GB60^{TM} CALCULUS OF VARIATIONS AND OPTIMIZATION

—

A CONFERENCE TO CELEBRATE THE 60TH BIRTHDAY OF GIUSEPPE BUTTAZZO

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2. Main talks

Luigi Ambrosio (Scuola Normale Superiore Pisa)

"*ODE's and Sobolev vector fields on metric measure spaces.*"

Abstract: In the talk I will illustrate some recent developments (joint work with D. Trevisan, and an independent work by N. Gigli) on calculus in metric measure spaces. I will show how intrinsic tools like the heat semigroup and the Bakry-Emery curvature condition can be used to prove existence and uniqueness for solutions to the continuity and transport equations. On the Lagrangian side, this leads to a well posed notion of flow, with existence, uniqueness and stability properties. In this way the DiPerna-Lions theory can be extended to an abstract framework, with a new point of view on the so-called commutator estimate.

Giovanni Alberti (Università di Pisa)

"*Examples of mixing for flows associated to divergence-free vector-fields.*"

Abstract: Let A_0 be one half of the n-dimensional torus M, and given a divergence-free vectorfield b on M, let A_t be the image of A_0 at time t according to the flow associated to b; since b is divergence-free, the flow is volume-preserving, and therefore the volume of A_t is half the volume of M for every t. Given $r > 0$, we say that A_t is uniformly distributed in M at the scale r if the intersection of A_t with every ball B with volume r has volume between, say, $1/4$ and $3/4$ of the volume of B; and when this happens we say that the flow is mixing on the scale r at time t. A. Bressan conjectured that mixing cannot occur faster than exponentially in time, that is, r must be larger than $exp(-ct)$ for some constant c that depends on certain quantities related to the vectorfield b. Even though much work has been done on this conjecture (e.g., by C. De Lellis and G. Crippa), very few examples are known of vectorfields b that actually exhibit such exponential decay of r (thus showing that the bound in the conjecture is optimal), and all are far from being smooth. In this talk I will describe some (more regular) examples obtained in a joint work with G. Crippa and A. Mazzucato; these constructions have been used to give examples of instantaneous loss of regularity (in space) for the solutions of the continuity equations associated to mildly regular vectorfields b.

Hedy Attouch (Université de Montpellier 2)

"*On gradient-like dynamic systems for multi-objective optimization in Hilbert spaces, and applications.*"

Abstract: In a general Hilbert framework, we consider continuous gradient-like dynamic systems for constrained multi-objective optimization, involving (possibly non-smooth) convex objective functions. We prove a descent property for each objective function, and the convergence of trajectories to weak Pareto minima. This approach provides a dynamical endogenous weighting of the criteria. Some extensions to the quasi-convex, and semi-algebraic cases are considered. By time discretization, we make the link with recent studies concerning gradient-like algorithms for multi-objective optimization. Applications are given in signal/imaging processing, and Pareto equilibrium in cooperative games.

Guy Bouchitté (Université du Sud Toulon-Var)

"*A duality principle for non convex variational problems*"

Abstract: In this talk I will develop a duality theory for classical problems of the Calculus of Variations of the kind

$$J(\Omega) := \inf \left\{ \int_\Omega (f(\nabla u) + g(u))\, dx + \int_{\Gamma_1} \gamma(u)\, dH^{d-1} \ ,\ u = 0 \text{ on } \Gamma_0 \right\}$$

where g, γ are possibly non convex functions with suitable growth conditions and f is a convex intergrand on $\mathbb{R}^d$. Here (Γ_0, Γ_1) is a partition of $\partial\Omega$. A challenging issue is to characterize the global minimizers of such a problem and the stability of the minimal value (with respect for instance to small deformations of the domain Ω).

We present a duality scheme in which the dual problem reads quite nicely as a linear programming problem. The solvability of this dual problem is a major issue. It can be achieved in the one dimensional case and in higher dimensions under special assumptions on f, g. Applications are given for a class of free boundary problems.

Dorin Bucur (Université de Savoie)

"*Spectral optimization problems: from shape optimization to a free boundary approach* "

Abstract: In this talk, I will review several joint results obtained with Giuseppe Buttazzo on questions related to shape optimization problems associated to spectral functionals and I will make the point on recent advances on this topic.

Guillaume Carlier (Université Paris Dauphine)

"*A shape optimization problem in congested transport*"

Abstract: Transport with congestion effects has been very much studied in discrete networks models since the 50's. Continuous counterparts have been introduced more recently and lead to variational problems with divergence constraints and degenerate anisotropic elliptic PDEs. After giving an overview of this theory, we will discuss models in which, the congestion can be lowered (by improving the road system, which of course has a cost) in some zones. We will then study the design of optimal low-congestion zones. This is a joint work with Giuseppe Buttazzo and Serena Guarino Lo Bianco.

Bernard Dacorogna (Ec. Pol. Tech. Fed. Lausanne)

"*Some recent results on the Jacobian equation*"

Abstract: Given two functions f and g, we want to find a map φ such that

$$\begin{cases} g\big(\varphi(x)\big)\,\det\nabla\varphi(x) &= f(x) \quad x\in\Omega \\ \varphi(x) &= x \quad x\in\partial\Omega. \end{cases}$$

(1) We first discuss the case where $g\cdot f>0$ and give three different ideas for the existence problem with optimal regularity.

(2) We then briefly comment on the case where $g>0$ but f is allowed to change sign.

(3) We finally consider the (local) existence, uniqueness and optimal regularity for the problem

$$g_i\big(\varphi(x)\big)\,\det\nabla\varphi(x)=f_i(x) \qquad \text{for every } 1\le i\le n$$

where $g_i\cdot f_i>0$.

Gianni Dal Maso (SISSA Trieste)

"*Quasistatic Evolution as Limit of Dynamic Evolutions: the Case of Perfect Plasticity*"

Abstract: We introduce a model of dynamic visco-elasto-plastic evolution in the linearly elastic regime and prove an existence and uniqueness result. Then we study the limit of (a rescaled version

of) the solutions when the data vary slowly. We prove that they converge, up to a subsequence, to a quasistatic evolution in perfect plasticity.

Irene Fonseca (Carnegie Mellon University)

"*Variational Methods for Crystal Surface Instability*"

Abstract: Using the calculus of variations it is shown that important qualitative features of the equilibrium shape of a material void in a linearly elastic solid may be deduced from smoothness and convexity properties of the interfacial energy.

In addition, short time existence, uniqueness, and regularity for an anisotropic surface diffusion evolution equation with curvature regularization are proved in the context of epitaxially strained two-dimensional films. This is achieved by using the H^{-1}-gradient flow structure of the evolution law, via De Giorgi's minimizing movements. This seems to be the first short time existence result for a surface diffusion type geometric evolution equation in the presence of elasticity.

Nicola Fusco (Università di Napoli "Federico II")

"*Stability of the isoperimetric inequality on the sphere*"

Abstract: We discuss a quantitative version of the isoperimetric inequality on the sphere with a constant independent of the volume of the set E.

Wilfrid Gangbo (Georgia Institut of Technology)

"*Existence of a solution to an equation arising from Mean Field Game*"

Abstract: We construct solutions to a non-local Hamilton-Jacobi, so-called Master Equation in Mean Field Game, starting from value functions which are metric viscosity solutions to Hamilton-Jacobi equations. The metric viscosity solutions have a representation formula from which solutions to the so-called First Order Mean Field equation can be derived. For the sake of illustration, the current study has been restricted to a special class Lagrangian. (This talk is based on a joint work with A. Swiech).

Antoine Henrot (Ecole des Mines, Nancy)

"*Shape Optimization problems with diameter constraints*"

Abstract: In this talk we consider some shape optimization problems with a constraint on the diameter of the sets. We will first give some general properties for that kind of problem. Then we will investigate more deeply the "toy-problem"

$$\min\{\gamma\, A(\Omega) - P(\Omega),\ \Omega \text{ convex},\ D(\Omega) = 1\}$$

where $\gamma > 0$ is a parameter, $A(\Omega)$ the area, $P(\Omega)$ the perimeter and $D(\Omega)$ the diameter of the convex set Ω. We will describe the solutions according to the value of the parameter γ and also gives some consequences on the Blaschke-Santaló diagram involving the three quantities $A(\Omega)$, $P(\Omega)$ and $D(\Omega)$. This is a joint work with J. Lamboley (Paris-Dauphine) and Y. Privat (Paris 6).

Bernd Kawohl (Universität zu Köln)

"*How to cut a cake*"

Abstract: Over 50 years ago Polya stated the following problem. Given a plane convex set K (a cake) find the shortest curve that bisects it into two pieces of equal area. Is it true that this curve is never longer than the diameter of a circular cake of the same area? Under the additional assumption that K is centrosymmetric (i.e. $K = -K$) he gave a simple proof that this is indeed the case. Without this assumption the proof is much harder, and I report on a joint paper with L. Esposito, V. Ferone, C. Nitsch and C. Trombetti containing the proof. It is remarkable that the answer to Polya's question is negative if only straight cuts are allowed. In that case N. Fusco and A. Pratelli were able to show that the Auerbach triangle and not the disc provides gives rise to "the longest shortest cut".

Edouard Oudet (Université "Joseph Fourier", Grenoble)

"*Giuseppe, the convexity and me*"

Abstract: We discuss several progresses in convex geometry and optimal transportation which have been initiated or developed by Giuseppe during the last decade.

Michel Pierre (ENS Rennes)

"*Reaction-Diffusion: singular limits and applications*"

Abstract: Modelling chemical reactions often requires to identify the limit of systems where rate constants of some components tend to infinity. These limit systems are more significant in chemical engineering. Justifying rigorously these passing to the limit often leads to challenging mathematical questions. We will discuss some of them together with surprisingly connected problems arising in optimal spectral partitions.

Eugene Stepanov (St. Petersburg State University)

"*Steiner networks and gas pipelines*"

Abstract: We will discuss network optimization problems which are intrinsically related to the Steiner minimal network problem, like, for instance, the problem of optimal shape of a gas pipeline. The regularity of solutions will be discussed, as well as some exact solutions will be presented.

Franco Tomarelli (Politecnico di Milano)

"*Inpainting, Segmentation, Free Discontinuity*"

Abstract: This talk focuses on a second order functional depending on free discontinuity and free gradient-discontinuity, whose minimizers provide a variational solution to contour detection problem in image segmentation and inpainting.

Enrique Zuazua (BCAM, Bilbao)

"*Optimal placement of sensors and actuators for waves*"

Abstract: In this lecture we address the problem of the optimal placement of sensors and actuators for wave propagation problems. Using Fourier series representation the problem can be recast as a

spectral optimal design problem, involving all the spectrum of the laplacian. We show that, depending on the complexity of the data to be observed/controlled, several scenarios have to be distinguished. Those in which the solution is a classical set constituted by a finite number of simply connected subdomains, others in which the optimal set is of Cantor type and those leading to relaxation phenomena. We also explain how closely this topic is related to the fine properties of the high frequency behavior of the eignefunctions of the laplacian which is intimately linked to the ergodicity properties of the dynamical system generated by the corresponding billiard. We shall also discuss the same problem for heat processes showing that, in that frame, according to intuition, the problem is governed by a finite number of Fourier modes. These results will be illustrated by numerical simulations. This work is based on recent joint work in collaboration with Y. Privat and E. Trélat.

3. Short talks

Andrea Davini (Università di Roma)

"*Convergence of the solutions of the discounted equation*"

Abstract: We consider the discounted equation

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M,$$

set on a compact and connected Riemannian manifold M, where λ is a positive parameter, H is a continuous Hamiltonian, coercive in the momentum, and c is the associated critical value. Under these assumptions, the corresponding solutions $u_\lambda : M \to \mathbb{R}$ are equi–bounded and equi–Lipschitz, hence they uniformly converge, along subsequences as the discount factor λ goes to 0, to a viscosity solution of the critical equation

$$H(x, d_x u) = c \qquad \text{in } M.$$

Due to the lack of a uniqueness result for the critical equation, it is not clear at this point that the solutions selected at the limit along different subsequences are the same. When H is additionally assumed convex in the momentum, we prove that the u_λ uniformly converge to a specific solution of the critical equation, characterized in terms of a class of probability measures introduced in the framework of weak KAM Theory.

This is a joint work with A. Fathi, R. Iturriaga and M. Zavidovique.

Lorenzo Freddi (Università di Udine)

"*From 3D non-linear elasticity to 1D elastic models for thin-walled beams*"

Abstract: Geometrically, a thin-walled beam is a slender structural element whose length is much larger than the diameter of the cross-section which, on its hand, is larger than the thickness of the thin wall. Beams of this kind have been used for a long time in civil and mechanical engineering and, most of all, in flight vehicle structures because of their high ratio between maximum strength and weight. Because of their slenderness thin-walled beams are quite easy to buckle and to deform and hence, in several circumstances, their study has to be conducted by means of nonlinear theories. In this talk, starting from three-dimensional nonlinear elasticity, we rigorously derive a hierarchy of one-dimensional models for a thin-walled beam, in the spirit of what has been done by Friesecke, James and Müller in [4] for plates. The different limit models are distinguished by the different scaling of the elastic energy, which, in turn, depends on the scaling of the applied loads. More precisely, denoting by h and δ_h the length of the sides of the cross section and by ε_h the scaling factor of the bulk elastic energy, we can identify three main regimes:

- subcritical: $\lim_{h\to 0} \delta_h/\varepsilon_h = 0$;
- critical: $\lim_{h\to 0} \delta_h/\varepsilon_h = 1$

- supercritical: $\lim_{h\to 0} \delta_h/\varepsilon_h = +\infty$

In the subcritical regime it is easy to see that, if $\varepsilon_h = O(1)$, then the limit model coincides with the nonlinear string model deduced by Acerbi, Buttazzo and Percivale in [1] for a beam of uniformly small cross-section. If instead $\lim_{h\to 0} \varepsilon_h = 0$, we obtain the energy of an inextensible string. In the critical regime, and under the additional assumption that $\lim_{h\to 0} h^2/\delta_h = 0$, we obtain a Cosserat model for a thin-walled beam. Finally, in the supercritical regime we deduce one dimensional linear/quasi-linear models for thin-walled beams.
The talk is based on two papers ([2] and [3]) in collaboration with Maria Giovanna Mora and Roberto Paroni.

- [1] E. Acerbi, G. Buttazzo and D. Percivale, A variational definition of the strain energy for an elastic string, *J. Elasticity*, **25** (1991), 137–148.
- [2] L. Freddi, M.G. Mora and R. Paroni, Nonlinear thin-walled beams with a rectangular cross-section - Part I, *Math. Mod. Meth. Appl. S.*, **22** (2012), 1150016 (34 pp).
- [3] L. Freddi, M.G. Mora and R. Paroni, Nonlinear thin-walled beams with a rectangular cross-section - Part II, *Math. Mod. Meth. Appl. S.*, **23** (2013), 743-775.
- [4] G. Friesecke, R.D. James, S. Müller, A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence., *Arch. Ration. Mech. Anal.* **180** (2006), 183–236.

Xin Yang Lu (Instituto Superior Técnico, Lisboa)

"*Average distance problem: regularity and applications*"

Abstract: Regularity of minimizers of the average distance problem is a delicate problem: it is known that Lipschitz regularity holds, while C^1 is false in general. For application in data analysis, the original version of the average distance problem exhibits several drawbacks, thus additional penalization terms (e.g. Willmore energy, L^2 norm on the density) should be considered. Moreover, due to computational cost, it is often advantageous to restrict the unknown to the family of parameterized curves. In this talk we will present some recent results on the regularity of minimizers, for both original and penalized problems.

Joint work with Dejan Slepčev.

Francesca Agnese Prinari (Università di Ferrara)

"*Power law approximation under differential constraints*"

Abstract: We study the Γ-convergence of the power-law functionals

$$F_p(V) = \left(\int_\Omega f(x, V(x))^p \, dx \right)^{1/p},$$

as p tends to $+\infty$ and $V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A}$ where $\mathcal{A}$ is a constant-rank partial differential operator. We show that the Γ-limit is represented in the supremal form

$$F(V) = \operatorname{ess\,sup}_{x\in\Omega} \tilde{f}(x, V(x)) \tag{1}$$

and we give an explicit representation formula for the function $\tilde{f}$. Moreover we discuss the weak* lower semicontinuity of functionals of the form (1) when $V \in L^\infty(\Omega; \mathbb{M}^{d\times N}) \cap \operatorname{Ker} \mathcal{A}$.

Joint work with Nadia Ansini.

Berardo Ruffini (Université de Grenoble) & **Bozhidar Velichkov** (Università di Pisa)

"*Optimal potentials for Schrödinger operators*"

Abstract: This is a two-parts talk, based on the works [1] and [2], in which we discuss some optimization problems for Schrödinger operators $-\Delta + V$.

We consider the spectral optimization problem

$$\min\left\{\mathcal{F}(V)\;\middle|\; V:\Omega\to[0,+\infty]\ \text{ such that }\ \int_\Omega \Phi(V)\,dx = 1\right\}, \tag{2}$$

where:

- the design region $\Omega\subset\mathbb{R}^d$ is a bounded open set or the entire space $\mathbb{R}^d$;
- $\Phi:[0,+\infty]\to[0,+\infty]$ is a function satisfying some monotonicity and convexity assumptions, which are fulfilled for example in the following model cases
$$\Phi(V) = V^{-1/2} \qquad \text{and} \qquad \Phi(V) = \exp(-V);$$
- $\mathcal{F}$ is a functional depending on the spectrum of the operator $-\Delta+V$ with Dirichlet boundary conditions on $\partial\Omega$, precisely
$$\mathcal{F}(V) = F\big(\lambda_1(V),\dots,\lambda_k(V)\big),$$
where $F:\mathbb{R}^k\to\mathbb{R}$ is a given increasing and lower semi-continuous function and $\lambda_k(V)$ is the $k-$th eigenvalue of the Schrödinger operator $-\Delta+V$ on Ω.

In the first part of the talk, we will prove the existence of an optimal potential for (2) in a bounded domain $\Omega\subset\mathbb{R}^d$ for very general constraints Φ and cost functionals $\mathcal{F}$.

In the second part we consider the case $\Omega=\mathbb{R}^d$, proving the existence of an optimal potential for

$$\Phi(V) = V^{-1/2} \qquad \text{and} \qquad \mathcal{F}(V) = \lambda_k(V).$$

- [1] G. Buttazzo, A. Gerolin, B. Ruffini, B. Velichkov: Optimal potentials for Schrödinger operators. *Available at* `http://cvgmt.sns.it/person/262/`
- [2] D. Bucur, G. Buttazzo, B. Velichkov: Spectral Optimization Problems for Potentials and Measures. *Available at* `http://cvgmt.sns.it/person/262/`