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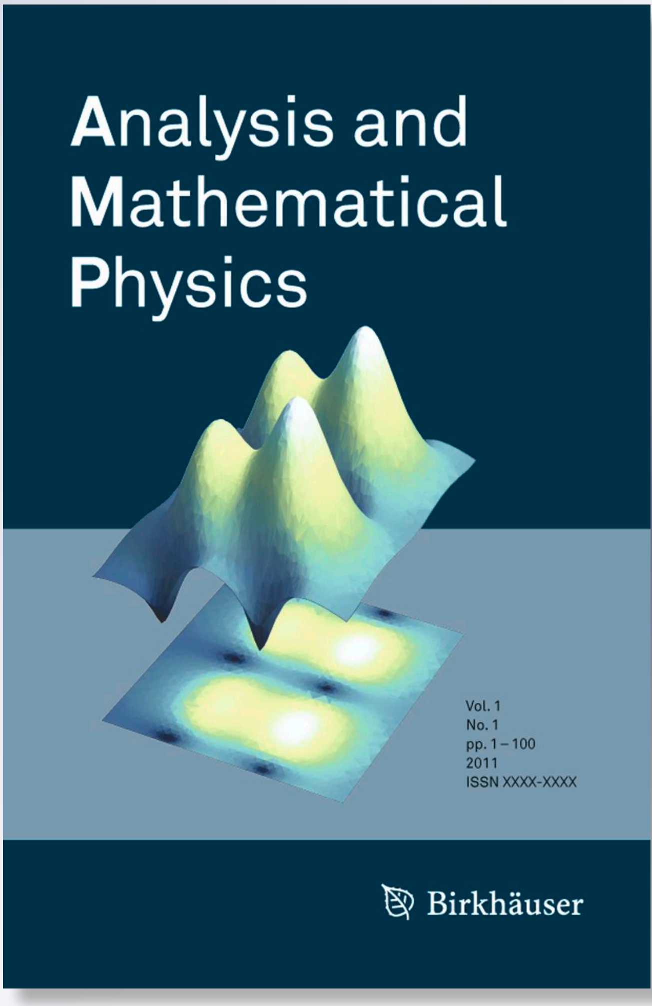

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Abstract This paper contains a selection, dictated by personal taste and by no means complete, of open problems in local discrete holomorphic dynamics.

Keywords Local discrete dynamical systems · Holomorphic dynamical systems · Normal form · Germs tangent to the identity · Parabolic dynamics

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1 Introduction

The aim of this paper is to collect and put in context a few open problems in the area of local holomorphic discrete dynamics; so let us begin by defining what is a discrete local holomorphic dynamical system.

Definition 1 Let M be a complex manifold, and $p \in M$. A *discrete holomorphic local dynamical system* at p is a holomorphic map $f: U \to M$ such that $f(p) = p$, where $U \subseteq M$ is an open neighborhood of p; we shall also always assume that $f \not\equiv \mathrm{id}_U$. We shall denote by $\mathrm{End}(M, p)$ the set of holomorphic local dynamical systems at p.

We shall be mainly concerned with the behavior of f nearby p, and thus $\mathrm{End}(M, p)$ actually is the set of germs of holomorphic self-maps of M at p; for this reason we shall often use the word "germ" as an abbreviation for "discrete holomorphic local dynamical system", and we shall allow us to restrict the domain of $f \in \mathrm{End}(M, p)$ to any suitable neighborhood of p whenever useful.

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In the next sections we shall discuss some specific open problems, selected according to our taste and with no pretense of completeness; but in this introduction we shall instead present the most basic questions one can ask about a local discrete dynamical system. Claiming to have completely understood a given local dynamical systems amounts to having complete answers to at least the first two questions below; and claiming to have completely understood a given *class* of discrete holomorphic local dynamical systems most of the times amounts to having answers to the remaining three questions.

First of all, a local dynamical system is not a dynamical system in the standard (global) sense of the word, because points can escape from the domain of definition. However, it is easy to associate a *bona-fide* dynamical system with any local dynamical system. The phase space of this new dynamical system is the stable set:

Definition 2 Let $f \in \mathrm{End}(M, p)$ be a (discrete holomorphic) local dynamical system defined on an open set $U \subseteq M$. Then the *stable set* K_f of f is

$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U).$$

In other words, the stable set of f is the set of all points $z \in U$ such that the *orbit* $\{f^k(z) \mid k \in \mathbb{N}\}$ is well-defined, where f^k denotes the kth iterate of f. If $z \in U \setminus K_f$, we shall say that z (or its orbit) *escapes* from U.

Clearly, $p \in K_f$, and so the stable set is never empty (but it can happen that $K_f = \{p\}$). It depends a priori on U, but again in most cases we shall be interested only in the behavior nearby p. Thus the first natural question in discrete holomorphic local dynamics is:

(Q1) *What is the topological structure of (the germ at* p *of)* K_f*?*

For instance, does K_f have non-empty interior? Is it locally connected at p? Is $K_f \setminus \{p\}$ connected? What is the topological (homological, cohomological) structure of $U \setminus K_f$? And so on.

Remark 1 Both the definition of stable set and Question 1 are topological in character; we might also state them for local dynamical systems which are continuous only. However, the *answers* might (and usually will) strongly depend on the holomorphicity of the dynamical system.

Clearly, the stable set K_f is completely f-invariant, and thus the pair (K_f, f) is the promised discrete global dynamical system, canonically associated with the given discrete local dynamical system. In particular, the second natural question in discrete holomorphic local dynamics is

(Q2) *What is the dynamical structure of* (K_f, f)*?*

For instance, what is the asymptotic behavior of the orbits? Do they converge to p, or have they a chaotic behavior? Is there a dense orbit? Do there exist proper f-invariant subsets, that is sets $L \subset K_f$ such that $f(L) \subseteq L$? If they do exist, what is the dynamics on them?

To answer all these questions, one of the most efficient ways is to replace f by a dynamically equivalent but simpler (e.g., linear) system g. In our context, "dynamically equivalent" means "locally conjugated"; and we have different kinds of conjugacy to consider.

Definition 3 Let $f_1\colon U_1 \to M_1$ and $f_2\colon U_2 \to M_2$ be two holomorphic local dynamical systems at $p_1 \in M_1$ and $p_2 \in M_2$ respectively. We shall say that f_1 and f_2 are *holomorphically* (respectively, *smoothly,* C^k with $k \in \mathbb{N}^*$, or *topologically*) *locally conjugated* if there are open neighborhoods $W_1 \subseteq U_1$ of p_1, $W_2 \subseteq U_2$ of p_2, and a biholomorphism (respectively, a C^∞ diffeomorphism, a C^k diffeomorphism, or a homeomorphism) $\varphi\colon W_1 \to W_2$ with $\varphi(p_1) = p_2$ such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi$$

on $\varphi^{-1}\big(W_2 \cap f_2^{-1}(W_2)\big) = W_1 \cap f_1^{-1}(W_1)$.

If $f_1\colon U_1 \to M_1$ and $f_2\colon U_2 \to M_2$ are locally conjugated we clearly have

$$K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}});$$

so the local dynamics of f_1 about p_1 is to all purposes equivalent (up to the order of smoothness of the local conjugation) to the local dynamics of f_2 about p_2.

In particular, using local coordinates centered at $p \in M$ it is easy to show that any holomorphic local dynamical system at p is holomorphically locally conjugated to a holomorphic local dynamical system at $O \in \mathbb{C}^n$, where $n = \dim M$; so from now on we shall mostly work only with $\mathrm{End}(\mathbb{C}^n, O)$.

Whenever we have an equivalence relation in a class of objects, classification problems come out. So the third natural question in local holomorphic dynamics is

(Q3) *Find a (possibly small) class $\mathcal{F}$ of holomorphic local dynamical systems at $O \in \mathbb{C}^n$ such that every holomorphic local dynamical system f at a point in an n-dimensional complex manifold is holomorphically (respectively, smoothly, C^k, or topologically) locally conjugated to a (possibly) unique element of $\mathcal{F}$, called holomorphic (respectively, smooth, C^k or topological) normal form of f.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathcal{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

(Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects, called invariants, to any holomorphic local dynamical system f at $O \in \mathbb{C}^n$ so that two holomorphic local dynamical systems at O can be holomorphically (respectively, smoothly, C^k, or topologically) locally conjugated only if they have the same invariants. The class of invariants is complete if two holomorphic local dynamical systems at O are holomorphically (respectively, smoothly, C^k, or topologically) locally conjugated if and only if they have the same invariants.*

Contrarily to the previous ones, our final general question makes sense only for *holomorphic* local dynamical systems. A discrete holomorphic local dynamical system at $O \in \mathbb{C}^n$ is given by an element of $\mathbb{C}_0\{z_1, \ldots, z_n\}^n$, the space of n-tuples of converging power series in $z_1, \ldots, z_n$ without constant terms, which is a subspace of the space $\mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ of n-tuples of formal power series without constant terms. It is well known that $\mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ is closed under composition of power series, and that an element $\Phi \in \mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ if and only if its linear part is a linear automorphism of $\mathbb{C}^n$.

Definition 4 We say that $f_1, f_2 \in \mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ are *formally conjugated* if there is an invertible $\Phi \in \mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$ such that

$$f_1 = \Phi^{-1} \circ f_2 \circ \Phi$$

in $\mathbb{C}_0[\![z_1, \ldots, z_n]\!]^n$.

Clearly, two holomorphically locally conjugated holomorphic local dynamical systems are both formally and topologically locally conjugated too. On the other hand, there are examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural general question in local holomorphic dynamics is

(Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at $O \in \mathbb{C}^n$.*

In the rest of this paper we shall describe a few specific open problems, both in one and in several complex variables; we refer to [3,4] and [14] for surveys on the theory of discrete holomorphic local dynamical systems, for more details on what is known about the previous questions, and for the background of the open problems we selected.

2 One complex variable

2.1 Linearization in specific families

A discrete holomorphic local dynamical system in one complex variable is given by a germ of holomorphic function fixing the origin of the form

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}\{z\},$$

where $\lambda = f'(0) \in \mathbb{C}^n$ is the *multiplier* of f. It is well-known (Kœnigs' theorem) that if $|\lambda| \neq 0, 1$ then f is *holomorphically linearizable,* that is locally holomorphically conjugated to the linear map $w \mapsto \lambda w$—and thus in this case questions (Q1)–(Q5) presented in the introduction are easily solved. If $a_1 = 0$, and thus we can write $f(z) = a_k z^k + o(z^k)$ with $k \geq 2$ and $a_k \neq 0$, it is also well-known (Böttcher's theorem) that f is locally holomorphically conjugated to $w \mapsto w^k$, and so in this case too the local dynamics is completely clear.

If $|\lambda| = 1$ and $\lambda = e^{2\pi i p/q}$ is a qth root of unity, then it is easy to prove that f is (topologically, holomorphically or formally) linearizable if and only if $f^q = \mathrm{id}$; therefore the *linearization problem*, that is deciding when f is linearizable, is meaningful only when $\lambda = e^{2\pi i\theta}$ with $\theta \notin \mathbb{Q}$. Note that it is not difficult to show that every germ with multiplier of this form is formally linearizable, and that it is topologically linearizable if and only if it is holomorphically linearizable; so the main question here is deciding whether a given germ is holomorphically linearizable or not—and, in particular, whether this question can be solved just by examining the multiplier.

The main result in this area is due to Brjuno [20–22] and Yoccoz [73,74]. To state it, let us introduce a bit of terminology and some notations.

Definition 5 We shall say that $f \in \mathrm{End}(\mathbb{C}, O)$ is *elliptic* if its multiplier has modulus one but is not a root of unity; and that the origin is a *Siegel point* (respectively, a *Cremer point*) if f is (respectively, is not) holomorphically linearizable.

Definition 6 For $\lambda \in S^1$ and $m \geq 1$ put

$$\Omega_\lambda(m) = \min_{1 \leq k \leq m} |\lambda^k - \lambda|.$$

Clearly, λ is a root of unity (or $\lambda = 0$) if and only if $\Omega_\lambda(m) = 0$ for all m greater than or equal to some $m_0 \geq 1$; furthermore, if $|\lambda| \neq 0, 1$ then $\Omega_\lambda(m)$ is bounded away from zero, whereas if $|\lambda| = 1$ then

$$\lim_{m \to +\infty} \Omega_\lambda(m) = 0.$$

We shall say that $\lambda \in S^1$, not a root of unity, satisfies the *Brjuno condition* (or that λ is a *Brjuno number*) if

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty. \tag{1}$$

Remark 2 There are several equivalent reformulations of the Brjuno condition, as the convergence of other series involving Ω_λ, or as the convergence of a series involving the continuous fraction expansion of θ; see, e.g., [61,67,74].

We can then state the famous *Brjuno–Yoccoz theorem:*

Theorem 1 (Brjuno [20–22], Yoccoz [73,74]) *Let $\lambda \in S^1$, not a root of unity. Then the following statements are equivalent:*

(i) *the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;*
(ii) *the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ;*
(iii) *the number λ satisfies the Brjuno condition.*

This theorem has two aspects. From one side, it gives a necessary and sufficient condition on λ ensuring that *all* local dynamical systems with multiplier λ are holomorphically linearizable. However, there clearly exist holomorphically linearizable germs whose multiplier does not satisfy the Brjuno condition: it suffices to take a germ of the form $f(z) = \varphi^{-1}\big(\lambda\varphi(z)\big)$, where φ is a local biholomorphism and λ is not a Brjuno number.

One the other hand, Theorem 1 says that in the family of quadratic polynomials a given germ is holomorphically linearizable *if and only if* its multiplier is a Brjuno number. This observation immediately leads to the first open problem of this survey:

(OP1) *Find families* $\{f_{\lambda,a}(z) = \lambda z + z^2 g_a(z)\}_{a\in M} \subset \mathrm{End}(\mathbb{C}, O)$ *of elliptic discrete holomorphic dynamical systems such that* $f_{\lambda,a}$ *is holomorphically linearizable if and only if* λ *is a Brjuno number. For instance, is it true that an elliptic polynomial is holomorphically linearizable if and only if its multiplier is a Brjuno number? Or, even more specifically, is it true that an elliptic cubic polynomial is holomorphically linearizable if and only if its multiplier is a Brjuno number?*

In this context, it might be useful to remember the following dichotomy due to Il'yashenko and Perez-Marco:

Theorem 2 (Il'yashenko [45], Perez-Marco [55]) *Given* $g \in \mathrm{End}(\mathbb{C}, O)$ *and* $\lambda \in \mathbb{C}^*$ *not a root of unity, put*

$$f_{\lambda,a} = \lambda z + azg(z)$$

for all $a \in \mathbb{C}$. *Then:*

(i) *either the origin is a Siegel point of* $f_{\lambda,a}$ *for all* $a \in \mathbb{C}$, *or*
(ii) *the origin is a Cremer point of* $f_{\lambda,a}$ *for all* $a \in \mathbb{C} \setminus K$, *where* $K \subset\subset \mathbb{C}$ *is a bounded exceptional set of capacity (and hence Lebesgue measure) zero.*

2.2 Regularity of the Brjuno function

There is a different way of expressing the Brjuno condition, leading to another interesting open problem.

Given $\theta \in [0, 1)$ set

$$r(\theta) = \inf\{r(f) \mid f \in \mathrm{End}(\mathbb{C}, 0) \text{ is defined and injective in } \Delta \text{ and has multiplier} e^{2\pi i\theta}\},$$

where $\Delta \subset \mathbb{C}$ is the unit disk and $r(f) \geq 0$ is the radius of convergence of the unique formal linearization of f with multiplier 1.

On the other hand, given an irrational number $\theta \in [0, 1)$ let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions, and put

$$\alpha_n = -\frac{q_n\theta - p_n}{q_{n-1}\theta - p_{n-1}}, \qquad \alpha_0 = \theta,$$
$$\beta_n = (-1)^n(q_n\theta - p_n), \qquad \beta_{-1} = 1.$$

Definition 7 The *Brjuno function* $B\colon [0,1)\setminus\mathbb{Q} \to (0,+\infty]$ is defined by

$$B(\theta) = \sum_{n=0}^{\infty} \beta_{n-1}\log\frac{1}{\alpha_n}.$$

Then Yoccoz has proved the following quantitative relationship between the infimum $r(\theta)$ of the radii of convergence and the Brjuno function:

Theorem 3 (Yoccoz [74])

(i) *$B(\theta) < +\infty$ if and only if $\lambda = e^{2\pi i\theta}$ is a Brjuno number;*
(ii) *there exists a universal constant $C > 0$ such that*

$$|\log r(\theta) + B(\theta)| \le C$$

for all $\theta \in [0,1)\setminus\mathbb{Q}$ such that $B(\theta) < +\infty$;
(iii) *if $B(\theta) = +\infty$ then there exists a non-linearizable $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier $e^{2\pi i\theta}$.*

The Brjuno function is clearly quite an irregular function, diverging at a G_δ-dense set of irrational numbers. A surprising fact is that, on the contrary, numerical experiments (see, e.g., [26] and references therein) as well as theoretical results (see [50]) suggest that $\log r + B$ is more regular, even better than continuous. More specifically, Marmi, Mattei and Yoccoz proposed the following problem:

(OP2) *Does $\log r + B$ admit a $1/2$-Hölder continuous extension to $[0, 1)$?*

2.3 Classification of Cremer points

The local dynamics about a Siegel point is completely clear. The local dynamics about a Cremer point, on the other hand, is extremely complicated. The best results up to now are due to Perez-Marco and Biswas:

Theorem 4 (Pérez-Marco [53,54]) *Assume that 0 is a Cremer point for an elliptic discrete holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$. Then:*

(i) *The stable set K_f is compact, connected, full (i.e., $\mathbb{C}\setminus K_f$ is connected), it is not reduced to $\{0\}$, and it is not locally connected at any point distinct from the origin.*
(ii) *Any point of $K_f\setminus\{0\}$ is recurrent (that is, a limit point of its orbit).*
(iii) *There is an orbit in K_f which accumulates at the origin, but no non-trivial orbit converges to the origin.*

Theorem 5 (Biswas [13]) *The multiplier and the conformal class of the stable set K_f are a complete set of holomorphic invariants for Cremer points. In other words, two elliptic non-linearizable holomorphic local dynamical systems f and g are holomorphically locally conjugated if and only if they have the same multiplier and there is a biholomorphism (not necessarily conjugating the dynamics) of a neighborhood of K_f with a neighborhood of K_g.*

Surprisingly enough, the topological classification of Cremer points is still open. Clearly, the homeomorphism class of the (germ at the origin of the) stable set is a topological invariant; moreover, a non-trivial theorem due to Naishul (see [54] for another proof) shows that the multiplier is another topological invariant:

Theorem 6 (Naishul [52]) *Let $f, g \in \mathrm{End}(\mathbb{C}, O)$ be two elliptic discrete holomorphic local dynamical systems. If f and g are topologically locally conjugated then $f'(0) = g'(0)$.*

Thus a natural open question in this context is:

(OP3) *Are the multiplier and the homeomorphism class of the stable set a complete set of topological invariants for discrete holomorphic local dynamical systems in $\mathrm{End}(\mathbb{C}, O)$ having a Cremer point at the origin?*

2.4 Effective classification of parabolic germs

Definition 8 A local dynamical system $f \in \mathrm{End}(\mathbb{C}, 0)$ is *parabolic* if its multiplier is a root of unity; it is *tangent to the identity* if its multiplier is 1.

Clearly, if $f \in \mathrm{End}(\mathbb{C}, 0)$ is parabolic then a suitable iterate f^q is tangent to the identity; so most dynamical questions for parabolic systems can be reduced to the study of tangent to the identity germs.

A qualitative description of the dynamics of tangent to the identity germs is given by the famous *Leau-Fatou flower theorem.* To state it we need to recall a couple of definitions:

Definition 9 Let $f \in \mathrm{End}(\mathbb{C}, 0) \setminus \{\mathrm{id}\}$ be tangent to the identity, and thus of the form

$$f(z) = z + a_{r+1}z^{r+1} + O(z^{r+2})$$

with $a_{r+1} \neq 0$, where $r + 1 \geq 2$ is the *multiplicity* of f. A unit vector $v \in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at the origin if $a_{r+1}v^r$ is real and negative (respectively, positive).

Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions. Furthermore, a repelling (attracting) direction for f is attracting (repelling) for f^{-1}, which is defined in a neighborhood of the origin.

It turns out that to every attracting direction is associated a connected component of $K_f \setminus \{0\}$.

Definition 10 Let $v \in S^1$ be an attracting direction for an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity. The *basin* centered at v is the set of points $z \in K_f \setminus \{0\}$ such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that, up to shrinking the domain of f, we can assume that $f(z) \neq 0$ for all $z \in K_f \setminus \{0\}$). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

An *attracting petal* centered at an attracting direction v of f is an open simply connected f-invariant set $P \subseteq K_f \setminus \{0\}$ such that a point $z \in K_f \setminus \{0\}$ belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

Then:

Theorem 7 (Leau [47], Fatou [35–37]) *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a discrete holomorphic local dynamical system tangent to the identity with multiplicity $r + 1 \geq 2$ at the fixed point. Let $v_1^+, \ldots, v_r^+ \in S^1$ be the r attracting directions of f at the origin, and $v_1^-, \ldots, v_r^- \in S^1$ the r repelling directions. Then*

(i) *for each attracting (repelling) direction $v_j^\pm$ there exists an attracting (repelling) petal $P_j^\pm$, so that the union of these $2r$ petals is a pointed neighborhood of the origin. Furthermore, the $2r$ petals are arranged cyclically so that two petals intersect if and only if the angle between their central directions is π/r.*

(ii) *$K_f \setminus \{0\}$ is the (disjoint) union of the basins centered at the r attracting directions.*

(iii) *If B is a basin centered at one of the attracting directions then there is a function $\varphi\colon B \to \mathbb{C}$ such that $\varphi \circ f(z) = \varphi(z) + 1$ for all $z \in B$. Furthermore, if P is the corresponding petal constructed in part* (i), *then $\varphi|_P$ is a biholomorphism with an open subset of the complex plane containing a right half-plane—and so $f|_P$ is holomorphically conjugated to the translation $z \mapsto z + 1$.*

Starting from this theorem, Camacho [24] and, independently, Shcherbakov [68] have completed the topological classification of germs tangent to the identity, showing that the multiplicity is a complete set of topological invariants:

Theorem 8 (Camacho [24], Shcherbakov [68]) *Assume that $f \in \mathrm{End}(\mathbb{C}, 0)$ is a holomorphic local dynamical system tangent to the identity with multiplicity $\nu \geq 2$. Then f is topologically locally conjugated to $g(z) = z + z^\nu$.*

Furthermore, the formal classification is obtained with not too difficult a computation, and a complete set of invariants is given by the multiplicity and another complex number, the *index,* explicitly computable.

On the other hand, the holomorphic classification is incredibly more difficult. Écalle [30–33] and, independently, Voronin [71] have given a complete set of invariants, consisting in the multiplicity, the index, and a functional invariant, an equivalence class of functions with specific properties constructed starting from the biholomorphisms introduced in Theorem 7.(iii). This set of invariants is not only complete but also *full,* in the sense that every possible value of the invariants is realized by a germ tangent to the identity; however, to explicitly compute Écalle-Voronin functional invariant is an almost impossible task. In particular, the following problem is still open:

(OP4) *Give an effective procedure for deciding whether two germs tangent to the identity are holomorphically locally conjugated.*

Clearly, a similar question can be asked for parabolic germs; in that case another (topological, formal and holomorphic) invariant is the multiplier, and one has to replace the multiplicity by a suitably defined parabolic multiplicity, and the index by a suitably defined parabolic index (or, better yet, by Écalle's *iterative residue*). Then the multiplier and the parabolic multiplicity are a complete set of topological invariants, the multiplier, the parabolic multiplicity and the iterative residue are a complete set of formal invariants, and adding suitably adjusted Écalle-Voronin functional invariants one obtains a complete set of holomorphic invariants. Again, most of the times the computation of the functional invariants is hopeless, and thus we have the following generalization of the previous question:

(OP4′) *Give an effective procedure for deciding whether two parabolic germs are holomorphically locally conjugated.*

See also [45,48,49] and [51] for alternative presentations of Écalle-Voronin invariants.

3 Several complex variables

We have seen how in one complex variable the multiplier (that is, the derivative at the fixed point) plays a fundamental role. In several complex variables instead of the multiplier we may consider the eigenvalues of the differential at the fixed point, and give a first classification of discrete holomorphic local dynamical systems based on them. Clearly there are many cases to consider, and correspondingly many ways to precise the five basic questions stated in the introduction. Here we shall limit ourselves to a selection of some important open problems focusing on four main classes of systems: *non-invertible, tangent to the identity, linearizable,* and *mixed* systems.

A piece of terminology we shall systematically use is the following:

Definition 11 The *homogeneous expansion* of a $f \in \mathrm{End}(\mathbb{C}^n, O)$ is the expansion

$$f(z) = \sum_{j \geq c(f)} P_j(z)$$

where $c(f) \geq 1$ is the *order* of f, and P_j is an n-tuple of homogeneous polynomials of degree j (and we are of course assuming that $P_{c(f)} \not\equiv O$). Furthermore, we shall say that f is *dominant* if $\det \mathrm{Jac}(f) \not\equiv O$.

3.1 Non-invertible systems: asymptotic attraction rate

A discrete holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ is invertible if and only if the differential df_O is; therefore f is non-invertible if and only if 0 is an eigenvalue of df_O. In particular, we shall also say that f is *superattracting* if $df_O \equiv O$, i.e., if $c(f) \geq 2$.

There are a couple of interesting open questions (suggested by Mattias Jonsson) about the sequence $\{c(f^k)\}_{k\in\mathbb{N}}$. In dimension 1 it is easy to see that $c(f^k) = c(f)^k$ for all $k \geq 1$, and thus $c(f^k)^{1/k} = c(f)$ for all $k \geq 1$. On the other hand, in dimension 2 or more we only have

$$c(f^{h+k}) \geq c(f^h)c(f^k) \tag{2}$$

for all $h, k \in \mathbb{N}$, and the inequality can be strict. Consider for instance a germ $f \in \mathrm{End}(\mathbb{C}^2, O)$ whose linear term is non-zero but nilpotent; then we have $c(f) = 1$ but $c(f^2) \geq 2$.

Nevertheless, (2) implies the existence of the *asymptotic attraction rate* defined by the limit

$$c_\infty(f) = \lim_{k\to+\infty} c(f^k)^{1/k},$$

which is a basic (formal and holomorphic, at least) invariant of f. Contrarily to the dimension one case, $c_\infty(f)$ is not necessarily an integer. However, Favre and Jonsson [39] have proved the following

Theorem 9 (Favre–Jonsson [39]) *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be non-invertible and dominant. Then $c_\infty = c_\infty(f)$ is a quadratic integer, i.e., there exist integers $a, b \in \mathbb{Z}$ such that $c_\infty^2 + ac_\infty + b = 0$. Moreover, there exists $\delta \in (0, 1]$ such that $\delta c_\infty^k \leq c(f^k) \leq c_\infty^k$ for all $k \geq 1$.*

This result clearly suggests an open problem:

(OP5) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be non-invertible and dominant. Is it true that $c_\infty = c_\infty(f)$ is an algebraic integer of order at most n, i.e., there are integers $a_0, \ldots, a_{n-1} \in \mathbb{Z}$ such that $c_\infty^n + a_{k-1}c_\infty^{k-1} + \cdots + a_0 = 0$? Moreover, does there exist $\delta \in (0, 1]$ such that $\delta c_\infty^k \leq c(f^k) \leq c_\infty^k$ for all $k \geq 1$?*

Roughly speaking, the order of a germ may be thought of as a sort of analog of the degree of a polynomial map: the former somewhat measure the rate of attraction of the origin, while the latter measure the rate of attraction of infinity. For the sequence of the degree of the iterates of a polynomial map, Favre and Jonsson [40] have proved the following

Theorem 10 (Favre–Jonsson [40]) *Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be a polynomial map. Then the sequence $\{\deg F^j\}_{j\in\mathbb{N}}$ satisfies a linear recursion formula with integer coefficients.*

We can then wonder whether a similar property holds for the sequence of orders of the iterates of a germ:

(OP6) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be non-invertible and dominant. Is it true that the sequence $\{c(f^k)\}_{k\in\mathbb{N}}$ satisfies, at least for k large enough, a linear recursion formula with integer coefficients?*

3.2 Non-invertible systems: classification

We have already remarked that in dimension 1 every non-invertible germ of order k is holomorphically conjugated to z^k, by Böttcher's theorem. In several variables, this is not true: for instance, as first remarked by Hubbard and Papadopol [44], the map $F(z, w) = (z^2 + w^3, w^2)$ cannot be, even topologically, locally conjugated to the homogeneous quadratic map $H(z, w) = (z^2, w^2)$. Indeed, both maps have as critical locus (which is topologically defined) the union of the two axes; however the union of the two axes is H-invariant but not F-invariant. More precisely, the critical value set of H is the union of the two axes, whereas the critical value set of F is the union of the z-axis with the curve $z^2 = w^3$.

Definition 12 Given $f \in \mathrm{End}(\mathbb{C}^n, O)$, we shall denote by

$$\mathrm{Crit}(f) = \{\det(df) = 0\}$$

the set of critical points of f, and by

$$\mathrm{PCrit}(f) = \bigcup_{k \geq 0} f^k\bigl(\mathrm{Crit}(f)\bigr)$$

the *postcritical set* of f.

The postcritical set of a homogeneous map is a cone; thus a superattracting germ f can be topologically (respectively, holomorphically) locally conjugated to a homogeneous map only if its postcritical set is a topological (respectively, analytic) cone, that is the image of a standard cone under a local homeomorphism (respectively, biholomorphism). Buff, Epstein and Koch [23] have proved that this condition is also sufficient when the homogeneous map is non-degenerate:

Theorem 11 (Buff-Epstein-Koch, 2011 [23]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a superattracting germ, and let $H: \mathbb{C}^n \to \mathbb{C}^n$ be the first non-vanishing term, of degree $c = c(f)$, in the homogeneous expansion of f. Assume that H is non-degenerate, that is $H^{-1}(O) = \{O\}$. Then the following assertions are equivalent:*

- (i) *f is holomorphically locally conjugated to H;*
- (ii) *there is a germ of holomorphic vector field ξ with $\xi(p) = p + o(\|p\|)$ as $p \to O$ and such that $df \circ \xi = c\, \xi \circ f$;*
- (iii) *there is a germ of holomorphic vector field ζ tangent to the postcritical set of f and such that $\zeta(p) = p + o(\|p\|)$ for $p \to O$;*
- (iv) *the postcritical set of f is an analytic cone.*

This result immediately prompts three open problems:

(OP7) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a superattracting germ, and let H be the first non-zero term in the homogeneous expansion of f. Assume that H is degenerate, that is there exists $v \in \mathbb{C}^n \setminus \{O\}$ such that $H(v) = O$. Is it still true that if the postcritical set of f is an analytic cone then f is holomorphically locally conjugated to H?*

(OP8) *Let* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *be a superattracting germ, and let* H *be the first non-zero term in the homogeneous expansion of* f. *Assume that the postcritical set of* f *is a topological cone nearby the origin. Under which conditions is then* f *topologically locally conjugated to* H? *In particular, is this true when* H *is non-degenerate?*

(OP9) *Let* $f, g \in \mathrm{End}(\mathbb{C}^n, O)$ *be two superattracting germs, with* $c(f) = c(g)$. *Is it true that* f *and* g *are topologically (respectively, holomorphically) locally conjugated if and only if their postcritical sets are topologically (respectively, holomorphically) conjugated, that is, there is a local homeomorphism (respectively, biholomorphism) sending the postcritical set of* f *onto the postcritical set of* g?

Concerning instead the classification problem with respect to the formal conjugacy, Abate and Raissy in [8] gave a formal classification of superattracting germs $f \in \mathrm{End}(\mathbb{C}^2, O)$ of order 2. The methods described there can in principle be used to attack the general formal classification problem; here we limit ourselves to a more specific question:

(OP10) *Classify with respect to formal conjugation the superattracting germs* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *of order 2 when* $n \geq 3$, *or of order 3 when* $n = 2$.

3.3 Non-invertible systems: rigidification

Asking for a holomorphic classification of non-invertible germs is possibly too much; on the other hand, a birational classification has been obtained by Favre, Jonsson and Ruggiero at least in dimension 2. Let us introduce a bit of terminology to explain their results.

Definition 13 Given $f \in \mathrm{End}(M, p)$, set

$$\mathrm{Crit}^\infty(f) = \bigcup_{k\geq 0} f^{-k}\bigl(\mathrm{Crit}(f)\bigr) = \bigcup_{k\geq 0} \mathrm{Crit}(f^k).$$

We say that f *weakly rigid* if $\mathrm{Crit}^\infty(f)$, as a germ at the origin, is composed by $0 \leq q \leq \dim M$ smooth irreducible components, having a simple normal crossing at the origin. If f is weakly rigid and $W_1, \ldots, W_q$ are the irreducible components of $\mathrm{Crit}^\infty(f)$, we shall say that f is *rigid* if moreover for each $j = 1, \ldots, q$ there exists $I_j \subseteq \{1, \ldots, q\}$ such that $f(W_j) = \bigcap_{i\in I_j} W_i$.

Invertible germ, because $\mathrm{Crit}^\infty(f) = \varnothing$, and non-dominant germs, because $\mathrm{Crit}^\infty(f) = M$, are trivially rigid; so this notion is interesting only for non-invertible dominant germs.

Definition 14 A *modification* of a complex n-dimensional manifold M is a surjective holomorphic map $\pi\colon \tilde{M} \to M$ obtained as composition of a finite number of blow-ups of submanifolds (or points); in particular, a modification is a birational isomorphism.

The modification is *based* at a point p_0 if the first blow-up is made along a submanifold of M containing p_0, and subsequent blow-ups are made along submanifolds intersecting the inverse image of p_0.

In dimension 2 Favre–Jonsson [39] and Ruggiero [66] have proved that every (non-invertible dominant) germ is birationally conjugated to a rigid germ:

Theorem 12 (Favre–Jonsson [39], Ruggiero [66]) *For every $f \in \mathrm{End}(\mathbb{C}^2, O)$ there exist a modification $\pi \colon \tilde{M} \to \mathbb{C}^2$ based at O, a point $p \in E = \pi^{-1}(O)$ and a rigid holomorphic germ $\tilde{f} \in \mathrm{End}(M, p)$ so that $\pi \circ \tilde{f} = f \circ \pi$.*

Since Favre [38] has classified 2-dimensional (non-invertible dominant) rigid germs, this result gives birational normal forms for non-invertible dominant germs.

Of course, one would like to extend these results at least to dimension 3, and thus we can add two more open problems to the list:

(OP11) *Classify 3-dimensional non-invertible dominant rigid germs.*
(OP12) *Under which conditions on $f \in \mathrm{End}(\mathbb{C}^3, O)$ there exist a modification $\pi \colon \tilde{M} \to \mathbb{C}^3$ based at O, a point $p \in E = \pi^{-1}(O)$ and a (possibly weakly) rigid holomorphic germ $\tilde{f} \in \mathrm{End}(M, p)$ so that $\pi \circ \tilde{f} = f \circ \pi$?*

It should be mentioned that the latter problem is somewhat related to a non-dynamical result by Cutkosky [28]:

Theorem 13 (Cutkosky [28]) *Let $f \colon X \to Y$ be a dominant morphism of algebraic 3-varieties over $\mathbb{C}$. Then there exist: modifications $\phi \colon \tilde{X} \to X$ and $\psi \colon \tilde{Y} \to Y$, with $\tilde{X}$ and $\tilde{Y}$ non-singular; simple normal crossing divisors $D_{\tilde{Y}}$ in $\tilde{Y}$ and $D_{\tilde{X}}$ in $\tilde{X}$; and a morphism $\tilde{f} \colon \tilde{X} \to \tilde{Y}$ such that the diagram*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\ \tilde{f}\ } & \tilde{Y} \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

commutes, $D_{\tilde{X}} = \tilde{f}^{-1}(D_{\tilde{Y}})$ and $\tilde{f}$ is toroidal with respect to $D_{\tilde{X}}$ and $D_{\tilde{Y}}$ (i.e., $\tilde{f}$ is locally given by monomials in suitable étale local parameters on $\tilde{X}$).

3.4 Parabolic systems: Fatou flower

Our next topic is the dynamics of *parabolic* systems, where the differential is unipotent; more precisely we are interested in *tangent to the identity* germs, that is holomorphic local dynamical systems $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots \in \mathbb{C}_0\{z_1, \ldots, z_n\}^n, \tag{3}$$

where P_ν is the first non-zero term in the homogeneous expansion of f, and $\nu \geq 2$ is the *multiplicity* of f.

Of course, the first thing one would like to do is to find a several variables version of the Leau-Fatou flower theorem. To describe the right generalization, we have to introduce a few concepts.

Definition 15 Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent at the identity and of multiplicity ν. A *characteristic direction* for f is a non-zero vector $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_\nu(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. If $P_\nu(v) = O$ (that is, $\lambda = 0$) we shall say that v is a *degenerate* characteristic direction; otherwise, (that is, if $\lambda \neq 0$) we shall say that v is *non-degenerate*.

Characteristic directions always exist, and it is not difficult to show (see, e.g., [9]) that a generic f has exactly $(\nu^n - 1)/(\nu - 1)$ characteristic directions, counted with respect to a suitable multiplicity.

The notion of characteristic directions has a dynamical origin. Indeed, it is possible to prove (see, e.g., [42]) that if an orbit $\{f^k(z_0)\}$ of a germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity converges to the origin *tangentially* to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$—that is $f^k(z_0) \to O$ in $\mathbb{C}^n$ and $[f^k(z_0)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$, where $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ is the canonical projection—then v is a characteristic direction of f.

We can now introduce the several variables analogue of petals: parabolic curves.

Definition 16 A *parabolic curve* for $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity of multiplicity $\nu \geq 2$ is an injective holomorphic map $\varphi\colon \Delta \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(b) φ is continuous at the origin, and $\varphi(0) = O$;
(c) $\varphi(\Delta)$ is f-invariant, and $(f|_{\varphi(\Delta)})^k \to O$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \to 0$ in Δ we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$.

Finally, a *Fatou flower* tangent to a direction $[v]$ is a set of $\nu - 1$ parabolic curves tangent to $[v]$, with domains the connected components of a set of the form $D_{\delta,\nu} = \{\zeta \in \mathbb{C} \mid |\zeta^{\nu-1} - \delta| < \delta\}$ for $\delta > 0$ small enough.

Then the first main generalization of the Leau-Fatou flower theorem to several complex variables is due to Écalle and Hakim (see also [72]):

Theorem 14 (Écalle, 1985 [33]; Hakim, 1998 [42]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a germ tangent to the identity, and $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ a non-degenerate characteristic direction. Then there exists (at least) a Fatou flower for f tangent to $[v]$.*

This result applies to germs tangent to the identity having non-degenerate characteristic directions; however, it is easy to find examples of germs having only degenerate characteristic directions. In dimension 2 it is possible to get Fatou flowers in this case too (see also [19]):

Theorem 15 (Abate [2]) *Every germ $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity, with the origin as an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

The proof works in dimension 2 only, and this leads to our next open problem:

(OP13) *Is it true that every germ* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *tangent to the identity, with the origin as an isolated fixed point, admits at least one Fatou flower tangent to some direction?*

Some partial results are presented in [9] and [63], showing however that the analogy with the local dynamics of holomorphic vector fields that guided the proof of Theorem 15 breaks down when $n \geq 3$; so apparently new ideas are needed.

3.5 Parabolic systems: classification

Theorems 14 and 15 describe the dynamics only on 1-dimensional subsets of an n-dimensional space, and so are very far from determining the dynamical behavior of a tangent to the identity germ in a full neighborhood of the origin.

Now, it is possible to attach to each characteristic direction (see [33] and [43] for the non-degenerate case, and [5–7, 11] for the general case) $n-1$ numbers, called *directors* or *indices* (actually, directors and indices are not the same numbers, but they are strongly related; see, again, [6] and [11]), that can be useful to describe the behavior in a neighborhood of the Fatou flowers. For instance, Hakim [43] (see also [12]) has proved that if all the directors at a non-degenerate characteristic direction have positive real part then the Fatou flower is attracting, that is there is an open neighborhood of the Fatou flower consisting of points whose orbit is converging to the origin tangentially to the given characteristic direction. On the other hand, there are examples (see, e.g., [11] and references therein) of germs having orbits converging to the origin without being tangent to any direction, as well as of germs having periodic orbits arbitrarily close to the origin (a phenomenon that cannot happen in dimension 1). Thus in general the stable set is larger than the set of points with orbits converging to the origin tangentially to some direction; however, in the known examples the presence of "anomalous" points in the stable set seems again related to the indices, and in particular to the existence of purely imaginary indices.

Anyway, the natural open problem here is:

(OP14) *Describe, using characteristic directions, directors, indices and possibly other invariants, the stable set of a germ tangent to the identity in* $\mathbb{C}^n$.

See also [18] for some results on this problem when all directions are characteristic.

In [11] we started a systematic study of the local dynamics of a particularly important class of dynamical systems tangent to the identity: time-1 maps of homogeneous vector fields. Indeed, if we identify, as we can, a homogenous vector field in $\mathbb{C}^n$ of degree $\nu \geq 2$ with a n-tuple P_ν of homogeneous polynomials of degree ν then its time-1 map is of the form

$$g(z) = z + P_\nu(z) + O(\|z\|^{\nu+1}),$$

and thus it is tangent to the identity of multiplicity ν.

These germs are particularly important because of the following reformulation of Camacho's Theorem 8:

Corollary 1 *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity ν at the fixed point. Then f is topologically locally conjugated to the time-1 map of the homogeneous vector field $z^\nu \frac{\partial}{\partial z}$.*

Thus in dimension one time-1 maps of homogeneous vector fields provide a complete set of topological normal forms for germs tangent to the identity. This, and the work done in [11], suggests the following open problem:

(OP15) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be given by*

$$f(z) = z + P_\nu(z) + O(\|z\|^{\nu+1}), \tag{4}$$

and assume that all characteristic directions of f are non-degenerate. Is it true that f is topologically locally conjugated to the time-1 map of the homogeneous vector field P_ν?

The assumption on the characteristic directions is necessary. If v is a degenerate characteristic direction for the time-1 map g of a homogeneous vector field P_ν (that is, $P_\nu(v) = O$) then the whole complex line $\mathbb{C}v$ consists of zeroes of the vector field, and thus it is pointwise fixed by g, whereas the fixed point set of a generic germ of the form (4) consists of the origin only.

Concerning the formal classification, in his monumental work [33] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with at least one non-degenerate characteristic direction. For instance, he has proved the following

Theorem 16 (Écalle [33,34]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of multiplicity $\nu \geq 2$. Assume that*

(a) *f has exactly $(\nu^n - 1)/(\nu - 1)$ distinct non-degenerate characteristic directions and no degenerate characteristic directions;*
(b) *the directors of any non-degenerate characteristic direction are irrational and mutually independent over $\mathbb{Z}$.*

Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, and denote by $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ its directors. Then there exist a unique $\rho \in \mathbb{C}$ and unique (up to dilations) formal series $R_1, \ldots, R_n \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$, where each R_j contains only monomials of total degree at least $\nu + 1$ and of partial degree in z_j at most $\nu - 2$, such that f is formally conjugated to the time-1 map of the formal vector field

$$X = \frac{1}{(\nu - 1)(1 + \rho z_n^{\nu-1})} \left\{ [-z_n^\nu + R_n(z)] \frac{\partial}{\partial z_n} + \sum_{j=1}^{n-1} [-\alpha_j z_n^{\nu-1} z_j + R_j(z)] \frac{\partial}{\partial z_j} \right\}.$$

A natural question is how to complete the formal classification:

(OP16) *Find formal normal forms and invariants for germs tangent to the identity having (possibly only) degenerate characteristic directions.*

Partial results on this problem can be find in [8,10,18].

3.6 Parabolic systems: irregular singularities

In [11] we showed that the study of the dynamics of time-1 maps of n-dimensional homogeneous vector fields can be reduced to the study of singular holomorphic foliations in Riemann surfaces of $\mathbb{P}^{n-1}(\mathbb{C})$ and of geodesics for meromorphic connections on Riemann surfaces (notice that a singular holomorphic foliation in $\mathbb{P}^1(\mathbb{C})$ is completely determined by its finite set of singular points, and so when $n = 2$ the problem reduces to the study of geodesics for a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$; this might be the moral reason why the dynamics of germs tangent to the identity seems to be substantially more complicated in dimension 3 or more than in dimension 2), and in particular we were able to give a complete description of the dynamics in a full neighborhood of the origin for a large class of 2-dimensional examples. Our study suggested several questions worth of further study; let us just mention one of them.

We have already remarked that there are two kinds of characteristic directions: non-degenerate and degenerate. Actually, as already noticed in [2] and exploited in [6,11], we have to refine this classification because there are different types of degenerate characteristic directions. To make things simpler, let us assume that $n = 2$ and take a homogeneous vector field $Q = (Q_1, Q_2)$ of degree $\nu \geq 2$. In [11] it is shown that we can reduce the study of the dynamics of Q to the study of the dynamics of another vector field, the *geodesic field* G, defined on the total space of a suitable line bundle on $\mathbb{P}^1(\mathbb{C})$. In coordinates induced by the usual non-homogeneous coordinates centered in $[1 : 0] \in \mathbb{P}^1(\mathbb{C})$ the geodesic field can be written as

$$G = g_1(z)v\frac{\partial}{\partial z} + (\nu - 1)g_2(z)v^2\frac{\partial}{\partial v} \tag{5}$$

where $g_1(z) = Q_2(1, z) - zQ_1(1, z)$ and $g_2(z) = Q_1(1, z)$. In particular, $v_0 = (1, \zeta_0)$ is a characteristic direction of Q if and only if $g_1(\zeta_0) = 0$, and it is a degenerate characteristic direction if and only if $g_1(\zeta_0) = g_2(\zeta_0) = 0$.

Let $\mu_j(\zeta_0) \in \mathbb{N}$ be the order of vanishing of g_j at ζ_0. Then we shall say that the point with (z, v) coordinates given by $(\zeta_0, 0)$—which corresponds to the point $[1 : \zeta_0] \in \mathbb{P}^1(\mathbb{C})$—is

– an *apparent singularity* if $\mu_1(\zeta_0) \leq \mu_2(\zeta_0)$;
– a *Fuchsian singularity* if $\mu_1(\zeta_0) = \mu_2(\zeta_0) + 1$; and
– an *irregular singularity* if $\mu_1(\zeta_0) > \mu_2(\zeta_0) + 1$.

Then it is easy to see that non-degenerate characteristic directions are either Fuchsian or irregular (whereas degenerate characteristic directions can be apparent, Fuchsian or irregular), and it turns out that Fuchsian singularities with $\mu_1 = 1$ can be characterized as the non-degenerate characteristic directions with non-zero director.

In [11] we found formal normal forms for G around all kinds of singularities, but holomorphic normal forms only for apparent and Fuchsian singularities. Using the holomorphic normal forms we were able to study in detail the local dynamics of G about apparent and Fuchsian singularities (and thus the dynamics of Q around the corresponding characteristic directions); but it is still open the study of the local dynamics about irregular singularities. Thus we have another open problem:

(OP17) *Describe the local dynamics of a vector field G of the form* (5) *about an irregular singularity* $(\zeta_0, 0)$.

See [70] for some results related to this topic.

Remark 3 We limited our presentation to germs tangent to the identity; but similar results can be obtained and similar questions can be asked for germs whose differential is represented by a Jordan matrix with 1 as only eigenvalues. However, in [1] is shown that such germs are birationally conjugated (through a modification based at the origin) to germs tangent to the identity, and so many questions can be reduced to the latter case.

3.7 Linearizable systems: Brjuno condition

In one variable we saw that every germ whose multiplier was not a root of unity is formally linearizable. Now, $\lambda \in \mathbb{C}^*$ is a root of unity if and only if $\lambda^q = \lambda$ for some $q \geq 2$; a similar (but more widespread) phenomenon might prevent formal linearization in several variables too.

Definition 17 Given $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, a *resonance* for $\boldsymbol{\lambda}$ is a relation of the form

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \tag{6}$$

for some $1 \leq j \leq n$ and some $k_1, \ldots, k_n \in \mathbb{N}$ with $k_1 + \cdots + k_n \geq 2$. If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of the differential at the origin of $f \in \mathrm{End}(\mathbb{C}^n, O)$ we shall say that (6) is a *resonance of* f.

Resonances are the obstruction to formal linearization. Indeed, a standard computation shows that the coefficients of a formal linearization have in the denominators quantities of the form $\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j$; in particular it follows that a germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ with no resonances is always formally conjugated to its differential df_O, that is, it is formally linearizable. It should be mentioned that however a given germ can be formally (and even holomorphically) linearizable even in presence of resonances; see, e.g., [57,58].

The Brjuno problem in several variables consists in deciding when a formal linearization is actually convergent, keeping in mind that in absence of resonance the formal linearization is unique, but in presence of resonances the formal linearization if it exists is in general not unique; see [59–61] for a discussion (and much more) of this problem in a general setting.

To describe the main results known on the Brjuno problem, let us introduce the following definition:

Definition 18 For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ and $m \geq 2$ set

$$\Omega_{\boldsymbol{\lambda}}(m) = \min\bigl\{|\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j| \bigm| k_1, \ldots, k_n \in \mathbb{N},\ 2 \leq k_1 + \cdots + k_n \leq m,\ 1 \leq j \leq n,\ \lambda_1^{k_1} \cdots \lambda_n^{k_n} \neq \lambda_j\bigr\}. \tag{7}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of df_O, we shall write $\Omega_f(m)$ for $\Omega_{\boldsymbol{\lambda}}(m)$.

For some $f \in \mathrm{End}(\mathbb{C}^n, O)$ it might well happen that

$$\lim_{m \to +\infty} \Omega_f(m) = 0,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case.

Up to now the best result ensuring the convergence of a formal linearization is the analogue of the Brjuno theorem, that is the implication (iii)$\Longrightarrow$(i) in Theorem 1, proved by Brjuno [21,22] in absence of resonances and generalized by Rüssmann [67] and Raissy [60] to the formally linearizable case:

Theorem 17 (Brjuno [21,22], Rüssmann [67], Raissy [60]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a discrete holomorphic local dynamical system formally linearizable (e.g., without resonances) and with df_O diagonalizable. Assume that*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_f(2^{k+1})} < +\infty. \tag{8}$$

Then f is holomorphically linearizable.

Remark 4 The assumption of diagonalizable differential is necessary. Indeed, Yoccoz [74] has proved that for every $A \in GL(n, \mathbb{C})$ such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$ with $df_O = A$ which is not holomorphically linearizable.

Recalling Theorem 1 it is natural to ask whether condition (8) is necessary for the holomorphic linearization of all germs having diagonalizable differential with given eigenvalues. We can then state the following

(OP18) *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ be such that*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_{\boldsymbol{\lambda}}(2^{k+1})} = +\infty.$$

> *Is it possible to find $f \in \mathrm{End}(\mathbb{C}^n, O)$, with diagonalizable differential having eigenvalues $\lambda_1, \ldots, \lambda_n$, which is formally linearizable but not holomorphically linearizable?*

There are a couple of cases where a positive answer is known. For instance, it is possible to adapt the classical one-variable construction of Cremer [27] and prove the following

Theorem 18 *Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}$ be without resonances and such that*

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\Omega_{\boldsymbol{\lambda}}(m)} = +\infty. \tag{9}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$*, with* $df_O = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$*, not holomorphically linearizable.*

This does not answer (OP17) because, exactly as in one variable, it is possible to find $\boldsymbol{\lambda} \in \mathbb{C}^n$ such that the limsup in (9) is finite but the series in (8) diverges.

Another easy situation is when one of the components of $\boldsymbol{\lambda}$ does not satisfy the one-dimensional Brjuno condition. Indeed, if $\lambda \in S^1$ does not satisfy (1) then any $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = \big(\lambda z_1 + z_1^2, g(z)\big)$$

is not holomorphically linearizable, because if $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$ were a holomorphic linearization of f then $\psi(\zeta) = \varphi(\zeta, O)$ would be a holomorphic linearization of the quadratic polynomial $\lambda\zeta + \zeta^2$, against Theorem 1.

This again is not enough to solve (OP17) because there are n-tuples $\boldsymbol{\lambda} \in \mathbb{C}^n$ formed by complex numbers all individually satisfying the one-dimensional Brjuno condition (1) but not satisfying (8): for instance, take $\lambda \in S^1$ not a Brjuno number, $0 < |\mu| < 1$ and put $\lambda_1 = \mu$ and $\lambda_2 = \lambda\mu^{-1}$. Clearly both λ_1 and λ_2 satisfy trivially (1), whereas

$$\lambda_1^k \lambda_2^{k+1} - \lambda_2 = \mu^{-1}(\lambda^{k+1} - \lambda),$$

and thus it is easy to see that (λ_1, λ_2) does not satisfy (8).

We end this subsection with an open problem which is a recasting in this context of our first open problem:

(OP19) *Given* $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$*, find families* $\{f_{\boldsymbol{\lambda},a}\}_{a \in M} \subset \mathrm{End}(\mathbb{C}^n, O)$ *of formally linearizable germs whose differential is represented by the diagonal matrix of eigenvalues* $(\lambda_1, \ldots, \lambda_n)$ *such that* $f_{\boldsymbol{\lambda},a}$ *is holomorphically linearizable if and only if* $\boldsymbol{\lambda}$ *satisfies* (8)*. For instance, is it true that a germ of the form* $f(z) = (\lambda_1 z_1, \lambda_2 z_2) + Q(z)$*, where* Q *is a pair of quadratic homogeneous polynomials and* (λ_1, λ_2) *have no resonances, is holomorphically linearizable if and only if* (λ_1, λ_2) *satisfy* (8)?

3.8 Mixed systems

Another feature telling the several variable case apart from the one variable case is the presence of mixed systems, where the eigenvalues of the differential are part hyperbolic, part elliptic and part parabolic. There has been a fair amount of work in this area, mostly concerned with situations where the eigenvalues were contained in the closed unit disk of the complex plane; see, e.g., [15–17,25,29,41,62,64,65,69] and references therein. Among the many open problems in this area, four might be particularly interesting.

The first two arise from the work of Bracci–Molino [15] and Rong [64]; for simplicity I shall state them in dimension 2, but similar questions can be asked in any dimension. Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be such that the eigenvalues of df_O are 1 and $\lambda = e^{2\pi i\theta} \neq 1$. Pöschel [56] has shown that if λ satisfies the Brjuno condition (1) then there exists a 1-dimensional f-invariant complex disk centered at the origin and tangent to the eigenspace of λ where f is conjugated to the irrational rotation of angle $2\pi\theta$; Bracci and Molino (and Rong in dimension higher than 2) have instead studied the existence of parabolic behavior (and, in particular, of parabolic curves) for such germs. To summarize their results we need a definition:

Definition 19 Let $f = (f_1, f_2) \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic germ such that $df_O = \mathrm{diag}(1, \lambda)$ with $\lambda = e^{2\pi i\theta} \neq 1$. We shall say that f is in *ultra-resonant normal form* if the order of vanishing of $f_2(z_1, 0)$ is at least equal to the order of vanishing of $f_1(z_1, 0) - z_1$. If f is in ultra-resonant normal form, we denote by $\nu(f) \geq 2$ the order of vanishing of $f_1(z_1, 0) - z_1$, and we say that f is *dynamically separating* if $\nu(f) - 1$ is less than or equal to the order of vanishing of $f_2^o(z_1, 0)$, where $f_2^o(z_1, z_2) = z_2^{-1}\big[f_2(z_1, z_2) - \lambda z_2 - f_2(z_1, 0)\big]$.

We can summarize their result as follows:

Theorem 19 (Bracci–Molino [15]) *Let $f = (f_1, f_2) \in \mathrm{End}(\mathbb{C}^2, O)$ be a holomorphic germ such that $df_O = diag(1, \lambda)$ with $\lambda = e^{2\pi i\theta} \neq 1$. Then:*

(i) *up to local holomorphic conjugation we can always assume that f is in ultra-resonant normal form;*
(ii) *if f is in ultra-resonant normal form, then $\nu(f)$ and the property of being dynamically separating are local holomorphic invariants of f;*
(iii) *if f is in ultra-resonant normal form and dynamically separating then there exist (at least) $\nu(f) - 1$ parabolic curves for f at O tangent to the eigenspace of* 1.

Thus we can have the coexistence of invariant disks (elliptic behavior) and parabolic curves. However, this begs the question of how the dynamics changes moving from the one-dimensional invariant disk to the one-dimensional parabolic curves. More precisely, our next open problem is

(OP20) *Describe the local dynamics in a full neighborhood of the origin of a dynamically separating $f \in \mathrm{End}(\mathbb{C}^2, O)$ in ultra-resonant normal form.*

Clearly, a similar question can be asked for dynamically separating germs in $\mathbb{C}^n$ with $n \geq 3$; see [64] for the necessary definitions.

A closely related open problem is the following:

(OP20′) *Describe the local dynamics in a full neighborhood of the origin of a germ $f \in \mathrm{End}(\mathbb{C}^2, O)$ in ultra-resonant normal form but not dynamically separating.*

An interesting example has been worked out in detail in [64].

We have thus seen a situation where parabolic and elliptic behavior coexist, because of the presence of an eigenvalue equal to 1. However, recently Bracci et al. [16] and Bracci and Zaitsev [17] have made the very interesting discovery that, somewhat surprising, more generally the existence of resonances might cause the appearance of parabolic dynamics even when no eigenvalue is a root of unity. Their approach applies to the so-called m-resonant germs.

Definition 20 Given $m \in \mathbb{N}^*$, we say that $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n$ is *m-resonant* if there exist m multi-indices $P^1, \ldots, P^m \in \mathbb{N}^n$ linearly independent over $\mathbb{Q}$ generating over $\mathbb{N}$ all resonances for $\boldsymbol{\lambda}$, in the sense that $\lambda_s = \prod_{j=1}^n \lambda_j^{k_j}$ if and only if there are $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ such that

$$(k_1, \ldots, k_n) = \sum_{h=1}^m \alpha_h P^h + e_s,$$

where e_s is the s-th vector in the canonical basis of $\mathbb{C}^n$.

A germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ with df_O diagonalizable is *m-resonant* if the set of eigenvalues of df_O is.

If $f \in \mathrm{End}(\mathbb{C}^n, O)$ is m-resonant, then it has a formal Poincaré–Dulac normal form $g = (g_1, \ldots, g_n)$ of the following kind:

$$g_j(z) = \lambda_j z_j + \sum_{|\boldsymbol{\alpha}\cdot\mathbf{P}|\geq 1} a_{\boldsymbol{\alpha},j} z^{\boldsymbol{\alpha}\cdot\mathbf{P}} z_j, \tag{10}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ ranges in $\mathbb{N}^m$, $\boldsymbol{\alpha}\cdot\mathbf{P} = \sum_{h=1}^m \alpha_h P^h$, and we are using the usual multi-index notations $z^Q = z_1^{q_1} \cdots z_n^{q_n}$ and $|Q| = q_1 + \cdots + q_n$ for $Q \in \mathbb{N}^n$. The *weighted order* k_0 of f is the minimal $|\boldsymbol{\alpha}|$ such that $a_{\boldsymbol{\alpha},j} \neq 0$ for some $1 \leq j \leq n$; it is a local formal (and hence holomorphic) invariant.

Expression (10) suggests introducing the holomorphic map $\pi\colon \mathbb{C}^n \to \mathbb{C}^m$ given by $\pi(z) = (z^{P^1}, \ldots, z^{P^m})$. Indeed, we can find $\Phi \in \mathrm{End}(\mathbb{C}^m, O)$ whose homogeneous expansion is of the form

$$\Phi(u) = u + H_{k_0+1}(u) + O(\|u\|^{k_0+2})$$

such that $\pi \circ g = \Phi \circ \pi$. In particular, g is semi-conjugated to Φ, and g acts on the foliation given by the level sets of π; thus the dynamics of g must reflect the (parabolic) dynamics of Φ.

The exact expression of Φ might depend on the choice of the Poincaré–Dulac formal normal form g, that furthermore might not be holomorphically conjugated to f; but since f is holomorphically conjugated to g up to any fixed order, the dynamics of the *parabolic shadow* of f

$$\phi(u) = u + H_{k_0+1}(u)$$

still can say a lot on the dynamics of f. For instance, ϕ describes the action of f on the leaf space of (a foliation conjugated to) the foliation of the level sets of π up to order $k_0 + 1$.

As indicated before, to describe the local dynamics of a parabolic shadow one needs the characteristic directions. A particularly important case is when a parabolic shadow ϕ admits an *attracting* non-degenerate characteristic direction v, that is a non-degenerate characteristic direction whose directors have all positive real part (and thus the corresponding Fatou flower is attracting; see [43]). It turns out that this is a property of the germ f, in the sense that if a parabolic shadow ϕ has an attracting non-degenerate characteristic direction then every parabolic shadow of f has one.

If v is a non-degenerate characteristic direction for a parabolic shadow ϕ, up to scaling we can assume that $H_{k_0+1}(v) = -(1/k_0)v$. We can then introduce the following definition:

Definition 21 Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be m-resonant. We say that f is *parabolically attracting* if it admits a parabolic shadow ϕ having an attracting non-degenerate characteristic direction $[v]$ such that

$$\mathrm{Re}\left(\sum_{|\boldsymbol{\alpha}|=k_0} \frac{a_{\boldsymbol{\alpha},j}}{\lambda_j} v^{\boldsymbol{\alpha}}\right) < 0$$

for $j = 1, \ldots, n$, where v is the representative of $[v]$ normalized so that $H_{k_0+1}(v) = -(1/k_0)v$.

It turns out that if the condition in this definition is satisfied for one parabolic shadow it is satisfied by all of them, and thus being parabolically attracting is a property of the germ f and not of a particular parabolic shadow.

We are now able to state (a particular case of) the main result of [16]:

Theorem 20 (Bracci–Raissy–Zaitsev [16]) *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be m-resonant of weighted order k_0, and parabolically attracting. Assume that all the eigenvalues of df_O have modulus one. Then there exist (at least) k_0 disjoint basins of attraction with the origin in the boundary.*

In analogy with previous problems, this result suggests the following open problem:

(OP21) *Describe the local dynamics of a m-resonant parabolically attracting germ $f \in \mathrm{End}(\mathbb{C}^n, O)$ in a full neighborhood of the origin.*

In [16] this has been done for 1-resonant parabolically attracting germs in Poincaré–Dulac normal form; but it still open for 1-resonant parabolically attracting germs not in Poincaré–Dulac normal form, as well as for 2-resonant parabolically attracting germs in Poincaré–Dulac normal form.

If $\boldsymbol{\lambda} \in (\mathbb{C}^*)^n$ is m-resonant, and $(k_1, \ldots, k_n) \in \mathbb{N}$ are such that $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_s$, we in particular have $k_s \geq 1$. Let us give a name to this situation:

Definition 22 We shall say that $\boldsymbol{\lambda} \in (\mathbb{C}^*)^n$ is *resonance effective* if for every $(k_1, \ldots, k_n) \in \mathbb{N}^n$ and $1 \leq s \leq n$ such that $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_s$ we have $k_s \geq 1$.

If $\boldsymbol{\lambda} \in (\mathbb{C}^*)^n$ is resonance effective, then the set

$$\mathrm{Rese}(\boldsymbol{\lambda}) = \{(h_1, \ldots, h_n) \in \mathbb{N}^n \mid \lambda_1^{h_1} \cdots \lambda_n^{h_n} = 1\}$$

gives all the resonances of $\boldsymbol{\lambda}$, in the sense that $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_s$ if and only if $(k_1, \ldots, k_s - 1, \ldots, k_n) \in \mathrm{Rese}(\boldsymbol{\lambda})$.

The set $\mathrm{Rese}(\boldsymbol{\lambda})$ is clearly closed under addition; then it is possible to prove (see [59]) that there exist a finite number of elements $P^1, \ldots, P^m \in \mathrm{Rese}(\boldsymbol{\lambda})$, called *minimal* (not necessarily $\mathbb{Q}$-linearly independent), and a finite number (possibly equal to zero) of different elements $C_1, \ldots, C_r \in \mathrm{Rese}(\boldsymbol{\lambda})$, called *cominimal,* such that every element of $\mathrm{Rese}(\boldsymbol{\lambda})$ can be written either in the form $\alpha_1 P^1 + \cdots + \alpha_m P^m$ or in the form $C_j + \alpha_1 P^1 + \cdots + \alpha_m P^m$ for suitable $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ and (in the second case) $1 \leq j \leq r$. So the m-resonant case corresponds to resonance effective, no cominimal elements and $\mathbb{Q}$-linearly independent minimal elements. Our last open problem is then the following:

(OP22) *Given* $f \in \mathrm{End}(\mathbb{C}^n, O)$ *locally invertible, let* $\boldsymbol{\lambda} \in (\mathbb{C}^*)^n$ *be the set of eigenvalues of* df_O, *and assume that* $\boldsymbol{\lambda}$ *is resonance effective. Is it possible to extend Bracci–Raissy–Zaitsev results to the case when* $\mathrm{Rese}(\boldsymbol{\lambda})$ *has no cominimal elements but has* $\mathbb{Q}$*-linearly dependent minimal elements? And to the case when* $\mathrm{Rese}(\boldsymbol{\lambda})$ *has cominimal elements?*

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