# AREA IMPLIES COAREA

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Abstract. We regard one side of the coarea formula as a measure and compute its density by an area-type formula. As an application, we show the first nontrivial example of coarea formula for vector-valued sub-Riemannian Lipschitz mappings.

## Contents



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## 1. Introduction

Coarea formula is an important tool in different areas of Analysis, as for instance Geometric Measure Theory and Partial Differential Equations. It has been found by H. Federer, [7]. As a small historical record, here the author remarks: "The original motivation leading to the discovery of this theorem was the simplification of certain arguments in [6]". Federer proves the coarea formula for Lipschitz mappings of Riemannian manifolds. Extensions of this formula to various classes of Sobolev mappings have been recently established by J. Malý, D. Swanson and W. P. Ziemer in [24], where one can find further references. In this paper we are mainly concerned with coarea formula in the case of sub-Riemannian manifolds, whose distance does not satisfy a biLipschitz estimate with respect to any Riemannian distance.

More precisely, we study the coarea formula when the Riemannian distance in the source space is replaced by the sub-Riemannian distance. This yields two substantial differences with respect to the Riemannian context. First, the Hausdorff dimensions of level sets may be strictly greater than their topological dimensions. Second, sub-Riemannian Lipschitz mappings may not be Lipschitz with respect to any Riemannian distance. According to this picture, we have two possible choices to state a sub-Riemannian coarea formula. One possibility is to consider Riemannian Lipschitz mappings, whereas level sets are measured by the sub-Riemannian distance. In this case the coarea formula is a consequence of three main ingredients: the Riemannian coarea formula, an area-type formula for higher codimensional smooth submanifolds and a weak Sard-type theorem, [16], [18], [19], [22], [26]. However, it is rather clear that the point of coarea formula in the sub-Riemannian setting is that of including also Lipschitz mappings with respect to the sub-Riemannian distance in the source space. This is a new difficulty with respect to the Riemannian setting, since sub-Riemannian Lipschitz mappings may not be differentiable on a set of positive measure, [20]. As a consequence, we cannot apply the Riemannian coarea formula and this is the first important obstacle.

On the other hand, for real-valued sub-Riemannian Lipschitz mappings on stratified groups the coarea formula holds, [20]. The proof is based on a general coarea formula for functions of bounded variations with respect to vector fields, [12], [15], [23], [25]. Here the key point is to show that we can replace the perimeter measure of upper level sets with the one codimensional sub-Riemannian Hausdorff measure of level sets. Unfortunately, this method cannot be extended to vector-valued sub-Riemannian Lipschitz mappings, since level sets can be seen as boundaries of finite perimeter sets only in the case of real-valued mappings.

Due to the special symmetries of Euclidean spaces, the proof of the classical coarea formula for vector-valued Lipschitz mappings relies on a sort of "extension procedure" by orthogonal projections, that enlarge the dimension of the target to make it locally bi-Lipschitz equivalent to the source space. This allows for applying a linearization argument to the extended mapping, along with estimates on its Jacobian, [8]. The application of this approach to stratified groups seems to entail several drawbacks. In fact, the target extension should be also an algebraic extension that respects the sub-Riemannian distance. Moreover, it is not true that all projections are Lipschitz continuous with respect to the sub-Riemannian distance.

We overcome these difficulties following a different scheme. We regard one side of the coarea formula as a set function and then compute its density. This is the original approach used by Federer in [7], where he first shows that this set function is countably additive on measurable sets of the source space. In our case, this set function is the integral of the sub-Riemannian spherical Hausdorff measures of level sets, instead of either Riemannian Hausdorff measures of 3.1 in [7] or metric Hausdorff measures of 2.10.26 in [8]. Since our method requires that this set function is a Borel regular measure, in Theorem 2.2 we prove this fact for a Lipschitz mapping of boundedly compact metric spaces, where the source metric space is countably finite with respect to a general Carathéodory measure (Definition 2.1). Then we call this set function the *coarea measure*.

Our technique differs from [7], since we compute the density of the coarea measure by a blow-up method in the same way as in [18] we established an upper density estimate for a family of integrals of "size $\delta$ approximating measures". This leads us to a coarea inequality for sub-Riemannian Lipschitz mappings of stratified groups. Let us point out that in the Euclidean context this blow-up approach can be found in Lemma 2.96 of [1]. After the density of the coarea measure is computed, then its Borel regularity allows us to apply classical differentiability theorems for measures in metric spaces, [8].

The main observation of this work is that once any area-type formula is available for the Hausdorff measure of level sets, then we are able to obtain a sharp lower density estimate of the coarea measure, hence establishing the coarea formula. Recently B. Franchi, R. Serapioni and F. Serra Cassano have established area-type formulae for the spherical Hausdorff measure of intrinsic regular submanifolds of Heisenberg groups $\mathbb{H}^n$ with codimension higher than one, [14]. As an application of the method described above, we use these results to establish the first nontrivial example of coarea formula for vector-valued sub-Riemannian Lipschitz mappings.

**Theorem 1.1** (Coarea formula). *Let $f : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ be a Lipschitz mapping with $1 \leq k \leq n$. Then we have*

$$\int_{\mathbb{H}^n} u(x)\, J_H f(x)\, dx = \int_{\mathbb{R}^k} \left( \int_{f^{-1}(y)} u(x)\, d\mathcal{S}_d^{2n+2-k}(x) \right) dy\,, \tag{1}$$

*where $u : \mathbb{H}^n \longrightarrow [0, +\infty]$ is any nonnegative measurable function.*

However, this coarea formula should not be regarded as an isolated case, but rather as a special instance of the general method described above. For instance, one could check that our approach provides a unified method to get different proofs of either the Euclidean coarea formulae or the sub-Riemannian coarea formulae of [20]. As a sort of meta-statement, we may expect that whenever area-type formulae are available in general stratified groups, then new sub-Riemannian coarea formulae will follow.

Turning to the proof of Theorem 1.1, given in Subsection 4.1, the concrete application of our scheme has to face further technical issues. In fact, the coarea measure corresponds to an integral of measures of level sets, hence the area-type formula of [14] has to be applied to a parametrized family of implicit mappings that around regular points foliate the source space. In particular, the implicit mappings must be defined on balls of common radius. This requires a "uniform version" of the implicit function theorem of [14]. More precisely, we have to make explicit the geometric constant that multiplies the radius of the balls on which the implicit mappings are defined. Section 3 is devoted to the proof of this quantitative version of the implicit function theorem. The argument of the proof has some interesting variants with respect to the classical one, due to noncommutativity of the group law and to the use of the key Hölder estimate (13) for right translations applied to the left invariant homogeneous distance. This is a typical feature of the non-abelian case. We also remark that the blow-up of the coarea measure and the local foliation by implicit mappings give rise to their "double scale differentiability", that is expressed in the limit (29). In fact, we consider a sort of "nonlinear" difference quotient, in the direction $\delta_\rho w$, as $\rho \to 0$, that is applied to a foliating implicit mapping that also depends on $\rho$.

Finally, we wish to point out that area-type formulae and implicit function theorems for $\mathbb{R}^k$-valued mappings on Heisenberg groups are still to be understood for $k > n$. In fact, the known results for $k \le n$ rely on a representation of the Heisenberg group $\mathbb{H}^n$ as semidirect product of a commutative $k$-dimensional homogeneous subgroup by a $k$-codimensional homogeneous normal subgroup. This factorization is not possible for $k > n$. Although under Euclidean regularity area-type formulae and coarea formulae hold for all codimensions $k = 1, \dots, 2n$, [21], in the case of sub-Riemannian Lipschitz mappings, the coarea formula (1) for $k > n$ is not clear. In particular, the simplest version of this formula for $n = 1$ and $k = 2$ is already an intriguing open question. The study of this case is the object of recent investigations, [17].

## 2. Coarea measure in metric spaces

In the sequel, we will denote by $X$ and $Y$ two boundedly compact metric spaces, namely bounded sequences in these spaces admit converging subsequences.

**Definition 2.1** (Carathéodory measure)**.** Let $a \ge 0$ and let $\mathcal{O}$ be a family of open sets of $X$. For each $E \subset X$, we define the set function

$$\Gamma^a_t(E) = \beta_a \ \inf \left\{ \sum_{j=1}^{\infty} \operatorname{diam}(F_j)^a : E \subset \bigcup_{j=1}^{\infty} F_j, \ \operatorname{diam}(F_j) \le t, \ F_j \in \mathcal{O} \right\}.$$

The positive number $\beta_a$ plays the role of the geometric constant. According to [8], we define the *Carathéodory measure*

$$\Gamma^a(E) = \sup_{t>0} \Gamma^a_t(E)\,,$$

that is a Borel regular measure on $X$. In addition, our definition assumes that

$$\theta_a^{-1}\, \Gamma^a \le \mathcal{H}^a \le \theta_a\, \Gamma^a \tag{2}$$

for some $\theta_a > 0$. Recall that when $\mathcal{O}$ coincides with all open sets of $X$ and $2^p\, \beta_p$ equals the volume of the unit Euclidean ball in $\mathbb{R}^p$, we have $\Gamma^p = \mathcal{H}^p$ that is the classical $p$-dimensional Hausdorff measure.

Since our measures $\Gamma^a$ are constructed from family of open sets, we have

$$\limsup_{n\to\infty} \Gamma^a_t(K_n) \le \Gamma^a_t\Big(\bigcap_{n=1}^{\infty} \overline{\bigcup_{j\ge n} K_j}\Big),$$

for every equibounded sequence $(K_n)$ of compact sets and for each $t > 0$. Thus, for each continuous mapping $f : A \longrightarrow Y$ on a closed set $A \subset X$, the function $Y \ni y \longrightarrow \Gamma^a_t\big(K \cap f^{-1}(y)\big)$ is upper semicontinuous, whenever $K$ is a compact set. It follows that

$$Y \ni y \longrightarrow \Gamma^a\big(K \cap f^{-1}(y)\big) \quad \text{is Borel.} \tag{3}$$

Since $X$ is boundedly compact, closed sets are union of increasing sequences of compact sets and this easily leads us to the following

**Proposition 2.1.** *Let $F$ and $A$ be closed sets of $X$ and let $f : A \longrightarrow Y$ be continuous. Then $y \longrightarrow \Gamma^a\big(F \cap f^{-1}(y)\big)$ is Borel measurable.*

Taking into account 2.10.25 of [8], we get the following inequality

$$\int_Y^* \Gamma^{Q-P}\left(E \cap f^{-1}(\xi)\right) d\Gamma^P(\xi) \le \mathrm{Lip}(f)\, \frac{\omega_{Q-P}\,\omega_P\,\beta_{Q-P}\,\beta_P\,\beta_Q}{\omega_Q}\, \Gamma^Q(E)\,, \tag{4}$$

where $\int^*$ denotes the upper integral, $E \subset X$ and $0 \le P \le Q$. Since $\Gamma^P$ is a regular measure, it follows that for every $\Gamma^Q$ negligible set $N \subset X$

$$\Gamma^{Q-P}\big(N \cap f^{-1}(y)\big) = 0 \quad \text{for } \Gamma^P\text{-a.e. } y \in Y. \tag{5}$$

**Theorem 2.1.** *Let $X$ be $\Gamma^Q$ countably finite and let $f : A \longrightarrow Y$ be a Lipschitz mapping, where $A \subset X$. Then for every $\Gamma^Q$-measurable set $G$ of $X$, we have that*

$$Y \ni y \longrightarrow \Gamma^{Q-P}\big(G \cap f^{-1}(y)\big) \quad \textit{is } \Gamma^P \textit{ measurable.} \tag{6}$$

Proof. Let $B$ be a Borel set of $X$. Since $X$ is $\Gamma^Q$ countably finite, and $\Gamma^Q$ is Borel regular, taking into account 2.2.2 of [8] we get a countable union of closed sets $T = \cup_j F_j$ such that $\Gamma^Q(B \setminus T) = 0$. Taking into account Proposition 2.1 and (5), it follows that

$$y \longrightarrow \Gamma^{Q-P}\big(B \cap f^{-1}(y)\big)$$

is $\Gamma^Q$ measurable. Again, by $\Gamma^Q$ countably finiteness of $X$ and Borel regularity of $\Gamma^Q$, 2.2.3 of [8] gives us a countable union of Borel sets $E = \cup_j G_j$, with $\Gamma^Q(G_j) < \infty$ such that $\Gamma^Q(G \setminus E) = 0$. Finally, by (5), we get (6). □

**Definition 2.2** (Coarea measure)**.** Let $X$ be $\Gamma^Q$ countably finite and let $f : A \longrightarrow Y$ be a Lipschitz mapping defined on a closed set of $X$. The *coarea measure* is defined for every $S \subset X$ as

$$\nu_f(S) = \inf\left\{ \int_Y \Gamma^{Q-P}\big(G \cap f^{-1}(y)\big)\, d\Gamma^P(y) :\ G \supset S \text{ is } \Gamma^Q \text{ measurable}\right\}.$$

**Theorem 2.2.** *The coarea measure $\nu_f$ is a Borel regular measure on $X$ such that*

$$\nu_f(G) = \int_Y \Gamma^{Q-P}\big(G \cap f^{-1}(y)\big)\, d\Gamma^P(y) \tag{7}$$

*for any $\Gamma^Q$ measurable set $G$ of $X$. Moreover, any $\Gamma^Q$ measurable set is also $\nu_f$ measurable.*

Proof. Clearly, (7) is straightforward from definition of coarea measure. Taking into account Theorem 2.1, one easily observes that $\nu$ is countably subadditive. Now, we prove that $\Gamma^Q$ measurable sets of $X$ are also $\nu$ measurable, hence in particular $\nu$ is a Borel measure. Let $E$ be a $\Gamma^Q$ measurable set, let $S \subset X$ and let $G \supset S$ be any $\Gamma^Q$ measurable set. We have $\nu(G) = \nu(G \setminus E) + \cap(G \cap E) \geq \nu(S \cap E) + \nu(S \cap E)$, therefore the arbitrary choice of $G \supset S$ and the definition of $\nu$ imply that $E$ is also $\nu$ measurable. To show Borel regularity, we choose any $S \subset X$, hence Lebesgue's theorem and the definition of $\nu$ give us a $\Gamma^Q$-measurable set $G \supset S$ such that $\nu(S) = \nu(G)$. Arguing as in the proof of Theorem 2.1, $\Gamma^Q$ countably finiteness of $X$ gives us a countable union of Borel sets $T = \cup_j B_j$ such that $T \supset G$ and $\Gamma^Q(T \setminus G) = 0$. By (4) and (7) we have $\nu(T \setminus G) = 0$, then $\nu(S) \leq \nu(T) \leq \nu(T \setminus G) + \nu(G) = \nu(G) = \nu(S)$, hence $\nu$ is also Borel regular. □

## 3. A uniform implicit function theorem

We can think of the Heisenberg group as a $2n+1$ dimensional Hilbert space $\mathbb{H}^n$ equipped with orthogonal subspaces $V_1$ and $V_2$, where $\dim(V_1) = 2n$ and $\dim(V_2) = 1$. Notice that this notation is not conventional, since $V_j$ commonly denote the layers of the associated stratified Lie algebra. We have the canonical linear orthogonal projections $\pi_j : \mathbb{H}^n \longrightarrow V_j$ and the components $x_j = \pi_j(x)$, where $x = x_1 + x_2$. The group operation in $\mathbb{H}^n$ is defined as follows

$$x \cdot y = x + y + \omega(x_1, y_1)\,,$$

where $\omega : V_1 \times V_1 \longrightarrow V_2$ is bilinear antisymmetric and non-degenerate, namely, $\omega(x_1, \cdot)$ is non-vanishing for all $x_1 \in V_1 \setminus \{0\}$.

**Remark 3.1.** One should also observe that the Lie algebra of $\mathbb{H}^n$ can be identified with $\mathbb{H}^n$ itself, where $V_1$ and $V_2$ exactly coincide with the canonical layers of the stratified algebra. Moreover, the group operation along with the fixed scalar product on $\mathbb{H}^n$ also yields a left invariant Riemannian metric on $\mathbb{H}^n$ that makes its Lie algebra isometric to $\mathbb{H}^n$ equipped with the fixed scalar product.

The so-called "intrinsic dilations" in $\mathbb{H}^n$ are defined as $\delta_r(x) = rx_1 + r^2 x_2$ and then form a one-parameter group family of group isomorphisms on $\mathbb{H}^n$. A continuous, left invariant distance $d$ such that $d(\delta_r x, \delta_r y) = r d(x,y)$ for all $x, y \in \mathbb{H}^n$ and $r > 0$ is called *homogeneous distance* of $\mathbb{H}^n$. This left invariant metric on $\mathbb{H}^n$ gives the well known *sub-Riemannian distance* $\varrho$ on $\mathbb{H}^n$, see for instance [4]. This is an important example of homogeneous distance. In the sequel, for every homogeneous distance $d$, we will use the abbreviation $d(v) = d(0, v)$. Recall that 0 is the unit element of $\mathbb{H}^n$.

Next we introduce the natural class of "smooth mappings" adapted to the stratified structure of the Heisenberg group. We will follow notation and results of [10, 11], where these mappings have been introduced in arbitrary stratified groups.

**Definition 3.1.** Let $(e_1, \ldots, e_{2n})$ be an orthonormal basis of $V_1$ and consider the left invariant vector fields $(X_1, \ldots, X_{2n})$ of $\mathbb{H}^n$ such that $X_j(0) = e_j$ for $j = 1, \ldots, 2n$.

Let $\Omega$ be open set of $\mathbb{H}^n$. A mapping $f:\Omega\longrightarrow\mathbb{R}^k$ is of class $\mathcal{C}^1$, or it belongs to $\mathcal{C}^1(\Omega,\mathbb{R}^k)$, if the distributional derivatives $X_jf^i$ exist and are continuous in $\Omega$ for all $i=1,\dots,k$ and $j=1,\dots,2n$.

Translating the "stratified mean value theorem" (1.41) of [11] into our notation, we have the estimate

$$|f(x\cdot y)-f(x)|\le C\,d(y)\sup_{d(z)\le b\,d(y),j=1,\dots,2n}|X_jf(xz)|\,. \tag{8}$$

Clearly, introducing the $k\times(2n+1)$ matrix $\nabla_Hf(x)$ of entries $X_jf^i$ for $i=1,\dots,k$ and $j=1,\dots,2n$, whose $(2n+1)$-th column is vanishing, and replacing $f$ with the $y\longrightarrow f(y)-\nabla_Hf(x)(x^{-1}y)$ in (8) we immediately have

$$f(x\cdot y)-f(x)-\nabla_Hf(x)(y)=o\big(d(y)\big)\quad\text{as}\quad d(y)\to0, \tag{9}$$

where $\nabla_Hf(x)$ is identified with a linear mapping through the fixed orthonormal basis $(e_1,\dots,e_{2n+1})$ of $\mathbb{H}^n$. We say that $f$ satisfying (9) is *differentiable* at $x$, hence $f\in\mathcal{C}^1(\Omega,\mathbb{R}^k)$ is everywhere differentiable and $\nabla_Hf(x)$ is continuous. Conversely, it is easy to check that an everywhere differentiable mapping $f$ such that $x\longrightarrow\nabla_Hf(x)$ is continuous belongs to $\mathcal{C}^1(\Omega,\mathbb{R}^k)$. In fact, in particular

$$x\longrightarrow X_jf^i(x)=\lim_{t\to0}\frac{1}{t}\big(f^i(x\cdot e^{tX_j})-f^i(x)\big)$$

and are continuous functions on $\Omega$. Recall that $e^{tX}$ is the unique curve $\gamma$ such that $\gamma(0)=0$ and $\dot\gamma(t)=X(\gamma(t))$, where $X$ is a left invariant vector field. As a consequence, continuity of $x\longrightarrow X_jf^i(x)$ and the obvious formula

$$f^i(x\cdot e^{tX_j})-f^i(x)=\int_0^tX_jf^i(x\cdot e^{sX_j})\,ds$$

imply that $f\in\mathcal{C}^1(\Omega,\mathbb{R}^k)$.

**Remark 3.2.** Although a proof of the previous characterization of $\mathcal{C}^1$ smoothness can be found at Proposition 5.8 of [13], we have decided to add another simple proof that only uses the stratified mean value theorem of the Folland and Stein's book, [11]. This should stress how this theorem immediately imply the pointwise properties of $\mathcal{C}^1$ functions, as soon as the right notion of differentiability (9) is considered. We also recall that this differentiability is a special instance of the more general and well known notion of "Pansu differentiability" that includes group-valued mappings, [27].

**Remark 3.3.** Let us stress that all $C^1$ smooth mappings in the standard sense are also $\mathcal{C}^1$, but the converse does not hold, see for instance Remark 5.9 of [13]. Furthermore, it is possible to construct mappings of $\mathcal{C}^1(\Omega,\mathbb{R})$ that are not even approximately differentiable in the Euclidean sense on a set of positive measure, [20].

**Definition 3.2.** The *horizontal Jacobian* of a differentiable mapping $f:\Omega\longrightarrow\mathbb{R}^k$ at $x\in\Omega$ is the standard Jacobian of the matrix $\big(X_jf^i(x)\big)^{i=1,\dots,k}_{j=1,\dots,2n}$. We denote this number by $J_Hf(x)$. If $V_l=\sum_{j=1}^{2n}c_l^jX_j$ for $l=1,\dots k$ and $c_l=(c_l^i)$ are orthonormal vectors of $\mathbb{R}^{2n}$, we denote by $J_Vf(x)$ the number $|\det\big(V_lf^i(x)\big)|$.

**Remark 3.4.** Notice that $J_H f(x)$ only depends on the group operation and on the fix scalar product on $\mathbb{H}^n$. The restricted Jacobian $J_V f(x)$ only depends on the horizontal subspace $V$ of $\mathrm{span}\{X_1, \dots, X_{2n}\}$.

**Definition 3.3.** Let $x \to F(x)$ be a mapping of metric spaces. The modulus of continuity of $F$ at $\overline{x}$ is the function $]0, +\infty[ \ni t \to \omega_{\overline{x}}(t)$ defined as $\max_{\{y: |\overline{x}-y| \le t\}} d\big(F(\overline{x}), F(y)\big)$. The modulus of continuity of $F$ on a compact set $K$ is set as $]0, +\infty[ \ni t \to \omega_K(t)$, that is defined as $\max_{\{x,y \in K:\, d(x,y) \le t\}} d\big(F(x), F(y)\big)$.

The scalar product of $\mathbb{H}^n$ defines a norm in the linear space of homogeneous linear mappings $\|L\| = \max_{\{x \in V_1: |x|=1\}} |L(x)|$, where $L : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ and the Euclidean norm is fixed on $\mathbb{R}^k$. We will use this norm for the modulus of continuity of the differential $x \to \nabla_H f(x)$. Taking into account that geodesics in $\mathbb{H}^n$ are smooth, see for instance [4], then the following lemma can be easily proved by standard arguments.

**Lemma 3.1.** *Let $\Omega$ be an open set of $\mathbb{H}^n$ and let $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$. Let $K$ and $K_0$ be compact sets of $\Omega$ such that $K \subset K_0$ and any couple of points in $K$ are connected by at least one geodesic contained in $K_0$. Then for all $\delta > 0$, we get*

$$\sup\left\{ \frac{|f(y) - f(x) - \nabla_H f(x)(y-x)|}{\varrho(x,y)} : x, y \in K,\ 0 < \varrho(x,y) \le \delta \right\} \le \omega_{K_0}(\delta)\,,$$

*where $\omega_{K_0}$ is the modulus of continuity of $x \longrightarrow \nabla_H f(x)$ on $K_0$.*

In the sequel, $\Omega$ will always denote an open set of $\mathbb{H}^n$. The closed ball $D^{cc}_{x,r}$ of center $x$ and radius $r$ will refer to the sub-Riemannian distance. Couples of compact sets $K$ and $K_0$ as in the previous lemma are given in the following elementary

**Lemma 3.2.** *Any geodesic connecting two points of $D^{cc}_{x,r}$ is contained in $D^{cc}_{x,2r}$.*

Notice that a more general version of Lemma 3.1 is a consequence of the "stratified mean value theorem" (1.41) of [11], applied to the Heisenberg group.

**Definition 3.4.** We say that a commutative subgroup of $\mathbb{H}^n$ contained in $V_1$ is a *horizontal subgroup*. Every subgroup of $\mathbb{H}^n$ containing $V_2$ that is closed with respect to dilations is a *vertical subgroup*.

**Definition 3.5** (Differentiation with respect to a subgroup)**.** Let $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$ and let $V$ be horizontal subgroup. Then for $x \in \Omega$ the partial differential with respect to $V$ $\partial_V f(x) : V \longrightarrow \mathbb{R}^k$ is defined as

$$\lim_{d(v) \to 0} \frac{f(xv) - f(x) - \partial_V f(x)(v)}{d(v)} = 0\,.$$

**Definition 3.6.** Let $V$ be a horizontal subgroup. If $V'$ is any linear subspace of $V_1$ such that $V_1 = V \oplus V'$, then we define the associated mapping $J : N \times V \longrightarrow \mathbb{H}^n$, $J(u,v) = u \cdot v$, where $N = V' + V_2$. We say that $V$ is *complementary* to $N$ and that $(N, V)$ is a *factorization* of $\mathbb{H}^n$. The *angle between $N$ and $V$* is defined as the length of $n \wedge v$, where $n$ is a unit $(2n+1-k)$-vector of $N$ and $v$ is a unit $k$ vector of $V$. We denote this number by $|N \wedge V|$. If $N$ is the orthogonal complement of $V$, that is also a vertical subgroup, we say that $(N, V)$ is an *orthogonal factorization* of $\mathbb{H}^n$.

The *canonical projections* $\pi_N : \mathbb{H}^n \longrightarrow N$ and $\pi_V : \mathbb{H}^n \longrightarrow V$ with respect to $(N,V)$ are defined by the identity $J^{-1}(x) = \big(\pi_N(x), \pi_V(x)\big)$.

**Remark 3.5.** The mapping $J$ of the previous definition is an analytic diffeomorphism. In fact, let $x \in \mathbb{H}^n$, $x = x_1 + x_2$ and denote by $x_V$ the canonical projection of $x_1$ into $V$ with respect to the direct product $V_1 = V \oplus V'$. Then we have the explicit formula $J^{-1}(x) = \big(x - x_V - \frac{1}{2}\omega(x_1, x_V), x_V\big)$.

**Definition 3.7.** Let $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$, where $1 \le k \le n$. Let $x_0 \in \Omega$ and assume that $\nabla_H f(x_0) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ is surjective. Let $V$ be any $k$-dimensional horizontal subgroup such that $\nabla_H f(x_0)_{|V}$ is invertible. Then for any vertical subgroup $N$ complementary to $V$ we say that $(N,V)$ is a factorization of $\mathbb{H}^n$ that is *adapted* to $\nabla_H f(x_0)$.

**Remark 3.6.** For every vertical subgroup $N$ of $\mathbb{H}^n$ with $\dim(N) \ge n+1$, it is possible to find a horizontal subgroup $V$ that is complementary to $N$, see Lemma 3.26 of [14]. Thus, once $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$ has surjective differential $\nabla_H f(x_0) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$, the kernel $\ker \nabla_H f(x_0)$ is a vertical subgroup of dimension greater than or equal to $n+1$ and we have a complementary horizontal subgroup $V$, where $\nabla_H f(x_0)_{|V}$ is surjective. Furthermore, the linear space $N$ orthogonal to $V$ is also a vertical subgroup, that does not necessarily coincide with the kernel of the differential. This shows that the previous definition is well posed, namely, we can always find an orthogonal factorization adapted to some surjective differential.

**Definition 3.8.** When a homogeneous distance is fixed on $\mathbb{H}^n$, the open and the closed ball of center $x$ and radius $r$ is denoted by $B_{x,r}$ and $D_{x,r}$, respectively. When a horizontal subgroup $V$ is complementary to $N$, we also introduce the closed ball $D^N_{u,r} = D_{u,r} \cap N$ for all $u \in N$ and the closed ball $D^V_{v,r} = D_{v,r} \cap V$ for all $v \in V$. By the mapping $J : N \times V \longrightarrow \mathbb{H}^n$ associated to $N$ and $V$ we introduce a special notation to denote the group products of closed balls $D^N_{u,r} \cdot D^V_{v,r} = J(D^N_{u,r} \times D^V_{v,r})$.

**Theorem 3.1.** *Let $k$ be an integer such that $1 \le k \le n$ and consider $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$. Let $x_0 \in \Omega$, let $\nabla_H f(x_0) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ be surjective and let $(N,V)$ be a factorization of $\mathbb{H}^n$ adapted to $\nabla_H f(x_0)$. Let $(u_0, v_0) \in N \times V$ be the unique element such that $x_0 = J(u_0, v_0)$. Let $R > 0$ be such that $D^{cc}_{x_0,2R} \subset \Omega$ and let $s, r > 0$ be such that the compact set $D^N_{u_0,3s} \cdot D^V_{v_0,3r}$ is contained in $D^{cc}_{x_0,R}$ and $(\partial_V f)(x)$ is invertible for all $x \in D^N_{u_0,3s} \cdot D^V_{v_0,3r}$. We denote by $\omega$ the modulus of continuity of $x \to \nabla_H f(x)$ on the compact $D^{cc}_{x_0,2R}$. Then for all $\bar{x} \in D^N_{u_0,s} \cdot D^V_{v_0,r}$, selecting $0 < \delta_{\bar{x}} < \min\{1, s, r\}$ such that*

$$2\,\omega\big((1+c_0)\delta_{\bar{x}}\big)\,\|\big(\partial_V f(\bar{x})\big)^{-1}\| < 1\,, \tag{10}$$

*where $c_0$ only depends on $R$, $s$ and $\varrho(x_0)$ and introducing the constant*

$$\kappa(\bar{x}) = \min\left\{1, \left(\frac{1 - 2\,\omega\big((1+c_0)\delta_{\bar{x}}\big)\,\|\big(\partial_V f(\bar{x})\big)^{-1}\|}{\|\big(\partial_V f(\bar{x})\big)^{-1}\|\,\mathrm{Lip}_R(f)}\right)^2\right\}, \tag{11}$$

*where $\mathrm{Lip}_R(f)$ denotes the Lipschitz constant of $f$ on $D^{cc}_{x_0,R}$ with respect to $\varrho$, there exists a unique continuous mapping $\varphi^{\bar{x}} : D^N_{\bar{u},\kappa(\bar{x})\delta^2_{\bar{x}}} \longrightarrow D^V_{\bar{v},\delta_{\bar{x}}}$ such that $\varphi^{\bar{x}}(\bar{u}) = \bar{v}$ and* *for all* $u\in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}}$ *there holds*

$$f\big(u\cdot\varphi^{\bar x}(u)\big)=f(\bar x)\,. \tag{12}$$

Proof. Let us fix $(\bar u,\bar v)\in D^N_{u_0,s}\times D^V_{v_0,r}$ and set $\bar x=\bar u\cdot\bar v$. We first define $L_{u,v}=(\partial_V f)(u\cdot v):V\longrightarrow\mathbb{R}^k$, $\bar L=L_{\bar u,\bar v}$ and recall that $L_{u,v}$ is invertible for all $u\cdot v\in D^N_{u_0,3s}\cdot D^V_{v_0,3r}$. We introduce $g(u,v)=f(u\cdot v)-f(\bar u\cdot\bar v)$ and define the mapping

$$\alpha_u(h)=h-\bar T\big(g(u,\bar v\cdot h)\big),$$

where $\bar T=\bar L^{-1}:\mathbb{R}^k\longrightarrow V$ and $v=\bar v\cdot h$. Recall that

$$D^N_{\bar u,s}\cdot D^V_{\bar v,r}\subset D^N_{\bar u_0,s}\cdot D^V_{\bar v_0,r}\subset D^{cc}_{x_0,R},$$

then $\alpha_u$ is well defined for all $h\in D^V_r$ and $u\in D^N_{\bar u,s}$. We observe that the group operation restricted to $V$ coincides with the sum of vector spaces, then

$$\begin{aligned}\alpha_u(h)-\alpha_u(h') &= h-h'+\bar T\big(g(u,\bar v\cdot h')-g(u,\bar v\cdot h)\big)\\ &= h-h'+\bar T\big(L_{u,\bar v\cdot h}(h'-h)+E_{u,\bar v\cdot h}(h,h')\big),\end{aligned}$$

where Lemma 3.1 and Lemma 3.2 imply that

$$|E_{u,\bar v\cdot h}(h,h')|\le\varrho(h,h')\,\omega(\varrho(h,h'))$$

whenever $h,h'\in D^V_r$ and $u\in D^N_{\bar u,s}$. Therefore taking into account the equality

$$\alpha_u(h)-\alpha_u(h')=\bar T\big(L_{u,\bar v\cdot h}-L\big)(h'-h)+\bar T\big(E_{u,\bar v\cdot h}(h,h')\big)\,,$$

for $\delta<r$ and every $h,h'\in D^V_\delta$ and every $u\in D^N_{\bar u,s}$, we have

$$|\alpha_u(h)-\alpha_u(h')|\le\|\bar T\|\Big(\|L_{u,\bar v\cdot h}-\bar L\|+\omega(2\delta)\Big)\;\varrho(h,h')\,.$$

By homogeneity of $\varrho$, there is a geometric constant $c>0$ only depending on both $\varrho$ and the subspaces $N$ and $V$ such that $\varrho(v)\le c\,\varrho(u\cdot v)$ for all $(u,v)\in N\times V$. Then $\varrho(\bar vh)\le c\,\big(R+\varrho(x_0)\big)+1$, where we have chosen $\delta<\min\{1,r\}$. Then there exists $c_0\ge1$ depending on $R$, $s$ and $\varrho(x_0)$ such that

$$\varrho\left((\bar v\cdot h)^{-1}\cdot(\bar u^{-1}\cdot u)\cdot(\bar v\cdot h)\right)\le c_0\;\varrho(\bar u,u)^{1/2}\,. \tag{13}$$

We let $\delta<\min\{1,r,s\}$ and observe that for all $u\in D^N_{\bar u,\delta^2}$, we have

$$\varrho(u\cdot\bar v\cdot h,\bar u\cdot\bar v)\le c_0\;\varrho(\bar u,u)^{1/2}+\delta\le(1+c_0)\,\delta.$$

This proves that

$$|\alpha_u(h)-\alpha_u(h')|\le2\,\|\bar T\|\,\omega\big((1+c_0)\delta\big)\;\varrho(h,h')\,.$$

Now, we make $\delta=\delta_{\bar x}$ smaller, depending on $\bar x$, such that (10) holds and consider the constant $\kappa(\bar x)$ defined in (11). Analogously, for all $u\in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}}$, we obtain

$$|\alpha_u(0)|\le\|\bar T\|\,\mathrm{Lip}_R(f)\,\varrho(u\cdot\bar v,\bar u\cdot\bar v)\le c_0\,\sqrt{\kappa(\bar x)}\,\|\bar T\|\,\mathrm{Lip}_R(f)\;\delta_{\bar x}\,.$$

Thus, taking into account (11), for all $h \in D^V_{\delta_{\bar x}}$, we have achieved

$$\begin{aligned} |\alpha_u(h)| &\leq |\alpha_u(0)| + 2\,\|\bar T\|\,\omega\big((1+c_0)\delta_{\bar x}\big)\,\varrho(h) \\ &\leq c_0\sqrt{\kappa(\bar x)}\|\bar T\|\mathrm{Lip}_R(f)\delta_{\bar x} + 2\|\bar T\|\omega\big((1+c_0)\delta_{\bar x}\big)\delta_{\bar x} \\ &\leq \delta_{\bar x}\,. \end{aligned}$$

It follows that for all $u \in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}}$, the mapping $\alpha_u : D^V_{\delta(\bar x)} \longrightarrow D^V_{\delta(\bar x)}$ is well defined and it is a contraction, hence we get the unique point $\psi^{\bar x}(u) \in D^V_{\delta(\bar x)}$ such that $f\big(u\cdot\bar v\cdot\psi^{\bar x}(u)\big) = f(\bar x)$. Clearly, $\psi^{\bar x}(\bar u) = 0$ and the argument above gives continuity of $\psi^{\bar x}$ at $\bar u$, since we can choose $\delta$ arbitrarily small. Then the translated mapping $\varphi^{\bar x} = \bar v\cdot\psi^{\bar x} : D^N_{\bar u,\kappa\delta^2} \longrightarrow D^V_{\bar v,\delta}$ is continuous at $\bar u$, takes value $\bar v$ at $\bar u$ and it is the unique mapping such that $f\big(u\cdot\varphi^{\bar x}(u)\big) = f(\bar x)$ for all $u \in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}}$. Let us pick any $u' \in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}}$, hence

$$\big(u',\varphi^{\bar x}(u')\big) \in D^N_{\bar u,\kappa(\bar x)\delta^2_{\bar x}} \times D^V_{\bar v,\delta_{\bar x}} \subset D^N_{\bar u,s}\times D^V_{\bar v,r} \subset D^N_{u_0,2s}\times D^V_{v_0,2r}$$

and we can repeat all the previous arguments replacing $(\bar u,\bar v)$ with $\big(u',\varphi^{\bar x}(u')\big)$, since

$$D^N_{u',s}\cdot D^V_{\varphi^{\bar x}(u'),r} \subset D^N_{u_0,3s}\cdot D^V_{v_0,3r} \subset D^{cc}_{x_0,R}\,.$$

This gives in particular the continuity of $\varphi^{\bar x}$ at $u'$ and concludes the proof. □

As a simple consequence of Theorem 3.1, we obtain the main result of this section.

**Theorem 3.2** (Uniform implicit function theorem). *Let $k$ be a positive integer such that $k \leq n$ and consider $f \in \mathcal{C}^1(\Omega,\mathbb{R}^k)$. Let $x_0 \in \Omega$, let $\nabla_H f(x_0) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ be surjective and let $(N,V)$ be a factorization of $\mathbb{H}^n$ adapted to $\nabla_H f(x_0)$. Let $(u_0,v_0) \in N\times V$ be the unique element such that $x_0 = J(u_0,v_0)$. Let $R > 0$ be such that $D^{cc}_{x_0,2R} \subset \Omega$ and let $s,r > 0$ be such that $D^N_{u_0,3s}\cdot D^V_{v_0,3r} \subset D^{cc}_{x_0,R}$ and $(\partial_V f)(x)$ is invertible for all $x \in D^N_{u_0,3s}\cdot D^V_{v_0,3r}$. We denote by $\omega$ the modulus of continuity of $x \to \nabla_H f(x)$ on the compact $D^{cc}_{x_0,2R}$. Let $0 < \delta < \min\{1,s,r\}$ be such that*

$$2\,\omega\big((1+c_0)\delta\big)\max_{x\in D^N_{u_0,s}\cdot D^V_{v_0,r}} \|\big(\partial_V f(x)\big)^{-1}\| < 1\,, \tag{14}$$

*where $c_0$ only depends on $R$, $s$ and $\varrho(x_0)$ and let*

$$\kappa = \min\left\{1, \left(\frac{1-2\,\omega\big((1+c_0)\delta\big)\displaystyle\max_{x\in D^N_{u_0,3s}\cdot D^V_{v_0,3r}}\big\|\big(\partial_V f(x)\big)^{-1}\big\|}{\mathrm{Lip}_R(f)\displaystyle\max_{x\in D^N_{u_0,3s}\cdot D^V_{v_0,3r}}\big\|\big(\partial_V f(x)\big)^{-1}\big\|}\right)^2\right\}, \tag{15}$$

*where $\mathrm{Lip}_R(f)$ denotes the Lipschitz constant of $f$ on $D^{cc}_{x_0,R}$ with respect to $\varrho$. Then for all $(\bar u,\bar v) \in D^N_{u_0,s}\times D^V_{v_0,r}$ there exists a unique mapping $\varphi^{\bar u,\bar v} : D^N_{\bar u,\kappa\delta^2} \longrightarrow D^V_{\bar v,\delta}$ such that it is continuous, $\varphi^{\bar u,\bar v}(\bar u) = \bar v$ and $f\big(u\cdot\varphi^{\bar u,\bar v}(u)\big) = f(\bar u\cdot\bar v)$ for all $u \in D^N_{\bar u,\kappa\delta^2}$.*

## 4. Coarea formula in Heisenberg groups

We first construct the coarea measure in Heisenberg groups, where we rely on notation and terminology of the previous sections.

**Lemma 4.1.** *Let $0 \le k \le 2n+1$ be an integer and let $U$ be a bounded open set of $\mathbb{H}^n$ containing the unit element $0$. Let $\mathcal{O} = \{x \cdot \delta_r U : x \in \mathbb{H}^n,\ r > 0\}$ be the family of translated and dilated copies of $U$. Then the corresponding measure $\Gamma^{2n+2-k}$ with respect to a homogeneous distance is a Carathéodory measure.*

Proof. Let $c_0 > 0$ be such that the closed ball of radius $c_0$ and centered at $0$ is contained in $U$. Let $E \subset \mathbb{H}^n$ and choose any $\varepsilon > 0$, so that

$$\frac{\omega_{2n+2-k}}{2^{2n+2-k}} \sum_{j=0}^{\infty} \operatorname{diam}(F_j)^{2n+2-k} - \varepsilon < \mathcal{H}_t^{2n+2-k}(E)\,,$$

where $\omega_{2n+2-k}$ is the Lebesgue measure of the unit ball in $\mathbb{R}^{2n+2-k}$, for a suitable family of open sets $\{F_j\}_{j\in\mathbb{N}}$ that cover $E$ and such that $\operatorname{diam}(F_j) \le t$ for all $j \in \mathbb{N}$. We choose $\xi_j \in F_j$ and observe that $F_j \subset \xi_j \delta_{o_j/c_0} U$, where $o_j = \operatorname{diam}(F_j)$. Then

$$\begin{aligned}
\Gamma^{2n+2-k}_{t\,\operatorname{diam}(U)/c_0}(E) \;&\le\; \beta_{2n+2-k} \left(\frac{\operatorname{diam}(U)}{c_0}\right)^{2n+2-k} \sum_{j=0}^{\infty} \operatorname{diam}(F_j)^{2n+2-k} \\
&\le\; \frac{2^{2n+2-k}\beta_{2n+2-k}}{\omega_{2n+2-k}} \left(\frac{\operatorname{diam}(U)}{c_0}\right)^{2n+2-k} \big(\mathcal{H}^{2n+2-k}(E) + \varepsilon\big).
\end{aligned}$$

Thus, letting $\varepsilon \to 0^+$ and $t \to 0^+$, we have obtained a metric constant $\theta_{2n+2-k} > 0$ such that $\Gamma^{2n+2-k} \le \theta_{2n+2-k}\,\mathcal{H}^{2n+2-k}$. Taking $\theta_{2n+2-k}$ sufficiently large, then estimate (2) follows. Thus, we have proved that $\Gamma^{2n+2-k}$ is a Carathéodory measure. □

**Definition 4.1.** We denote by $\mathcal{S}^a$ the Carathéodory measure constructed with open $\mathcal{O} = \{B_{x,r} : x \in \mathbb{H}^n,\ r > 0\}$ with respect to a fixed homogeneous distance. This is the well known *spherical Hausdorff measure*.

**Remark 4.1.** Let $0 \le k \le 2n+1$ be an integer. Clearly the spherical Hausdorff measure $\mathcal{S}^{2n+2-k}$ is a Carathéodory measure. Moreover, following Definition 2.1, it is an elementary verification to observe that replacing $\mathcal{O} = \{B_{x,r} : x \in \mathbb{H}^n,\ r > 0\}$ with $\mathcal{F} = \{D_{x,r} : x \in \mathbb{H}^n,\ r > 0\}$ yields the same spherical Hausdorff measure $\mathcal{S}^{2n+2-k}$.

**Remark 4.2.** Let $0 \le k \le 2n+1$ be an integer and consider the Carathéodory measure $\Gamma^{2n+2-k}$ constructed with open balls $\mathcal{O} = \{B_{x,r} : x \in \mathbb{H}^n,\ r > 0\}$ with respect to a fixed homogeneous distance $d$. Since $d$ is left invariant and 1-homogeneous with respect to dilations, arguing as in Proposition 1.10 of [18], it is easy to check that

$$\mathcal{S}^{2n+2-k}(x \cdot \delta_r E) = r^{2n+2-k}\, \mathcal{S}^{2n+2-k}(E)\,, \tag{16}$$

for any $E \subset \mathbb{H}^n$ and all $x \in \mathbb{H}^n$ and $r > 0$.

**Remark 4.3.** In particular, by (16) it follows that $\mathcal{S}^{2n+2}$ is left invariant and scales like the Lebesgue measure $\mathcal{L}^{2n+1}$ on metric balls, hence it is proportional to $\mathcal{L}^{2n+1}$ as Haar measure of the same group $\mathbb{H}^n$. In particular, $\mathcal{S}^{2n+2}$ is countably finite.

The next lemma will use the notion of rescaled set $E_{x,\rho} = \delta_{1/\rho}\big(x^{-1}\cdot E\big)$ and of rescaled function $f_{x,\rho}(\xi) = [f(x\delta_\rho\xi) - f(x)]/\rho$.

**Lemma 4.2** (Rescaling)**.** *Let $A$ be closed in $\mathbb{H}^n$ and let $f : A \longrightarrow \mathbb{R}^k$ be Lipschitz. Then the coarea measure*

$$\nu_f(E) = \int_{\mathbb{R}^k} \mathcal{S}^{2n+2-k}\big(E\cap f^{-1}(y)\big)\, dy\,. \tag{17}$$

*is well defined on sets $E$ of $\mathbb{H}^n$ and satisfies $\nu_f(E) = \rho^{2n+2}\nu_{f_{x,\rho}}(E_{x,\rho})$ for all $\rho>0$.*

Proof. By Remark 4.3, $\mathcal{S}^{2n+2}$ is countably finite, hence Theorem 2.2 ensures that $\nu_f$ defines a Borel regular measure on $\mathbb{H}^n$, whose measurable sets contain $\mathcal{S}^{2n+2}$ measurable sets. The rescaling property of $\nu_f$ immediately follows by a change of variable in the integral (17) and applying formula (16). □

From now on, $\mathcal{H}^p$ will denote the $p$-dimensional Hausdorff measure with respect to the fixed scalar product of $\mathbb{H}^n$.

**Definition 4.2.** Following Definition 6.1 of [22], we say that a homogeneous distance $d$ on $\mathbb{H}^n$ is *symmetric on all layers* if there exists $\sigma : [0,+\infty[^2 \longrightarrow [0,+\infty[$ such that $d(x,0) = \sigma(|x_1|,|x_2|)$, where $|\cdot|$ denotes the norm arising from the fixed scalar product on $\mathbb{H}^n$ and $x = x_1 + x_2$ where $x_i \in V_i$.

One can construct many examples of homogeneous distances that are symmetric on all layers.

**Example 1.** Consider $c \ge 1$ such that $|\omega(x,y)| \le c|x|\,|y|$. It is easy to check that

$$\|x\| = \max\left\{|x_1|, \left(\frac{|x_2|}{c}\right)^{1/2}\right\}$$

defines a homogenous norm that satisfies the triangle inequality $\|x\cdot y\| \le \|x\| + \|y\|$ with respect to the group operation. Clearly, $d(x,y) = \|x^{-1}y\|$ is a homogeneous distance on $\mathbb{H}^n$ that is homogeneous on all layers with $\sigma(t_1,t_2) = \max\{t_1, \sqrt{t_2/c}\}$.

**Example 2.** Let us equip $\mathbb{H}^n$ with a structure of H-type group, where the scalar product induces a norm $|\cdot|$ such that the generalized complex structure $J_z : V_1 \longrightarrow V_1$ satisfies $|J_z(x)| = |z||x|$, $J_z^2 = -|z|^2\mathrm{Id}_{V_1}$ and $V_1\times V_2 \ni (x,z) \longrightarrow J_z(x) \in V_1$ is bilinear. Then defining the group operation setting $\omega(x,y) = [x,y]/2$, the Cygan norm, [5], is defined as follows

$$\|x\| = \sqrt[4]{|x_1|^4 + 16|x_2|^2}$$

and it satisfies the triangle inequality $\|x\cdot y\| \le \|x\| + \|y\|$. This defines a homogeneous distance that is symmetric on all layers, where $\sigma(t_1,t_2) = \sqrt[4]{t_1^4 + 16t_2^2}$.

In view of the previous examples, we denote by $\|\cdot\|_d$ the homogeneous norm arising from a homogeneous distance, namely, $\|x\|_d = d(x,0)$ for all $x \in \mathbb{H}^n$.

**Definition 4.3** (Metric factor)**.** Let $d$ be a homogeneous distance on $\mathbb{H}^n$ and let $N$ be a $p$-dimensional vertical subgroup of $\mathbb{H}^n$. We denote by $D_1$ the closed ball of $\mathbb{H}^n$ with respect to $d$, that is centered at the origin and it has radius equal to one. We define the *metric factor of $d$* as the function $\theta_{p+1}(N) = \mathcal{H}^p(D_1 \cap N)$.

**Definition 4.4.** According to Proposition 6.1 of [22], whenever a homogeneous distance $d$ on $\mathbb{H}^n$ is symmetric on all layers, the metric factor $\theta_{p+1}(N)$ is constantly equal to a geometric constant for all $p$-dimensional vertical subgroups $N$ of $\mathbb{H}^n$. We denote by $\alpha_{p+1}$ this geometric constant and by $\mathcal{S}_d^{p+1}$ the spherical Hausdorff measure corresponding to the Carathéodory measure of Definition 2.1 with $\beta_{p+1} = \alpha_{p+1}/2^{p+1}$.

In the next theorem, we recall the following area-type formula.

**Theorem 4.1** (Franchi, Serapioni and Serra Cassano, [14]). *Let $k$ be a positive integer such that $k \le n$ and let $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$. Let $\bar{x} \in \Omega$, let $\nabla_H f(\bar{x}) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ be surjective and let $(N, V)$ be a factorization of $\mathbb{H}^n$ adapted to $\nabla_H f(\bar{x})$. Let $(\bar{u}, \bar{v}) \in N \times V$ be the unique element such that $\bar{x} = J(\bar{u}, \bar{v})$ and let $s, r > 0$ be such that $D^N_{\bar{u},s} \cdot D^V_{\bar{v},r} \subset \Omega$, $(\partial_V f)(x)$ is invertible for all $x \in D^N_{\bar{u},s} \cdot D^V_{\bar{v},r}$ and there exists a unique continuous mapping $\varphi : D^N_{\bar{u},s} \longrightarrow D^V_{\bar{v},r}$ such that $f\big(u \cdot \varphi(u)\big) = f(\bar{u} \cdot \bar{v})$ for all $u \in D^N_{\bar{u},s}$. Then defining the graph mapping $\Phi(u) = u \cdot \varphi(u)$, we have*

$$\mathcal{S}_d^{2n+2-k}\big(\Phi(A)\big) = |N \wedge V| \int_A \frac{J_H f\big(\Phi(u)\big)}{J_V f\big(\Phi(u)\big)} \, d\mathcal{H}^{2n+1-k}(u)\,, \tag{18}$$

*for all measurable sets $A \subset D^N_{\bar{u},s}$, where $|N \wedge V|$ is introduced in Definition 3.6.*

The coarea factor has been introduced for linear mappings of linear spaces in [2] and then extended to homogeneous homomorphisms of stratified groups in [18].

**Definition 4.5** (Coarea factor). Let $L : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ be a linear mapping that satisfies $L(\delta_r x) = rL(x)$ for all $x \in \mathbb{H}^n$ and $r > 0$. Then the coarea factor is the unique number $\mathbf{C}_k(L)$ such that

$$\mathcal{S}_d^{2n+2}(D_1)\, \mathbf{C}_k(L) = \int_{\mathbb{R}^k} \mathcal{S}_d^{2n+2-k}\big(D_1 \cap L^{-1}(y)\big)\, dy\,. \tag{19}$$

**Remark 4.4.** By classical coarea formula, the previous definition yields

$$\mathbf{C}_k(L) = \frac{\mathcal{S}_d^{2n+2-k}(D_1 \cap N)}{\mathcal{H}^{2n+1-k}(D_1 \cap N)} \frac{\mathcal{L}^{2n+1}(D_1)}{\mathcal{S}_d^{2n+2}(D_1)}\, JL$$

where $JL$ is the Jacobian of $L$ with respect to the scalar product of $\mathbb{H}^n$ and $N$ is the kernel of $L$. On the other hand, by standard covering arguments, the same that show equality between Hausdorff and Lebesgue measure in $\mathbb{R}^n$, see for instance [3], we have $\mathcal{S}_d^{2n+2-k} = \mathcal{H}^{2n+1-k}$. This shows that

$$\mathbf{C}_k(L) = \frac{\mathcal{L}^{2n+1}(D_1)}{\mathcal{S}_d^{2n+2}(D_1)}\, JL\,. \tag{20}$$

In the next theorem, we adapt results of [18] to our setting.

**Theorem 4.2** (Coarea inequality). *Let $A \subset \mathbb{H}^n$ be a measurable set and consider a Lipschitz map $f : A \longrightarrow \mathbb{R}^k$. Then we have*

$$\int_{\mathbb{R}^k} \mathcal{S}_d^{2n+2-k}\left(A \cap f^{-1}(y)\right)\, dy \le \int_A \mathbf{C}_k\big(\nabla_H f(x)\big)\, d\mathcal{S}_d^{2n+2}(x)\,.$$

4.1. **Proof of the coarea formula.** This subsection is entirely devoted to the proof of Theorem 1.1. First of all, by standard arguments, approximating $u$ by increasing step functions, it is not restrictive to prove (1) in the case $u$ equals the characteristic function of a bounded measurable set of $\mathbb{H}^n$. Then we restrict our attention to the case where $f \in \mathcal{C}^1(\Omega, \mathbb{R}^k)$. Let us choose $x_0 \in \Omega$ such that $\nabla_H f(x_0) : \mathbb{H}^n \longrightarrow \mathbb{R}^k$ is surjective and consider

$$\frac{\nu_f(D_{x_0,\rho})}{\rho^{2n+2}} = \int_{\mathbb{R}^k} \mathcal{S}_d^{2n+2-k}\big(D_1 \cap f_{x_0,\rho}^{-1}(y)\big)\, dy\,, \tag{21}$$

due to Lemma 4.2. Let $(N, V)$ be a factorization of $\mathbb{H}^n$ adapted to $\nabla_H f(x_0)$, in the special case where $N$ coincides with the kernel of $\nabla_H f(x_0)$. We have $x_0 = u_0 \cdot v_0$, where $(u_0, v_0) \in N \times V$, hence $(\partial_V f)(x_0) : V \longrightarrow \mathbb{R}^k$ is invertible and it corresponds to the differential of $v \to f(u_0 \cdot v)$ with respect to any norm on $V$. We apply the classical inverse mapping theorem, according to which there exist $\rho_0, \rho_1 > 0$ such that $D^V_{v_0,\rho_0} \ni v \to f(u_0 \cdot v)$ is invertible onto its image. Moreover, this image contains the Euclidean closed ball $D^E_{f(x_0),\rho_1}$ of center $f(x_0)$ and radius $\rho_1$. Let us define $L = \nabla_H f(x_0)$ and choose any $y \in L(D_1)$. We have $\xi \in D_1$ such that $L(\xi) = y$ and $|y| \le \|L\|\,|\xi|$. We set $c(d) = \sup_{0<\|\eta\|_d \le 1} |\eta|\, \|\eta\|_d^{-1}$. Then for all $\rho < \rho_1/\big(c(d)\|L\|\big)$, we obtain a unique $v^{\rho y} \in D^V_{v_0,\rho_0}$ such that

$$f(u_0 \cdot v^{\rho y}) = f(x_0) + \rho y\,. \tag{22}$$

The inverse mapping theorem also implies that $v^{\rho y} \to v_0$ as $\rho y \to 0$, hence the previous convergence is uniform with respect to $|y| \le \|L\|\, c(d)$ as $\rho \to 0^+$. We are now in the position to apply Theorem 3.2. Let $s, r, R, \kappa, \delta > 0$ be as in this theorem. Then for all $\rho < \min\{\rho_0, r, \rho_1\big(c(d)\|L\|\big)^{-1}\}$ we have a unique $v^{\rho y} \in D^V_{v_0,\rho_0}$ satisfying (22) and we have $\varphi^{\rho y} : D^N_{u_0,\kappa\delta^2} \longrightarrow D^V_{v^{\rho y},\delta}$ continuous that satisfies the conditions $\varphi^{\rho y}(u_0) = v^{\rho y}$ and

$$f\big(u \cdot \varphi^{\rho y}(u)\big) = f(x_0) + \rho y$$

for every $u \in D^N_{u_0,\kappa\delta^2}$. The convergence $v^{\rho y} \to v_0$ implies that

$$D^N_{u_0,\kappa\delta^2} \cdot D^V_{v_0,\delta/2} \subset D^N_{u_0,\kappa\delta^2} \cdot D^V_{v^{\rho y},\delta}$$

for $\rho$ sufficiently small and all $|y| \le \|L\|\, c(d)$. We wish to emphasize here the independence of $\delta$ from $\rho y$, as a consequence of the "uniform implicit function theorem". It follows that

$$f^{-1}\big(f(x_0) + \rho y\big) \cap D^N_{u_0,\kappa\delta^2} \cdot D^H_{v_0,\delta/2} = \Phi^{\rho y}(D^N_{u_0,\kappa\delta^2}) \cap D^N_{u_0,\kappa\delta^2} \cdot D^H_{v_0,\delta/2}\,,$$

where we have introduced the graph function $\Phi^{\rho y} : D^N_{u_0,\kappa\delta^2} \longrightarrow D^N_{u_0,\kappa\delta^2} \cdot D^V_{v^{\rho y},\delta}$, $\Phi^{\rho y}(u) = u \cdot \varphi^{\rho y}(u)$. There exists $\sigma > 0$ such that $D_{x_0,\sigma} \subset D^N_{u_0,\kappa\delta^2} \cdot D^V_{v_0,\delta/2}$, therefore $\delta_{1/\rho}\left(x_0^{-1} f^{-1}\big(f(x_0) + \rho y\big)\right) \cap D_{\sigma/\rho} = \delta_{1/\rho}\left(x_0^{-1} \cdot \Phi^{\rho y}(D^N_{u_0,\kappa\delta^2})\right) \cap D_{\sigma/\rho}$. Clearly, for $\rho > 0$ small, less than $\sigma$, we achieve

$$D_1 \cap \delta_{1/\rho}\left(x_0^{-1} f^{-1}\big(f(x_0) + \rho y\big)\right) = D_1 \cap f^{-1}_{x_0,\rho}(y) = D_1 \cap \delta_{1/\rho}\left(x_0^{-1} \cdot \Phi^{\rho y}(D^N_{u_0,\kappa\delta^2})\right)\,.$$

Now, we define the subset

$$E_{u_0,\rho} = \{u \in D^N_{u_0,\kappa\delta^2} : d(x_0, \Phi^{\rho y}(u)) \le \rho\} \subset D^N_{u_0,\kappa\delta^2}\,, \tag{23}$$

hence for $\rho>0$ small we have proved that

$$\delta_{1/\rho}\left(x_0^{-1}\cdot\Phi^{\rho y}(E_{u_0,\rho})\right)=D_1\cap f_{x_0,\rho}^{-1}(y) \tag{24}$$

for every $y\in L(D_1)$. Formula (24) is the key point to represent the $\mathcal{S}_d^{2n+2-k}$ in integral form and then to compute its limit as $\rho\to 0^+$. In fact, due to (16) and applying the area type formula (18), we get

$$\mathcal{S}_d^{2n+2-k}\big(D_1\cap f_{x_0,\rho}^{-1}(y)\big)=\frac{1}{\rho^{2n+2-k}}\,|N\wedge V|\int_{E_{u_0,\rho}}\frac{J_Hf\big(\Phi^{\rho y}(u)\big)}{J_Vf\big(\Phi^{\rho y}(u)\big)}\,d\mathcal{H}^{2n+1-k}(u)\,. \tag{25}$$

Next, we perform some suitable changes of variable on $E_{u_0,\rho}$ that will lead to a sort of "uniform differentiability" of the family $\{\Phi^{\rho y}\}_{\rho y}$, as we will see below. We first define

$$x_0^{-1}\cdot\Phi^{\rho y}(u)=w\cdot\varphi^{x_0,\rho y}(w),$$

where $w=x_0^{-1}\cdot u\cdot v_0$ and $\varphi^{x_0,\rho y}(w)=\varphi^{\rho y}(x_0\cdot w\cdot v_0^{-1})-v_0$. We define $\tau_{x_0^{-1}}(u)=x_0^{-1}\cdot u\cdot v_0$ and notice that $\tau_{x_0^{-1}}:N\longrightarrow N$. Since $\tau_{x_0^{-1}}$ preserves $\mathcal{H}^{2n+1-k}$ defined on $N$, we have

$$\int_{E_{u_0,\rho}}\frac{J_Hf\big(\Phi^{\rho y}(u)\big)}{J_Vf\big(\Phi^{\rho y}(u)\big)}\,d\mathcal{H}^{2n+1-k}(u)=\int_{\tau_{x_0^{-1}}(E_{u_0,\rho})}\frac{J_Hf\big(\Phi^{\rho y}(\tau_{x_0}(w))\big)}{J_Vf\big(\Phi^{\rho y}(\tau_{x_0}(w))\big)}\,d\mathcal{H}^{2n+1-k}(w)\,,$$

where $\tau_{x_0}=(\tau_{x_0^{-1}})^{-1}$. We observe that $(\Phi^{\rho y}\circ\tau_{x_0})(w)=x_0\cdot w\cdot\varphi^{x_0,\rho y}(w)$, then

$$\tau_{x_0^{-1}}(E_{u_0,\rho})=\left\{w\in c_{v_0^{-1}}(D^N_{\kappa\delta^2}):\|w\cdot\varphi^{x_0,\rho y}(w)\|_d\le\rho\right\}\,,$$

where $c_x:N\to N$ is defined as $c_x(n)=x\cdot n\cdot x^{-1}$. We define the rescaled set $F_{u_0,\rho}=\delta_{1/\rho}\big(\tau_{x_0^{-1}}(E_{u_0,\rho})\big)$, so that we have

$$F_{u_0,\rho}=\left\{w\in\delta_{1/\rho}\big(c_{v_0^{-1}}(D_{\kappa\delta^2})\cap N\big):\|w\cdot\delta_{1/\rho}(\varphi^{x_0,\rho y}(\delta_\rho w))\|_d\le 1\right\}$$

and then

$$\int_{E_{u_0,\rho}}\frac{J_Hf\big(\Phi^{\rho y}(u)\big)}{J_Vf\big(\Phi^{\rho y}(u)\big)}\,d\mathcal{H}^{2n+1-k}(u)=\rho^{2n+2-k}\int_{F_{u_0,\rho}}\frac{J_Hf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}{J_Vf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}d\mathcal{H}^{2n+1-k}(w)\,,$$

where we have set $\Phi^{x_0,\rho v}(w)=w\cdot\varphi^{\rho v}(w)$. Taking into account (25), we get

$$\mathcal{S}_d^{2n+2-k}\big(D_1\cap f_{x_0,\rho}^{-1}(y)\big)=|N\wedge V|\int_{F_{u_0,\rho}}\frac{J_Hf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}{J_Vf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}\,d\mathcal{H}^{2n+1-k}(w)\,. \tag{26}$$

Now, we wish to study the limit of $\delta_{1/\rho}(\varphi^{x_0,\rho y}(\delta_\rho w))$ as $\rho\to 0^+$. Let us point out that that this limit amounts the above mentioned "uniform differentiability", since we are considering the family of functions $\{\varphi^{x_0,\rho y}\}_{\rho y}$ that varies with $\rho y$. We set $L_0=L_{|V}=\partial_Vf(x_0):V\longrightarrow\mathbb{R}^k$ and $T=L_0^{-1}$, hence taking into account Lemma 3.1

$$\begin{aligned}|\rho y-L_0(\varphi^{x_0,\rho y}(w))| &= |f(x_0\cdot\Phi^{x_0,\rho y}(w))-f(x_0)-L_0(\varphi^{x_0,\rho y}(w))|\\ &\le \omega(\|\Phi^{x_0,\rho y}(w)\|_\varrho)\,\|\Phi^{x_0,\rho y}(w)\|_\varrho\,,\end{aligned}$$

since $w \in c_{v_0^{-1}}(D^N_{\kappa\delta^2})$ belongs in particular to the kernel of $L$ and $x_0 \cdot \Phi^{x_0,\rho y}(w) \in D^{cc}_{x_0,R}$. Recall that $\varrho$ is the sub-Riemannian distance and $\omega$ is the modulus of continuity of $\nabla_H f$ on $D^{cc}_{x_0,2R}$, according to Theorem 3.2. This in turn implies that

$$|\varphi^{x_0,\rho y}(w) - \rho\, T(y)| \le \|T\|\, \omega(\|\Phi^{x_0,\rho y}(w)\|_\varrho)\, \|\Phi^{x_0,\rho y}(w)\|_\varrho\,. \tag{27}$$

Now, we wish to make $\|\Phi^{x_0,\rho y}(w)\|_\varrho$ uniformly small in $\rho y$ and $w$. To do this, we observe that a uniform Hölder continuity for $w \to \Phi^{x_0,\rho y}(w)$ holds with respect to $\rho y$. Let $w, w' \in c_{v_0^{-1}}(D^N_{\kappa\delta^2})$ and recall that $c_{v_0^{-1}}(D^N_{\kappa\delta^2})$ is contained in the kernel of $L$. Then the equalities

$$\begin{aligned}
&L_0(\varphi^{x_0,\rho y}(w)^{-1} \cdot \varphi^{x_0,\rho y}(w')) \\
&= L(\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w')) \\
&= f\big(x_0 \cdot \Phi^{x_0,\rho y}(w')\big) - f\big(x_0 \cdot \Phi^{x_0,\rho y}(w)\big) - L(\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w'))\,,
\end{aligned}$$

along with the estimate

$$\begin{aligned}
&|f\big(x_0 \cdot \Phi^{x_0,\rho y}(w')\big) - f\big(x_0 \cdot \Phi^{x_0,\rho y}(w)\big) - L(\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w'))| \\
&\le 2\, \|\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w')\|_\varrho\, \omega(\|\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w')\|_\varrho)
\end{aligned}$$

lead us to

$$\begin{aligned}
&|\varphi^{x_0,\rho y}(w)^{-1} \cdot \varphi^{x_0,\rho y}(w')| = |\varphi^{x_0,\rho y}(w') - \varphi^{x_0,\rho y}(w)| \\
&\le 2\, \|T\|\, \|\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w')\|_\varrho\, \omega(\|\Phi^{x_0,\rho y}(w)^{-1} \cdot \Phi^{x_0,\rho y}(w')\|_\varrho)\,,
\end{aligned}$$

where we have used the fact that $\varrho(v, v') = |v - v'|$ for all $v, v' \in V$. Since $\Phi^{x_0,\rho y}(w)$ belongs in particular to $D^{cc}_R$ for all $w \in c_{v_0^{-1}}(D^N_{\kappa\delta^2})$, we are allowed to assume that $R > 0$ sufficiently small such that

$$4\, \|T\|\, \omega(t) < 1 \quad \text{for all} \quad 0 < t < 2R.$$

It follows that

$$|\varphi^{x_0,\rho y}(w') - \varphi^{x_0,\rho y}(w)| \le C\, \|w^{-1} w'\|_\varrho^{1/2}\,, \tag{28}$$

for all $w, w' \in c_{v_0^{-1}}(D^N_{\kappa\delta^2})$, where $C > 0$ is such that

$$\|v w^{-1} w' v\|_\varrho \le C\, \|w^{-1} w'\|_\varrho^{1/2}$$

for all $v \in D^V_\delta$ and $w \in c_{v_0^{-1}}(D^N_{\kappa\delta^2})$. We now fix any compact set $K$ of $\mathbb{H}^n$ and replace $w$ with $\delta_\rho w$ in (27), where $w$ varies in $K$. Then

$$\begin{aligned}
|\varphi^{x_0,\rho y}(\delta_\rho w)| &\le |\varphi^{x_0,\rho y}(\delta_\rho w) - \varphi^{x_0,\rho y}(0)| + |\varphi^{x_0,\rho y}(0)| \\
&\le C\, \rho^{1/2}\, \|w\|_\varrho^{1/2} + |v^{\rho y} - v_0|\,.
\end{aligned}$$

It follows that $\Phi^{x_0,\rho y}(\delta_\rho w)$ goes to zero as $\rho \to 0^+$, uniformly in $w \in K$ and $y \in L(D_1)$. The estimate (27) gives

$$\frac{|\varphi^{x_0,\rho y}(\delta_\rho w) - \rho\, T(y)|}{\rho} \le 2\, \|T\|\, \omega\big(\Phi^{x_0,\rho y}(\delta_\rho w)\big)\, (\|w\|_\varrho + |T(y)|)\,.$$

The following "uniform differentiability" follows

$$\sup_{w \in K,\, y \in L(D_1)} \left| \frac{\varphi^{x_0,\rho y}(\delta_\rho w)}{\rho} - T(y) \right| \longrightarrow 0 \quad \text{as} \quad \rho \to 0^+. \tag{29}$$

This limit is the key for the final steps of the proof. For technical reasons, we have to introduce

$$F^{\lambda}_{u_0,\rho}=\Big\{w\in\delta_{1/\rho}\big(c_{v_0^{-1}}(D_{\kappa\delta^2})\cap N\big):\|w\cdot\delta_{1/\rho}(\varphi^{x_0,\rho y}(\delta_\rho w))\|_d\le\lambda\Big\}\,,$$

then we observe that

$$c_1(\|w\|+\|\delta_{1/\rho}(\varphi^{x_0,\rho y}(\delta_\rho w))\|_d)\le\|w\cdot\delta_{1/\rho}(\varphi^{x_0,\rho y}(\delta_\rho w))\|_d\le\lambda$$

for some constant $c_1>0$ implies that $F_{u_0,\rho}\subset D^N_{\lambda/c_1}$. Let us consider

$$S^{\lambda}_N(y)=\{w\in N:w\cdot T(y)\in D_\lambda\}.$$

We pick $\lambda'<1$ such that

$$\mathcal{H}^{2n+1-k}_{|\cdot|}(\partial S^{\lambda'}_N)=\mathcal{H}^{2n+1-k}_{|\cdot|}(\{w\in N:\|w\cdot T(y)\|_d=\lambda'\})=0\,. \tag{30}$$

The conjunction of the previous equalities and (29) immediately yields

$$\mathbf{1}_{F^{\lambda'}_{u_0,\rho}}(w)\to\mathbf{1}_{S^{\lambda'}_N(y)}(w)$$

for $\mathcal{H}^{2n+1-k}_{|\cdot|}$-a.e. $w\in N$ as $\rho\to0^+$. It follows that

$$\begin{aligned}&\liminf_{\rho\to0^+}\int_{F_{u_0,\rho}}\frac{J_Hf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}{J_Vf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}\,d\mathcal{H}^{2n+1-k}(w)\\&\ge\liminf_{\rho\to0^+}\int_{F^{\lambda'}_{u_0,\rho}}\frac{J_Hf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}{J_Vf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}\,d\mathcal{H}^{2n+1-k}(w)\\&=\int_{S^{\lambda'}_N(y)}\frac{J_Hf(x_0)}{J_Vf(x_0)}\,d\mathcal{H}^{2n+1-k}(w)\,.\end{aligned}$$

Since we can choose $\lambda'<1$ satisfying (30) that is arbitrarily close to one, we get

$$\liminf_{\rho\to0^+}\int_{F_{u_0,\rho}}\frac{J_Hf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}{J_Vf\big(x_0\cdot\Phi^{x_0,\rho y}(\delta_\rho w)\big)}\,d\mathcal{H}^{2n+1-k}(w)\ge\int_{A^1_N(y))}\frac{J_Hf(x_0)}{J_Vf(x_0)}\,d\mathcal{H}^{2n+1-k}(w)\,.$$

where $A^1_N(y)=\{w\in N:\|w\cdot T(y)\|_d<1\}$. Taking into account (26) and the classical coarea formula applied to the canonical projection $\pi_V:\mathbb{H}^n\longrightarrow V$, we have

$$\liminf_{\rho\to0^+}\int_{L(D_1)}\mathcal{S}^{2n+2}_d\big(D_1\cap f^{-1}_{x_0,\rho}(y)\big)\,dy\ge\frac{|N\wedge V|\,J_Hf(x_0)}{J_Vf(x_0)}\int_{L(D_1)}\mathcal{H}^{2n+1-k}\big(S^1_N(y)\big)\,dy\,.$$

We define $\Phi^y_0:N\longrightarrow\mathbb{H}^n$ as $\Phi^y_0(n)=n\cdot T(y)$ and observe that

$$D_1\cap L^{-1}(y)=\{\Phi^y_0(n):n\in N\}\cap D_1=\Phi^y_0\big(S^1_N(y)\big).$$

Then the area-type formula (18) gives

$$\liminf_{\rho\to0^+}\int_{L(D_1)}\mathcal{S}^{2n+2-k}_d\big(D_1\cap f^{-1}_{x_0,\rho}(y)\big)\,dy\ge\int_{L(D_1)}\mathcal{S}^{2n+2-k}_d\big(D_1\cap L^{-1}(y)\big)\,dy\,. \tag{31}$$

Due to (21), we have proved that

$$\liminf_{\rho\to0^+}\frac{\nu_f(D_{x_0,\rho})}{\rho^{2n+2}}\ge\int_{\mathbb{R}^k}\mathcal{S}^{2n+2-k}_d\big(D_1\cap L^{-1}(y)\big)\,dy\,.$$

Taking into account Definition 4.5, we have the lower density estimate

$$\liminf_{\rho\to 0^+}\frac{\nu_f(D_{x_0,\rho})}{\mathcal{S}_d^{2n+2}(D_{x_0,\rho})}\geq \mathbf{C}_k\big(\nabla_H f(x_0)\big)\,. \tag{32}$$

Approximating by an increasing sequence of measurable step functions converging to $x\longrightarrow \mathbf{C}_k\big(\nabla_H f(x)\big)$, the conjunction of (32) and both 2.9.4 and 2.9.2 of [8] gives

$$\int_{\mathbb{R}^k}\mathcal{S}_d^{2n+2-k}\big(G_0\cap f^{-1}(y)\big)\;dy\geq \int_{G_0}\mathbf{C}_k\big(\nabla_H f(x)\big)\;d\mathcal{S}_d^{2n+2}(x)\,,$$

where $G_0=\{x\in\Omega:\nabla_H f(x)\,\text{is surjective}\}$. The weak Sard type theorem, [18], is an immediate of Theorem 4.2, namely, we have

$$\int_{\mathbb{R}^k}\mathcal{S}_d^{2n+2-k}\big((\Omega\setminus G_0)\cap f^{-1}(y)\big)\;dy=0\,.$$

Furthermore, Theorem 4.2 also provides the opposite inequality, hence

$$\int_{\mathbb{R}^k}\mathcal{S}_d^{2n+2-k}\big(\Omega\cap f^{-1}(y)\big)\;dy=\int_{\Omega}\mathbf{C}_k\big(\nabla_H f(x)\big)\;d\mathcal{S}_d^{2n+2}(x)\,.$$

Taking into account (20) and observing that the classical Jacobian $J\nabla_H f(x)$ coincides with $J_H f(x)$ we get

$$\int_{\mathbb{R}^k}\mathcal{S}_d^{2n+2-k}\big(\Omega\cap f^{-1}(y)\big)\;dy=\int_{\Omega}J_H f(x)\;dx\,.$$

Outer approximation of measurable sets by open sets extends the previous formula to all measurable sets. Finally, in the case $f:\mathbb{H}^n\longrightarrow\mathbb{R}^k$ is Lipschitz, taking into account the validity of a Whitney type theorem, see Theorem 6.8 of [13], then the proof follows by standard arguments, see for instance Theorem 3.5 of [20]. □

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