Hyperbolic Dehn filling in dimension four

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(joint work with Stefano Riolo)

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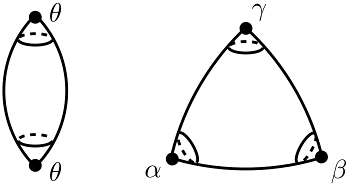

Natural stratification:

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for some *prime* B (does not decompose as $B = S^0 * C$).

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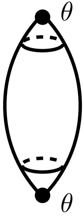
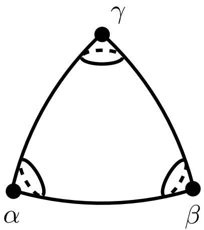
Every connected $(n-2)$-stratum has $N = S^{n-3} * C_\alpha$ and we say that it has *cone angle* α.

Some spherical cone-surfaces:



$S^0 * C_\theta$



$S^2(\alpha, \beta, \gamma)$

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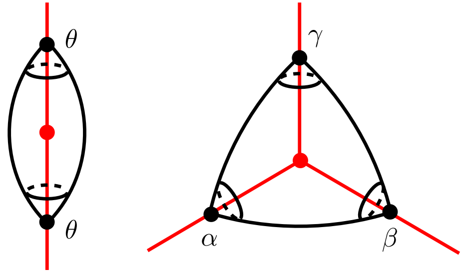

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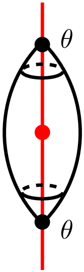

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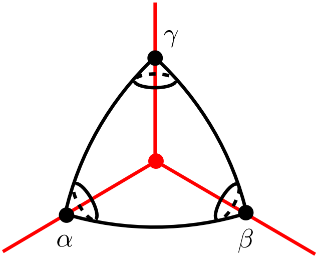
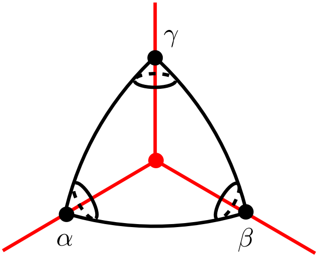


$S^2(\alpha, \beta, \gamma)$

Suppose cone angles are $< 2\pi$.

In a locally orientable cone 3-manifold, the underlying space is a 3-manifold and the singular locus Σ is a 1-complex.

Some spherical cone-surfaces:



$S^0 * C_\theta$ $\qquad$ $S^2(\alpha, \beta, \gamma)$

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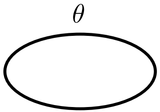
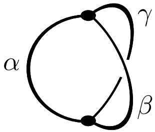
If cone angles are $\leq \pi$, vertices have valence 3.

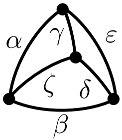
Some spherical cone 3-manifolds:

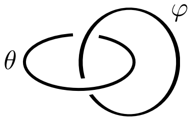


$S^1 * C_\theta$



$S^0 * S^2(\alpha, \beta, \gamma)$

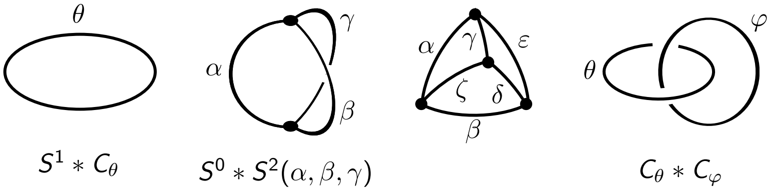




$C_\theta * C_\varphi$

The underlying space here is S^3.

Some spherical cone 3-manifolds:



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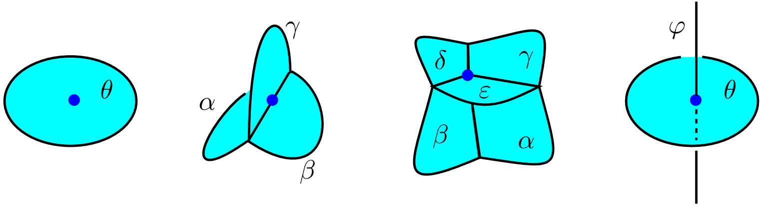
The corresponding strata in a cone 4-manifold:



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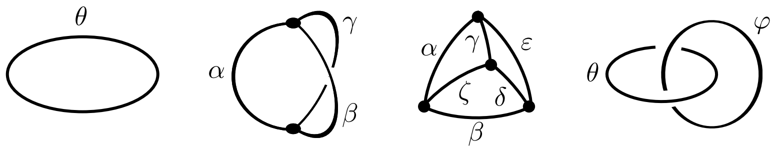

$S^1 * C_\theta$ $S^0 * S^2(\alpha, \beta, \gamma)$ $C_\theta * C_\varphi$

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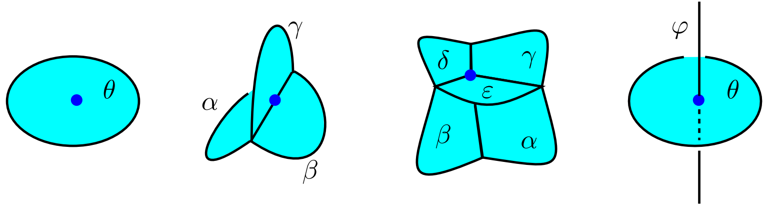
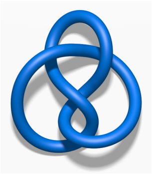
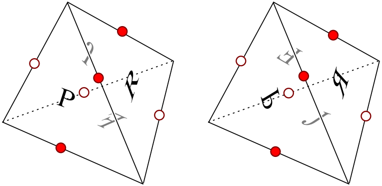


Example: double of a simple polytope.

Hyperbolic Dehn filling





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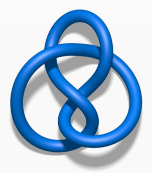

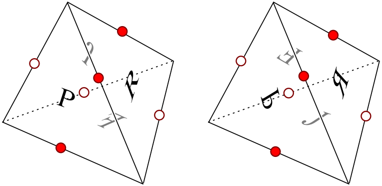

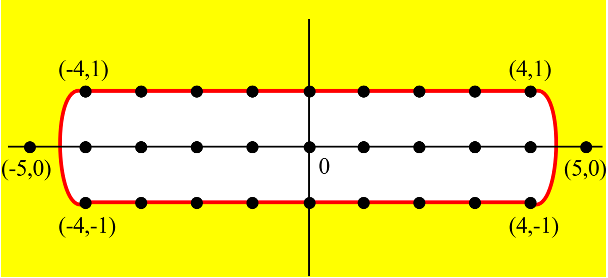

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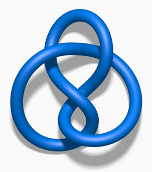

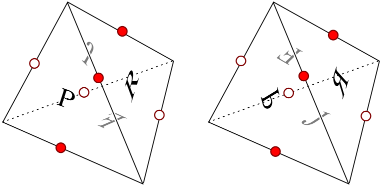

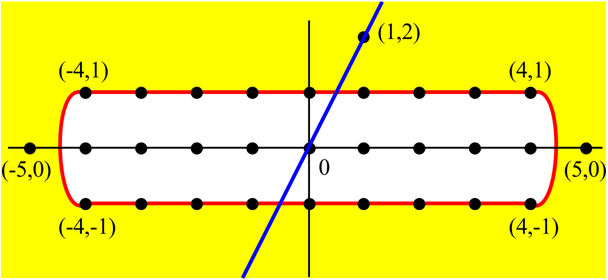

Slope $(1,2)$: hyperbolic with cone angles $< 2\pi + K$

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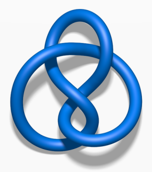
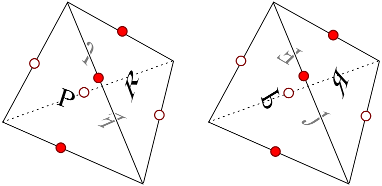
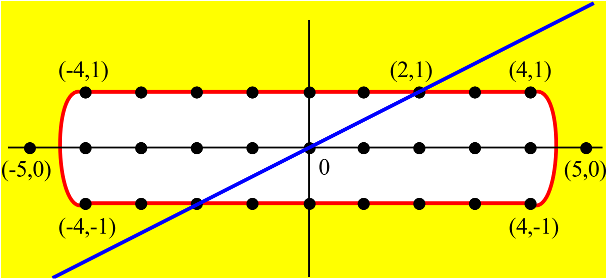
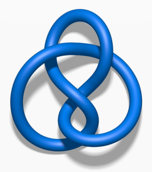


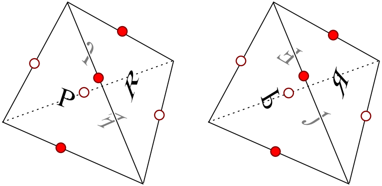


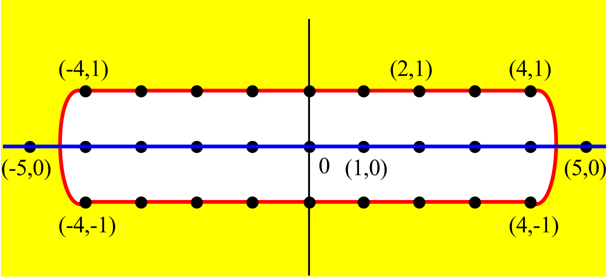


Slope $(2, 1)$: hyperbolic with cone angles $< 2\pi$

Hyperbolic Dehn filling







Slope $(1, 0)$: hyperbolic with cone angles $< 2\pi - K$

Theorem (M, Riolo)

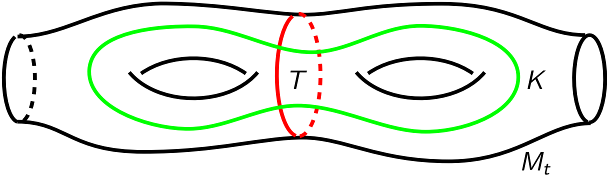
There is an analytic path M_t *with* $t \in [0,1]$ *of finite-volume complete hyperbolic cone four-manifolds with singular set*

$$\Sigma = T \cup K$$

where T *is a torus and* K *a Klein bottle, with cone angles* α *and* β *respectively, intersecting transversely in two points. We have*

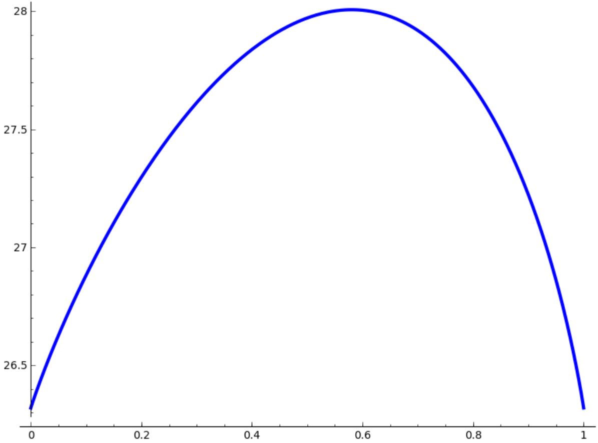
$$\alpha(0) = 0, \quad \alpha(1) = 2\pi, \quad \beta(0) = 2\pi, \quad \beta(1) = 0.$$

The angles α *and* β *vary strictly monotonically in* t*.*

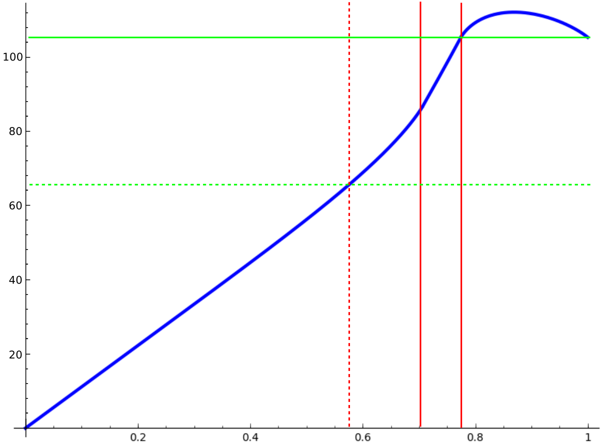


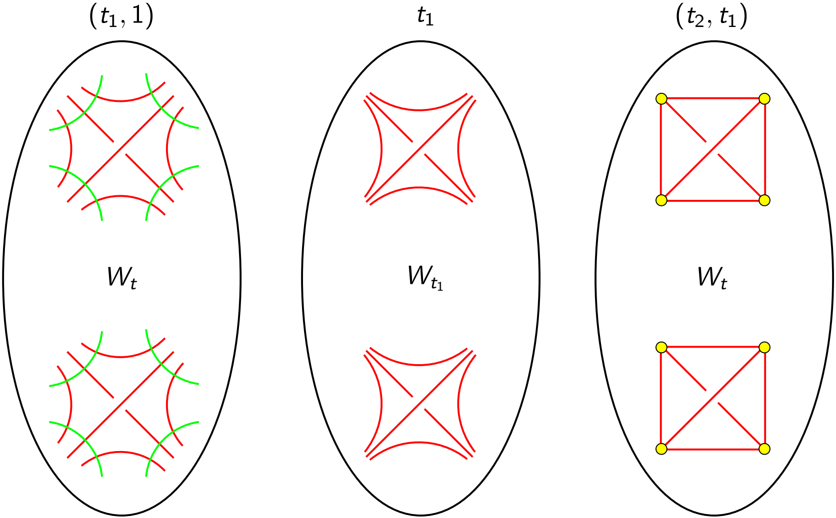
The hyperbolic manifolds M_0 and M_1 have no singularities.

The volume of M_t:



A similar deformation W_t:





$(t_1, 1)$
t_1
(t_2, t_1)
W_t
W_{t_1}
W_t

(Kerchoff – Storm) Let the polytope $P_t \subset \mathbb{H}^4$ be the intersection of the following 24 half-spaces:

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$$
\begin{aligned}
\mathbf{0}^+ &= \left(\sqrt{2}, 1, 1, 1, 1/t\right), & \mathbf{0}^- &= \left(\sqrt{2}, 1, 1, 1, -t\right), \\
\mathbf{1}^+ &= \left(\sqrt{2}, 1, -1, 1, -1/t\right), & \mathbf{1}^- &= \left(\sqrt{2}, 1, -1, 1, t\right), \\
\mathbf{2}^+ &= \left(\sqrt{2}, 1, -1, -1, 1/t\right), & \mathbf{2}^- &= \left(\sqrt{2}, 1, -1, -1, -t\right), \\
\mathbf{3}^+ &= \left(\sqrt{2}, 1, 1, -1, -1/t\right), & \mathbf{3}^- &= \left(\sqrt{2}, 1, 1, -1, t\right), \\
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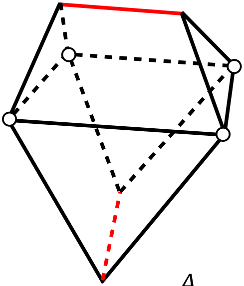
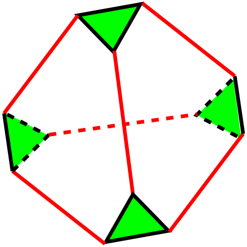
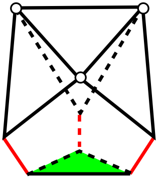
\mathbf{5}^+ &= \left(\sqrt{2}, -1, 1, 1, -1/t\right), & \mathbf{5}^- &= \left(\sqrt{2}, -1, 1, 1, t\right), \\
\mathbf{6}^+ &= \left(\sqrt{2}, -1, -1, 1, 1/t\right), & \mathbf{6}^- &= \left(\sqrt{2}, -1, -1, 1, -t\right), \\
\mathbf{7}^+ &= \left(\sqrt{2}, -1, -1, -1, -1/t\right), & \mathbf{7}^- &= \left(\sqrt{2}, -1, -1, -1, t\right), \\
A &= \left(1, \sqrt{2}, 0, 0, 0\right), & B &= \left(1, 0, \sqrt{2}, 0, 0\right), \\
C &= \left(1, 0, 0, \sqrt{2}, 0\right), & D &= \left(1, 0, 0, -\sqrt{2}, 0\right), \\
E &= \left(1, 0, -\sqrt{2}, 0, 0\right), & F &= \left(1, -\sqrt{2}, 0, 0, 0\right), \\
G &= \left(1, 0, 0, 0, -\sqrt{2}t\right), & H &= \left(1, 0, 0, 0, \sqrt{2}t\right).
\end{aligned}
$$



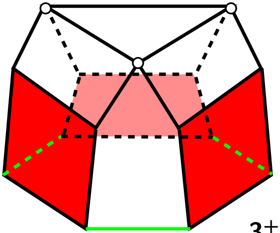
A



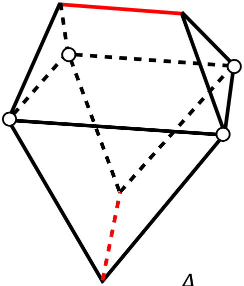
G



$\mathbf{0^{-}}$



$\mathbf{3^{+}}$



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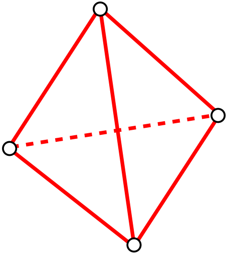

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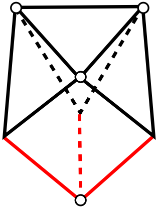
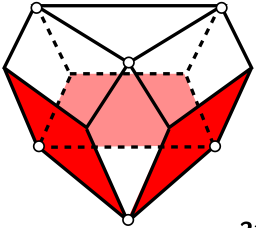


$\mathbf{0}^-$



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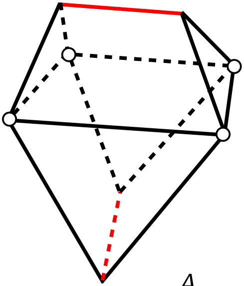

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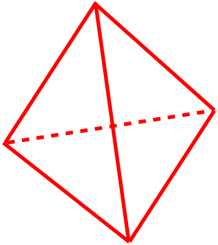

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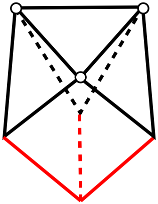

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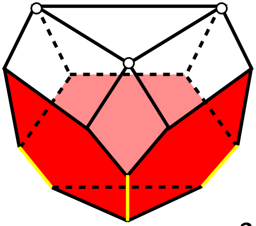

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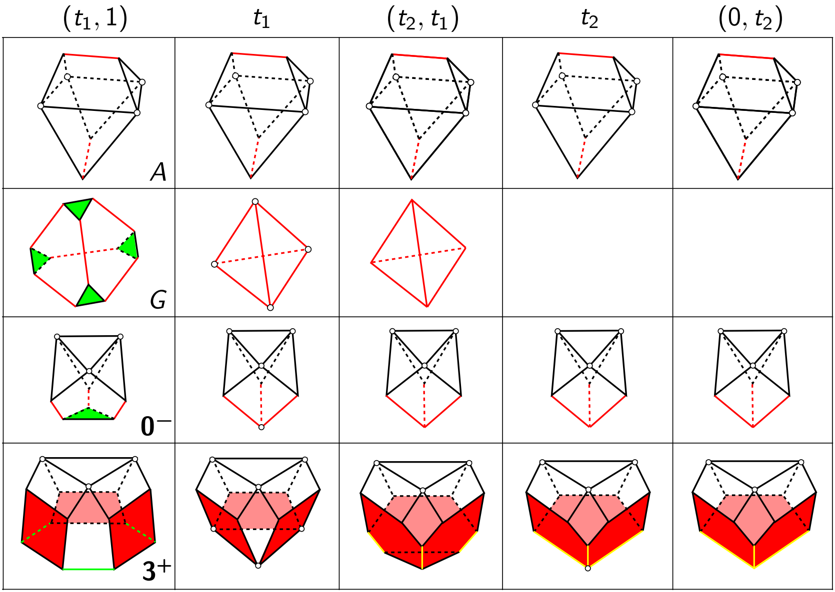

$(t_1, 1)$
t_1
(t_2, t_1)
t_2
$(0, t_2)$
A
G
$\mathbf{0}^-$
$\mathbf{3}^+$