ETA 21/11/13

$\mathbb{P}^n(\mathbb{C}) = \mathbb{C}^n \cup \mathbb{P}^{n-1}(\mathbb{C})$

$D^{2n} \ni y \longmapsto \dfrac{y}{1-\|y\|}$ $\qquad (\partial D^{2n} \longrightarrow \mathbb{P}^{n-1}(\mathbb{C}))$

$\mathbb{P}^n(\mathbb{C}) = \mathbb{P}^{n-1}(\mathbb{C})$ con attaccata una $2n$-cella

$\Rightarrow C(\mathbb{P}^n(\mathbb{C}))$:

$$\begin{array}{ccccccccccccc} 5 & & 4 & & 3 & & 2 & & 1 & & 0 & & \\ 0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to & 0 & \to & \mathbb{Z} & \to & 0 \end{array}$$

$$\Rightarrow H_k(\mathbb{P}^n(\mathbb{C})) = \begin{cases} \mathbb{Z} & 0 \le k \le 2n,\ k \equiv 0\,(2) \\ 0 & \text{altrimenti} \end{cases}$$

$\mathbb{P}^n(\mathbb{R})$ = ottenuto da $\mathbb{P}^{n-1}(\mathbb{R})$ attaccando una n-cella

$$C(\mathbb{P}^n(\mathbb{R})) : \quad \overset{n+1}{0} \quad \overset{n}{\mathbb{Z}} \to \overset{n-1}{\mathbb{Z}} \to \quad \cdots \quad \overset{2}{\mathbb{Z}} \to \overset{1}{\mathbb{Z}} \to \overset{0}{\mathbb{Z}} \to 0$$

$$\mathbb{P}^n(\mathbb{R}) = \{ [x_1 : \dots : x_{n+1}] \ : \ \dots \}$$

$$\mathbb{P}^k(\mathbb{R}) = \{[x_1 : \dots : x_{k+1} : 0 : \dots : 0]\}$$

$$G_k : D^k \longrightarrow \mathbb{P}^k(\mathbb{R}) \subset \mathbb{P}^n(\mathbb{R})$$

$$(y_1, \dots, y_k) \longmapsto [y_1 : \dots : y_k : \sqrt{1 - \|y\|} : 0 \dots 0]$$

$g_k = G_k|_{S^{k-1}}$; devo calcolare

$$\varepsilon(g_{k+1}, g_k) = \deg\left(S^k \xrightarrow[(1)]{} \mathbb{P}^k(\mathbb{R}) \xrightarrow[(2)]{} \mathbb{P}^k(\mathbb{R}) / \mathbb{P}^{k-1}(\mathbb{R})\right.$$

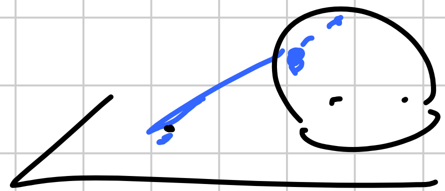
$$\quad \Big| (3)$$

$$\downarrow G_k^{-1}$$

$$\left(S^k \xleftarrow[(4)]{\varphi_k} D^k/S^{k-1} \right.$$

(4) $\varphi_k : D^k/S^{k-1} \to S^k$

$$[x] \longmapsto \frac{x}{1-\|x\|} \longmapsto \frac{\left(2x(1-\|x\|),\, 2\|x\|-1\right)}{2\|x\|^2 - 2\|x\| + 1}.$$



(3) $\bar{G}_k^{-1}$

$$G_k : D^k \longrightarrow \mathbb{P}^k(\mathbb{R})$$
$$(y_1, \dots y_k) \longmapsto [y_1 : \dots : y_k : \sqrt{1-\|y\|^2} : 0 \dots 0]$$

$$\begin{tikzcd}
D^k \arrow[r, "G_k"] \arrow[d] & \mathbb{P}^k(\mathbb{R}) \arrow[d] & {[z]} \arrow[d, mapsto] \\
D^k/S^{k-1} \arrow[r, "\bar{G}_k"] & \mathbb{P}^k(\mathbb{R})/\mathbb{P}^{k-1}(\mathbb{R}) & [\![z]\!]
\end{tikzcd}$$

$\bar{G}_k^{-1}([\![z]\!]) = [y]$ se

$$[y_1 : \dots : y_k : \sqrt{1-\|y\|^2}\,] = [z_1 : \dots : z_k : z_{k+1}]$$

cioè $\dfrac{y}{\sqrt{1-\|y\|^2}} = \dfrac{(z_1 \dots z_k)}{z_{k+1}} \Rightarrow \dots \Rightarrow \sqrt{1-\|y\|^2} = \dfrac{|z_{k+1}|}{\|z\|}$

$$\Rightarrow [y] = \left[(z_1 \dots z_k) \cdot \frac{|z_{k+1}|}{z_{k+1}} \cdot \frac{1}{\|z\|}\right]$$

$$= \left[\frac{(z_1 \dots z_k)}{\|z\|} \cdot \mathrm{sgn}(z_{k+1})\right]$$

Oss: è ok perché per $z_{k+1} = 0$ trovo $\dfrac{z_1 \dots z_k}{\|z\|} \in S^{k-1}$.

$$S^k \ni (y_1, \dots, y_{k+1}) \xmapsto{(1)} [y_1 : \dots : y_{k+1} : \sqrt{1 - \|y\|^2} : 0 \dots]$$
$$= [y_1 : \dots : y_{k+1} : 0 : 0 \dots]$$
$$\xmapsto{(2)} [\![y_1 : \dots : y_{k+1}]\!]$$
$$\xmapsto{(3)} [(y_1, \dots, y_k) \cdot \operatorname{sgn}(y_{k+1})]$$

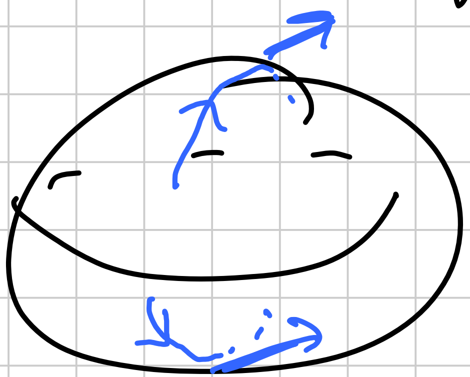
Oss: tutte le S^{k-1} equatoriale di equaz $y_{k+1} = 0$ finisce in $(0,1) \in S^k$; le restrizioni a

$y_{k+1} \geq 0$ e

$y_{k+1} \leq 0$

$$\overset{(4)}{\longmapsto} \quad \frac{\left(\pm 2(y_1 \cdots y_k)\left(1-\sqrt{y_1^2 \cdots y_k^2}\right),\ 2\sqrt{y_1^2+\cdots+y_k^2}-1\right)}{2\left(y_1^2+\cdots+y_k^2\right)-2\sqrt{y_1^2+\cdots y_k^2}+1}$$

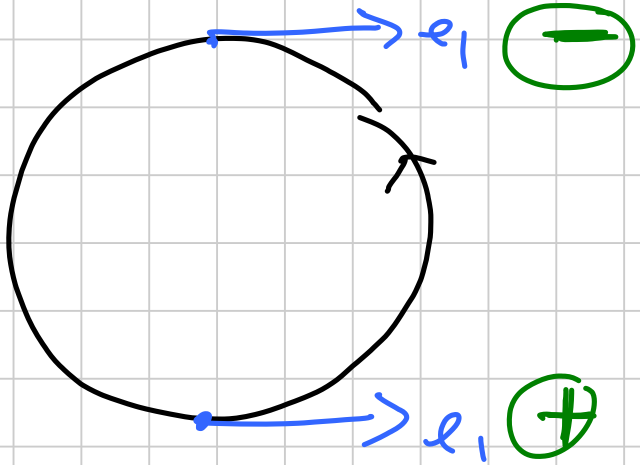
Il valore $(0,-1)$ è assunto precisamente una volta da ciascuna delle due restrizioni, risp. in $(0,1)$ e in $(0,-1)$. Se ristringo alla curva $y_j \longmapsto (0,\dots,y_j,0\dots0,\pm\sqrt{1-y_j^2})$

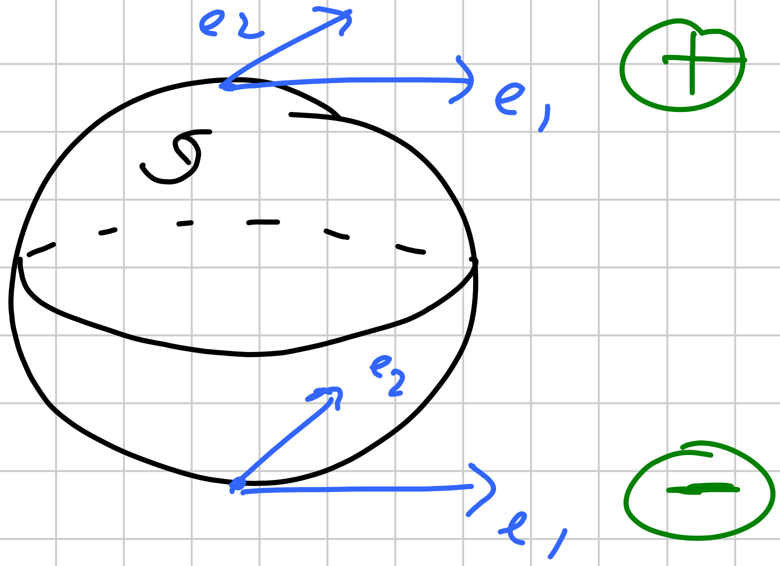


trovo

$$\left(0\dots0,\ \frac{\pm 2y_j(1-|y_j|)}{\underbrace{2y_j^2-2y_j+1}_{\pm 2y_j+o(y_j)}},0\dots,0,\ \frac{2|y_j|-1}{\underbrace{2y_j^2-2y_j+1}_{1+o(y_j)}}\right)$$

$\Rightarrow$ $e_1, \dots, e_k$ viene mandato in $2e_1, \dots, 2e_k$ dalla prima restrizione, $-2e_1, \dots, -2e_k$ dalla seconda. Notiamo che $e_1, \dots, e_k$ è base di segno $(-1)^k$ in $(0,1)$ e $(-1)^{k+1}$ in $(0,-1)$:





Grado I restrit:

$e_1, \dots e_k$ in $(0,1)$ $\longrightarrow$ $2e_1, \dots, 2e_k$ in $(0,-1)$

segno $(-1)^k$ segno $(-1)^{k+1}$

-1

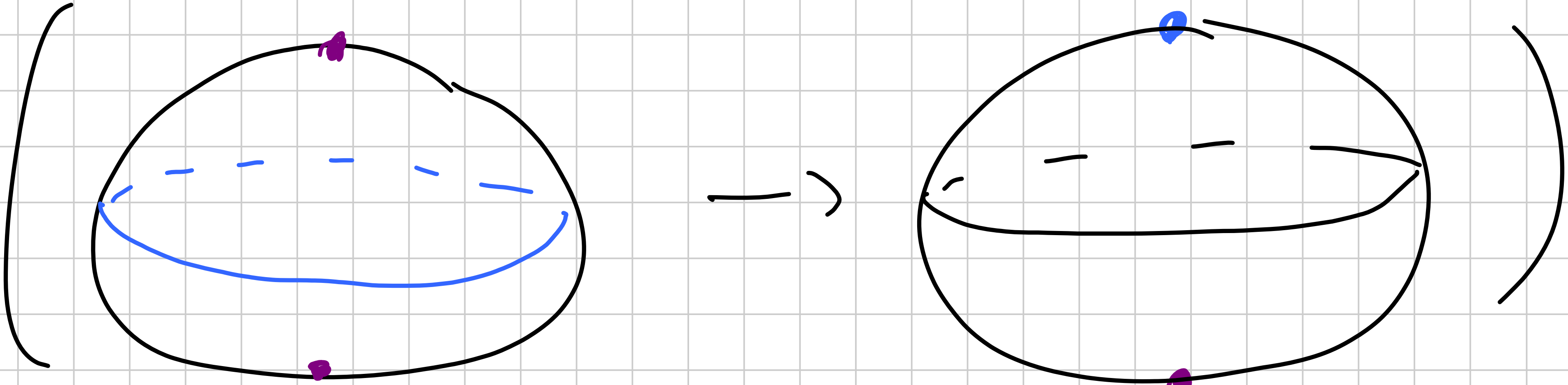
Grado II esercizi:

$e_1 \dots e_n \longmapsto -2e_1, \dots, -2e_n$

in $(0,-1)$ in $(0,-1)$

$(-1)^k$

$$\Longrightarrow \mathcal{E}(g_{k+1}, g_k) = (-1) + (-1)^k = \begin{cases} 0 & k \text{ pari} \\ -2 & k \text{ dispari} \end{cases}$$



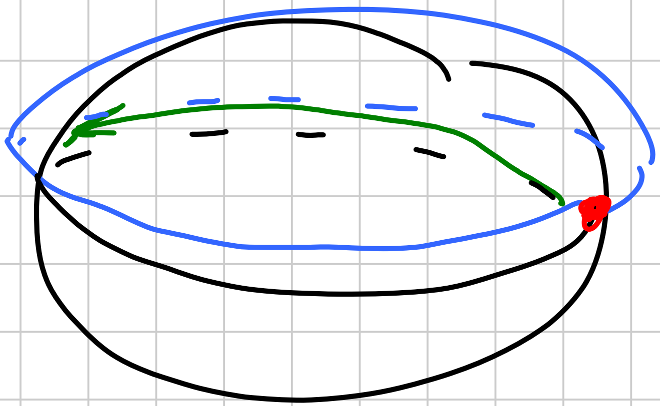
$$
\begin{array}{llllllll}
 & 4 & & 3 & & 2 & & 1 & & 0 & & \\
C: & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{2\cdot} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 \\
H & & & \mathbb{Z}/2 & & 0 & & \mathbb{Z}/2 & & \mathbb{Z} & &
\end{array}
$$

$$
\Longrightarrow H_k(\mathbb{P}^n(\mathbb{R})) = \begin{cases} \mathbb{Z} & k=0 \text{ e se } k=n \text{ dispari} \\ 0 & k \text{ pari} \\ \mathbb{Z}/2 & \text{altrimenti.} \end{cases}
$$

Più informalmente: $\quad P^n(\mathbb{R}) = S^n/_{x \sim -x}$

$G_k : D^k \to P^k(\mathbb{R})$ si solleva a

$\tilde{G}_k : D^k \to S^n$ $\quad \tilde{G}_k(x_1 \dots x_k) = \left(x_1 \dots x_k, \sqrt{1-\|x\|^2}, 0 \dots\right)$



Vedo S^k come unione di due D^k

$E_k^{\pm} : D^k \to S^k$

$E_k(x_1 \dots x_k) = \left(x_1 \dots x_k, \pm\sqrt{1-\|x\|^2}\right)$

$$\tilde{g}_{k+1} \circ E_k^+ = \tilde{G}_k$$

(E_k^+ ha orientamento giusto, E_k^- sbagliato)

$$\tilde{g}_{k+1} \circ E_k^- = \tilde{G}_k \circ (-id_{D^k})$$

$$\Rightarrow \varepsilon(g_{k+1}, g_k) = 1 + \underbrace{(-1)}_{\text{da } E_k^-} \cdot \underbrace{(-1)^k}_{\deg(-id_{D^k})}$$

$\Rightarrow$ stesso risultato.

$\blacksquare$

Fatto: Se vedo un complesso simpliciale (o un Δ-complesso) come CW complesso, l'omologia simpliciale è uguale all'omologia cellulare.

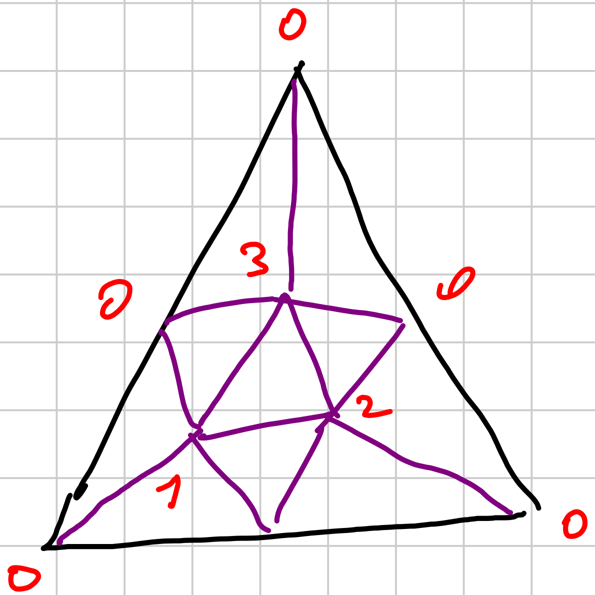
I complessi di catene sono gli stessi: devo vedere che i bordi sono uguali. Serve:

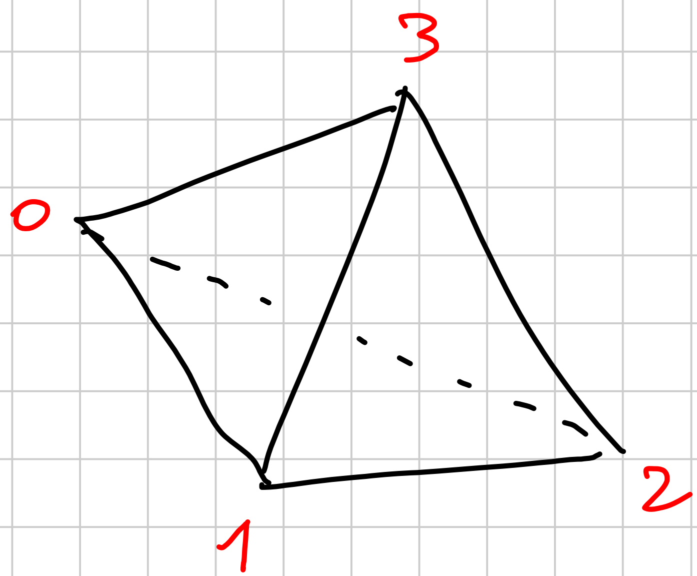
versione PL dell'omeo $D^k/S^{k-1} \longrightarrow S^k$,

ovvero

$$\Delta_k/\partial\Delta_k \longrightarrow \partial\Delta_{k+1}$$

$k=2$





<u>Esercizio</u>: usando questo fatto verificare che

$$\deg\left(\tau^{-1}\circ\sigma\big|_{\partial}\right) = \varepsilon(\sigma,\tau)$$

indico con σ, τ sia i simplessi sia le loro parametrizzazioni.

Caratteristica di Eulero

Lem: Sia $\mathcal{C} = \{(C_n, \partial_n)\}$ un complesso di catene con C_n $\mathbb{F}$-spazio vett. e $C_n = 0$ per $n \gg 0$.

(Oss: gli $H_n(\mathcal{C})$ sono $\mathbb{F}$-spazi vett.)

$$\Longrightarrow \sum_{i=0}^{\infty} (-1)^i \dim(C_i) = \sum_{i=0}^{\infty} (-1)^i \dim(H_i(\mathcal{C})).$$

$\underline{Dim}$: $\to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$

$Z_n = \operatorname{Ker} \partial_n \qquad B_n = \operatorname{Im} \partial_{n+1}$

$\Rightarrow \dim C_n = \dim Z_n + \dim B_{n-1}$;

$H_n(\mathcal{C}) = Z_n / B_n \Rightarrow \dim H_n = \dim Z_n - \dim B_n$

$$\sum_i (-1)^i \dim C_i = \sum (-1)^i \dim Z_i + \underbrace{\sum_i (-1)^i \dim B_{i-1}}_{-\sum (-1)^i \dim B_i}$$

$$= \sum (-1)^i \dim H_i . \quad \boxtimes$$

Def: $\mathrm{rank}(\mathbb{Z}^p) = p$; A gruppo abeliano

$\mathrm{Tor}(A) = \{ a \in A : \exists n \in \mathbb{Z} \text{ t.c. } n \cdot a = 0 \}$;

$\mathrm{rank}(A) := \mathrm{rank}(A/\mathrm{Tor}(A))$.

Teorema: Se $\mathcal{C} = \{(C_n, \partial_n)\}_n$ è un complesso di gruppi abeliani finitamente generati e liberi, e $C_n = 0$ per $n >> 0$ allora

$$\sum_i (-1)^i \mathrm{rank}(C_i) = \sum_i (-1)^i \mathrm{rank}(H_i(\mathcal{C})).$$

Cor: se X sp. top è realizzato come complesso simpliciale finito o CW complesso finito,

$\chi(X) := \sum (-1)^i \left(\# X^{[i]}\right)$ è un invariante di X.

Es: $\chi(S^n) = 1 + (-1)^n = \begin{cases} 2 & n \equiv 0 \ (2) \\ 0 & \text{---} \end{cases}$

$\phantom{Es: \chi(S^n) = {}} \underset{\text{una 0-cella}}{1} \underset{\text{una } n\text{-cella}}{(-1)^n}$

$\chi(\mathbb{P}^n(\mathbb{R})) = 1 - 1 + 1 \dots (-1)^n = \begin{cases} 1 & n \equiv 0 \ (2) \\ 0 & \text{---} \end{cases}$

Cor: $S^3 \cong P^3$ (anche senza usare π_1).

Dimo Teo: Come per sp. vett. basta vedere che

1) $\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$

(2) $\operatorname{rank}(H_i) = \operatorname{rank}(Z_i) - \operatorname{rank}(B_i)$

(1) facile: $B_{i-1} \cong C_i/Z_i$ ma sono abeliani liberi

$C_i \cong Z_i \oplus B_{i-1} \Rightarrow Z_i, B_{i-1}$ liberi

$\Rightarrow \dots$

Per (2) serve la nozione di $\otimes$ di gruppi.

G, H gruppi abeliani:

$G \otimes H :=$ gruppo libero generato da $G \times H$

modulo $(g_1 + g_2, h) \sim (g_1, h) + (g_2, h)$ $\forall \ldots$

$(g, h_1 + h_2) \sim (g, h_1) + (g, h_2)$

Indico l'immagine di (g, h) con $g \otimes h$. Facile:

Prop: $G \otimes H$ a meno di isomorfismo è caratterizzato dalla proprietà universale:

- $\exists \, \varphi_0 : G \times H \to G \otimes H$ omomorfismo dx/sx
- $\forall \, \varphi : G \times H \to Z$ omomorfismo dx/sx esiste unico un omomorfismo $\tilde{\varphi} : G \otimes H \to Z$ t.c.

$$\begin{tikzcd} G \times H \arrow[r, "\varphi_0"] \arrow[dr, "\varphi"'] & G \otimes H \arrow[d, "\tilde{\varphi}"] \\ & Z \end{tikzcd}$$

commuta

Dim: $\varphi_0(g,h) = g \otimes h$; $\tilde{\varphi}$ definito estendendo
$\tilde{\varphi}(g \otimes h) = \varphi(g,h)$. [...]. $\boxtimes$

Estensione: • R anello commutativo con 1
(come campo senza inversi moltiplicativi)

• M modulo su R (come sp. vett. su R); anche N

• $M \otimes_R N = R\langle M \times N \rangle = \left\{ \sum_{i=1}^{k} r_i (m_i, n_i) \right\}$
modulo

$$(r_1 m_1 + r_2 m_2, n) \sim r_1(m_1, n) + r_2(m_2, n)$$

$$(\quad \cdot \quad - \quad - \quad - \quad \sim (\quad) \quad (\quad))$$

Proprietà univ: • $\exists \varphi_0 : M \times N \to M \otimes_R N$

R-bilineare

• $\forall \varphi : M \times N \to Z$ R-bil $\exists ! \tilde{\varphi} : M \otimes_R N \to Z$

R-lin t.c. $\varphi = \tilde{\varphi} \circ \varphi_0$.

<u>Oss</u>: $\mathbb{Z}$-modulo $\Longleftrightarrow$ gruppo abeliano.

Oss: G, H gruppi ab. $G \otimes H = G \otimes_{\mathbb{Z}} H$.

Proprietà: (1) $G \otimes H \cong H \otimes G$

(2) $(G_1 \oplus G) \otimes H \cong (G_1 \otimes H) \oplus (G_2 \otimes H)$

(3) $G \otimes \mathbb{Z} \cong G$ (4) $\mathbb{Z}/_m \otimes \mathbb{Q} = 0$

Dim: (1),(2) ✓

(3)
$$
\begin{aligned}
G \otimes \mathbb{Z} &\longleftrightarrow G \\
g \otimes n &\longmapsto n \cdot g \\
g \otimes 1 &\longleftarrow\!\shortmid\ g
\end{aligned}
$$

(4)
$$
[j] \otimes x = [j] \otimes m \cdot \frac{x}{m} = m \cdot \left([j] \otimes \frac{x}{m} \right) = \underbrace{[m \cdot j]}_{0} \otimes \frac{x}{m} . \square
$$

Cor: G abeliano fin. generato

$$\Rightarrow \operatorname{rank}(G) = \dim_{\mathbb{Q}} (G \otimes \mathbb{Q}).$$

Infatti: $G = \mathbb{Z}^r \oplus \underbrace{\mathrm{Tor}(G)}_{\mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_p\mathbb{Z}}$

$$\Rightarrow G \otimes \mathbb{Q} = \big(\underbrace{\mathbb{Z} \otimes \mathbb{Q}}_{\mathbb{Q}}\big)^r \oplus_i \big(\underbrace{\mathbb{Z}/m_i\mathbb{Z} \otimes \mathbb{Q}}_{0}\big) = \mathbb{Q}^r$$

Dovevamo provare: $\mathrm{rank}(H_i) = \mathrm{rank}(Z_i) - \mathrm{rank}(B_i)$ dove $H_i = Z_i / B_i$; allora basta vedere:

<u>Prop</u>: $(G/H) \otimes \mathbb{Q} \cong (G \otimes \mathbb{Q}) / (H \otimes \mathbb{Q})$

<u>Dim</u>: $H \otimes \mathbb{Q} \subset G \otimes \mathbb{Q}$. Sia:

$$\varphi : G \otimes \mathbb{Q} \longrightarrow (G/H) \otimes \mathbb{Q}$$

$$g \otimes x \longmapsto [g] \otimes x$$ estesa sulle somme (formalmente $(g, x) \longmapsto [g] \otimes x$

+ usare prop. univ) .

Ben def, $\mathbb{Q}$-lin, surgettiva; $\operatorname{Ker}\psi \supset H \otimes \mathbb{Q}$.

Sia $\sum_{i=1}^{n} g_i \otimes x_i \in \operatorname{Ker}\psi$. Sia $N \in \mathbb{Z}$ t.c. $N x_i = p_i \in \mathbb{Z}$;

$$\sum g_i \otimes x_i = \sum g_i \otimes \frac{1}{N} p_i = \frac{1}{N} \sum g_i \otimes p_i$$

$$= \frac{1}{N} \sum p_i \cdot g_i \otimes 1 = \frac{1}{N} \sum (p_i \cdot g_i) \otimes 1$$

$$= \sum (p_i g_i) \otimes \frac{1}{N} = \left(\sum p_i g_i \right) \otimes \frac{1}{N}$$

Poiché è in $\operatorname{Ker}\psi$ ho $\sum p_i g_i \in H$

$$\Rightarrow \underbrace{\left(\sum_i (R_i g_i)\right) \otimes \frac{1}{n}}_{\text{è quello iniziale.}} \in H \otimes \mathbb{Q}$$

$\boxtimes$