Basins of attraction in quadratic dynamical systems with a Jordan fixed point

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Abstract

In this note we study the dynamics of the family of maps $f(z,w) = (z+w+\alpha z^2+\beta w^2, w+w^2)$, both on $\mathbb{R}^2$ and on $\mathbb{C}^2$. All these maps have the origin as an isolated non-hyperbolic fixed point where the differential is not diagonalizable. We shall give sufficient conditions on the parameters for the existence of an open set attracted by the origin.

Keywords: Discrete complex dynamics, parabolic fixed point, Jordan fixed point, basin of attraction.

0. Introduction

Recently, a number of papers studying the behavior of holomorphic discrete dynamical systems about a non-hyperbolic fixed point in several variables have appeared (see, e.g., [1–3, 7–9, 10–13]). Their main concern was to determine the existence of complex submanifolds attracted by the fixed point under the action of the dynamical system.

To be more precise, let us fix a (germ of) holomorphic self-map f of $\mathbb{C}^n$ fixing the origin, and such that the spectrum of df_O is contained in $\Delta \cup \{1\}$ (the more extensively studied case up to now), where Δ is the open unit disk in the plane. A *parabolic d-manifold* for f *at the origin* is a complex d-manifold $M \subset \mathbb{C}^n$ such that $O \in \overline{M} \setminus M$ (where the closure is taken with respect to the topology of $\mathbb{C}^n$), $f(M) \subset M$ and $(f|_M)^k \to O$ as $k \to +\infty$; they are a natural several variables generalization of the petals appearing in the classical Leau-Fatou flower theorem. A parabolic n-manifold will be called a *basin of attraction* of the origin.

We shall limit ourselves here to recall what is known for $n = 2$, which is enough to put the results of the present note in perspective. We shall always assume that the origin is an isolated fixed point. If $(df_O) = \{1, \lambda\}$ with $|\lambda| < 1$ (the so-called *semiattractive* situation), Ueda [11, 12] and Hakim [7] proved the existence of a basin of attraction of the origin. If, on the other hand, $(df_O) = \{1\}$, there are two cases to consider. When $df_O = \mathrm{id}$, then there always exists a parabolic 1-manifold (i.e., a parabolic curve) at the origin [2]; furthermore, Hakim [8, 9] and Weickert [13] gave sufficient conditions for the existence of a basin of attraction of the origin.

When $df_O = J_2$, where J_2 is the Jordan canonical matrix associated to the eigenvalue 1 (and then we say that the origin is a *Jordan fixed point*), the situation has been studied in [1]. In this case the map f can be written as

$$f_1(z,w) = z + w + a^1_{11}z^2 + 2a^1_{12}zw + a^1_{22}w^2 + \cdots,$$
$$f_2(z,w) = w + a^2_{11}z^2 + 2a^2_{12}zw + a^2_{22}w^2 + a^2_{111}z^3 + \cdots.$$

In [1] it is proved that (assuming that the origin is an isolated fixed point) if at least one of the quantities a^2_{11}, $\varepsilon = a^1_{11} + a^2_{12}$, $\eta = (a^1_{11} - a^2_{12})^2 + 2a^2_{111}$ is different from zero then the map f has at least one parabolic curve at the origin. But it turns out that this is always true, even when $a^2_{11} = \varepsilon = \eta = 0$. Indeed, in the latter case blowing up the origin the germ f lifts to a germ of the form

$$\tilde{f}(z_1,z_2) = \big(z_1 + \alpha z_1^2 + z_1z_2 + O(\|z\|^3), z_2 - 2\alpha^2 z_1^2 - 3\alpha z_1z_2 - z_2^2 + O(\|z\|^3)\big),$$

for some $\alpha \in \mathbb{C}$. Using the terminology introduced in [2], it is easy to see that this map has two singular directions, $[1 : -\alpha]$ and $[0 : 1]$. The latter gives rise to a parabolic curve that should be discarded, because it is contained in the exceptional divisor of the blow-up. But the former, even if it is a degenerate characteristic direction in the sense of Hakim, has residual index $-1/2$ and thus, thanks to [2, Corollary 3.3], it also gives rise to a parabolic curve, which is transversal to the exceptional divisor and thus it can be projected down producing a parabolic curve at the origin for our map f. Thus, we have proved the existence of a parabolic curve in all cases when $df_O = J_2$ and the origin is an isolated fixed point (it should be remarked that this result is not a consequence of Hakim's theory, but it can be obtained only using the techniques introduced in [2]).

In [1] we also applied Hakim's results to get sufficient conditions for the existence of basins of attraction when $df_O = J_2$. The aim of this short note is to provide an example of a family of quadratic holomorphic self maps of $\mathbb{C}^2$, with the origin as isolated fixed point, such that $df_O = J_2$ and with a basin of attraction of the origin even if they do not satisfy the sufficient conditions described in [1]. The family is the following:

$$f(z,w) = \big(z + w + \alpha z^2 + \beta w^2, w + w^2\big), \tag{0.1}$$

with $\alpha \neq 0$ and $\beta \in \mathbb{C}$. When $\operatorname{Re}\alpha > 0$ this map has a basin of attraction of the origin (Theorem 1.3) but it does not satisfy the criterion described in [1, Remark 3.5].

We shall also study the action of map (0.1) on the real plane $\mathbb{R}^2$ when α and β are real; we shall obtain a fairly complete description of the dynamics, which is interesting because, as far as I know, there exist only a few papers devoted to real dynamical systems with a fixed point where the differential is not diagonalizable (the only ones I am aware of, that is [5, 6], do not study the family (0.1), and deal only with the existence of invariant curves).

1. Complex dynamics

We begin recalling a couple of results about the well-known map $g(w) = w + w^2$, which is the standard example of holomorphic map of one variable with a parabolic basin (at its unique fixed point, the origin); see, e.g., [4] for all unproved assertions. The basin of attraction to the origin is a cauliflower-like bounded set $C \subset \mathbb{C}$; the orbits of points outside $\overline{C}$ go to infinity at an exponential rate; the boundary ∂C is the Julia set of g, and it is a closed completely invariant set containing the origin.

The following Lemma, whose proof is elementary, describes the behavior of g restricted to $\mathbb{R}$:

Lemma 1.1: *For $\tilde u_0 \in \mathbb{R}$ set $\tilde u_n = g^n(\tilde u_0)$. Then:*

(i) *For all $\tilde u_0 \in \mathbb{R} \setminus \{0, -1\}$ the sequence $\{\tilde u_n\}$ is strictly increasing.*
(ii) *If $\tilde u_0 \in [-1, 0]$ then $\tilde u_n \to 0$; otherwise $\tilde u_n \to +\infty$.*
(iii) *If $\tilde u_0 \in (-1, 0)$ then*

$$\forall n \geq 1 \qquad -\frac{1}{n} \leq \tilde u_n \leq \frac{\tilde u_1}{n}, \tag{1.1}$$

that is $|\tilde u_n| = O(1/n)$.

We shall also need a quantitative extimate on the way orbits inside C approach the origin:

Lemma 1.2: *For all $w_0 \in C$ set $w_n = u_n + iv_n = g^n(w_0)$. Then*

$$\lim_{n\to+\infty} nw_n = \lim_{n\to+\infty} nu_n = -1 \qquad \text{and} \qquad \lim_{n\to\infty} nv_n = 0. \tag{1.2}$$

More precisely, there are c_1, $c_2 > 0$ depending on w_0 such that

$$|1 + nu_n| \leq |1 + nw_n| \leq \frac{c_1}{n} \log n \tag{1.3}$$

and

$$|y_n| \leq \frac{c_2}{n^2} \left(1 + \frac{c_1}{n} \log n\right)^2 \tag{1.4}$$

for all $n \geq 1$.

Proof: For all $w_0 \in C$ and $j \geq 1$ we can write

$$\frac{1}{w_j} = \frac{1}{w_{j-1}} - 1 + \frac{w_{j-1}}{1 + w_{j-1}}.$$

Adding for $j = 1, \ldots, n$ and dividing by n we find

$$\frac{1}{nw_n} = \frac{1}{nw_0} - 1 + \frac{1}{n}\sum_{j=1}^{n} \frac{w_{j-1}}{1+w_{j-1}},$$

and thus (1.2) follows by the convergence of the averages of a converging sequence. In particular, we get a $k_1 > 0$ (depending on w_0) such that $|w_j| \le k_1/j$ for all $j \ge 1$. Thus there exists $k_2 > 0$ such that

$$\forall n \ge 1 \qquad \frac{1}{n}\left|\sum_{j=1}^{n} \frac{w_{j-1}}{1+w_{j-1}}\right| \le \frac{1}{n}\sum_{j=1}^{n} \frac{|w_{j-1}|}{1-|w_{j-1}|} \le \frac{k_2}{n}\log n.$$

Therefore we can find a suitable $c_1 > 0$ so that

$$|1 + nu_n| \le |1 + nw_n| \le n|w_n| \left[\frac{1}{n|w_0|} + \frac{1}{n}\left|\sum_{j=1}^{n} \frac{w_{j-1}}{1+w_{j-1}}\right|\right] \le \frac{c_1}{n}\log n$$

for all $n \ge 1$, and (1.3) is proved.

Now let $F = \{\operatorname{Re} w < -3|w|^2\}$, and for every $c > 0$ set $H_c = \{|\operatorname{Im} w| < c|\operatorname{Re} w|^2|\} \cap F$. The set F is a disk of center $-1/6$ and radius $1/6$, and it is well-known that for every $w_0 \in C$ there is $n_0 \ge 0$ such that $g^n(w_0) \in F$ for all $n \ge n_0$. Furthermore, it is easy to check that H_c is g-invariant for all $c > 0$. In particular, the g-invariance of $H_{|v_{n_0}|/|u_{n_0}|^2}$ implies

$$\forall n \ge n_0 \qquad |v_n| \le \frac{|v_{n_0}|}{|u_{n_0}|^2}\,|u_n|^2 \le \frac{|v_{n_0}|}{|u_{n_0}|^2}\frac{1}{n^2}\left(1 + \frac{c_1}{n}\log n\right)^2,$$

and so we can find $c_2 > 0$ such that (1.4) is satisfied for all $n \ge 1$. □

As described in the introduction, we are interested in the dynamics of maps of the form

$$f(z, w) = (z + w + \alpha z^2 + \beta w^2, w + w^2), \tag{1.5}$$

with $\alpha \neq 0$ and $\beta \in \mathbb{C}$, whose only fixed point is the origin, which is a Jordan fixed point.

We first of all remark that they are conjugated to maps of the form

$$\tilde{f}(z, w) = (z + z^2 + \alpha(w + \beta w^2), w + w^2), \tag{1.6}$$

via the map $(z, w) \mapsto (z/\alpha, w)$. We set $g_{\alpha,\beta}(w) = \alpha(w + \beta w^2)$; in particular, $g_{1,1} = g$.

We shall write $z_0 = x_0 + iy_0$, $w_0 = u_0 + iv_0$ and $(z_n, w_n) = \tilde{f}^n(z_0, w_0)$. We now prove the existence of a basin of attraction of the origin when $\operatorname{Re}\alpha > 0$ and $\beta \in \mathbb{C}$:

Theorem 1.3: *Let $\tilde{f}\colon \mathbb{C}^2 \to \mathbb{C}^2$ be given by (1.6). Assume $\operatorname{Re}\alpha > 0$, and choose $k_0 > 0$. Then there are c_1, c_2, $c_3 > 0$, continuous functions a_1, $a_2\colon [-1/2, 0) \to \mathbb{R}^+$ and a (discontinuous) function $n_0\colon [-1/2, 0) \to \mathbb{N}$ such that setting*

$$D = \left\{(z_0, w_0) \in \mathbb{C}^2 \;\middle|\; x_0 \in [-1/2, 0),\ |y_0| < a_1(x_0),\ -a_2(x_0) < u_0 < 0,\ |v_0| < k_0|u_0|^2\right\}$$

then for every $(z_0, w_0) \in D$ we have

$$\forall n \ge 1 \qquad x_n \le -\frac{c_1|x_1|}{n^{1/2}}, \tag{1.7}$$

and

$$\forall n \ge n_0(x_0) \qquad |x_n| \le \frac{c_2}{|x_1|n^{1/2}} \quad \text{and} \quad |y_n| \le \frac{c_3}{|x_1|n^{1/2}}. \tag{1.8}$$

In particular, if we denote by D' the symmetric of D with respect to the plane $z=-1/2$, then the set $D\cup D'$ is contained into the basin of attraction of the origin.

Proof: It is easy to check that in the set $\{|v_0|\le k|u_0|^2\}$ one has

$$\operatorname{Re} g_{\alpha,\beta}(w_0)=(\operatorname{Re}\alpha)u_0+O(|u_0|^2)\qquad\text{and}\qquad \operatorname{Im} g_{\alpha,\beta}(w_0)=(\operatorname{Im}\alpha)u_0+O(|u_0|^2).$$

Therefore, recalling Lemma 1.2, we can find a_3, k_1, k_2, $k_3>0$ (with $k_1<1<k_2$) such that if $u_0\in(-a_3,0)$ and $|v_0|\le k_0|u_0|^2$ then

$$-\frac{k_1}{n}\ge \operatorname{Re} g_{\alpha,\beta}(w_n)\ge -\frac{k_2}{n}\qquad\text{and}\qquad |\operatorname{Im} g_{\alpha,\beta}(w_n)|\le\frac{k_3}{n}$$

for all $n\ge 1$.

Now set $c_1=\sqrt{k_1}$, $c_3=k_3/c_1$ and $c_2=\sqrt{2(k_2+c_3^2)}$. For $x_0\in[-1/2,0)$ let $n_0=n_0(x_0)\ge 1$ be the least integer greater than $|\tilde x_1|^{-2}\max\{(4c_1^2)^{-1},4c_2^2\}$, where $\tilde x_1=x_0+x_0^2=g(x_0)$. Notice that $|x_1|\ge|\tilde x_1|$ for any $y_0\in\mathbb{R}$ and $u_0\in(-a_3,0)$, and thus

$$n_0>\frac{1}{|x_1|^2}\max\left\{\frac{1}{4c_1^2},4c_2^2\right\}. \tag{1.9}$$

Set $\tilde x_n=g^n(x_0)$, so that $(\tilde x_n,0)=\tilde f^n(x_0,0)$. By Lemma 1.1 we have $\tilde x_n\in[-1/4,0)$ and

$$|\tilde x_n|\le\frac{1}{n}<\frac{c_2}{n^{1/2}}$$

for all $n\ge1$. Therefore we can choose $a_1(x_0)$, $a_2(x_0)>0$ (with $a_2(x_0)<a_3$), depending continuously on x_0, such that if $|y_0|<a_1(x_0)$, $-a_2(x_0)<u_0<0$ and $|v_0|<k_0|u_0|^2$ then $|x_{n_0}|\le c_2/n_0^{1/2}$, $|y_{n_0}|\le c_3/n_0^{1/2}$ and $x_k\in(-1/2,0)$ for $k=0,\dots,n_0$. In particular, (1.7) holds for $n=1$ and (1.8) holds for $n=n_0$.

We now show, by induction, that (1.7) holds for $n=1,\dots,n_0$. Assume it holds for some $1\le n<n_0$; since, by assumption, $x_n\in(-1/2,0)$ and g is increasing in that interval, we have

$$\begin{aligned} x_{n+1}&=x_n+x_n^2-y_n^2+\operatorname{Re} g_{\alpha,\beta}(w_n)\le-\frac{c_1|x_1|}{n^{1/2}}+\frac{c_1^2|x_1|^2}{n}-\frac{k_1}{n}\\ &=-\frac{c_1|x_1|}{(n+1)^{1/2}}\left(1+\frac{1}{n}\right)^{1/2}\left(1+\frac{k_1(1-|x_1|^2)}{c_1|x_1|n^{1/2}}\right)\le-\frac{c_1|x_1|}{(n+1)^{1/2}}. \end{aligned}\tag{1.10}$$

So (1.7) holds for $n\le n_0$.

Now we prove simultaneously both (1.7) and (1.8) by induction for $n\ge n_0$. First of all, notice that for any $A>0$ we have

$$n\ge\frac{1}{4A^2}\qquad\Longrightarrow\qquad\left(1+\frac{1}{n}\right)^{1/2}\left(1-\frac{A}{n^{1/2}}\right)<1. \tag{1.11}$$

Assume then that (1.7) and (1.8) hold for some $n\ge n_0$; in particular,

$$0>-\frac{c_1|x_1|}{n^{1/2}}\ge x_n\ge-\frac{c_2}{|x_1|n^{1/2}}\ge-\frac{c_2}{|x_1|n_0^{1/2}}>-1/2,$$

by (1.9), and we can repeat the computations in (1.10) to get (1.7) for $n+1$.

Next

$$\begin{aligned} |y_{n+1}|&\le|y_n|(1-2|x_n|)+|\operatorname{Im} g_{\alpha,\beta}(w_n)|\le\frac{c_3}{|x_1|n^{1/2}}\left(1-\frac{2c_1|x_1|}{n^{1/2}}\right)+\frac{k_3}{n}\\ &=\frac{c_3}{|x_1|(n+1)^{1/2}}\left(1+\frac{1}{n}\right)^{1/2}\left(1-\frac{c_1|x_1|}{n^{1/2}}\right)\le\frac{c_3}{|x_1|(n+1)^{1/2}}, \end{aligned}$$

where the last inequality holds because of (1.11) and (1.9).

We are ready for the last inductive step. We have

$$
\begin{aligned}
|x_{n+1}| &= |x_n + x_n^2| + |y_n|^2 + |\operatorname{Re} g_{\alpha,\beta}(w_n)| \le \frac{c_2}{|x_1|n^{1/2}}\left(1 - \frac{c_2}{|x_1|n^{1/2}}\right) + \frac{c_3^2}{|x_1|^2 n} + \frac{k_2}{n} \\
&= \frac{c_2}{|x_1|(n+1)^{1/2}}\left(1+\frac{1}{n}\right)^{1/2}\left(1 - \frac{c_2^2 - |x_1|^2k_2 - c_3^2}{c_2|x_1|n^{1/2}}\right) \le \frac{c_2}{(n+1)^{1/2}}\left(1+\frac{1}{n}\right)^{1/2}\left(1 - \frac{c_2}{2|x_1|n^{1/2}}\right) \\
&\le \frac{c_2}{|x_1|(n+1)^{1/2}},
\end{aligned}
$$

again by (1.11), because if $A = c_2/2|x_1|$ then $1/4A^2 = |x_1|^2/c_2^2 < 1$.

Finally the final assertion follows from the fact that the basin of attraction to the origin is symmetric with respect to the plane $z_0 = -1/2$. Indeed, if we conjugate $\tilde{f}$ by $(z,w) \mapsto (z - 1/2, w)$ we get $(z^2 + 1/4 + g_{\alpha,\beta}(w), w + w^2)$ which is symmetric with respect to the w-axis. □

Remark. The criterion proved in [1] applying Hakim's results for the existence of basins of attraction for 2-dimensional maps with a Jordan fixed point is the following. Write $f = (f_1, f_2)$,

$$
\begin{aligned}
f_1(z,w) &= z + w + a_{11}^1 z^2 + 2a_{12}^1 zw + a_{22}^1 w^2 + \cdots, \\
f_2(z,w) &= w + a_{11}^2 z^2 + 2a_{12}^2 zw + a_{22}^2 w^2 + a_{111}^2 z^3 + \cdots,
\end{aligned}
$$

and set $\varepsilon = a_{11}^1 + a_{12}^2$ and $\eta = (a_{11}^1 - a_{12}^2)^2 + 2a_{111}^2$. Then there is a basin of attraction of the origin if $a_{11}^2 = 0$, $\eta \neq 0$ and $|\operatorname{Re}(\varepsilon/\sqrt{\eta})| > 1$. In our case we have $a_{11}^2 = 0$ but $\varepsilon = \eta = 1$ (independently of α and β), and so this criterion does not apply. See Figures 1.a and 1.b for two bidimensional slices of the basin of attraction when $\alpha = 1$ and $\beta = 0$.

2. Real dynamics

We now study the restriction of $\tilde{f}$ to $\mathbb{R}^2$ when both α and β are real, so that $\tilde{f}(\mathbb{R}^2) \subseteq \mathbb{R}^2$. The results in this case are fairly complete and interesting in their own right.

Take $(\tilde{x}_0, \tilde{u}_0) \in \mathbb{R}^2$ and set $(\tilde{x}_n, \tilde{u}_n) = \tilde{f}^n(\tilde{x}_0, \tilde{u}_0)$ as usual. First of all, if $\tilde{u}_0 \notin [-1, 0]$ then $\tilde{u}_n \to +\infty$. If $\tilde{u}_0 = -1, 0$, then $\tilde{u}_n = 0$ for all $n \ge 1$ and therefore $\tilde{x}_n \to 0$ iff $\tilde{x}_1 \in [-1, 0]$, and $\tilde{x}_n \to +\infty$ otherwise. Now, if $\tilde{u}_0 = 0$ then $\tilde{x}_1 = g(\tilde{x}_0) \in [-1, 0]$ iff $\tilde{x}_0 \in [-1, 0]$. On the other hand, if $\tilde{u}_0 = -1$ then $\tilde{x}_1 = g(\tilde{x}_0) + g_{\alpha,\beta}(-1)$, and again we find the exact conditions ensuring $\tilde{x}_n \to 0$.

This is enough to determine the behavior of $\tilde{x}_n$ when $\alpha < 0$:

Proposition 2.1: *Let $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by* (1.6) *with $\alpha < 0$ and $\beta \in \mathbb{R}$. Take $(\tilde{x}_0, \tilde{u}_0) \in \mathbb{R} \times (-1, 0)$. Then $\tilde{x}_n \to +\infty$; more precisely, $\tilde{x}_n = O(\log n)$. In particular, we have $(\tilde{x}_n, \tilde{u}_n) \to (0, 0)$ iff $\tilde{u}_0 = 0$ and $\tilde{x}_0 \in [-1, 0]$ or $\tilde{u}_0 = -1$ and $\tilde{x}_0 \in g^{-1}\big([-1 - g_{\alpha,\beta}(-1), -g_{\alpha,\beta}(-1)]\big)$.*

Proof: The point is that when $\alpha < 0$ there is $b > 0$ such that $g_{\alpha,\beta}$ is (positive and) decreasing in $[-b, 0]$; to be precise, $b = 1/2\beta$ if $\beta > 0$, and any $b > 0$ works if $\beta \le 0$. Choose $n_0 \ge 0$ so that $\tilde{u}_n \in (-b, 0)$ for all $n \ge n_0$. Then

$$
\begin{aligned}
\tilde{x}_n &= \tilde{x}_{n_0} + \sum_{j=n_0}^{n-1} (\tilde{x}_{j+1} - \tilde{x}_j) = \tilde{x}_{n_0} + \sum_{j=n_0}^{n-1} \big(\tilde{x}_j^2 + g_{\alpha,\beta}(\tilde{u}_j)\big) \\
&\ge \tilde{x}_{n_0} + \sum_{j=n_0}^{n-1} g_{\alpha,\beta}\left(\frac{\tilde{u}_1}{j}\right) = \tilde{x}_{n_0} + |\alpha||\tilde{u}_1| \sum_{j=n_0}^{n-1} \frac{1}{j} - |\alpha||\tilde{u}_1|^2 \beta \sum_{j=n_0}^{n-1} \frac{1}{j^2},
\end{aligned}
$$

thanks to Lemma 1.1.(ii), and we are done. □

To study the case $\alpha > 0$ we need the following observation:

Lemma 2.2: *Let* $\tilde f:\mathbb{R}^2\to\mathbb{R}^2$ *be given by* (1.6) *with* $\alpha>0$ *and* $\beta\in\mathbb{R}$. *Then there is* $b_0=b_0(\alpha,\beta)\in(0,1]$ *such that if* $(\tilde x_0,\tilde u_0)\in[-1,0]\times[-b_0,0]$ *then* $\tilde x_n\in[-1,0]$ *for all* $n\ge 0$ *and there is* $n_0=n_0(\alpha,\beta)\ge 0$ *such that* $\tilde x_n\in[-1/2,0]$ *for all* $n\ge n_0$.

Proof: Let

$$b_0=\max\bigl\{b\in[0,1]\bigm| g_{\alpha,\beta}\bigl([-b,0]\bigr)\subseteq[-3/4,0]\bigr\},$$

and define $b_1<b_0$ in the same way replacing $-3/4$ by $-1/4$. Let $n_0\ge 0$ be the minimum integer such that $g^{n_0}([-b_0,0])\subseteq[-b_1,0]$ (and hence $g^n([-b_0,0])\subseteq[-b_1,0]$ for all $n\ge n_0$). The assertion then follows by remarking that $(\tilde x,\tilde u)\in[-1,0]\times[-b_0,0]$ implies $g(\tilde x)+g_{\alpha,\beta}(\tilde u)\in[-1,0]$, and that $(\tilde x,\tilde u)\in[-1,0]\times[-b_1,0]$ implies $g(\tilde x)+g_{\alpha,\beta}(\tilde u)\in[-1/2,0]$. □

Set $D_{\mathbb{R}}=(-1,0)\times(-b_0,0)$, where $b_0>0$ is given by the previous lemma. Then:

Lemma 2.3: *Let* $\tilde f:\mathbb{R}^2\to\mathbb{R}^2$ *be given by* (1.6) *with* $\alpha>0$ *and* $\beta\in\mathbb{R}$. *Take* $(\tilde x_0,\tilde u_0)\in D_{\mathbb{R}}$. *Then there are* $c_1=c_1(\alpha,\beta)>0$ *and* $c_2=c_2(\alpha,\beta,\tilde x_0,\tilde u_0)>0$ *such that*

$$\forall n\ge 1 \qquad \frac{c_2}{n^{1/2}}\le|\tilde x_n|\le\frac{c_1}{n^{1/2}}. \tag{2.1}$$

In particular, $(\tilde x_n,\tilde u_n)\to(0,0)$.

Proof: Let us start with the right-hand side of (2.1). Let $0<a_0\le b_0$ be such that $g_{\alpha,\beta}$ is (negative and) increasing on $[-a_0,0)$, and choose $n_1\ge n_0$ (where n_0 is given by the previous lemma) such that $-1/n$ (and hence $\tilde u_n$) belongs to $[-a_0,0)$ for all $n\ge n_1$. Set

$$c_1=\max\left\{\frac14\left(1+\sqrt{1+16\alpha(1+|\beta|)}\right),1,\dots,n_1^{1/2}\right\}.$$

Clearly the right-hand side of (2.1) holds for $1\le n\le n_1$. Assume it holds for some $n\ge n_1$; then (recalling Lemmas 2.2 and 1.1)

$$0\ge\tilde x_{n+1}=\tilde x_n+\tilde x_n^2+g_{\alpha,\beta}(\tilde u_n)\ge-\frac{c_1}{n^{1/2}}+\frac{c_1^2}{n}+g_{\alpha,\beta}\left(-\frac1n\right)\ge-\frac{c_1}{(n+1)^{1/2}},$$

because $(1+n^{-1})^{1/2}-1<1/2n$ implies

$$c_1\left[\frac{1}{(n+1)^{1/2}}-\frac{1}{n^{1/2}}\right]+\frac{c_1^2-\alpha}{n}+\frac{\alpha\beta}{n^2}\ge\frac1n\left[c_1^2-\alpha-\frac{c_1}{2(n+1)^{1/2}}-\frac{\alpha|\beta|}{n}\right]>\frac1n\left[c_1^2-\frac12c_1-\alpha(1+|\beta|)\right]=0,$$

by the choice of c_1, and we are done.

To prove the left-hand side of (2.1), choose $a_0>0$ as before, and $n_1\ge n_0$ such that $\tilde u_n$ (and hence $\tilde u_1/n$) belongs to $[-a_0,0)$ for all $n\ge n_1$; we moreover require that $n_1>|\beta||\tilde u_1|$. Set

$$c_2=\min\left\{\frac{1}{\sqrt2}\sqrt{\alpha|\tilde u_1|(1-|\beta||\tilde u_1|/n_1)},|\tilde x_1|,\dots,n_1^{1/2}|\tilde x_{n_1}|\right\}.$$

Clearly the left-hand side of (2.1) holds for $1\le n\le n_1$. Assume it holds for some $n\ge n_1$; then (recalling again Lemmas 2.2 and 1.1)

$$\begin{aligned}\tilde x_{n+1}&=\tilde x_n+\tilde x_n^2+g_{\alpha,\beta}(\tilde u_n)\le-\frac{c_2}{n^{1/2}}+\frac{c_2^2}{n}+g_{\alpha,\beta}\left(\frac{\tilde u_1}{n}\right)\\&=-\frac{c_2}{(n+1)^{1/2}}\left(\frac{n+1}{n}\right)^{1/2}\left[1+\frac{1}{c_2n^{1/2}}\left(\alpha|\tilde u_1|\left(1-\frac{\beta|\tilde u_1|}{n}\right)-c_2^2\right)\right]\le-\frac{c_2}{(n+1)^{1/2}},\end{aligned}$$

again by the choice of c_2. □

Set

$$\Omega_{\mathbb{R}} = \bigcup_{n\geq 0} \left(\tilde{f}^{-n}(D_{\mathbb{R}}) \cap \mathbb{R}^2\right).$$

Clearly, $(\tilde{x}_n, \tilde{u}_n) \to (0,0)$ for all $(\tilde{x}_0, \tilde{u}_0) \in \Omega_{\mathbb{R}}$. We now prove that $\Omega_{\mathbb{R}}$ is exactly the basin of attraction to the origin for $\tilde{f}$ restricted to $\mathbb{R}^2$; in the proof we shall describe it precisely.

Theorem 2.4: *Let $\tilde{f}\colon\mathbb{R}^2 \to \mathbb{R}^2$ be given by* (1.6) *with $\alpha > 0$ and $\beta \in \mathbb{R}$. Then $\Omega_{\mathbb{R}}$ is the basin of attraction of $(0,0)$ for $\tilde{f}|_{\mathbb{R}^2}$. Furthermore, $(\tilde{x}_n, \tilde{u}_n) \to (0,0)$ for all $(\tilde{x}_0, \tilde{u}_0) \in \overline{\Omega_{\mathbb{R}}}$, whereas $\{(\tilde{x}_n, \tilde{u}_n)\}$ diverges if $(\tilde{x}_0, \tilde{u}_0) \notin \overline{\Omega_{\mathbb{R}}}$.*

Proof: First of all, notice again that both $\Omega_{\mathbb{R}}$ and the basin of attraction are symmetric with respect to the axis $\tilde{x}_0 = -1/2$.

Now take $(\tilde{x}_0, \tilde{u}_0) \in \mathbb{R} \times [-b_0, 0)$. We have the following possibilities:

(a) $\tilde{x}_0 \in [-1, 0]$;
(b) $\tilde{x}_0 > 0$;
(c) $\tilde{x}_0 < -1$.

In case (a) we have $(\tilde{x}_1, \tilde{u}_1) \in D_{\mathbb{R}}$, and thus $(\tilde{x}_0, \tilde{u}_0) \in \Omega_{\mathbb{R}}$. Case (c) is equivalent to case (b), because of the simmetry with respect to $\tilde{x}_0 = -1/2$; so we are left with the latter.

Take then $(\tilde{x}_0, \tilde{u}_0) \in \mathbb{R}^2$ so that $\tilde{x}_0 > 0$ and $\tilde{u}_0 \in [-b_0, 0)$. There are only two possibilities: either there is a first $j_0 \geq 1$ such that $\tilde{x}_{j_0} < 0$, or $\tilde{x}_j > 0$ for all $j \geq 0$ (remark that $\tilde{x}_{j_0} = 0$ forces $\tilde{x}_{j_0+1} < 0$). In the first case we have $\tilde{x}_{j_0} > g_{\alpha,\beta}(\tilde{u}_{j_0-1}) > -1$; therefore $(\tilde{x}_{j_0}, \tilde{u}_{j_0}) \in D_{\mathbb{R}}$ and $(\tilde{x}_0, \tilde{u}_0) \in \Omega_{\mathbb{R}}$.

So we are interested in understanding when the whole sequence $\{\tilde{x}_n\}$ stays positive. Since $\tilde{x}_{j+1} < \tilde{x}_j$ iff $|\tilde{x}_j| < \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|}$, it follows that if we have $\tilde{x}_{j_0+1} \geq \tilde{x}_{j_0} > 0$ for some j_0 large enough as to have $\tilde{u}_{j_0} \in [-a_0, 0]$, where a_0 is as in the proof of Lemma 2.3, then

$$\tilde{x}_{j_0+1} \geq \tilde{x}_{j_0} \geq \sqrt{|g_{\alpha,\beta}(\tilde{u}_{j_0})|} > \sqrt{|g_{\alpha,\beta}(\tilde{u}_{j_0+1})|},$$

and so $\tilde{x}_{j_0+2} \geq \tilde{x}_{j_0+1}$. This means that, if it stays positive, the sequence $\{\tilde{x}_n\}$ is either eventually decreasing to 0 or eventually increasing to $+\infty$ (remember that if the sequence $\{(\tilde{x}_n, \tilde{u}_n)\}$ is converging it must converge to a fixed point of $\tilde{f}$).

We shall now prove that there is a function $\eta\colon[-b_0, 0] \to \mathbb{R}^+$ such that:

– if $0 < \tilde{x}_0 < \eta(\tilde{u}_0)$ then $(\tilde{x}_0, \tilde{u}_0) \in \Omega_{\mathbb{R}}$;
– if $\tilde{x}_0 = \eta(\tilde{u}_0)$ then $(\tilde{x}_0, \tilde{u}_0) \in \partial\Omega_{\mathbb{R}}$ and $\tilde{x}_n \to 0^+$;
– if $\tilde{x}_0 > \eta(\tilde{u}_0)$ then $\tilde{x}_n \to +\infty$.

Set $\chi_\pm(x) = \frac{1}{2}(\pm\sqrt{1+4x} - 1)$. It is easy to check that $\tilde{x}_{j+1} > a \geq 0$ iff $\tilde{x}_j > \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)| + a)$. In particular,

$$\begin{aligned}
\tilde{x}_{j+1} > 0 &\iff \tilde{x}_j > \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)|) \iff \tilde{x}_{j-1} > \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)|)\big) \iff \cdots \\
&\iff \tilde{x}_0 > \chi_+\Big(|g_{\alpha,\beta}(\tilde{u}_0)| + \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_1)| + \cdots + \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)|)\big)\cdots\big)\Big) = l_j; \\
\tilde{x}_{j+1} < \tilde{x}_j &\Longrightarrow \tilde{x}_j < \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|} \Longrightarrow \tilde{x}_{j-1} < \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|}\big) \Longrightarrow \cdots \\
&\Longrightarrow \tilde{x}_0 < \chi_+\Big(|g_{\alpha,\beta}(\tilde{u}_0)| + \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_1)| + \cdots + \chi_+(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|})\cdots\big)\Big) = m_j.
\end{aligned}$$

It is easy to check that $l_j < m_j$ for all $j \geq 0$ (because χ_+ is increasing and $\chi_+(x) \leq \sqrt{x}$). Furthermore, the sequence $\{l_j\}$ is strictly increasing (this is obvious) and the sequence $\{m_j\}$ is eventually strictly decreasing (this follows from $\chi_+(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|}) < \sqrt{|g_{\alpha,\beta}(\tilde{u}_{j-1})|}$, which is a consequence of $|g_{\alpha,\beta}(\tilde{u}_j)| < |g_{\alpha,\beta}(\tilde{u}_{j-1})|$, which in turns holds as soon as j is large enough). Finally, since $0 < \chi'_+(x) < 1$ for all $x > 0$, we have

$$\begin{aligned}
|m_j - l_j| \leq \Big|&\chi_+\big(|g_{\alpha,\beta}(\tilde{u}_1)| + \cdots + \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|}\big)\cdots\big) \\
&- \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_1)| + \cdots \chi_+\big(|g_{\alpha,\beta}(\tilde{u}_{j-1})| + \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)|)\big)\cdots\big)\Big| \\
\leq \cdots \leq \big|&\sqrt{|g_{\alpha,\beta}(\tilde{u}_j)|} - \chi_+(|g_{\alpha,\beta}(\tilde{u}_j)|)\big| \to 0
\end{aligned}$$

as $j \to +\infty$; therefore the two sequences converge to the same limit, which we shall denote by $\eta(\tilde u_0)$.

Now consider our $\tilde x_0$. We have three possibilities:

- $0 < \tilde x_0 < \eta(\tilde u_0)$. This means that $\tilde x_0 < l_j$ eventually, that is $\tilde x_j < 0$ eventually, and we have seen that this implies $(\tilde x_0, \tilde u_0) \in \Omega_{\mathbb{R}}$.
- $\tilde x_0 > \eta(\tilde u_0)$. Then $\tilde x_0 > l_j$ for all j's, that is $\tilde x_j > 0$ always, and $\tilde x_0 > m_j$ if j is greater than some j_0. This means that $\tilde x_{j+1} \ge \tilde x_j$ eventually, and this forces $\tilde x_n \to +\infty$, as already remarked.
- $\tilde x_0 = \eta(\tilde u_0)$. Then $l_j < \tilde x_0 < m_j$ for all j large enough, and thus $\{\tilde x_j\}$ is positive and eventually strictly decreasing, necessarily to 0. Thus $(\tilde x_0, \tilde u_0) \in \partial\Omega_{\mathbb{R}}$ and $(\tilde x_n, \tilde u_n) \to (0,0)$.

Clearly, $\eta(0) = 0 = \chi_+(0)$; but $l_1 > \chi_+(|g_{\alpha,\beta}(\tilde u_0)|)$ as soon as $|g_{\alpha,\beta}(\tilde u_1)| \ne 0$, and thus in general we have

$$\eta(\tilde u_0) > \chi_+(|g_{\alpha,\beta}(\tilde u_0)|) \ge 0.$$

Finally, take a generic $(\tilde x_0, \tilde u_0) \in \mathbb{R} \times [-1, 0]$. Obviously, there is an $n_0 \ge 0$ (it can be chosen depending only on α and β) such that $\tilde u_{n_0} \in [-b_0, 0]$. Then there are again only three possibilities:

(i) $-1 - \eta(\tilde u_{n_0}) < \tilde x_{n_0} < \eta(\tilde u_{n_0})$: then $(\tilde x_{n_0}, \tilde u_{n_0}) \in \Omega_{\mathbb{R}}$ — and thus $(\tilde x_0, \tilde u_0) \in \Omega_{\mathbb{R}}$ too.
(ii) $\tilde x_{n_0} = -1 - \eta(\tilde u_{n_0})$ or $\tilde x_{n_0} = \eta(\tilde u_{n_0})$: then $(\tilde x_{n_0}, \tilde u_{n_0}) \in \partial\Omega_{\mathbb{R}}$ — and thus $(\tilde x_0, \tilde u_0) \in \partial\Omega_{\mathbb{R}}$, because $(\tilde x_n, \tilde u_n) \to (0,0)$.
(iii) $\tilde x_{n_0} < -1 - \eta(\tilde u_{n_0})$ or $\tilde x_{n_0} > \eta(\tilde u_{n_0})$: then $\tilde x_n \to +\infty$.

Summing up, we have shown that the basin of attraction of the origin for $\tilde f|_{\mathbb{R}^2}$ is $\Omega_{\mathbb{R}}$, that $(\tilde x_n, \tilde u_n) \to (0,0)$ iff $(\tilde x_0, \tilde u_0) \in \overline{\Omega_{\mathbb{R}}}$, and that $\{(\tilde x_n, \tilde u_n)\}$ diverges iff $(\tilde x_0, \tilde u_0) \notin \overline{\Omega_{\mathbb{R}}}$. See Figure 2 for a typical example of what $\Omega_{\mathbb{R}}$ looks like. □

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Figure legend 1.a: **Figure 1.a.** Bidimensional section $(z = w)$ of the basin of attraction when $\alpha = 1$ and $\beta = 0$.

Figure legend 1.b: **Figure 1.b.** Bidimensional section $(\operatorname{Re} z = \operatorname{Re} w = 0)$ of the basin of attraction when $\alpha = 1$ and $\beta = 0$.

Figure legend 2: **Figure 2.**