Soul Searchin'

Marco Abate

Abstract What happens to mathematicians working in the middle of the night?

It's late, I'm drinking a bloody Mary and I'm listening to Solomon Burke. I have a paper to write, places to go, theorems to prove. The paper is, as the hour but more, late. I cannot go to the places I would like to (or even have to) go. The theorems are on strike, they refuse to prove themselves. Do not believe mathematicians saying they are working; they are not. They are waiting for theorems kind enough to prove themselves. If not a theorem, at least a proposition. In nights like this one, even a lemma would be welcome. Not a corollary, no, a corollary would not be welcome, not ever. Corollaries ambush you behind the corner, after you've been distracted by a beautiful theorem passing by winking at you. You turn your head expecting — or at least hoping — that it will slowly stop, that it will wait a little meditating about the meaning of the universe, and that then it will quietly turn around, with a gentle smile, a welcoming gesture and it will be yours forever and ever — but it won't. No slow stopping, no turning around, no smiles or gestures, welcoming or else; it won't be yours. And you get stuck with that obnoxious corollary, completely useless without the theorem, but still nagging at your elbow, not even clean enough to be sold out as a conjecture. A useless little brat. A pitiful reminder of the theorem that could — should! — have been mine and it didn't, and won't ever ever be because I'm not good enough, I cannot be good enough, only dirty corollaries are what I deserve, I'm not a real mathematician, not even an applied one, I must leave right now, in shame, I will go and live in a shanty town, surviving collecting garbage from waste dumps, with dirty corollaries as only company, to constantly remind me that I am a failure, no shining theorems for me, ever.

Darn, I forgot. I cannot leave. We are in lockdown. The only place I can go to is the kitchen, to fill up this bloody Mary sorely needing filling up. Done. I prepare bloody killing bloody Marys, if you don't mind me saying so. The right amount

Marco Abate
Dipartimento di Matematica, Università di Pisa, Largo Pontecorvo 5, 56127 Pisa, Italy, e-mail: marco.abate@unipi.it

of vodka. And the right amount is a lot: this is a true theorem, Banach *docet.* Not a dirty corollary. A true bright theorem in its shining armor. A sip — even better two, perchance three — and its light will dispel the darkness in my mind where all those propositions and lemmas are trying to hide, are trying to hide themselves and the true path to the true theorem, the greatest theorem, the theorem to end all theorems… and. It. Will. Be. Mine! Another sip and I'm sure I will start glimpsing it! Another sip and now that I think about it, really think about it — ah, bloody Marys really help you think, yes they do — you know, this dirty corollary is not that dirty after all. A little cleansing here, a bit of scraping there and it might pass for a half-decent conjecture. Why bother to chase ungrateful theorems when you can sell your colleagues a nice enticing conjecture, promising a garden of delights that nobody will enter, but it won't be your fault, you just suggested the way and were so generous to leave it to others to follow, it will be their fault if the conjecture will remain unproved and undeflowered! And yet I have the feeling that something is off… another sip of bloody Mary will surely clear my mind… Mmm…

What? What did you say? That I got carried away? That I promised to write something meaningful? Or at least more meaningful? And that the Solomon Burke album has ended? Decisions are called for. We must decide. Uplifting music: Sergio Cammariere. I do not need a waiter to uplift my glass to toast my deciding strength, but it helps anyways. Lift the music and lift the glass, let's have a musical toast to meaningful mathematics, to sound mathematics! Or at least to mathematics sounding sound! Or to… Whatever, let's start.

1 The algebraic playground

Everything will take place in a big large wonderfully smooth complex manifold called M for "mom". Inside it (her?) it lies a slightly mischievous possibly singular subvariety called S for — you guessed it — "son". It (he?) might possibly be singular, but he promised his mom to stay reduced, connected (he'll try to be irreducible too, but no promises on that count) and, of course, pure (dimensional).

Our S comes with a whole box full of toys, that he mischievously drops on the floor. And, lo and behold, this is the pattern the toys drew on the floor:

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\begin{array}{ccccccccc}
 & & & \mathscr{I}_S & \overset{\iota_0}{\hookrightarrow} & \mathscr{O}_M & \overset{d}{\longrightarrow} & \Omega_M & \\
 & & \swarrow \pi & \swarrow D & \swarrow \pi_2 & \downarrow \pi_1 & & & \\
O \longrightarrow & \mathscr{I}_S/\mathscr{I}_S^2 & \underset{\tilde{\rho}}{\overset{\iota_1}{\rightleftarrows}} & \mathscr{O}_M/\mathscr{I}_S^2 & \underset{\rho}{\overset{\theta}{\rightleftarrows}} & \mathscr{O}_M/\mathscr{I}_S & \longrightarrow & O & \\
 & \tau^* \uparrow\downarrow d_{\mathscr{I}} & \swarrow d_2 & & & & & & \\
 & \Omega_M \otimes_{\mathscr{O}_S} \mathscr{O}_S & & & & & & & \\
 & \downarrow & & & & & & & \\
 & \Omega_S & & & & & & & \\
 & \downarrow & & & & & & & \\
 & O & & & & & & &
\end{array}
$$

(Ok, ok, the arrow with a D attached to it should be more slanted toward the left, because it goes from $\mathscr{O}_M$ to $\mathscr{I}_S/\mathscr{I}_S^2$, but even my LaTeX-pertise has its limits...). What are all these horrible typographical misfits, I hear you muttering. They are not horrible, the son answers, they actually were showcased in a real art show a few years ago... Hush son, says the mom, let me explain to our honored guests.

Of course, $\mathscr{O}_M$ is the sheaf of germs of holomorphic functions on myself. Very important, the origin of the world. Instead, $\mathscr{I}_S$ is the sheaf of ideals of my son (full of ideals, my son, at least one at each point, I'm very proud of him), ideals of germs of holomorphic functions vanishing on him, and you have no idea how much scrubbing and cleaning is needed to keep those germs vanishing, we don't want any dirty subvariety, oh no. Ω_M is the sheaf of holomorphic differentials, that are wonderful things when you have them, but for the life of me don't ask what they are. Ω_S is my son's sheaf of holomorphic differentials. Since he is so singular sometimes, the mad French mathematician told me that I should never mention it in his presence; it is defined by the diagram, so he said to me. I'm only a standard smooth mom, if the mad French mathematician says so it is so.

And then there are all those funny little letters... ι_0 and ι_1 are just inclusions: yes, son, send your ideals into me and call me Giocasta I'll be your victim. And then those annoying three little pigs... I mean, pi's: π, π_1 and π_2 are just projections, sort of canonical if you are of the religious type. θ too is a projection, but it lives at a lower level, and the pigs didn't want it to mix with them. And look to that forlorn arrow with no name, almost at the base of the social echelon: it is a projection too, but of such a lower class that the pi's didn't want it to be named. And the mad French mathematician, nodding madly, confirmed that there is no need to name it; it is defined by the diagram. Yes, sir.

The d's... The first d is the usual d, sending f into df. Ok, it is not much as explanations go, but that's something you already know, don't you? Derivatives and such. To explain d_2 and $d_{\mathscr{I}}$ let me tell you that I don't like that much those pi's, always bringing mud, bricks and wolves into the house... never mind that, but anyway the fact is that sometimes I like to trick them a bit, and I've devised a little notation of my own, writing $[f]_j$ instead of $\pi_j(f)$ just to confuse them. Neat, don't you think? Where f is a little germ of mine, of course, but a very clean one, I assure you, not even a snooze will come out of it. Well, having said that, d_2 is defined as follows:

$$d_2[f]_2 = df \otimes [1]_1,$$

and $d_{\mathscr{I}} = d_2 \circ \iota_1$, of course. A little thought will show, honored guests, that the definition is well-posed. In case of doubt, my son will be glad to explain everything up to the last little microlocal detail. His father was an analytic space, you know.

The capital D... well, let's say that it is a derivation and leave it at that. It is not always there, you know; it'd have to be dashed, but there is no way to find a decent dashed oblique arrow these days. Same thing for those radical arrows going contrarily-wise, ρ, $\tilde{\rho}$ and $\tau^\star$ (why wasting such a good-looking star when there is no τ around is beyond me): they're not always there, thanks God in this country we are still writing left-to-right-top-to-bottom, but when they are around they'll make themselves noticed, no doubt, those rascals.

I think that's all... You asking...? Oh well yes of course, the diagram is commutative with exact rows and columns, I like to keep my home tidy and clean, thank you very much. Oh well, except the first row, yes you're right, but tell me where should I put that Ω_M, just tell me, what a poor smooth manifold has to do? And now, if you excuse me, I'm beginning to have a tiny little headache, I've better go to rest. Where did I leave that covering, it was right here last time I checked... (fade)

Disclaimer

No definition has been harmed in the preparation of this paper. The co-authors of [1] strongly deny any responsibility for this rant, and sternly advise young mathematicians to never drink and write.

References

[1] M. Abate, F. Bracci, F. Tovena: *Index theorems for holomorphic maps and foliations*, Indiana Univ. Math. J. **57** (2008), 2999–3048.