On a Lemma by Ansel and Stricker

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Let S be a local martingale with values in $\mathbb{R}^d$, and let H be a d-dimensional predictable process, such that the stochastic integral $H \cdot S$ does exist: if the process $(H \cdot S)_t$ is uniformly bounded from below by a constant (or, more in general, by an integrable random variable), then $H \cdot S$ is a local martingale, hence a supermartingale.

This result, which is inspired from a proposition by Emery in [8] for the case $d = 1$, is due to Ansel and Stricker ([1], Corollary 3.5). Though obtained as a corollary to a more general proposition, it has become a fundamental result in mathematical finance. For instance, it was stated (as Theorem 2.9) and widely used by Delbaen and Schachermayer in their seminal paper on the fundamental theorem of asset pricing [5].

The purpose of this short note is to provide a different proof of the Ansel and Stricker's lemma, which also allows us to give a formulation of this result for the stochastic integral of measure-valued processes with respect to a family of semimartingales, indexed by a continuous parameter.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leqslant t \leqslant T}, \mathbb{P})$ be a filtered probability space, which satisfies the usual conditions.

Theorem 1 *Let X be an adapted càdlàg process and (M^n) a sequence of martingales such that*

(i) $\sup_{t \leqslant T} |M^n_t - X_t|$ *tends to 0 in probability as $n \to \infty$;*

(ii) there exist an increasing sequence (η_k) of stopping times which converges stationarily to T and a sequence θ_k of integrable random variables, such that $X_{t \wedge \eta_k} \geqslant \theta_k$.

(iii) for every stopping time τ, we have that $(\Delta M^n_\tau)^- \leqslant (\Delta X_\tau)^-$ and $(\Delta M^n_\tau)^+ \leqslant (\Delta X_\tau)^+$ (where $\Delta X_t = X_t - X_{t-}$).

Then, X is a local martingale.

Proof. We can assume, for simplicity, that $M_0^n = X_0 = 0$. Define a sequence (τ_n) of stopping times as follows:

$$\tau_n = \inf\{t > 0 : X_t > n \text{ or } M_t^n > X_t + 1 \text{ or } M_t^n < X_t - 1\} \wedge T\,.$$

Because of (i), we have that $\lim_n \tau_n = T$, $\mathbb{P}$-a.s. Possibly up to a subsequence we can assume that $\sum_n \mathbb{P}(\tau_n < T) < \infty$. We then define the stopping times $\sigma_n = (\inf_{m \geqslant n} \tau_m) \wedge \eta_n$: the sequence σ_n is increasing and converges to T.

We will show that for all m, the stopped process $X_t^{\sigma_m} = X_{t\wedge\sigma_m}$ is a martingale. For every t, the sequence $M^n_{t\wedge\sigma_m}$ goes to $X_{t\wedge\sigma_m}$ in probability. Thanks to (ii) and the definition of σ_m, the jump ΔX_{σ_m} is such that $(\Delta X_{\sigma_m})^- \leqslant m - \theta_m$; condition (iii) implies that $(\Delta M^n_{\sigma_m})^- \leqslant m - \theta_m$ as well. Since $M_t^n \geqslant X_t - 1$ for $n \geqslant m$ and $t < \sigma_m$, we have that

$$M^n_{t\wedge\sigma_m} \geqslant \theta_m - 1 - (m - \theta_m) = 2\theta_m - m - 1\,.$$

We can then apply Fatou's lemma to find that

$$\mathbb{E}\left[X_{t\wedge\sigma_m}\right] \leqslant \liminf_{n\to\infty} \mathbb{E}\left[M^n_{t\wedge\sigma_m}\right] = 0\,.$$

This shows that $X_{t\wedge\sigma_m}$ is integrable: in particular, taking $t = T$, we obtain that X_{σ_m} is integrable and, as a consequence, ΔX_{σ_m} is integrable.

In an analogous way, we find that, for $n \geqslant m$,

$$M^n_{t\wedge\sigma_m} \leqslant m + 1 + (\Delta M^n_{\sigma_m})^+ \leqslant m + 1 + (\Delta X_{\sigma_m})^+\,.$$

So, we can apply Lebesgue theorem and obtain that for every fixed m and t, the sequence of random variables $M^n_{t\wedge\sigma_m}$ converges to $X_{t\wedge\sigma_m}$ in $L^1(\mathbb{P})$: this implies that $X_t^{\sigma_m}$ is a martingale. □

As a corollary, we deduce the lemma of Ansel and Stricker:

Corollary 2 *Let S be a d-dimensional local martingale and let H be a S-integrable predictable process. If there exists some constant $C > 0$ such that $(H \cdot S) \geqslant -C$ for all t, then $H \cdot S$ is a local martingale.*

Proof. We set $X = H \cdot S$, $H^n = H\mathbf{1}_{\{\|H\|\leqslant n\}}$ and $M^n = H^n \cdot S$. Every M^n is a local martingale, hence we can find an increasing sequence (τ_m) of stopping times such that $\lim_m \tau_m = \infty$ and $M^n_{\tau_m}$ is a martingale ([6], Theorem 3). So, up to a standard localization, we can assume that every M^n is a martingale.

The claim follows from Proposition 1 as soon as we check that conditions (i) and (iii) are fulfilled (condition (ii) is contained in the assumptions of the corollary). It is well-known that if H is integrable with respect to S, then $\sup_{t\leqslant T} |M_t^n - X_t|$, tends to 0 in probability, whence condition (i). Condition (iii) follows trivially once we have observed that $\Delta M^n_\tau = \Delta(H\cdot S)_\tau \mathbf{1}_{\{\|H\|_\tau \leqslant n\}}$. Hence the claim is proved. □

Now we briefly show how the previous arguments can be applied to the case of *measure-valued* integrands.

Let $\mathbf{M} = (M^x)_{x\in I}$ be a family of locally square integrable martingales, where I is a compact subset of $\mathbb{R}$. We denote by $\mathcal{P}$ the predictable σ-field and suppose that $\mathbf{M}$ satisfies the following:

Assumption 1. *There exist an increasing predictable process A_t and a function Q defined on $\Omega \times [0,T] \times I \times I$, measurable with respect to $\mathcal{P}\otimes\mathcal{B}(I)\otimes\mathcal{B}(I)$, such that, for almost all $(\omega, s) \in \Omega \times [0,T]$:*

(i) the function $(x,y) \mapsto Q_{\omega,s}(x,y)$ is symmetric, non-negative definite and continuous;
(ii) the function $(x,y) \mapsto \int_0^t Q_{\omega,s}(x,y)dA_s(\omega)$, is symmetric, non-negative definite and continuous;
(iii) for fixed $x, y \in I$ and for all $t \in [0,T]$, we have that:

$$\langle M^x, M^y\rangle_t(\omega) = \int_0^t Q_{\omega,s}(x,y)dA_s(\omega) \qquad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

With this assumption, a stochastic integral with respect to $\mathbf{M}$ can be defined on an appropriate class of measure-valued processes, by making use of a theory on cylindrical integration developed by Mikulevicius and Rozovskii [9] (see also [3], Section 3). More in details, consider a stochastic process ϕ with values in the set of the Radon measures on I (the dual set of the space of continuous functions $\mathcal{C} = \mathcal{C}(I,\mathbb{R})$), such that, for every $f \in \mathcal{C}$, the process $\langle\phi_s, f\rangle_{\mathcal{M},\mathcal{C}}$ is predictable. We indicate by $\langle\phi_s, Q_s\psi_s\rangle = \langle\phi_s, Q_s\psi_s\rangle_{\mathcal{M},\mathcal{C}}$ the bilinear form $\int_I \phi_s(\mathrm{d}x)\int_I Q_s(x,y)\psi_s(\mathrm{d}y)$.
Suppose that

$$\mathbb{E}\left[\int_0^T \langle\phi_s, Q_s\phi_s\rangle\,\mathrm{d}A_s\right] < \infty :$$

then it is possible to define the stochastic integral $\phi\cdot\mathbf{M}$ which is a square-integrable martingale (see [9] for details). Moreover, if $\int_0^t\langle\phi_s, Q_s\phi_s\rangle\,\mathrm{d}A_s$ is locally integrable, the stochastic integral $\phi\cdot\mathbf{M}$ is defined and is a locally square-integrable martingale.

More general stochastic integrals can be defined, in a similar way to what happens for the finite dimensional case (see [2] page 130).

Let, for every n, $\phi^n = \phi\mathbf{1}_{\{\langle\phi,Q\phi\rangle\leqslant n\}}$: we say that ϕ is $\mathbf{M}$-*integrable* if the sequence of square-integrable martingales $\phi^n\cdot\mathbf{M}$ is convergent for the semimartingale topology (see [7] for the definition of this topology) and by definition $\phi\cdot\mathbf{M} = \lim_{n\to\infty}\phi^n\cdot\mathbf{M}$. Note that, if $X = \phi\cdot\mathbf{M}$, then $\phi^n\cdot\mathbf{M} = \mathbf{1}_{\{\langle\phi,Q\phi\rangle\leqslant n\}}\cdot X$.

Exactly as for the finite-dimensional case, the process $\phi\cdot\mathbf{M}$ might not be a local-martingale (see for instance, [8]), but the analogue of Corollary 2 holds (whit a proof similar to that of Corollary 2).

Proposition 3 *Let ϕ be a measure-valued integrable process. If there exists some constant C such that $(\phi \cdot \mathbf{M})_t \geqslant -C$ for all t, then $\phi \cdot \mathbf{M}$ is a local martingale.*

Remark: We point out that a stochastic integral **H**.**M** has been defined in [4] for a wider class of integrands **H**, and that in this more general framework the analogue of the Ansel–Stricker's lemma is false (see [4], Example 2.1).

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