ETA – 5/11/13

Complesso di catene $\{(C_n, \partial_n)\}_{n=0}^{\infty} = \mathcal{C}$

$\partial_n : C_n \to C_{n-1}$ $\quad \partial_n \circ \partial_{n+1} = 0$

$Z_n = \operatorname{Ker}(\partial_n)$ cicli $\quad B_n = \operatorname{Im}(\partial_{n+1})$ bordi

$H_n(\mathcal{C}) = Z_n / B_n$.

Def: una succ $\quad \cdots \to A_{n+1} \xrightarrow{\varphi_{n+1}} A_n \xrightarrow{\varphi_n} A_{n-1} \to \cdots$

di gruppi abeliani e omomorfismi è esatta se $\mathrm{Im}(\varphi_{n+1}) = \mathrm{Ker}(\varphi_n)$ - Dunque e è un complesso di catene con omologia nulla -

<u>Oss</u>: $0 \longrightarrow K \xrightarrow{i} A$ esatta $\Longleftrightarrow$ i iniettiva

$A \xrightarrow{p} Q \longrightarrow 0$ esatta $\Longleftrightarrow$ p suriettiva

<u>Def</u>: $0 \longrightarrow K \xrightarrow{i} A \xrightarrow{p} Q \longrightarrow 0$

è detta esatta corta ; equivale a dire

$K < A$ e $Q = A/K$.

<u>Def</u>: Una successione esatta di mappe tra catene è una successione di mappe tra catene esatte a ogni livello; cioè $\mathcal{C}^{(k)}$ complessi di catene; $\varphi^{(k)} : \mathcal{C}^{(k)} \longrightarrow \mathcal{C}^{(k-1)}$ mappe tra complessi di catene, cioè

$$
\begin{tikzcd}
C_n^{(k)} \arrow[r, "\varphi_m^{(k)}"] \arrow[d, "\partial_n^{(k)}"'] \arrow[dr, phantom, "\circlearrowleft"] & C_m^{(k-1)} \arrow[d, "\partial_m^{(k-1)}"] \\
C_{n-1}^{(k)} \arrow[r, "\varphi_{m-1}^{(k)}"] & C_{m-1}^{(k-1)}
\end{tikzcd}
\quad ,
$$

e per ogni m

$$
\dots \longrightarrow C_m^{(k+1)} \xrightarrow{\varphi_m^{(k+1)}} C_m^{(k)} \xrightarrow{\varphi_m^{(k)}} C_m^{(k-1)} \longrightarrow \dots
$$

è esatta.

Omologia relativa: K complesso simpliciale
L sottocomplesso

$$C_n(K,L) = \langle K^{[n]} \setminus L^{[n]} \rangle$$

È definito: $\partial_n^{(K,L)} : C_n(K,L) \to C_{n-1}(K,L)$

$$\sigma \longmapsto \sum_{\tau \in K^{[n-1]} \setminus L^{[n-1]},\ \tau < \sigma} \varepsilon(\sigma,\tau) \cdot \tau$$

Oss: $0 \longrightarrow C_n(L) \xrightarrow{in} C_n(K) \xrightarrow{Pr} C_n(K,L) \longrightarrow 0$

$$\sigma \longmapsto \sigma \qquad \sigma \longmapsto \begin{cases} 0 & \text{se } \sigma \in L^{[n]} \\ \sigma & \text{altrimenti} \end{cases}$$

dà una successione esatta corta di mappe tra complessi di catene.

Esattezza $\forall n$: ovvio

Mappe tra complessi:

$$\begin{tikzcd}
0 \arrow[r] & C_n(L) \arrow[r, "i_n"] \arrow[d, "\partial_n^L"] & C_n(K) \arrow[r, "p_n"] \arrow[d, "\partial_n^K"] & C_n(K,L) \arrow[r] \arrow[d, "\partial_n^{(K,L)}"] & 0 \\
0 \arrow[r] & C_{n-1}(L) \arrow[r, "i_{n-1}"] & C_{n-1}(K) \arrow[r, "p_{n-1}"] & C_{n-1}(K,L) \arrow[r] & 0
\end{tikzcd}$$

Esercizio

Teo: Sia $0 \to \mathcal{C}' \xrightarrow{i} \mathcal{C} \xrightarrow{p} \mathcal{C}'' \to 0$ succ. esatta corta di mappe tra complessi di catene.

(Scrivo $C_n, C_n', C_n'', Z_n, Z_n', Z_n'', B_n, B_n', B_n''$, H_n, H_n', H_n''.)

Allora c'è una successione esatta lunga

$$\cdots \longrightarrow H_n' \xrightarrow{i_{n*}} H_n \xrightarrow{p_{n*}} H_n'' \xrightarrow{d_n} H_{n-1}' \longrightarrow \cdots$$

Cor: se K è c.s. e $L \subset K$ ho

$$\cdots \longrightarrow H_n(L) \xrightarrow{i_{n*}} H_n(K) \xrightarrow{p_{n*}} H_n(K,L) \xrightarrow{d_n} H_{n-1}(L) \longrightarrow \cdots$$

(∂_n costruito algebricamente in astratto; vedremo nel caso $H_n(K,L) \to H_{n-1}(L)$ geom).

__Dimo__: $i_{n_*} : H_n' \to H_n$ ben def.

$$\begin{tikzcd}
C_n' \arrow[r, "i_n"] \arrow[d, "\partial_n'"] & C_n \arrow[d, "\partial_n"] \\
C_{n-1}' \arrow[r, "i_{n-1}"] & C_{n-1}
\end{tikzcd}$$

$[z] \in H_n'$

$z \in Z_n'$; $z \in C_n'$, $\partial_n' z = 0$

$i_{n_*}([z]) = [i_n(z)]$

Devo vedere: •) $i_n(z) \in Z_n$: $\partial_n(i_n(z)) =$

$= i_{n-1}(\partial'_n z) = i_{n-1}(0) = 0$

•) indip da z: Se $[z_1] = [z_2]$ ho

$$\begin{tikzcd}
w & C'_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "\partial'_{n+1}"] & C_{n+1} \arrow[d, "\partial_{n+1}"] \\
z_1, z_2 & C'_n \arrow[r, "i_n"] & C_n
\end{tikzcd}$$

$z_1 - z_2 \in B'_n$ cioè

$z_1 - z_2 = \partial'_{n+1}(w)$

$\Rightarrow i_n(z_1) - i_n(z_2) =$

$= i_n(\partial'_{n+1} w) = \partial_{n+1}(i_{n+1}(w))$

$\Rightarrow i_n(z_1) - i_n(z_2) \in B_n$

$\Rightarrow [i_n(z_1)] = [i_n(z_2)]$ —

$p_{n*} : H_n \to H_n''$ ben def

uguale

Esattezza al livello H_n

$$H_n' \xrightarrow{i_{n*}} H_n \xrightarrow{p_{n*}} H_n''$$

$p_{n*} \circ i_{n*} = 0$ cioè $\mathrm{Im}(i_{n*}) \subset \mathrm{Ker}(p_{n*})$

In generale $f_* \circ g_* = (f \circ g)_*$

ma $p_n \circ i_n = 0 \Rightarrow (p_n \circ i_n)_* = 0$

$\mathrm{Ker}(p_{n*}) \subset \mathrm{Im}(i_{n*})$

$[z] \in \mathrm{Ker}(p_{n*}) \Rightarrow \exists w \in C''_{n+1}$ t.c. $p_n(z) = \partial''_{n+1}(w)$

$\Rightarrow \exists u \in C_{n+1}$ t.c. $\partial''_{n+1}(p_{n+1}(u)) = p_n(z)$

$\Longrightarrow \exists u \in C_{n+1}$ t.c. $p_n(\partial_{n+1}(u)) = p_n(z)$

$\Longrightarrow \exists u \in C_{n+1}$ t.c. $z - \partial_{n+1} u \in \operatorname{Ker} p_n = \operatorname{Im} i_n$

$\Longrightarrow \exists u \in C_{n+1},\ v \in C'_n$ t.c. $z - \partial_{n+1} u = i_n(v)$

$\Longrightarrow [z] = [z - \partial_{n+1} u] = [i_n(v)] = i_{n*}([v])$

OK se

so che $v \in Z'_n$

cioè $\partial'_n v = 0$

$$i_{n-1}(\partial'_n(v)) = \partial_n(i_n(v))$$

$$= \partial_n(z - \partial_{n+1} u) = 0 + 0 = 0$$

$$\Rightarrow \partial'_n v = 0$$

$$\begin{tikzcd}
 & & u\ C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}"] & w\ C''_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"] & 0 \\
 & v\ C'_n \arrow[r, "i_n"] \arrow[d, "\partial'_n"] & z\ C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & C''_n & \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] & C_{n-1} & &
\end{tikzcd}$$

Definizione di $\partial_n : H_n'' \longrightarrow H_{n-1}'$

$z \in Z_n'' \subset C_n''$; $\exists w \in C_n$ t.c. $z = p_n(w)$;

$p_{n-1}(\partial_n w) = \partial_n''(p_n(w)) = \partial_n''(z) = 0$

$\Longrightarrow \partial_n w \in \mathrm{Ker}(p_{n-1}) = \mathrm{Im}(i_{n-1})$

$\Longrightarrow \exists!\, u \in C_{n-1}'$ t.c. $\partial_n w = i_{n-1} u$ _

Voglio definire $\partial_n([z]) = [u]$ _

$$
\begin{tikzcd}
 & & C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & C''_n \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & C'_{n-1} \arrow[r, "i_{n-1}"] & C_{n-1} \arrow[r, "p_{n-1}"] & C''_{n-1} &
\end{tikzcd}
$$

w z

u

$\partial'_{n-1} u = 0$

$$\begin{tikzcd}
 & & {}_{w}C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & {}_{z}C''_n \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & {}_{u}C'_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "\partial'_{n-1}"] & C_{n-1} \arrow[r, "p_{n-1}"] \arrow[d, "\partial_{n-1}"] & C''_{n-1} & \\
0 \arrow[r] & C'_{n-2} \arrow[r, "i_{n-2}"] & C_{n-2} & &
\end{tikzcd}$$

$$i_{n-2}(\partial'_{n-1} u) = \partial_{n-1}(i_{n-1} u) = \partial_{n-1}\partial_n w = 0$$

$\Longrightarrow \partial'_{n-1} u = 0 \longrightarrow [u] \in H'_{n-1}$ ben def.

Indipendenza da w

$p_n(w_1) = p_n(w_2) = z \Longrightarrow p_n(w_1 - w_2) = 0$

$\Longrightarrow \exists\, v \in C'_n$ t.c. $w_1 - w_2 = i_n(v)$.

Siano $u_1, u_2 \in C'_{n-1}$ t.c. $i_{n-1}(u_j) = \partial_n(w_j)$;

$i_{n-1}(u_2 + \partial'_n v) = \partial w_2 + \partial(w_1 - w_2) = \partial w_1$

$$\Longrightarrow u_1 = u_2 + \partial_n' v \Longrightarrow [u_1] = [u_2] \in H_{n-1}'$$

$$\begin{tikzcd}
 & v \ C_n' \arrow[r, "i_n"] \arrow[d, "\partial_n'"] & C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] \arrow[l, phantom, "w_1 - w_2"] & C_n'' \arrow[r] \arrow[d, "\partial_n''"] & 0 \\
0 \arrow[r] & C_{n-1}' \arrow[r, "i_{n-1}"] & C_{n-1} \arrow[r, "p_{n-1}"] & C_{n-1}'' &
\end{tikzcd}$$

Indipendenza da Z

Sia $z_2 = z_1 + \partial''_{n+1} x$; sia $y \in C_{n+1}$ t.c.

$x = p_{n+1} y$

Sia w_1 con $p_n(w_1) = z_1$;

$$
\begin{tikzcd}
C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}"] & C''_{n+1} \arrow[r] \arrow[d, "\partial''_{n+1}"] & 0 \\
C_n \arrow[r, "p_n"] & C''_n \arrow[r] & 0
\end{tikzcd}
$$

y x z_1, z_2

$p_n(w_1 + \partial_{n+1} y)$

$= z_1 + \partial''_{n+1} x = z_2$

$\Rightarrow$ posso scegliere $w_2 = w_1 + \partial_{n+1} y$

$$\begin{tikzcd} & C_n \arrow[r, "\pi_n"] \arrow[d, "\partial_n"] & C''_n \arrow[r] & 0 \\ C'_{n-1} \arrow[r, "i_{n-1}"] & C_{n-1} & & \end{tikzcd}$$

w_1

$w_1 + \partial_{n+1} y$

z_1

z_2

M

$\partial_n w_1 = \partial_n (w_1 + \partial_{n+1} y)$

Si può usare lo stesso u.

Esattezza in H_n''; $H_n \xrightarrow{p_{n*}} H_n'' \xrightarrow{d_n} H_{n-1}'$

$\operatorname{Im} p_{n*} \subset \operatorname{Ker} d_n$ cioè $d_n \circ p_{n*} = 0$

Devo calcolare $d_n(\underbrace{p_{n*}([W])}_{[Z]})$

Ho $\partial_n W = 0 \implies$ posso (devo) scegliere $u = 0$

$$
\begin{tikzcd}
 & & w \; C_n \arrow[r, "p_n"] \arrow[d, "\partial_n"] & z \; C_n'' \arrow[r] & 0 \\
0 \arrow[r] & v \; C_{n-1}' \arrow[r, "i_{n-1}"] & C_{n-1} & &
\end{tikzcd}
$$

$$\Rightarrow d_n(p_{n*}([w])) = 0$$

$$\mathrm{Ker}\, d_n \subset \mathrm{Im}\, p_{n*}$$

$[z] \in \operatorname{Ker} d_n \Rightarrow \exists\, v \in C_n' ,\ w \in C_n ,\ u \in C_{n-1}'$

$p_n(w) = z,\ i_{n-1}(u) = \partial_n w,\ u = \partial_n' v$

$$\begin{tikzcd}
0 \arrow[r] & \underset{v}{C_n'} \arrow[r, "i_n"] \arrow[d, "\partial_n'"] & \underset{w}{C_n} \arrow[r, "p_n"] \arrow[d, "\partial_n"] & \underset{z}{C_n''} \arrow[r] & 0 \\
0 \arrow[r] & \underset{u}{C_{n-1}'} \arrow[r, "i_{n-1}"] & C_{n-1} & &
\end{tikzcd}$$

$\Rightarrow \partial_n w = i_{n-1}(\partial_n'(v)) = \partial_n(i_n(v))$

$\partial_n(w - i_n(v)) = 0 \qquad p_n(w - i_n v) = z$

$\Rightarrow [z] = p_{n_*}([w - i_n v])$

Esattezza in H_n': $H_n'' \xrightarrow{d_{n+1}} H_n' \xrightarrow{i_{n*}} H_n$

$\underline{I}m(d_{n+1}) \subset \operatorname{Ker} i_{n*}$

Sia $d_{n+1}([z]) = [u]$

$$\Longrightarrow \quad i_{n*}([u]) = [i_n(u)] = [\partial_{n+1} w] = 0$$

$$\begin{tikzcd} & w \, C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}"] & z \, C''_{n+1} \\ u \, C'_n \arrow[r, "i_n"] & C_n & \end{tikzcd}$$

$\mathrm{Ker}(i_{n*}) \subset \mathrm{Im}(d_{n+1})$

Sia $i_{n*}([u]) = 0 \Longrightarrow$

$\exists x \in C_{n+1}$ t.c. $i_n(u) = \partial_{n+1}(x)$

Dico che $\partial''_{n+1}(p_{n+1}(x)) = 0$ da cui

$[u] = d_{n+1}(\underbrace{[p_{n+1}(x)]}_{\in H''_{n+1}})$

Infatti: $\partial''_{n+1}(p_{n+1}(x)) = p_n(\partial_{n+1} x) = p_n(i_n(x)) = 0$

$$
\begin{tikzcd}
 & x \; C_{n+1} \arrow[r, "p_{n+1}"] \arrow[d, "\partial_{n+1}"] & C''_{n+1} \arrow[d, "\partial''_{n+1}"] \\
x \; C'_n \arrow[r, "i_n"] & C_n \arrow[r, "p_n"] & C''_n
\end{tikzcd}
$$

$\blacksquare$

Successione esatta lunga di omologie