# Rigorous interval computations of hyperbolic tetrahedral shapes

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# Outline:

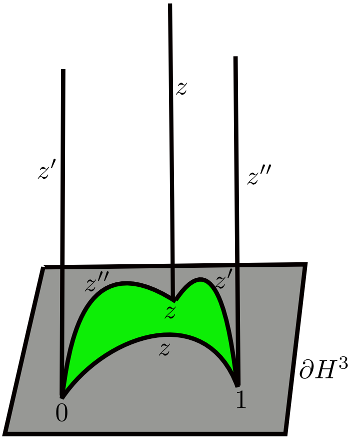
- Hyperbolic gluing equations
- Interval arithmetic
- Two ways to verify Newton's method
- Applications to 3-manifolds

# Main Question

How can we use a computer to rigorously verify a hyperbolic structure on a 3-manifold?

# Hyperbolic ideal tetrahedra

Up to similarity, each ideal tetrahedron in $H^3$ can be parametrized by a complex number.




Here, $z' = \frac{z-1}{z}$ and $z'' = \frac{1}{1-z}$.

# Gluing equations

We want to show that there is a solution the hyperbolic gluing equations.

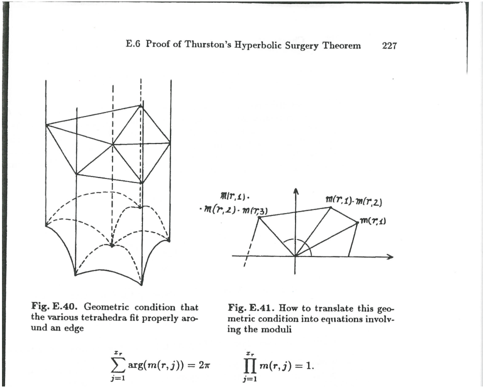





**Fig. E.40.** Geometric condition that the various tetrahedra fit properly around an edge

**Fig. E.41.** How to translate this geometric condition into equations involving the moduli

$$\sum_{j=1}^{z_r} \arg(m(r,j)) = 2\pi \qquad \prod_{j=1}^{z_r} m(r,j) = 1.$$

Figure: Benedetti and Petronio: Lectures on Hyperbolic Geometry page 227

# Three types of gluing equations

These equations are implemented and solved using software like Snappy (based on the Snappea Kernel). For a manifold $M$ with $n$ tetrahedra, $m$ unfilled cusps and $c$ filled cusps, we have:

Edge equations (n of these):

$$\sum_{j=1}^{n}(a_{j,k}\log(z_j) + b_{j,k}\log(\frac{1}{1-z_j}) + c_{j,k}\log(\frac{z_j - 1}{z_j})) = 0 + 2\pi i$$

Cusp equations (2 m of these):

$$\sum_{j=1}^{n}(a_{j,k}\log(z_j) + b_{j,k}\log(\frac{1}{1-z_j}) + c_{j,k}\log(\frac{z_j - 1}{z_j})) = 0 + 0\pi i$$

Dehn surgery equations (c of these):

$$\sum_{j=1}^{n}(a_{j,k}\log(z_j) + b_{j,k}\log(\frac{1}{1-z_j}) + c_{j,k}\log(\frac{z_j - 1}{z_j})) = 0 + 2\pi i$$

Note: here $arg(z) \in (-\pi, \pi]$ and valid solutions must have $arg(z_j) > 0$ for all $j$.

# Dehn Surgery Equations

If a manifold $M$ is obtained from a $p/q$ filling the cusp of $M'$, then we need to solve:

$$p\sum_{j=1}^{n}(a_{j,k}\log(z_k)+b_{j,k}\log(\frac{1}{1-z_j})+c_{j,k}\log(\frac{z_j-1}{z_j}))+$$

$$q\sum_{j=1}^{n}(a_{j,k+1}\log(z_k)+b_{j,k+1}\log(\frac{1}{1-z_j})+c_{j,k+1}\log(\frac{z_{j+1}-1}{z_{j+1}}))=0+2\pi i$$

Here: equation $k$ corresponds to the meridian equation for this cusp and $k+1$ the longitude equation.

# Restated goal

Goal: 1) Show that an approximated solution to the gluing equations is in a small neighborhood of an actual solution.

2) Precisely define small.

# Interval Arithmetic

Basic Idea: A computer can not easily deal with $\log 3$ as an exact number, but it can compute

$$1.098612 \approx \log 3 \in [1.09765625, 1.1015625].$$

Note: in binary that interval is [1.00011001,1.00011010].
We say $x \in [x]$ and $[x] = [\underline{x}, \bar{x}]$.

# Operations in interval arithmetic

- $+ : [a] + [b] = [\underline{a} + \underline{b} - \epsilon, \bar{a} + \bar{b} + \epsilon']$.
- $- : [a] - [b] = [\underline{a} - \bar{b} - \epsilon, \bar{a} - \underline{b} + \epsilon']$.
- $\cdot : [a] \cdot [b] = [min(a_i \cdot b_j) - \epsilon, max(a_i \cdot b_j) + \epsilon']$.
- $\div : [a] \div [b] = [min(\frac{a_i}{b_j}) - \epsilon, max(\frac{a_i}{b_j}) + \epsilon']$ if $0 \notin [\underline{b}, \bar{b}]$

Here $a_i \in \{\underline{a}, \bar{a}\}$ and $b_j \in \{\underline{b}, \bar{b}\}$ and $\epsilon$, $\epsilon'$ account for a processor's rounding error.

# Rounding for basic arithmetic

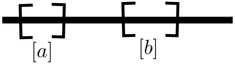
Let $O : \mathbb{R} \to \mathbb{F}$ be a rounding function set be the computer language one uses.
There are four types of rounding $O(x) = x_{approx}$

- $+\infty$ : overestimates the number $O(x) = \lceil x \rceil$
- $-\infty$ : underestimates the number $O(x) = \lfloor x \rfloor$
- nearest : $|x - x_{approx}|$ is smallest
- chopping: $O(x) = \lceil x \rceil$ for $x < 0$ and $\lfloor x \rfloor$ for $x > 0$

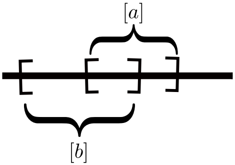
Note: chopping gives best allocation of memory and yields memory overflow errors least often.

# Inequalities

$>, <$ can return *TRUE*, *FALSE*, and *UNDETERMINED*.
Examples:

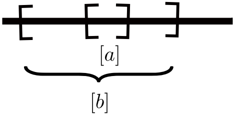



$[a] < [b]$ and $[b] > [a]$




$[a] \not< [b]$ and $[a] \not> [b]$

# Interval inclusion




$$[a] \subset [b]$$

# Approximating functions

To get values of functions we need to give the computer a Taylor (or Maclaurin) series and error bound. For example our program uses log, $\sqrt{}$, $e$ and arctan ($D$ is the domain of convergence).

$$f([z]) = \sum_{k=1}^{n} \frac{f^k([z])}{k!} \text{ for } [z] \subset D$$

and error

$$|E_k([z])| < \frac{M \cdot |[z]|^{k+1}}{(k+1)!}, |f^{k+1}([z])| < M$$

# Exact arithmetic

Given a number field $k$ such that $[k : \mathbb{Q}] = d$, we can represent $k$ as a $d$-dimensional vector space over $\mathbb{Q}$. Here, $\mathbb{Q}[x]/(x^d = a_{d-1}x^{d-1} + \ldots + a_0)$ $a_i \in \mathbb{Q}$.

Recording $\zeta \in \mathbb{Q}$ as $(\zeta_1, \zeta_2, \ldots, \zeta_d)$, $\zeta_i \in \mathbb{Z} \times \mathbb{Z}$. Given enough precision for $\zeta$, it is possible to verify an exact solution to the gluing equations with enough memory (and luck).

Used by snap.

# Advantages of interval arithmetic

- Fast (especially compared to exact arithmetic)
- Uses less memory than exact arithmetic
- Relatively easy to program
- Overwrites the $+.-, \cdot, \div$ functions
- Extends to functions naturally.
- Keeps track of accumulated error by itself
- There are rigorous verification test of Newton's method for interval arithmetic

# Krawczyk Test (statement due to Rump(1983))

**Theorem**

*Given a continuously differentiable $f : D \to \mathbb{R}^{2n}$, $\tilde{x} \in \mathbb{R}^{2n}$, $X = [x_1] \times [x_2] \times ...[x_{2n}]$ with $\vec{0} \in X$ and $\tilde{x} + X \subseteq D$, and $R \in \mathbb{R}^{2n} \times \mathbb{R}^{2n}$. Suppose*

$$S(X, \tilde{x}) = -Rf(\tilde{x}) + \{I - RJ_f(\tilde{x} + X)\}X \subset Int(X).$$

*Then $R$ and all matrices $M \in J_f(\tilde{x} + X)$ are non-singular and there is a unique root $\hat{x}$ of $f$ in $\tilde{x} + S(X, \tilde{x})$.*

Note: here $R \approx J_f^{-1}$.

# Krawczyk Test vs. Kantorovich test

The Krawczyk test requires fewer computations. Consequently, it is faster and requires less memory than the Kantorovich test.

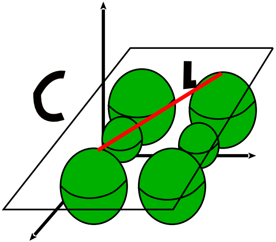
# Implementation

These methods have been used to verify:
the $\leq 5, 6, 7, 8$ tetrahedral censuses (5,6,7 due to Callahan, Hildebrand, and Weeks, 8 due to Thistlethwaite)
and
all but 439 of the manifolds in the closed census (Hodgson and Weeks).

# Implemented in

- MATLAB - big function library, requires a license
- c++ - fast, accurate and free, but hard to use
- python - not fast, not as accurate, but free and easy

# Parabolic length




For $M = \mathbb{H}^3/\Gamma$ with one cusp, we measure the length of a parabolic $p \in \Gamma$ that fixes $\infty$, by measuring it's displacement in the boundary of a maximal horoball.

In general, the length of a parabolic will be the length of its conjugate that fixes $\infty$.

# 6 - Theorem

**Theorem (Agol,Lackenby + Perelman)**

*Let $M$ be a 1-cusped hyperbolic manifold. If $\gamma$ is a parabolic element of length $\geq 6$, then $M(\gamma)$ is hyperbolic.*

Given a solution to the gluing equations, length is a function of the $z_i$.

# Martelli, Petronio, Roukema + $\epsilon$

Martelli, Petronio and Roukema (2012) created an algorithm that used approximate tetrahedral shapes to measure peripheral length.
With $\epsilon$ more work, this can be promoted to a rigorous computation.
This was used to verify 429 of the outstanding 439 closed manifolds have hyperbolic structures.
And is currently being used to identify,

# Currently rigorously Snappy

- Fundamental group presentation, homology, etc.
- *M.is_isometric_to*(*N*) returns TRUE only if a simplicial map is found.

# Further work

- Incorporate into Snappy
- Use these methods to verify topological invariants like: volume, tilt parameter, length spectrum.

Thank you!