationary increments and also

$$E[N_t] = \lambda t$$

which explains why λ is called the *rate* of the process.

4 Poisson process: third definition

To determine if a given counting process is actually a Poisson process, we must show that conditions (i), (ii) and (iii) of Section 3 are satisfied. Conditions (i) and (ii) are usually easily verified from our knowledge of the process. Condition (iii) is more difficult. For this reason an equivalent definition of a Poisson process is useful.

Definition 4.1. The counting process $(N_t)_{t\geq 0}$ is said to be a *Poisson process* having rate λ if

(i) $N_0 = 0$;

(ii) The process has stationary and independent increments;

(iii) $P\big(N_\delta = 1\big) = \lambda\delta + o(\delta)$;

(iv) $P\big(N_\delta \geq 2\big) = o(\delta)$.

Theorem 4.2. *Definitions 2.1, 3.1 and 4.1 are equivalent.*

Proof. We show that definition 4.1 implies definition 3.1. Put

$$P_n(t) = P\big(N_t = n\big).$$

We may have n events at time $t+\delta$ if

(a) we have n events at time t and no event beteween t and $t+\delta$;

(b) we have $n-1$ events at time t and 1 event beteween t and $t+\delta$;

(c) we have less than $n-1$ events at time t and more than 1 event beteween t and $t+\delta$.

So, for $n=0$, by independence (assumption (ii)) we have

$$P_0\big(t+\delta\big) = P\big(N_{t+\delta} = 0\big) = P\big(N_t = 0, N_{t+\delta} - N_t = 0\big) = P_0(t)P\big(N_{t+\delta} - N_t = 0\big)$$

By stationarity (assumption (ii)) we have

$$P\big(N_{t+\delta} - N_t = 0\big) = P(N_\delta = 0) = 1 - P\big(N_\delta > 0\big) = 1 - P\big(N_\delta = 1\big) - P\big(N_\delta \geq 2\big) = 1 - \lambda\delta + o(\delta).$$

where the last equality follows from assumptions (iii) and (iv). Hence we have obtained

$$P_0\big(t+\delta\big) = P_0(t)\Big(1 - \lambda\delta + o(\delta)\Big).$$

Rearranging and dividing by δ we get

$$\frac{P_0\big(t+\delta\big) - P_0(t)}{\delta} = -\lambda P_0(t) + \frac{o(\delta)}{\delta}$$

and letting $\delta \to 0$

$$P_0'(t) = -\lambda P_0(t).$$

By integrating this simple differential equation we obtain

$$P_0(t) = ce^{-\lambda t}, \qquad c \in \mathbb{R}.$$

We have the initial condition $P_=(0) = P\big(N_0 = 0\big) = 1$ by assumption (i), which yields

$$P_0(t) = e^{-\lambda t}.$$

For $n > 0$, we obtain

$$\begin{aligned} P_n\big(t+\delta\big) &= P\big(N_{t+\delta} = n\big) \\ &= P\big(N_t = n, N_{t+\delta} - N_t = 0\big) + P\big(N_t = n-1, N_{t+\delta} - N_t = 1\big) + \sum_{k=2}^{n} P\big(N_t = n-k, N_{t+\delta} - N_t = k\big) \\ &= P\big(N_t = n)P\big(N_{t+\delta} - N_t = 0\big) + P\big(N_t = n-1\big)P\big(N_{t+\delta} - N_t = 1\big) + \sum_{k=2}^{n} P\big(N_t = n-k\big)P\big(N_{t+\delta} - N_t = k\big) \end{aligned}$$

by the independence of the increments (assumptions (ii)). Continuing, we notice that, by assumption (iv)

$$\begin{aligned} 0 &\leq \sum_{k=2}^{n} P\big(N_t = n-k\big)P\big(N_{t+\delta} - N_t = k\big) = \sum_{k=2}^{n} o(\delta)P\big(N_t = n-k\big) \leq o(\delta)\sum_{k=0}^{n} P\big(N_t = n-k\big) \\ &= o(\delta)P\big(N_t \leq n\big) \leq o(\delta). \end{aligned}$$

which means that

$$\sum_{k=2}^{n} P\big(N_t = n-k\big) P\big(N_{t+\delta} - N_t = k\big) = o(\delta).$$

Hence, by assumptions (i) and (ii)

$$\begin{aligned} &P_n\big(t+\delta\big) = P_n(t)\big(1-\lambda\delta + o(\delta)\big) + P_{n-1}(t)\big(\lambda\delta + o(\delta)\big) + o(\delta) \\ &= (1-\lambda h)P_n(t) + \lambda\delta P_{n-1}(t) + o(\delta). \end{aligned}$$

Thus, rearranging and dividing by δ, we obtain

$$\frac{P_n(t+\delta) - P_n(t)}{\delta} = -\lambda P_n(t) + \lambda P_{n-1}(t) + \frac{o(\delta)}{\delta};$$

letting $\delta \to 0$ yields

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

and multiplying both members by $e^{\lambda t}$,

$$e^{\lambda t}\{P_n'(t) + \lambda P_n(t)\} = \lambda e^{\lambda t} P_{n-1}(t).$$

or

$$\frac{d}{dt}\big(e^{\lambda t} P_n(t)\big) = \lambda e^{\lambda t} P_{n-1}(t).$$

We already know that $P_0(t) = e^{-\lambda t}$; from this and the preceding formula we deduce

$$\frac{d}{dt}\big(e^{\lambda t} P_1(t)\big) = \lambda;$$

integrating

$$e^{\lambda t} P_1(t) = \lambda t + c, \Longrightarrow P_1(t) = (\lambda t + c)e^{-\lambda t}$$

and, since $P_1(0) = P(N_0 = 1) = 0$, we conclude with

$$P_1(t) = \lambda t e^{-\lambda t}.$$

We proceed by induction: assume that

$$P_{n-1}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} \mathrm{e}^{-\lambda t}.$$

Then (see above)

$$\frac{d}{dt}\big(e^{\lambda t} P_n(t)\big) = \lambda e^{\lambda t} P_{n-1}(t) = \frac{\lambda^n t^{n-1}}{(n-1)!};$$

integrating

$$e^{\lambda t} P_n(t) = \frac{\lambda^n t^{n-1}}{(n-1)!} + c, \Longrightarrow P_n(t) = \Big(\frac{\lambda^n t^n}{n!} + c\Big) e^{\lambda t};$$

since $P_n(0) = P(N_0 = n) = 0$, we conclude with

$$P_n(t) = \frac{\lambda^n t^n}{n!} e^{\lambda t},$$

as claimed. □

5 Examples

Typical examples of Poisson processes are

(a) customers that arrive to the checkout counter of a convenience store;

(b) atoms emitted by a radioactive substance (eg. uranium);

(c) spikes fired by a neuron;

(d) cars that arrive to the toll barrier of a highway...

References

[1] M. Dwass, Probability and Statistics. Benjamin, New York, 1970.

[2] Sheldon M. Ross, Introduction to Probability Models, Academic Press, 2014