Contents

Discrete holomorphic local dynamical systems

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1 Introduction

Let us begin by defining the main object of study in this survey.

Definition 1.1. Let M be a complex manifold, and $p \in M$. A *(discrete) holomorphic local dynamical system* at p is a holomorphic map $f\colon U \to M$ such that $f(p) = p$, where $U \subseteq M$ is an open neighbourhood of p; we shall also assume that $f \not\equiv \mathrm{id}_U$. We shall denote by $\mathrm{End}(M, p)$ the set of holomorphic local dynamical systems at p.

Remark 1.2. Since we are mainly concerned with the behavior of f nearby p, we shall sometimes replace f by its restriction to some suitable open neighbourhood of p. It is possible to formalize this fact by using germs of maps and germs of sets at p, but for our purposes it will be enough to use a somewhat less formal approach.

Remark 1.3. In this survey we shall never have the occasion of discussing continuous holomorphic dynamical systems (i.e., holomorphic foliations). So from now on all dynamical systems in this paper will be discrete, except where explicitly noted otherwise.

To talk about the dynamics of an $f \in \mathrm{End}(M, p)$ we need to define the iterates of f. If f is defined on the set U, then the second iterate $f^2 = f \circ f$ is defined on $U \cap f^{-1}(U)$ only, which still is an open neighbourhood of p. More generally, the k-th iterate $f^k = f \circ f^{k-1}$ is defined on $U \cap f^{-1}(U) \cap \cdots \cap f^{-(k-1)}(U)$. This suggests the next definition:

Definition 1.4. Let $f \in \mathrm{End}(M, p)$ be a holomorphic local dynamical system defined on an open set $U \subseteq M$. Then the *stable set* K_f of f is

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$$K_f = \bigcap_{k=0}^{\infty} f^{-k}(U) \ .$$

In other words, the stable set of f is the set of all points $z \in U$ such that the *orbit* $\{f^k(z) \mid k \in \mathbb{N}\}$ is well-defined. If $z \in U \setminus K_f$, we shall say that z (or its orbit) *escapes* from U.

Clearly, $p \in K_f$, and so the stable set is never empty (but it can happen that $K_f = \{p\}$; see the next section for an example). Thus the first natural question in local holomorphic dynamics is:

(Q1) *What is the topological structure of* K_f*?*

For instance, when does K_f have non-empty interior? As we shall see in Proposition 4.1, holomorphic local dynamical systems such that p belongs to the interior of the stable set enjoy special properties.

Remark 1.5. Both the definition of stable set and Question 1 (as well as several other definitions and questions we shall see later on) are topological in character; we might state them for local dynamical systems which are continuous only. As we shall see, however, the *answers* will strongly depend on the holomorphicity of the dynamical system.

Definition 1.6. Given $f \in \mathrm{End}(M,p)$, a set $K \subseteq M$ is *completely* f*-invariant* if $f^{-1}(K) = K$ (this implies, in particular, that K is f*-invariant,* that is $f(K) \subseteq K$).

Clearly, the stable set K_f is completely f-invariant. Therefore the pair (K_f, f) is a discrete dynamical system in the usual sense, and so the second natural question in local holomorphic dynamics is

(Q2) *What is the dynamical structure of* (K_f, f)*?*

For instance, what is the asymptotic behavior of the orbits? Do they converge to p, or have they a chaotic behavior? Is there a dense orbit? Do there exist proper f-invariant subsets, that is sets $L \subset K_f$ such that $f(L) \subseteq L$? If they do exist, what is the dynamics on them?

To answer all these questions, the most efficient way is to replace f by a "dynamically equivalent" but simpler (e.g., linear) map g. In our context, "dynamically equivalent" means "locally conjugated"; and we have at least three kinds of conjugacy to consider.

Definition 1.7. Let $f_1 \colon U_1 \to M_1$ and $f_2 \colon U_2 \to M_2$ be two holomorphic local dynamical systems at $p_1 \in M_1$ and $p_2 \in M_2$ respectively. We shall say that f_1 and f_2 are *holomorphically* (respectively, *topologically*) *locally conjugated* if there are open neighbourhoods $W_1 \subseteq U_1$ of p_1, $W_2 \subseteq U_2$ of p_2, and a biholomorphism (respectively, a homeomorphism) $\varphi \colon W_1 \to W_2$ with $\varphi(p_1) = p_2$ such that

$$f_1 = \varphi^{-1} \circ f_2 \circ \varphi \qquad \text{on} \qquad \varphi^{-1}\big(W_2 \cap f_2^{-1}(W_2)\big) = W_1 \cap f_1^{-1}(W_1) \ .$$

If $f_1\colon U_1 \to M_1$ and $f_2\colon U_2 \to M_2$ are locally conjugated, in particular we have

$$f_1^k = \varphi^{-1} \circ f_2^k \circ \varphi \quad \text{on} \quad \varphi^{-1}\bigl(W_2 \cap \cdots \cap f_2^{-(k-1)}(W_2)\bigr) = W_1 \cap \cdots \cap f_1^{-(k-1)}(W_1)$$

for all $k \in \mathbb{N}$ and thus

$$K_{f_2|_{W_2}} = \varphi(K_{f_1|_{W_1}})\ .$$

So the local dynamics of f_1 about p_1 is to all purposes equivalent to the local dynamics of f_2 about p_2.

Remark 1.8. Using local coordinates centered at $p \in M$ it is easy to show that any holomorphic local dynamical system at p is holomorphically locally conjugated to a holomorphic local dynamical system at $O \in \mathbb{C}^n$, where $n = \dim M$.

Whenever we have an equivalence relation in a class of objects, there are classification problems. So the third natural question in local holomorphic dynamics is

(Q3) *Find a (possibly small) class $\mathscr{F}$ of holomorphic local dynamical systems at $O \in \mathbb{C}^n$ such that every holomorphic local dynamical system f at a point in an n-dimensional complex manifold is holomorphically (respectively, topologically) locally conjugated to a (possibly) unique element of $\mathscr{F}$, called the* holomorphic *(respectively,* topological*)* normal form *of f.*

Unfortunately, the holomorphic classification is often too complicated to be practical; the family $\mathscr{F}$ of normal forms might be uncountable. A possible replacement is looking for invariants instead of normal forms:

(Q4) *Find a way to associate a (possibly small) class of (possibly computable) objects, called* invariants*, to any holomorphic local dynamical system f at $O \in \mathbb{C}^n$ so that two holomorphic local dynamical systems at O can be holomorphically conjugated only if they have the same invariants. The class of invariants is furthermore said* complete *if two holomorphic local dynamical systems at O are holomorphically conjugated if and only if they have the same invariants.*

As remarked before, up to now all the questions we asked made sense for topological local dynamical systems; the next one instead makes sense only for holomorphic local dynamical systems.

A holomorphic local dynamical system at $O \in \mathbb{C}^n$ is clearly given by an element of $\mathbb{C}_0\{z_1,\dots,z_n\}^n$, the space of n-uples of converging power series in $z_1,\dots,z_n$ without constant terms. The space $\mathbb{C}_0\{z_1,\dots,z_n\}^n$ is a subspace of the space $\mathbb{C}_0[\![z_1,\dots,z_n]\!]^n$ of n-uples of formal power series without constant terms. An element $\Phi \in \mathbb{C}_0[\![z_1,\dots,z_n]\!]^n$ has an inverse (with respect to composition) still belonging to $\mathbb{C}_0[\![z_1,\dots,z_n]\!]^n$ if and only if its linear part is a linear automorphism of $\mathbb{C}^n$.

Definition 1.9. We say that two holomorphic local dynamical systems f_1, $f_2 \in \mathbb{C}_0\{z_1,\dots,z_n\}^n$ are *formally conjugated* if there is an invertible $\Phi \in \mathbb{C}_0[\![z_1,\dots,z_n]\!]^n$ such that $f_1 = \Phi^{-1} \circ f_2 \circ \Phi$ in $\mathbb{C}_0[\![z_1,\dots,z_n]\!]^n$.

It is clear that two holomorphically locally conjugated holomorphic local dynamical systems are both formally and topologically locally conjugated too. On the other hand, we shall see examples of holomorphic local dynamical systems that are topologically locally conjugated without being neither formally nor holomorphically locally conjugated, and examples of holomorphic local dynamical systems that are formally conjugated without being neither holomorphically nor topologically locally conjugated. So the last natural question in local holomorphic dynamics we shall deal with is

(Q5) *Find normal forms and invariants with respect to the relation of formal conjugacy for holomorphic local dynamical systems at $O \in \mathbb{C}^n$.*

In this survey we shall present some of the main results known on these questions, starting from the one-dimensional situation. But before entering the main core of the paper I would like to heartily thank François Berteloot, Kingshook Biswas, Filippo Bracci, Santiago Diaz-Madrigal, Graziano Gentili, Giorgio Patrizio, Mohamad Pouryayevali, Jasmin Raissy and Francesca Tovena, without whom none of this would have been written.

2 One complex variable: the hyperbolic case

Let us then start by discussing holomorphic local dynamical systems at $0 \in \mathbb{C}$. As remarked in the previous section, such a system is given by a converging power series f without constant term:

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots \in \mathbb{C}_0\{z\} .$$

Definition 2.1. The number $a_1 = f'(0)$ is the *multiplier* of f.

Since $a_1 z$ is the best linear approximation of f, it is sensible to expect that the local dynamics of f will be strongly influenced by the value of a_1. For this reason we introduce the following definitions:

Definition 2.2. Let $a_1 \in \mathbb{C}$ be the multiplier of $f \in \mathrm{End}(\mathbb{C}, 0)$. Then

- if $|a_1| < 1$ we say that the fixed point 0 is *attracting;*
- if $a_1 = 0$ we say that the fixed point 0 is *superattracting;*
- if $|a_1| > 1$ we say that the fixed point 0 is *repelling;*
- if $|a_1| \neq 0, 1$ we say that the fixed point 0 is *hyperbolic;*
- if $a_1 \in S^1$ is a root of unity, we say that the fixed point 0 is *parabolic* (or *rationally indifferent*);
- if $a_1 \in S^1$ is not a root of unity, we say that the fixed point 0 is *elliptic* (or *irrationally indifferent*).

As we shall see in a minute, the dynamics of one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point is pretty elementary; so we start with this case.

Remark 2.3. Notice that if 0 is an attracting fixed point for $f \in \mathrm{End}(\mathbb{C},0)$ with non-zero multiplier, then it is a repelling fixed point for the inverse map $f^{-1} \in \mathrm{End}(\mathbb{C},0)$.

Assume first that 0 is attracting for the holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C},0)$. Then we can write $f(z) = a_1 z + O(z^2)$, with $0 < |a_1| < 1$; hence we can find a large constant $M > 0$, a small constant $\varepsilon > 0$ and $0 < \delta < 1$ such that if $|z| < \varepsilon$ then

$$|f(z)| \le (|a_1| + M\varepsilon)|z| \le \delta |z| \,. \tag{1}$$

In particular, if Δ_ε denotes the disk of center 0 and radius ε, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$ for $\varepsilon > 0$ small enough, and the stable set of $f|_{\Delta_\varepsilon}$ is Δ_ε itself (in particular, a one-dimensional attracting fixed point is always stable). Furthermore,

$$|f^k(z)| \le \delta^k |z| \to 0$$

as $k \to +\infty$, and thus every orbit starting in Δ_ε is attracted by the origin, which is the reason of the name “attracting” for such a fixed point.

If instead 0 is a repelling fixed point, a similar argument (or the observation that 0 is attracting for f^{-1}) shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ reduces to the origin only: all (non-trivial) orbits escape.

It is also not difficult to find holomorphic and topological normal forms for one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point, as shown in the following result, which can be considered as the beginning of the theory of holomorphic dynamical systems:

Theorem 2.4 (Kœnigs, 1884 [Kœ]). *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a one-dimensional holomorphic local dynamical system with a hyperbolic fixed point at the origin, and let $a_1 \in \mathbb{C}^* \setminus S^1$ be its multiplier. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to its linear part $g(z) = a_1 z$. The conjugation φ is uniquely determined by the condition $\varphi'(0) = 1$.*
(ii) *Two such holomorphic local dynamical systems are holomorphically conjugated if and only if they have the same multiplier.*
(iii) *f is topologically locally conjugated to the map $g_<(z) = z/2$ if $|a_1| < 1$, and to the map $g_>(z) = 2z$ if $|a_1| > 1$.*

Proof. Let us assume $0 < |a_1| < 1$; if $|a_1| > 1$ it will suffice to apply the same argument to f^{-1}.

(i) Choose $0 < \delta < 1$ such that $\delta^2 < |a_1| < \delta$. Writing $f(z) = a_1 z + z^2 r(z)$ for a suitable holomorphic germ r, we can clearly find $\varepsilon > 0$ such that $|a_1| + M\varepsilon < \delta$, where $M = \max\limits_{z \in \overline{\Delta_\varepsilon}} |r(z)|$. So we have

$$|f(z) - a_1 z| \le M|z|^2 \qquad \text{and} \qquad |f^k(z)| \le \delta^k |z|$$

for all $z \in \overline{\Delta_\varepsilon}$ and $k \in \mathbb{N}$.

Put $\varphi_k = f^k/a_1^k$; we claim that the sequence $\{\varphi_k\}$ converges to a holomorphic map $\varphi\colon \Delta_\varepsilon \to \mathbb{C}$. Indeed we have

$$\begin{aligned} |\varphi_{k+1}(z) - \varphi_k(z)| &= \frac{1}{|a_1|^{k+1}} \big| f\big(f^k(z)\big) - a_1 f^k(z) \big| \\ &\le \frac{M}{|a_1|^{k+1}} |f^k(z)|^2 \le \frac{M}{|a_1|} \left(\frac{\delta^2}{|a_1|} \right)^k |z|^2 \end{aligned}$$

for all $z \in \overline{\Delta_\varepsilon}$, and so the telescopic series $\sum_k (\varphi_{k+1} - \varphi_k)$ is uniformly convergent in Δ_ε to $\varphi - \varphi_0$.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi\big(f(z)\big) = \lim_{k\to+\infty} \frac{f^k\big(f(z)\big)}{a_1^k} = a_1 \lim_{k\to+\infty} \frac{f^{k+1}(z)}{a_1^{k+1}} = a_1 \varphi(z) \; ,$$

that is $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local holomorphic function such that $\psi'(0) = 1$ and $\psi^{-1} \circ g \circ \psi = f$, it follows that $\psi \circ \varphi^{-1}(\lambda z) = \lambda \psi \circ \varphi^{-1}(z)$; comparing the expansion in power series of both sides we find $\psi \circ \varphi^{-1} \equiv \mathrm{id}$, that is $\psi \equiv \varphi$, as claimed.

(ii) Since $f_1 = \varphi^{-1} \circ f_2 \circ \varphi$ implies $f_1'(0) = f_2'(0)$, the multiplier is invariant under holomorphic local conjugation, and so two one-dimensional holomorphic local dynamical systems with a hyperbolic fixed point are holomorphically locally conjugated if and only if they have the same multiplier.

(iii) Since $|a_1| < 1$ it is easy to build a topological conjugacy between g and $g_<$ on Δ_ε. First choose a homeomorphism χ between the annulus $\{|a_1|\varepsilon \le |z| \le \varepsilon\}$ and the annulus $\{\varepsilon/2 \le |z| \le \varepsilon\}$ which is the identity on the outer circle and given by $\chi(z) = z/(2a_1)$ on the inner circle. Now extend χ by induction to a homeomorphism between the annuli $\{|a_1|^k\varepsilon \le |z| \le |a_1|^{k-1}\varepsilon\}$ and $\{\varepsilon/2^k \le |z| \le \varepsilon/2^{k-1}\}$ by prescribing

$$\chi(a_1 z) = \tfrac{1}{2}\chi(z) \; .$$

Putting finally $\chi(0) = 0$ we then get a homeomorphism χ of Δ_ε with itself such that $g = \chi^{-1} \circ g_< \circ \chi$, as required. □

Remark 2.5. Notice that $g_<(z) = \frac{1}{2}z$ and $g_>(z) = 2z$ cannot be topologically conjugated, because (for instance) $K_{g_<}$ is open whereas $K_{g_>} = \{0\}$ is not.

Remark 2.6. The proof of this theorem is based on two techniques often used in dynamics to build conjugations. The first one is used in part (i). Suppose that we would like to prove that two invertible local dynamical systems f, $g \in \mathrm{End}(M,p)$ are conjugated. Set $\varphi_k = g^{-k} \circ f^k$, so that

$$\varphi_k \circ f = g^{-k} \circ f^{k+1} = g \circ \varphi_{k+1} \; .$$

Therefore if we can prove that $\{\varphi_k\}$ converges to an invertible map φ as $k \to +\infty$ we get $\varphi \circ f = g \circ \varphi$, and thus f and g are conjugated, as desired. This is exactly the way we proved Theorem 2.4.(i); and we shall see variations of this technique later on.

To describe the second technique we need a definition.

Definition 2.7. Let $f\colon X \to X$ be an open continuous self-map of a topological space X. A *fundamental domain* for f is an open subset $D \subset X$ such that

(i) $f^h(D) \cap f^k(D) = \emptyset$ for every $h \neq k \in \mathbb{N}$;
(ii) $\bigcup_{k \in \mathbb{N}} f^k(\overline{D}) = X$;
(iii) if $z_1, z_2 \in \overline{D}$ are so that $f^h(z_1) = f^k(z_2)$ for some $h > k \in \mathbb{N}$ then $h = k+1$ and $z_2 = f(z_1) \in \partial D$.

There are other possible definitions of a fundamental domain, but this will work for our aims.

Suppose that we would like to prove that two open continuous maps $f_1\colon X_1 \to X_1$ and $f_2\colon X_2 \to X_2$ are topologically conjugated. Assume we have fundamental domains $D_j \subset X_j$ for f_j (with $j = 1, 2$) and a homeomorphism $\chi\colon \overline{D_1} \to \overline{D_2}$ such that

$$\chi \circ f_1 = f_2 \circ \chi \tag{2}$$

on $\overline{D_1} \cap f_1^{-1}(\overline{D_1})$. Then we can extend χ to a homeomorphism $\tilde{\chi}\colon X_1 \to X_2$ conjugating f_1 and f_2 by setting

$$\tilde{\chi}(z) = f_2^k\big(\chi(w)\big) , \tag{3}$$

for all $z \in X_1$, where $k = k(z) \in \mathbb{N}$ and $w = w(z) \in \overline{D}$ are chosen so that $f_1^k(w) = z$. The definition of fundamental domain and (2) imply that $\tilde{\chi}$ is well-defined. Clearly $\tilde{\chi} \circ f_1 = f_2 \circ \tilde{\chi}$; and using the openness of f_1 and f_2 it is easy to check that $\tilde{\chi}$ is a homeomorphism. This is the technique we used in the proof of Theorem 2.4.(iii); and we shall use it again later.

Thus the dynamics in the one-dimensional hyperbolic case is completely clear. The superattracting case can be treated similarly. If 0 is a superattracting point for an $f \in \mathrm{End}(\mathbb{C}, 0)$, we can write

$$f(z) = a_r z^r + a_{r+1} z^{r+1} + \cdots$$

with $a_r \neq 0$.

Definition 2.8. The number $r \geq 2$ is the *order* (or *local degree*) of the superattracting point.

An argument similar to the one described before shows that for $\varepsilon > 0$ small enough the stable set of $f|_{\Delta_\varepsilon}$ still is all of Δ_ε, and the orbits converge (faster than in the attracting case) to the origin. Furthermore, we can prove the following

Theorem 2.9 (Böttcher, 1904 [Bö]). *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a one-dimensional holomorphic local dynamical system with a superattracting fixed point at the origin, and let $r \geq 2$ be its order. Then:*

(i) *f is holomorphically (and hence formally) locally conjugated to the map $g(z) = z^r$, and the conjugation is unique up to multiplication by an $(r-1)$-root of unity;*
(ii) *two such holomorphic local dynamical systems are holomorphically (or topologically) conjugated if and only if they have the same order.*

Proof. First of all, up to a linear conjugation $z \mapsto \mu z$ with $\mu^{r-1} = a_r$ we can assume $a_r = 1$.

Now write $f(z) = z^r h_1(z)$ for a suitable holomorphic germ h_1 with $h_1(0) = 1$. By induction, it is easy to see that we can write $f^k(z) = z^{r^k} h_k(z)$ for a suitable holomorphic germ h_k with $h_k(0) = 1$. Furthermore, the equalities $f \circ f^{k-1} = f^k = f^{k-1} \circ f$ yield

$$h_{k-1}(z)^r h_1\big(f^{k-1}(z)\big) = h_k(z) = h_1(z)^{r^{k-1}} h_{k-1}\big(f(z)\big) \; . \tag{4}$$

Choose $0 < \delta < 1$. Then we can clearly find $1 > \varepsilon > 0$ such that $M\varepsilon < \delta$, where $M = \max\limits_{z \in \overline{\Delta_\varepsilon}} |h_1(z)|$; we can also assume that $h_1(z) \neq 0$ for all $z \in \overline{\Delta_\varepsilon}$. Since

$$|f(z)| \le M|z|^r < \delta |z|^{r-1}$$

for all $z \in \overline{\Delta_\varepsilon}$, we have $f(\Delta_\varepsilon) \subset \Delta_\varepsilon$, as anticipated before.

We also remark that (4) implies that each h_k is well-defined and never vanishing on $\overline{\Delta_\varepsilon}$. So for every $k \ge 1$ we can choose a unique ψ_k holomorphic in Δ_ε such that $\psi_k(z)^{r^k} = h_k(z)$ on Δ_ε and with $\psi_k(0) = 1$.

Set $\varphi_k(z) = z\psi_k(z)$, so that $\varphi_k'(0) = 1$ and $\varphi_k(z)^{r^k} = f^k(z)$ on Δ_ε; in particular, formally we have $\varphi_k = g^{-k} \circ f^k$. We claim that the sequence $\{\varphi_k\}$ converges to a holomorphic function φ on Δ_ε. Indeed, we have

$$\begin{aligned}\left|\frac{\varphi_{k+1}(z)}{\varphi_k(z)}\right| &= \left|\frac{\psi_{k+1}(z)^{r^{k+1}}}{\psi_k(z)^{r^{k+1}}}\right|^{1/r^{k+1}} = \left|\frac{h_{k+1}(z)}{h_k(z)^r}\right|^{1/r^{k+1}} = \left|h_1\big(f^k(z)\big)\right|^{1/r^{k+1}} \\ &= \left|1 + O\big(|f^k(z)|\big)\right|^{1/r^{k+1}} = 1 + \frac{1}{r^{k+1}} O\big(|f^k(z)|\big) = 1 + O\left(\frac{1}{r^{k+1}}\right) ,\end{aligned}$$

and so the telescopic product $\prod_k (\varphi_{k+1}/\varphi_k)$ converges to φ/φ_1 uniformly in Δ_ε.

Since $\varphi_k'(0) = 1$ for all $k \in \mathbb{N}$, we have $\varphi'(0) = 1$ and so, up to possibly shrink ε, we can assume that φ is a biholomorphism with its image. Moreover, we have

$$\varphi_k\big(f(z)\big)^{r^k} = f(z)^{r^k} \psi_k\big(f(z)\big)^{r^k} = z^{r^k} h_1(z)^{r^k} h_k\big(f(z)\big) = z^{r^{k+1}} h_{k+1}(z) = \big[\varphi_{k+1}(z)^r\big]^{r^k} ,$$

and thus $\varphi_k \circ f = [\varphi_{k+1}]^r$. Passing to the limit we get $f = \varphi^{-1} \circ g \circ \varphi$, as claimed.

If ψ is another local biholomorphism conjugating f with g, we must have $\psi \circ \varphi^{-1}(z^r) = \psi \circ \varphi^{-1}(z)^r$ for all z in a neighbourhood of the origin; comparing the series expansions at the origin we get $\psi \circ \varphi^{-1}(z) = az$ with $a^{r-1} = 1$, and hence $\psi(z) = a\varphi(z)$, as claimed.

Finally, (ii) follows because z^r and z^s are locally topologically conjugated if and only if $r = s$ (because the order is the number of preimages of points close to the origin). □

Therefore the one-dimensional local dynamics about a hyperbolic or superattracting fixed point is completely clear; let us now discuss what happens about a parabolic fixed point.

3 One complex variable: the parabolic case

Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a (non-linear) holomorphic local dynamical system with a parabolic fixed point at the origin. Then we can write

$$f(z) = e^{2i\pi p/q} z + a_{r+1} z^{r+1} + a_{r+2} z^{r+2} + \cdots , \tag{5}$$

with $a_{r+1} \neq 0$.

Definition 3.1. The rational number $p/q \in \mathbb{Q} \cap [0, 1)$ is the *rotation number* of f, and the number $r + 1 \geq 2$ is the *multiplicity* of f at the fixed point. If $p/q = 0$ (that is, if the multiplier is 1), we shall say that f is *tangent to the identity.*

The first observation is that such a dynamical system is never locally conjugated to its linear part, not even topologically, unless it is of finite order:

Proposition 3.2. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system with multiplier λ, and assume that $\lambda = e^{2i\pi p/q}$ is a primitive root of the unity of order q. Then f is holomorphically (or topologically or formally) locally conjugated to $g(z) = \lambda z$ if and only if $f^q \equiv \mathrm{id}$.*

Proof. If $\varphi^{-1} \circ f \circ \varphi(z) = e^{2\pi i p/q} z$ then $\varphi^{-1} \circ f^q \circ \varphi = \mathrm{id}$, and hence $f^q = \mathrm{id}$.

Conversely, assume that $f^q \equiv \mathrm{id}$ and set

$$\varphi(z) = \frac{1}{q} \sum_{j=0}^{q-1} \frac{f^j(z)}{\lambda^j} .$$

Then it is easy to check that $\varphi'(0) = 1$ and $\varphi \circ f(z) = \lambda \varphi(z)$, and so f is holomorphically (and topologically and formally) locally conjugated to λz. □

In particular, if f is tangent to the identity then it *cannot* be locally conjugated to the identity (unless it was the identity to begin with, which is not a very interesting case dynamically speaking). More precisely, the stable set of such an f is never a neighbourhood of the origin. To understand why, let us first consider a map of the form

$$f(z) = z(1 + az^r)$$

for some $a \neq 0$. Let $v \in S^1 \subset \mathbb{C}$ be such that av^r is real and positive. Then for any $c > 0$ we have

$$f(cv) = c(1 + c^r a v^r) v \in \mathbb{R}^+ v;$$

moreover, $|f(cv)| > |cv|$. In other words, the half-line $\mathbb{R}^+ v$ is f-invariant and repelled from the origin, that is $K_f \cap \mathbb{R}^+ v = \emptyset$. Conversely, if av^r is real and negative then the segment $[0, |a|^{-1/r}]v$ is f-invariant and attracted by the origin. So K_f neither is a neighbourhood of the origin nor reduces to $\{0\}$.

This example suggests the following definition:

Definition 3.3. Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be tangent to the identity of multiplicity $r+1 \geq 2$. Then a unit vector $v \in S^1$ is an *attracting* (respectively, *repelling*) *direction* for f at the origin if $a_{r+1} v^r$ is real and negative (respectively, positive).

Clearly, there are r equally spaced attracting directions, separated by r equally spaced repelling directions: if $a_{r+1} = |a_{r+1}| e^{i\alpha}$, then $v = e^{i\theta}$ is attracting (respectively, repelling) if and only if

$$\theta = \frac{2k+1}{r}\pi - \frac{\alpha}{r} \qquad \left(\text{respectively, } \theta = \frac{2k}{r}\pi - \frac{\alpha}{r}\right).$$

Furthermore, a repelling (attracting) direction for f is attracting (repelling) for f^{-1}, which is defined in a neighbourhood of the origin.

It turns out that to every attracting direction is associated a connected component of $K_f \setminus \{0\}$.

Definition 3.4. Let $v \in S^1$ be an attracting direction for an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity. The *basin* centered at v is the set of points $z \in K_f \setminus \{0\}$ such that $f^k(z) \to 0$ and $f^k(z)/|f^k(z)| \to v$ (notice that, up to shrinking the domain of f, we can assume that $f(z) \neq 0$ for all $z \in K_f \setminus \{0\}$). If z belongs to the basin centered at v, we shall say that the orbit of z *tends to* 0 *tangent to* v.

A slightly more specialized (but more useful) object is the following:

Definition 3.5. An *attracting petal* centered at an attracting direction v of an $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity is an open simply connected f-invariant set $P \subseteq K_f \setminus \{0\}$ such that a point $z \in K_f \setminus \{0\}$ belongs to the basin centered at v if and only if its orbit intersects P. In other words, the orbit of a point tends to 0 tangent to v if and only if it is eventually contained in P. A *repelling petal* (centered at a repelling direction) is an attracting petal for the inverse of f.

It turns out that the basins centered at the attracting directions are exactly the connected components of $K_f \setminus \{0\}$, as shown in the *Leau-Fatou flower theorem:*

Theorem 3.6 (Leau, 1897 [L]; Fatou, 1919-20 [F1–3]). *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1 \geq 2$ at the fixed point. Let $v_1^+, \ldots, v_r^+ \in S^1$ be the r attracting directions of f at the origin, and $v_1^-, \ldots, v_r^- \in S^1$ the r repelling directions. Then*

(i) *for each attracting (repelling) direction $v_j^\pm$ there exists an attracting (repelling) petal $P_j^\pm$, so that the union of these $2r$ petals together with the origin forms* *a neighbourhood of the origin. Furthermore, the* $2r$ *petals are arranged ciclically so that two petals intersect if and only if the angle between their central directions is* π/r.

(ii) $K_f \setminus \{0\}$ *is the (disjoint) union of the basins centered at the* r *attracting directions.*

(iii) *If* B *is a basin centered at one of the attracting directions, then there is a function* $\varphi\colon B \to \mathbb{C}$ *such that* $\varphi \circ f(z) = \varphi(z) + 1$ *for all* $z \in B$. *Furthermore, if* P *is the corresponding petal constructed in part* (i), *then* $\varphi|_P$ *is a biholomorphism with an open subset of the complex plane containing a right half-plane — and so* $f|_P$ *is holomorphically conjugated to the translation* $z \mapsto z+1$.

Proof. Up to a linear conjugation, we can assume that $a_{r+1} = -1$, so that the attracting directions are the r-th roots of unity. For any $\delta > 0$, the set $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$ has exactly r connected components, each one symmetric with respect to a different r-th root of unity; it will turn out that, for δ small enough, these connected components are attracting petals of f, even though to get a pointed neighbourhood of the origin we shall need larger petals.

For $j = 1, \dots, r$ let $\Sigma_j \subset \mathbb{C}^*$ denote the sector centered about the attractive direction v_j^+ and bounded by two consecutive repelling directions, that is

$$\Sigma_j = \left\{ z \in \mathbb{C}^* \;\middle|\; \frac{2j-3}{r}\,\pi < \arg z < \frac{2j-1}{r}\,\pi \right\} .$$

Notice that each Σ_j contains a unique connected component $P_{j,\delta}$ of $\{z \in \mathbb{C} \mid |z^r - \delta| < \delta\}$; moreover, $P_{j,\delta}$ is tangent at the origin to the sector centered about v_j of amplitude π/r.

The main technical trick in this proof consists in transfering the setting to a neighbourhood of infinity in the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. Let $\psi\colon \mathbb{C}^* \to \mathbb{C}^*$ be given by

$$\psi(z) = \frac{1}{rz^r} \; ;$$

it is a biholomorphism between Σ_j and $\mathbb{C}^* \setminus \mathbb{R}^-$, with inverse $\psi^{-1}(w) = (rw)^{-1/r}$, choosing suitably the r-th root. Furthermore, $\psi(P_{j,\delta})$ is the right half-plane $H_\delta = \{w \in \mathbb{C} \mid \operatorname{Re} w > 1/(2r\delta)\}$.

When $|w|$ is so large that $\psi^{-1}(w)$ belongs to the domain of definition of f, the composition $F = \psi \circ f \circ \psi^{-1}$ makes sense, and we have

$$F(w) = w + 1 + O(w^{-1/r}) \; . \tag{6}$$

Thus to study the dynamics of f in a neighbourhood of the origin in Σ_j it suffices to study the dynamics of F in a neighbourhood of infinity.

The first observation is that when $\operatorname{Re} w$ is large enough then

$$\operatorname{Re} F(w) > \operatorname{Re} w + \frac{1}{2} \; ;$$

this implies that for δ small enough H_δ is F-invariant (and thus $P_{j,\delta}$ is f-invariant). Furthermore, by induction one has

$$\mathrm{Re}\, F^k(w) > \mathrm{Re}\, w + \frac{k}{2} \tag{7}$$

for all $w \in H_\delta$, which implies that $F^k(w) \to \infty$ in H_δ (and $f^k(z) \to 0$ in $P_{j,\delta}$) as $k \to \infty$.

Now we claim that the argument of $w_k = F^k(w)$ tends to zero. Indeed, (6) and (7) yield

$$\frac{w_k}{k} = \frac{w}{k} + 1 + \frac{1}{k}\sum_{l=0}^{k-1} O(w_l^{-1/r}) \; ;$$

hence Cesaro's theorem on the averages of a converging sequence implies

$$\frac{w_k}{k} \to 1 \; , \tag{8}$$

and so $\arg w_k \to 0$ as $k \to \infty$. Going back to $P_{j,\delta}$, this implies that $f^k(z)/|f^k(z)| \to v_j^+$ for every $z \in P_{j,\delta}$. Since furthermore $P_{j,\delta}$ is centered about v_j^+, every orbit converging to 0 tangent to v_j^+ must intersect $P_{j,\delta}$, and thus we have proved that $P_{j,\delta}$ is an attracting petal.

Arguing in the same way with f^{-1} we get repelling petals; unfortunately, the petals obtained so far are too small to form a full pointed neighbourhood of the origin. In fact, as remarked before each $P_{j,\delta}$ is contained in a sector centered about v_j^+ of amplitude π/r; therefore the repelling and attracting petals obtained in this way do not intersect but are tangent to each other. We need larger petals.

So our aim is to find an f-invariant subset P_j^+ of Σ_j containing $P_{j,\delta}$ and which is tangent at the origin to a sector centered about v_j^+ of amplitude strictly greater than π/r. To do so, first of all remark that there are $R, C > 0$ such that

$$|F(w) - w - 1| \le \frac{C}{|w|^{1/r}} \tag{9}$$

as soon as $|w| > R$. Choose $\varepsilon \in (0,1)$ and select $\delta > 0$ so that $4r\delta < R^{-1}$ and $\varepsilon > 2C(4r\delta)^{1/r}$. Then $|w| > 1/(4r\delta)$ implies

$$|F(w) - w - 1| < \varepsilon/2 \; .$$

Set $M_\varepsilon = (1+\varepsilon)/(2r\delta)$ and let

$$\tilde{H}_\varepsilon = \{ w \in \mathbb{C} \mid |\mathrm{Im}\, w| > -\varepsilon \,\mathrm{Re}\, w + M_\varepsilon \} \cup H_\delta \; .$$

If $w \in \tilde{H}_\varepsilon$ we have $|w| > 1/(2r\delta)$ and hence

$$\mathrm{Re}\, F(w) > \mathrm{Re}\, w + 1 - \varepsilon/2 \qquad \text{and} \qquad |\mathrm{Im}\, F(w) - \mathrm{Im}\, w| < \varepsilon/2 \; ; \tag{10}$$

it is then easy to check that $F(\tilde{H}_\varepsilon) \subset \tilde{H}_\varepsilon$ and that every orbit starting in $\tilde{H}_\varepsilon$ must eventually enter H_δ. Thus $P_j^+ = \psi^{-1}(\tilde{H}_\varepsilon)$ is as required, and we have proved (i).

To prove (ii) we need a further property of $\tilde{H}_\varepsilon$. If $w \in \tilde{H}_\varepsilon$, arguing by induction on $k \geq 1$ using (10) we get

$$k\left(1 - \frac{\varepsilon}{2}\right) < \operatorname{Re} F^k(w) - \operatorname{Re} w$$

and

$$\frac{k\varepsilon(1-\varepsilon)}{2} < |\operatorname{Im} F^k(w)| + \varepsilon \operatorname{Re} F^k(w) - \big(|\operatorname{Im} w| + \varepsilon \operatorname{Re} w\big) .$$

This implies that for every $w_0 \in \tilde{H}_\varepsilon$ there exists a $k_0 \geq 1$ so that we cannot have $F^{k_0}(w) = w_0$ for any $w \in \tilde{H}_\varepsilon$. Coming back to the z-plane, this says that any inverse orbit of f must eventually leave P_j^+. Thus every (forward) orbit of f must eventually leave any repelling petal. So if $z \in K_f \setminus \{O\}$, where the stable set is computed working in the neighborhood of the origin given by the union of repelling and attracting petals (together with the origin), the orbit of z must eventually land in an attracting petal, and thus z belongs to a basin centered at one of the r attracting directions — and (ii) is proved.

To prove (iii), first of all we notice that we have

$$|F'(w) - 1| \leq \frac{2^{1+1/r} C}{|w|^{1+1/r}} \tag{11}$$

in $\tilde{H}_\varepsilon$. Indeed, (9) says that if $|w| > 1/(2r\delta)$ then the function $w \mapsto F(w) - w - 1$ sends the disk of center w and radius $|w|/2$ into the disk of center the origin and radius $C/(|w|/2)^{1/r}$; inequality (11) then follows from the Cauchy estimates on the derivative.

Now choose $w_0 \in H_\delta$, and set $\tilde{\varphi}_k(w) = F^k(w) - F^k(w_0)$. Given $w \in \tilde{H}_\varepsilon$, as soon as $k \in \mathbb{N}$ is so large that $F^k(w) \in H_\delta$ we can apply Lagrange's theorem to the segment from $F^k(w_0)$ to $F^k(w)$ to get a $t_k \in [0,1]$ such that

$$\begin{aligned}\left|\frac{\tilde{\varphi}_{k+1}(w)}{\tilde{\varphi}_k(w)} - 1\right| &= \left|\frac{F\big(F^k(w)\big) - F^k\big(F^k(w_0)\big)}{F^k(w) - F^k(w_0)} - 1\right| = \left|F'\big(t_k F^k(w) + (1-t_k) F^k(w_0)\big) - 1\right| \\ &\leq \frac{2^{1+1/r} C}{\min\{|\operatorname{Re} F^k(w)|, |\operatorname{Re} F^k(w_0)|\}^{1+1/r}} \leq \frac{C'}{k^{1+1/r}} ,\end{aligned}$$

where we used (11) and (8), and the constant C' is uniform on compact subsets of $\tilde{H}_\varepsilon$ (and it can be chosen uniform on H_δ).

As a consequence, the telescopic product $\prod_k \tilde{\varphi}_{k+1}/\tilde{\varphi}_k$ converges uniformly on compact subsets of $\tilde{H}_\varepsilon$ (and uniformly on H_δ), and thus the sequence $\tilde{\varphi}_k$ converges, uniformly on compact subsets, to a holomorphic function $\tilde{\varphi}\colon \tilde{H}_\varepsilon \to \mathbb{C}$. Since we have

$$\begin{aligned}\tilde{\varphi}_k \circ F(w) &= F^{k+1}(w) - F^k(w_0) = \tilde{\varphi}_{k+1}(w) + F\big(F^k(w_0)\big) - F^k(w_0) \\ &= \tilde{\varphi}_{k+1}(w) + 1 + O\big(|F^k(w_0)|^{-1/r}\big) ,\end{aligned}$$

it follows that

$$\tilde{\varphi} \circ F(w) = \tilde{\varphi}(w) + 1$$

on $\tilde{H}_\varepsilon$. In particular, $\tilde{\varphi}$ is not constant; being the limit of injective functions, by Hurwitz's theorem it is injective.

We now prove that the image of $\tilde{\varphi}$ contains a right half-plane. First of all, we claim that

$$\lim_{\substack{|w|\to+\infty \\ w\in H_\delta}} \frac{\tilde{\varphi}(w)}{w} = 1 . \tag{12}$$

Indeed, choose $\eta > 0$. Since the convergence of the telescopic product is uniform on H_δ, we can find $k_0 \in \mathbb{N}$ such that

$$\left| \frac{\tilde{\varphi}(w) - \tilde{\varphi}_{k_0}(w)}{w - w_0} \right| < \frac{\eta}{3}$$

on H_δ. Furthermore, we have

$$\left| \frac{\tilde{\varphi}_{k_0}(w)}{w - w_0} - 1 \right| = \left| \frac{k_0 + \sum_{j=0}^{k_0-1} O(|F^j(w)|^{-1/r}) + w_0 - F^{k_0}(w_0)}{w - w_0} \right| = O(|w|^{-1})$$

on H_δ; therefore we can find $R > 0$ such that

$$\left| \frac{\tilde{\varphi}(w)}{w - w_0} - 1 \right| < \frac{\eta}{3}$$

as soon as $|w| > R$ in H_δ. Finally, if R is large enough we also have

$$\left| \frac{\tilde{\varphi}(w)}{w - w_0} - \frac{\tilde{\varphi}(w)}{w} \right| = \left| \frac{\tilde{\varphi}(w)}{w - w_0} \right| \left| \frac{w}{w_0} \right| < \frac{\eta}{3} ,$$

and (12) follows.

Equality (12) clearly implies that $(\tilde{\varphi}(w) - w^o)/(w - w^o) \to 1$ as $|w| \to +\infty$ in H_δ for any $w^o \in \mathbb{C}$. But this means that if $\operatorname{Re} w^o$ is large enough then the difference between the variation of the argument of $\tilde{\varphi} - w^o$ along a suitably small closed circle around w^o and the variation of the argument of $w - w^o$ along the same circle will be less than 2π — and thus it will be zero. Then the argument principle implies that $\tilde{\varphi} - w^o$ and $w - w^o$ have the same number of zeroes inside that circle, and thus $w^o \in \tilde{\varphi}(H_\delta)$, as required.

So setting $\varphi = \tilde{\varphi} \circ \psi$, we have defined a function φ with the required properties on P_j^+. To extend it to the whole basin B it suffices to put

$$\varphi(z) = \varphi\big(f^k(z)\big) - k\,, \tag{13}$$

where $k \in \mathbb{N}$ is the first integer such that $f^k(z) \in P_j^+$. □

A way to construct the conjugation φ as limit of hyperbolic linearizations given by Theorem 2.4 is described in [U3].

Remark 3.7. It is possible to construct petals that cannot be contained in any sector strictly smaller than Σ_j. To do so we need an F-invariant subset $\hat{H}_\varepsilon$ of $\mathbb{C}^* \setminus \mathbb{R}^-$ containing $\tilde{H}_\varepsilon$ and containing eventually every half-line issuing from the origin (but $\mathbb{R}^-$). For $M >> 1$ and $C > 0$ large enough, replace the straight lines bounding $\tilde{H}_\varepsilon$ on the left of $\mathrm{Re}\, w = -M$ by the curves

$$|\mathrm{Im}\, w| = \begin{cases} C \log|\mathrm{Re}\, w| & \text{if } r = 1\,, \\ C|\mathrm{Re}\, w|^{1-1/r} & \text{if } r > 1\,. \end{cases}$$

Then it is not too difficult to check that the domain $\hat{H}_\varepsilon$ so obtained is as desired (see [CG]).

So we have a complete description of the dynamics in the neighbourhood of the origin. Actually, Camacho has pushed this argument even further, obtaining a complete topological classification of one-dimensional holomorphic local dynamical systems tangent to the identity (see also [BH, Theorem 1.7]):

Theorem 3.8 (Camacho, 1978 [C]; Shcherbakov, 1982 [S]). *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is topologically locally conjugated to the map*

$$g(z) = z - z^{r+1}\,.$$

Remark 3.9. Camacho's proof ([C]; see also [Br2] and [J1]) shows that the topological conjugation can be taken smooth in a punctured neighbourhood of the origin. Jenkins [J1] also proved that if $f \in \mathrm{End}(\mathbb{C}, 0)$ is tangent to the identity with multiplicity 2 and the topological conjugation is actually real-analitic in a punctured neighbourhood of the origin, with real-analytic inverse, then f is locally holomorphically conjugated to $z - z^2$. Finally, Martinet and Ramis [MR] have proved that if a germ $f \in \mathrm{End}(\mathbb{C}, 0)$ tangent to the identity is C^1-conjugated (in a full neighbourhood of the origin) to $g(z) = z?z^{r+1}$, then the conjugation can be chosen holomorphic or antiholomorphic.

The formal classification is simple too, though different (see, e.g., Milnor [Mi]):

Proposition 3.10. *Let $f \in \mathrm{End}(\mathbb{C}, 0)$ be a holomorphic local dynamical system tangent to the identity with multiplicity $r+1$ at the fixed point. Then f is formally conjugated to the map*

$$g(z) = z - z^{r+1} + \beta z^{2r+1}\,, \tag{14}$$

where β is a formal (and holomorphic) invariant given by

$$\beta = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z - f(z)}\;, \tag{15}$$

where the integral is taken over a small positive loop γ about the origin.

Proof. An easy computation shows that if f is given by (14) then (15) holds. Let us now show that the integral in (15) is a holomorphic invariant. Let φ be a local biholomorphism fixing the origin, and set $F = \varphi^{-1}\circ f \circ \varphi$. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - f(z)} = \frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma} \frac{\varphi'(w)\,dw}{\varphi(w) - f\big(\varphi(w)\big)} = \frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma} \frac{\varphi'(w)\,dw}{\varphi(w) - \varphi\big(F(w)\big)}\;.$$

Now, we can clearly find M, $M_1 > 0$ such that

$$\begin{aligned}\left|\frac{1}{w-F(w)} - \frac{\varphi'(w)}{\varphi(w)-\varphi\big(F(w)\big)}\right| &= \frac{1}{\big|\varphi(w)-\varphi\big(F(w)\big)\big|}\left|\frac{\varphi(w)-\varphi\big(F(w)\big)}{w-F(w)} - \varphi'(w)\right| \\ &\le M\frac{|w-F(w)|}{\big|\varphi(w)-\varphi\big(F(w)\big)\big|} \le M_1\;,\end{aligned}$$

in a neighbourhood of the origin, where the last inequality follows from the fact that $\varphi'(0)\neq 0$. This means that the two meromorphic functions $1/\big(w - F(w)\big)$ and $\varphi'(w)/\big(\varphi(w) - \varphi(\big(F(w)\big)\big)$ differ by a holomorphic function; so they have the same integral along any small loop surrounding the origin, and

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z - f(z)} = \frac{1}{2\pi i}\int_{\varphi^{-1}\circ\gamma} \frac{dw}{w - F(w)},$$

as claimed.

To prove that f is formally conjugated to g, let us first take a local formal change of coordinates φ of the form

$$\varphi(z) = z + \mu z^d + O_{d+1} \tag{16}$$

with $\mu \neq 0$, and where we are writing O_{d+1} instead of $O(z^{d+1})$. It follows that $\varphi^{-1}(z) = z - \mu z^d + O_{d+1}$, $(\varphi^{-1})'(z) = 1 - d\mu z^{d-1} + O_d$ and $(\varphi^{-1})^{(j)} = O_{d-j}$ for all $j \geq 2$. Then using the Taylor expansion of φ^{-1} we get

$$
\begin{aligned}
&\varphi^{-1}\circ f\circ\varphi(z)\\
&\qquad=\varphi^{-1}\left(\varphi(z)+\sum_{j\geq r+1}a_j\varphi(z)^j\right)\\
&\qquad=z+(\varphi^{-1})'\big(\varphi(z)\big)\sum_{j\geq r+1}a_jz^j(1+\mu z^{d-1}+O_d)^j+O_{d+2r}\qquad(17)\\
&\qquad=z+[1-d\mu z^{d-1}+O_d]\sum_{j\geq r+1}a_jz^j(1+j\mu z^{d-1}+O_d)+O_{d+2r}\\
&\qquad=z+a_{r+1}z^{r+1}+\cdots+a_{r+d-1}z^{r+d-1}\\
&\qquad\qquad+[a_{r+d}+(r+1-d)\mu a_{r+1}]z^{r+d}+O_{r+d+1}.
\end{aligned}
$$

This means that if $d\neq r+1$ we can use a polynomial change of coordinates of the form $\varphi(z)=z+\mu z^d$ to remove the term of degree $r+d$ from the Taylor expansion of f without changing the lower degree terms.

So to conjugate f to g it suffices to use a linear change of coordinates to get $a_{r+1}=-1$, and then apply a sequence of change of coordinates of the form $\varphi(z)=z+\mu z^d$ to kill all the terms in the Taylor expansion of f but the term of degree z^{2r+1}.

Finally, formula (17) also shows that two maps of the form (14) with different β cannot be formally conjugated, and we are done. □

Definition 3.11. The number β given by (15) is called *index* of f at the fixed point. The *iterative residue* of f is then defined by

$$\mathrm{Resit}(f)=\frac{r+1}{2}-\beta\ .$$

The iterative residue has been introduced by Écalle [É1], and it behaves nicely under iteration; for instance, it is possible to prove (see [BH, Proposition 3.10]) that

$$\mathrm{Resit}(f^k)=\frac{1}{k}\,\mathrm{Resit}(f)\ .$$

The holomorphic classification of maps tangent to the identity is much more complicated: as shown by Écalle [É2–3] and Voronin [Vo] in 1981, it depends on functional invariants. We shall now try and roughly describe it; see [I2], [M1-2], [Ki], [BH], [MR] and the original papers for details.

Let $f\in\mathrm{End}(\mathbb{C},0)$ be tangent to the identity with multiplicity $r+1$ at the fixed point; up to a linear change of coordinates we can assume that $a_{r+1}=-1$. Let $P_j^{\pm}$ be a set of petals as in Theorem 3.6.(i), ordered so that P_1^+ is centered on the positive real semiaxis, and the others are arranged cyclically counterclockwise. Denote by φ_j^+ (respectively, φ_j^-) the biholomorphism conjugating $f|_{P_j^+}$ (respectively, $f|_{P_j^-}$) to the shift $z\mapsto z+1$ in a right (respectively, left) half-plane given by Theorem 3.6.(iii) — applied to f^{-1} for the repelling petals. If we moreover require that

$$\varphi_j^{\pm}(z)=\frac{1}{rz^r}\pm\mathrm{Resit}(f)\cdot\log z+o(1)\ ,\qquad(18)$$

then φ_j is uniquely determined.

Put now $U_j^+ = P_j^- \cap P_{j+1}^+$, $U_j^- = P_j^- \cap P_j^+$, and $S_j^\pm = \bigcup_{k\in\mathbb{Z}} U_j^\pm$. Using the dynamics as in (13) we can extend φ_j^- to $S_j^\pm$, and φ_j^+ to $S_{j-1}^+ \cup S_j^-$; put $V_j^\pm = \varphi_j^-(S_j^\pm)$, $W_j^- = \varphi_j^+(S_j^-)$ and $W_j^+ = \varphi_{j+1}^+(S_j^+)$. Then let $H_j^-\colon V_j^- \to W_J^-$ be the restriction of $\varphi_j^+ \circ (\varphi_j^-)^{-1}$ to V_j^-, and $H_j^+\colon V_j^+ \to W_j^+$ the restriction of $\varphi_{j+1}^+ \circ (\varphi_j^-)^{-1}$ to V_j^+.

It is not difficult to see that $V_j^\pm$ and $W_j^\pm$ are invariant by translation by 1, and that V_j^+ and W_j^+ contain an upper half-plane while V_j^- and W_j^- contain a lower half-plane. Moreover, we have $H_j^\pm(z+1) = H_j^\pm(z) + 1$; therefore using the projection $\pi(z) = \exp(2\pi i z)$ we can induce holomorphic maps $h_j^\pm\colon \pi(V_j^\pm) \to \pi(W_j^\pm)$, where $\pi(V_j^+)$ and $\pi(W_j^+)$ are pointed neighbourhoods of the origin, and $\pi(V_j^-)$ and $\pi(W_j^-)$ are pointed neighbourhoods of $\infty \in \mathbb{P}^1(\mathbb{C})$.

It is possible to show that setting $h_j^+(0) = 0$ one obtains a holomorphic germ $h_j^+ \in \mathrm{End}(\mathbb{C},0)$, and that setting $h_j^-(\infty) = \infty$ one obtains a holomorphic germ $h_j^+ \in \mathrm{End}\big(\mathbb{P}^1(C),\infty\big)$. Furthermore, denoting by λ_j^+ (respectively, λ_j^-) the multiplier of h_j^+ at 0 (respectively, of h_j^- at ∞), it turns out that

$$\prod_{j=1}^r (\lambda_j^+ \lambda_j^-) = \exp\big[4\pi^2 \mathrm{Resit}(f)\big] \,. \tag{19}$$

Now, if we replace f by a holomorphic local conjugate $\tilde{f} = \psi^{-1} \circ f \circ \psi$, and denote by $\tilde{h}_j^\pm$ the corresponding germs, it is not difficult to check that (up to a cyclic renumbering of the petals) there are constants α_j, $\beta_j \in \mathbb{C}^*$ such that

$$\tilde{h}_j^-(z) = \alpha_j h_j^-\left(\frac{z}{\beta_j}\right) \qquad \text{and} \qquad \tilde{h}_j^+(z) = \alpha_{j+1} h_j^+\left(\frac{z}{\beta_j}\right) \,. \tag{20}$$

This suggests the introduction of an equivalence relation on the set of $2r$-uple of germs $(h_1^\pm, \dots, h_r^\pm)$.

Definition 3.12. Let M_r denote the set of $2r$-uple of germs $\mathbf{h} = (h_1^\pm, \dots, h_r^\pm)$, with $h_j^+ \in \mathrm{End}(\mathbb{C},0)$, $h_j^- \in \mathrm{End}\big(\mathbb{P}^1(\mathbb{C}),\infty\big)$, and whose multipliers satisfy (19). We shall say that $\mathbf{h}$, $\tilde{\mathbf{h}} \in M_r$ are *equivalent* if up to a cyclic permutation of the indeces we have (20) for suitable α_j, $\beta_j \in \mathbb{C}^*$. We denote by $\mathscr{M}_r$ the set of all equivalence classes.

The procedure described above allows then to associate to any $f \in \mathrm{End}(\mathbb{C},0)$ tangent to the identity with multiplicity $r+1$ an element $\mu_f \in \mathscr{M}_r$.

Definition 3.13. Let $f \in \mathrm{End}(\mathbb{C},0)$ be tangent to the identity. The element $\mu_f \in \mathscr{M}_r$ given by this procedure is the *sectorial invariant* of f.

Then the holomorphic classification proved by Écalle and Voronin is

Theorem 3.14 (Écalle, 1981 [É2–3]; Voronin, 1981 [Vo]). *Let f, $g \in \mathrm{End}(\mathbb{C},0)$ be two holomorphic local dynamical systems tangent to the identity. Then f and g are holomorphically locally conjugated if and only if they have the same multiplicity,* *the same index and the same sectorial invariant. Furthermore, for any $r \geq 1$, $\beta \in \mathbb{C}$ and $\mu \in \mathscr{M}_r$ there exists $f \in \mathrm{End}(\mathbb{C},0)$ tangent to the identity with multiplicity $r+1$, index β and sectorial invariant μ.*

Remark 3.15. In particular, holomorphic local dynamical systems tangent to the identity give examples of local dynamical systems that are topologically conjugated without being neither holomorphically nor formally conjugated, and of local dynamical systems that are formally conjugated without being holomorphically conjugated. See also [Na] and [Tr].

We would also like to mention a result of Ribón appeared in the appendix of [CGBM]. It is known (see, e.g., [Br2]) that any germ $f \in \mathrm{End}(\mathbb{C},0)$ tangent to the identity is the time-one map of a unique formal (not necessarily holomorphic) vector field X singular at the origin, the *infinitesimal generator* of f. It is not difficult to see that f is holomorphically locally conjugated to its formal normal form (given by Proposition 3.10) if and only if X is actually holomorphic; Ribón has shown that this is equivalent to the existence of a real-analytic foliation invariant under f. More precisely, he has proved the following

Theorem 3.16 (Ribón, 2008 [CGBM]). *Let $f \in \mathrm{End}(\mathbb{C},0) \setminus \{\mathrm{id}\}$ be a germ tangent to the identity. If there exists a germ of real-analytic foliation $\mathscr{F}$, having an isolated singularity at the origin, such that $f^*\mathscr{F} = \mathscr{F}$, then the formal infinitesimal generator of f is holomorphic at the origin. In particular, f is holomorphically conjugated to its formal normal form.*

We end this section recalling a few result on parabolic germs not tangent to the identity. If $f \in \mathrm{End}(\mathbb{C},0)$ satisfies $a_1 = e^{2\pi i p/q}$, then f^q is tangent to the identity. Therefore we can apply the previous results to f^q and then infer informations about the dynamics of the original f, because of the following

Lemma 3.17. *Let f, $g \in \mathrm{End}(\mathbb{C},0)$ be two holomorphic local dynamical systems with the same multiplier $e^{2\pi i p/q} \in S^1$. Then f and g are holomorphically locally conjugated if and only if f^q and g^q are.*

Proof. One direction is obvious. For the converse, let φ be a germ conjugating f^q and g^q; in particular,

$$g^q = \varphi^{-1} \circ f^q \circ \varphi = (\varphi^{-1} \circ f \circ \varphi)^q .$$

So, up to replacing f by $\varphi^{-1} \circ f \circ \varphi$, we can assume that $f^q = g^q$. Put

$$\psi = \sum_{k=0}^{q-1} g^{q-k} \circ f^k = \sum_{k=1}^{q} g^{q-k} \circ f^k .$$

The germ ψ is a local biholomorphism, because $\psi'(0) = q \neq 0$, and it is easy to check that $\psi \circ f = g \circ \psi$. □

We list here a few results; see [Mi], [Ma], [C], [É2–3], [Vo], [MR] and [BH] for proofs and further details.

Proposition 3.18. *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $\lambda \in S^1$, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$. Then there exist $n \geq 1$ and $\alpha \in \mathbb{C}$ such that f is formally conjugated to*

$$g(z) = \lambda z - z^{nq+1} + \alpha z^{2nq+1} \; .$$

Definition 3.19. The number n is the *parabolic multiplicity* of f, and $\alpha \in \mathbb{C}$ is the *index* of f; the *iterative residue* of f is then given by

$$\mathrm{Resit}(f) = \frac{nq+1}{2} - \alpha \; .$$

Proposition 3.20 (Camacho). *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $\lambda \in S^1$, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$, and has parabolic multiplicity $n \geq 1$. Then f is topologically conjugated to*

$$g(z) = \lambda z - z^{nq+1} \; .$$

Theorem 3.21 (Leau-Fatou). *Let $f \in \mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $\lambda \in S^1$, and assume that λ is a primitive root of the unity of order q. Assume that $f^q \not\equiv \mathrm{id}$, and let $n \geq 1$ be the parabolic multiplicity of f. Then f^q has multiplicity $nq+1$, and f acts on the attracting (respectively, repelling) petals of f^q as a permutation composed by n disjoint cycles. Finally, $K_f = K_{f^q}$.*

Furthermore, it is possible to define the sectorial invariant of such a holomorphic local dynamical system, composed by $2nq$ germs whose multipliers still satisfy (19), and the analogue of Theorem 3.14 holds.

4 One complex variable: the elliptic case

We are left with the elliptic case:

(21) $$f(z) = e^{2\pi i\theta} z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\} \; ,$$

with $\theta \notin \mathbb{Q}$. It turns out that the local dynamics depends mostly on numerical properties of θ. The main question here is whether such a local dynamical system is holomorphically conjugated to its linear part. Let us introduce a bit of terminology.

Definition 4.1. We shall say that a holomorphic dynamical system of the form (21) is *holomorphically linearizable* if it is holomorphically locally conjugated to its linear part, the irrational rotation $z \mapsto e^{2\pi i\theta} z$. In this case, we shall say that 0 is a *Siegel point* for f; otherwise, we shall say that it is a *Cremer point*.

It turns out that for a full measure subset B of $\theta \in [0,1] \setminus \mathbb{Q}$ all holomorphic local dynamical systems of the form (21) are holomorphically linearizable. Conversely, the complement $[0,1]\setminus B$ is a G_δ-dense set, and for all $\theta\in[0,1]\setminus B$ the quadratic polynomial $z\mapsto z^2+e^{2\pi i\theta}z$ is not holomorphically linearizable. This is the gist of the results due to Cremer, Siegel, Brjuno and Yoccoz we shall describe in this section.

The first worthwhile observation in this setting is that it is possible to give a topological characterization of holomorphically linearizable local dynamical systems.

Definition 4.2. We shall say that p is *stable* for $f\in\mathrm{End}(M,p)$ if it belongs to the interior of K_f.

Proposition 4.3. *Let $f\in\mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system with multiplier $\lambda\in S^1$. Then f is holomorphically linearizable if and only if it is topologically linearizable if and only if 0 is stable for f.*

Proof. If f is holomorphically linearizable it is topologically linearizable, and if it is topologically linearizable (and $|\lambda|=1$) then it is stable. Assume that 0 is stable, and set

$$\varphi_k(z)=\frac{1}{k}\sum_{j=0}^{k-1}\frac{f^j(z)}{\lambda^j}\;,$$

so that $\varphi_k'(0)=1$ and

$$\varphi_k\circ f=\lambda\varphi_k+\frac{\lambda}{k}\left(\frac{f^k}{\lambda^k}-\mathrm{id}\right)\;. \tag{22}$$

The stability of 0 implies that there are bounded open sets $V\subset U$ containing the origin such that $f^k(V)\subset U$ for all $k\in\mathbb{N}$. Since $|\lambda|=1$, it follows that $\{\varphi_k\}$ is a uniformly bounded family on V, and hence, by Montel's theorem, it admits a converging subsequence. But (22) implies that a converging subsequence converges to a conjugation between f and the rotation $z\mapsto\lambda z$, and so f is holomorphically linearizable. □

The second important observation is that two elliptic holomorphic local dynamical systems with the same multiplier are always formally conjugated:

Proposition 4.4. *Let $f\in\mathrm{End}(\mathbb{C},0)$ be a holomorphic local dynamical system of multiplier $\lambda=e^{2\pi i\theta}\in S^1$ with $\theta\notin\mathbb{Q}$. Then f is formally conjugated to its linear part, by a unique formal power series tangent to the identity.*

Proof. We shall prove that there is a unique formal power series

$$h(z)=z+h_2z^2+\cdots\in\mathbb{C}[\![z]\!]$$

such that $h(\lambda z)=f\big(h(z)\big)$. Indeed we have

$$\begin{aligned}h(\lambda z)-f\big(h(z)\big)&=\sum_{j\ge2}\left\{\big[(\lambda^j-\lambda)h_j-a_j\big]z^j-a_j\sum_{\ell=1}^{j}\binom{j}{\ell}z^{\ell+j}\left(\sum_{k\ge2}h_kz^{k-2}\right)^{\ell}\right\}\\&=\sum_{j\ge2}\big[(\lambda^j-\lambda)h_j-a_j-X_j(h_2,\dots,h_{j-1})\big]z^j\;,\end{aligned}$$

where X_j is a polynomial in $j-2$ variables with coefficients depending on $a_2, \dots, a_{j-1}$. It follows that the coefficients of h are uniquely determined by induction using the formula

$$h_j = \frac{a_j + X_j(h_2, \dots, h_{j-1})}{\lambda^j - \lambda} \,. \tag{23}$$

In particular, h_j depends only on λ, $a_2, \dots, a_j$. □

Remark 4.5. The same proof shows that any holomorphic local dynamical system with multiplier $\lambda \neq 0$ and not a root of unity is formally conjugated to its linear part.

The formal power series linearizing f is not converging if its coefficients grow too fast. Thus (23) links the radius of convergence of h to the behavior of $\lambda^j - \lambda$: if the latter becomes too small, the series defining h does not converge. This is known as the *small denominators problem* in this context.

It is then natural to introduce the following quantity:

$$\Omega_\lambda(m) = \min_{1 \le k \le m} |\lambda^k - \lambda| \,,$$

for $\lambda \in S^1$ and $m \ge 1$. Clearly, λ is a root of unity if and only if $\Omega_\lambda(m) = 0$ for all m greater or equal to some $m_0 \ge 1$; furthermore,

$$\lim_{m \to +\infty} \Omega_\lambda(m) = 0$$

for all $\lambda \in S^1$.

The first one to actually prove that there are non-linearizable elliptic holomorphic local dynamical systems has been Cremer, in 1927 [Cr1]. His more general result is the following:

Theorem 4.6 (Cremer, 1938 [Cr2]). *Let $\lambda \in S^1$ be such that*

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\Omega_\lambda(m)} = +\infty \,. \tag{24}$$

Then there exists $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ which is not holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (24) *contains a G_δ-dense set.*

Proof. Choose inductively $a_j \in \{0, 1\}$ so that $|a_j + X_j| \ge 1/2$ for all $j \ge 2$, where X_j is as in (23). Then

$$f(z) = \lambda z + a_2 z^2 + \cdots \in \mathbb{C}_0\{z\} \,,$$

while (24) implies that the radius of convergence of the formal linearization h is 0, and thus f cannot be holomorphically linearizable, as required.

Finally, let $C(q_0) \subset S^1$ denote the set of $\lambda = e^{2\pi i \theta} \in S^1$ such that

$$\left| \theta - \frac{p}{q} \right| < \frac{1}{2^{q!}} \tag{25}$$

for some $p/q \in \mathbb{Q}$ in lowest terms, with $q \geq q_0$. Then it is not difficult to check that each $C(q_0)$ is a dense open set in S^1, and that all $\lambda \in \mathscr{C} = \bigcap_{q_0 \geq 1} C(q_0)$ satisfy (24). Indeed, if $\lambda = e^{2\pi i\theta} \in \mathscr{C}$ we can find $q \in \mathbb{N}$ arbitrarily large such that there is $p \in \mathbb{N}$ so that (25) holds. Now, it is easy to see that

$$|e^{2\pi i t} - 1| \leq 2\pi|t|$$

for all $t \in [-1/2, 1/2]$. Then let p_0 be the integer closest to $q\theta$, so that $|q\theta - p_0| \leq 1/2$. Then we have

$$|\lambda^q - 1| = |e^{2\pi i q\theta} - e^{2\pi i p_0}| = |e^{2\pi i(q\theta - p_0)} - 1| \leq 2\pi|q\theta - p_0| \leq 2\pi|q\theta - p| < \frac{2\pi}{2^{q!-1}}$$

for arbitrarily large q, and (24) follows. $\square$

On the other hand, Siegel in 1942 gave a condition on the multiplier ensuring holomorphic linearizability:

Theorem 4.7 (Siegel, 1942 [Si]). *Let $\lambda \in S^1$ be such that there exists $\beta > 1$ and $\gamma > 0$ so that*

$$\frac{1}{\Omega_\lambda(m)} \leq \gamma m^\beta \tag{26}$$

for all $m \geq 2$. Then all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ are holomorphically linearizable. Furthermore, the set of $\lambda \in S^1$ satisfying (26) *for some $\beta > 1$ and $\gamma > 0$ is of full Lebesgue measure in S^1.*

Remark 4.8. If $\theta \in [0,1) \setminus \mathbb{Q}$ is algebraic then $\lambda = e^{2\pi i\theta}$ satisfies (26) for some $\beta > 1$ and $\gamma > 0$. However, the set of $\lambda \in S^1$ satisfying (26) is much larger, being of full measure.

Remark 4.9. It is interesting to notice that for generic (in a topological sense) $\lambda \in S^1$ there is a non-linearizable holomorphic local dynamical system with multiplier λ, while for almost all (in a measure-theoretic sense) $\lambda \in S^1$ every holomorphic local dynamical system with multiplier λ is holomorphically linearizable.

Theorem 4.7 suggests the existence of a number-theoretical condition on λ ensuring that the origin is a Siegel point for any holomorphic local dynamical system of multiplier λ. And indeed this is the content of the celebrated *Brjuno-Yoccoz theorem:*

Theorem 4.10 (Brjuno, 1965 [Brj1–3], Yoccoz, 1988 [Y1–2]). *Let $\lambda \in S^1$. Then the following statements are equivalent:*

(i) *the origin is a Siegel point for the quadratic polynomial $f_\lambda(z) = \lambda z + z^2$;*
(ii) *the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ;*
(iii) *the number λ satisfies Brjuno's condition*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty \,. \tag{27}$$

Brjuno, using majorant series as in Siegel's proof of Theorem 4.7 (see also [He] and references therein) has proved that condition (iii) implies condition (ii). Yoccoz, using a more geometric approach based on conformal and quasi-conformal geometry, has proved that (i) is equivalent to (ii), and that (ii) implies (iii), that is that if λ does not satisfy (27) then the origin is a Cremer point for some $f \in \mathrm{End}(\mathbb{C},0)$ with multiplier λ — and hence it is a Cremer point for the quadratic polynomial $f_\lambda(z)$. See also [P9] for related results.

Remark 4.11. Condition (27) is usually expressed in a different way. Write $\lambda = e^{2\pi i\theta}$, and let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions. Then (27) is equivalent to

$$\sum_{k=0}^{+\infty} \frac{1}{q_k} \log q_{k+1} < +\infty \,,$$

while (26) is equivalent to

$$q_{n+1} = O(q_n^\beta) \,,$$

and (24) is equivalent to

$$\limsup_{k\to+\infty} \frac{1}{q_k} \log q_{k+1} = +\infty \,.$$

See [He], [Y2], [Mi], [Ma], [K], [P1] and references therein for details.

Remark 4.12. A clear obstruction to the holomorphic linearization of an elliptic $f \in \mathrm{End}(\mathbb{C},0)$ with multiplier $\lambda = e^{2\pi i\theta} \in S^1$ is the existence of *small cycles*, that is of periodic orbits contained in any neighbourhood of the origin. Perez-Marco [P2], using Yoccoz's techniques, has shown that when the series

$$\sum_{k=0}^{+\infty} \frac{\log\log q_{k+1}}{q_k}$$

converges then every germ with multiplier λ is either linearizable or has small cycles, and that when the series diverges there exists such germs with a Cremer point but without small cycles.

The complete proof (see [P1] and the original papers) of Theorem 4.10 is beyond the scope of this survey. We shall limit ourselves to describe a proof (adapted from [Pö]) of the implication (iii)$\Longrightarrow$(ii), to report two of the easiest results of [Y2], and to illustrate what is the connection between condition (27) and the radius of convergence of the formal linearizing map.

Let us begin with Brjuno's theorem:

Theorem 4.13 (Brjuno, 1965 [Brj1–3]). *Assume that $\lambda = e^{2\pi i\theta} \in S^1$ satisfies the Brjuno's condition*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_\lambda(2^{k+1})} < +\infty \,. \tag{28}$$

Then the origin is a Siegel point for all $f \in \mathrm{End}(\mathbb{C},0)$ *with multiplier* λ.

Proof. We already know, thanks to Proposition 4.4, that there exists a unique formal power series

$$h(z) = z + \sum_{k\geq 2} h_k z^k$$

such that $h^{-1} \circ f \circ h(z) = \lambda z$; we shall prove that h is actually converging. To do so it suffices to show that

$$\sup_k \frac{1}{k} \log |h_k| < \infty \,. \tag{29}$$

Since f is holomorphic in a neighbourhood of the origin, there exists a number $M > 0$ such that $|a_k| \leq M^k$ for $k \geq 2$; up to a linear change of coordinates we can assume that $M = 1$, that is $|a_l| \leq 1$ for all $k \geq 2$.

Now, $h(\lambda z) = f\big(h(z)\big)$ yields

$$\sum_{k\geq 2} (\lambda^k - \lambda) h_k z^k = \sum_{l\geq 2} a_l \left(\sum_{m\geq 1} h_m z^m \right)^l \,. \tag{30}$$

Therefore

$$|h_k| \leq \varepsilon_k^{-1} \sum_{\substack{k_1+\cdots+k_\nu = k \\ \nu \geq 2}} |h_{k_1}| \cdots |h_{k_\nu}| \,,$$

where

$$\varepsilon_k = |\lambda^k - \lambda| \,.$$

Define inductively

$$\alpha_k = \begin{cases} 1 & \text{if } k = 1\,, \\ \displaystyle\sum_{\substack{k_1+\cdots+k_\nu = k \\ \nu \geq 2}} \alpha_{k_1} \cdots \alpha_{k_\nu} & \text{if } k \geq 2\,, \end{cases}$$

and

$$\delta_k = \begin{cases} 1 & \text{if } k = 1\,, \\ \varepsilon_k^{-1} \displaystyle\max_{\substack{k_1+\cdots+k_\nu = k \\ \nu \geq 2}} \delta_{k_1} \cdots \delta_{k_\nu}\,, & \text{if } k \geq 2\,. \end{cases}$$

Then it is easy to check by induction that

$$|h_k| \leq \alpha_k \delta_k$$

for all $k \geq 2$. Therefore, to establish (29) it suffices to prove analogous estimates for α_k and δ_k.

To estimate α_k, let $\alpha = \sum_{k\geq 1} \alpha_k t^k$. We have

$$\alpha - t = \sum_{k\geq 2} \alpha_k t^k = \sum_{k\geq 2} \left(\sum_{j\geq 1} \alpha_j t^j \right)^k = \frac{\alpha^2}{1-\alpha} \;.$$

This equation has a unique holomorphic solution vanishing at zero

$$\alpha = \frac{t+1}{4} \left(1 - \sqrt{1 - \frac{8t}{(1+t)^2}} \right) \;,$$

defined for $|t|$ small enough. Hence,

$$\sup_k \frac{1}{k} \log \alpha_k < \infty \;,$$

as we wanted.

To estimate δ_k we have to take care of small divisors. First of all, for each $k \geq 2$ we associate to δ_k a specific decomposition of the form

$$\delta_k = \varepsilon_k^{-1} \delta_{k_1} \cdots \delta_{k_\nu} \;, \tag{31}$$

with $k > k_1 \geq \cdots \geq k_\nu$, $k = k_1 + \cdots + k_\nu$ and $\nu \geq 2$, and hence, by induction, a specific decomposition of the form

$$\delta_k = \varepsilon_{l_0}^{-1} \varepsilon_{l_1}^{-1} \cdots \varepsilon_{l_q}^{-1} \;, \tag{32}$$

where $l_0 = k$ and $k > l_1 \geq \cdots \geq l_q \geq 2$. For $m \geq 2$ let $N_m(k)$ be the number of factors ε_l^{-1} in the expression (32) of δ_k satisfying

$$\varepsilon_l < \frac{1}{4} \Omega_\lambda(m) \;.$$

Notice that $\Omega_\lambda(m)$ is non-increasing with respect to m and it tends to zero as m goes to infinity. The next lemma contains the key estimate.

Lemma 4.14. *For all $m \geq 2$ we have*

$$N_m(k) \leq \begin{cases} 0, & \text{if } k \leq m \;, \\ \frac{2k}{m} - 1, & \text{if } k > m \;. \end{cases}$$

Proof. We argue by induction on k. If $l \leq k \leq m$ we have $\varepsilon_l \geq \Omega_\lambda(m)$, and hence $N_m(k) = 0$.

Assume now $k > m$, so that $2k/m - 1 \geq 1$. Write δ_k as in (31); we have a few cases to consider.

Case 1: $\varepsilon_k \geq \frac{1}{4} \Omega_\lambda(m)$. Then

$$N(k) = N(k_1) + \cdots + N(k_\nu) ,$$

and applying the induction hypothesis to each term we get $N(k) \leq (2k/m) - 1$.

Case 2: $\varepsilon_k < \frac{1}{4}\,\Omega_\lambda(m)$. Then

$$N(k) = 1 + N(k_1) + \cdots + N(k_\nu) ,$$

and there are three subcases.

Case 2.1: $k_1 \leq m$. Then

$$N(k) = 1 \leq \frac{2k}{m} - 1 ,$$

and we are done.

Case 2.2: $k_1 \geq k_2 > m$. Then there is ν' such that $2 \leq \nu' \leq \nu$ and $k_{\nu'} > m \geq k_{\nu'+1}$, and we again have

$$N(k) = 1 + N(k_1) + \cdots + N(k_{\nu'}) \leq 1 + \frac{2k}{m} - \nu' \leq \frac{2k}{m} - 1 .$$

Case 2.3: $k_1 > m \geq k_2$. Then

$$N(k) = 1 + N(k_1) ,$$

and we have two different subsubcases.

Case 2.3.1: $k_1 \leq k - m$. Then

$$N(k) \leq 1 + 2\,\frac{k-m}{m} - 1 < \frac{2k}{m} - 1 ,$$

and we are done in this case too.

Case 2.3.2: $k_1 > k - m$. The crucial remark here is that $\varepsilon_{k_1}^{-1}$ gives no contribution to $N(k_1)$. Indeed, assume by contradiction that $\varepsilon_{k_1} < \frac{1}{4}\,\Omega_\lambda(m)$. Then

$$|\lambda^{k_1}| > |\lambda| - \frac{1}{4}\,\Omega_\lambda(m) \geq 1 - \frac{1}{2} = \frac{1}{2} ,$$

because $\Omega_\lambda(m) \leq 2$. Since $k - k_1 < m$, it follows that

$$\begin{aligned} \frac{1}{2}\,\Omega_\lambda(m) &> \varepsilon_k + \varepsilon_{k_1} = |\lambda^k - \lambda| + |\lambda^{k_1} - \lambda| \geq |\lambda^k - \lambda^{k_1}| \\ &= |\lambda^{k-k_1} - 1| \geq \Omega_\lambda(k - k_1 + 1) \geq \Omega_\lambda(m) , \end{aligned}$$

contradiction.

Therefore Case 1 applies to δ_{k_1} and we have

$$N(k) = 1 + N(k_{1_1}) + \cdots + N(k_{1_{\nu_1}}) ,$$

with $k > k_1 > k_{1_1} \geq \cdots \geq k_{1_{\nu_1}}$ and $k_1 = k_{1_1} + \cdots + k_{1_{\nu_1}}$. We can repeat the argument for this decomposition, and we finish unless we run into case 2.3.2 again. However, this loop cannot happen more than $m+1$ times, and we eventually have to land into a different case. This completes the induction and the proof. $\square$

Let us go back to the proof of Theorem 4.13. We have to estimate

$$\frac{1}{k}\log\delta_k = \sum_{j=0}^{q}\frac{1}{k}\log\varepsilon_{l_j}^{-1} .$$

By Lemma 4.14,

$$\operatorname{card}\left\{0 \leq j \leq q \;\middle|\; \frac{1}{4}\Omega_\lambda(2^{\nu+1}) \leq \varepsilon_{l_j} < \frac{1}{4}\Omega_\lambda(2^\nu)\right\} \leq N_{2^\nu}(k) \leq \frac{2k}{2^\nu}$$

for $\nu \geq 1$. It is also easy to see from the definition of δ_k that the number of factors $\varepsilon_{l_j}^{-1}$ is bounded by $2k-1$. In particular,

$$\operatorname{card}\left\{0 \leq j \leq q \;\middle|\; \frac{1}{4}\Omega_\lambda(2) \leq \varepsilon_{l_j}\right\} \leq 2k = \frac{2k}{2^0} .$$

Then

$$\frac{1}{k}\log\delta_k \leq 2\sum_{\nu\geq 0}\frac{1}{2^\nu}\log\bigl(4\,\Omega_\lambda(2^{\nu+1})^{-1}\bigr) = 2\log 4 + 2\sum_{\nu\geq 0}\frac{1}{2^\nu}\log\frac{1}{\Omega_\lambda(2^{\nu+1})} ,$$

and we are done. $\square$

The second result we would like to present is Yoccoz's beautiful proof of the fact that almost every quadratic polynomial f_λ is holomorphically linearizable:

Proposition 4.15. *The origin is a Siegel point of $f_\lambda(z) = \lambda z + z^2$ for almost every $\lambda \in S^1$.*

Proof. (Yoccoz [Y2]) The idea is to study the radius of convergence of the inverse of the linearization of $f_\lambda(z) = \lambda z + z^2$ when $\lambda \in \Delta^*$. Theorem 2.4 says that there is a unique map φ_λ defined in some neighbourhood of the origin such that $\varphi_\lambda'(0) = 1$ and $\varphi_\lambda \circ f = \lambda\varphi_\lambda$. Let ρ_λ be the radius of convergence of φ_λ^{-1}; we want to prove that φ_λ is defined in a neighbourhood of the unique critical point $-\lambda/2$ of f_λ, and that $\rho_\lambda = |\varphi_\lambda(-\lambda/2)|$.

Let $\Omega_\lambda \subset\subset \mathbb{C}$ be the basin of attraction of the origin, that is the set of $z \in \mathbb{C}$ whose orbit converges to the origin. Notice that setting $\varphi_\lambda(z) = \lambda^{-k}\varphi_\lambda\bigl(f_\lambda(z)\bigr)$ we can extend φ_λ to the whole of Ω_λ. Moreover, since the image of φ_λ^{-1} is contained in Ω_λ, which is bounded, necessarily $\rho_\lambda < +\infty$. Let $U_\lambda = \varphi_\lambda^{-1}(\Delta_{\rho_\lambda})$. Since we have

$$(\varphi_\lambda' \circ f)f' = \lambda\varphi_\lambda' \tag{33}$$

and φ_λ is invertible in U_λ, the function f cannot have critical points in U_λ.

If $z = \varphi_\lambda^{-1}(w) \in U_\lambda$, we have $f(z) = \varphi_\lambda^{-1}(\lambda w) \in \varphi_\lambda^{-1}(\Delta_{|\lambda|\rho_\lambda}) \subset\subset U_\lambda$; therefore

$$f(\overline{U_\lambda}) \subseteq \overline{f(U_\lambda)} \subset\subset U_\lambda \subseteq \Omega_\lambda,$$

which implies that $\partial U \subset \Omega_\lambda$. So φ_λ is defined on ∂U_λ, and clearly $|\varphi_\lambda(z)| = \rho_\lambda$ for all $z \in \partial U_\lambda$.

If f had no critical points in ∂U_λ, (33) would imply that φ_λ has no critical points in ∂U_λ. But then φ_λ would be locally invertible in ∂U_λ, and thus φ_λ^{-1} would extend across $\partial \Delta_{\rho_\lambda}$, impossible. Therefore $-\lambda/2 \in \partial U_\lambda$, and $|\varphi_\lambda(-\lambda/2)| = \rho_\lambda$, as claimed.

(Up to here it was classic; let us now start Yoccoz's argument.) Put $\eta(\lambda) = \varphi_\lambda(-\lambda/2)$. From the proof of Theorem 2.4 one easily sees that φ_λ depends holomorphically on λ; so $\eta\colon \Delta^* \to \mathbb{C}$ is holomorphic. Furthermore, since $\Omega_\lambda \subseteq \Delta_2$, Schwarz's lemma applied to $\varphi_\lambda^{-1}\colon \Delta_{\rho_\lambda} \to \Delta_2$ yields

$$1 = |(\varphi_\lambda^{-1})'(0)| \le 2/\rho_\lambda,$$

that is $\rho_\lambda \le 2$. Thus η is bounded, and thus it extends holomorphically to the origin.

So $\eta\colon \Delta \to \Delta_2$ is a bounded holomorphic function not identically zero; Fatou's theorem on radial limits of bounded holomorphic functions then implies that

$$\rho(\lambda_0) := \limsup_{r \to 1^-} |\eta(r\lambda_0)| > 0$$

for almost every $\lambda_0 \in S^1$. This means that we can find $0 < \rho_0 < \rho(\lambda_0)$ and a sequence $\{\lambda_j\} \subset \Delta$ such that $\lambda_j \to \lambda_0$ and $|\eta(\lambda_j)| > \rho_0$. This means that $\varphi_{\lambda_j}^{-1}$ is defined in Δ_{ρ_0} for all $j \ge 1$; up to a subsequence, we can assume that $\varphi_{\lambda_j}^{-1} \to \psi\colon \Delta_{\rho_0} \to \Delta_2$. But then we have $\psi'(0) = 1$ and

$$f_{\lambda_0}\bigl(\psi(z)\bigr) = \psi(\lambda_0 z)$$

in Δ_{ρ_0}, and thus the origin is a Siegel point for f_{λ_0}. □

The third result we would like to present is the implication (i) $\Longrightarrow$ (ii) in Theorem 4.10. The proof depends on the following result of Douady and Hubbard, obtained using the theory of quasiconformal maps:

Theorem 4.16 (Douady-Hubbard, 1985 [DH]). *Given $\lambda \in \mathbb{C}^*$, let $f_\lambda(z) = \lambda z + z^2$ be a quadratic polynomial. Then there exists a universal constant $C > 0$ such that for every holomorphic function $\psi\colon \Delta_{3|\lambda|/2} \to \mathbb{C}$ with $\psi(0) = \psi'(0) = 0$ and $|\psi(z)| \le C|\lambda|$ for all $z \in \Delta_{3|\lambda|/2}$ the function $f = f_\lambda + \psi$ is topologically conjugated to f_λ in $\Delta_{|\lambda|}$.*

Then

Theorem 4.17 (Yoccoz, 1988 [Y2]). *Let $\lambda \in S^1$ be such that the origin is a Siegel point for $f_\lambda(z) = \lambda z + z^2$. Then the origin is a Siegel point for every $f \in \mathrm{End}(\mathbb{C}, 0)$ with multiplier λ.*

Sketch of proof. Write

$$f(z) = \lambda z + a_2 z^2 + \sum_{k\geq 3} a_k z^k \ ,$$

and let

$$f^a(z) = \lambda z + a z^2 + \sum_{k\geq 3} a_k z^k \ ,$$

so that $f = f^{a_2}$. If $|a|$ is large enough then the germ

$$g^a(z) = a f^a(z/a) = \lambda z + z^2 + a \sum_{k\geq 3} a_k (z/a)^k = f_\lambda(z) + \psi^a(z)$$

is defined on $\Delta_{3/2}$ and $|\psi^a(z)| < C$ for all $z \in \Delta_{3/2}$, where C is the constant given by Theorem 4.16. It follows that g^a is topologically conjugated to f_λ. By assumption, f_λ is topologically linearizable; hence g^a is too. Proposition 4.3 then implies that g^a is holomorphically linearizable, and hence f^a is too. Furthermore, it is also possible to show (see, e.g., [BH, Lemma 2.3]) that if $|a|$ is large enough, say $|a| \geq R$, then the domain of linearization of g^a contains Δ_r, where $r > 0$ is such that Δ_{2r} is contained in the domain of linearization of f_λ.

So we have proven the assertion if $|a_2| \geq R$; assume then $|a_2| < R$. Since λ is not a root of unity, there exists (Proposition 4.4) a unique formal power series $\hat{h}^a \in \mathbb{C}[\![z]\!]$ tangent to the identity such that $g^a \circ \hat{h}^a(z) = \hat{h}^a(\lambda z)$. If we write

$$\hat{h}^a(z) = z + \sum_{k\geq 2} h_k(a) z^k$$

then $h_k(a)$ is a polynomial in a of degree $k-1$, by (30). In particular, by the maximum principle we have

$$|h_k(a_2)| \leq \max_{|a|=R} |h_k(a)| \tag{34}$$

for all $k \geq 2$. Now, by what we have seen, if $|a| = R$ then $\hat{h}^a$ is convergent in a disk of radius $r(a) > 0$, and its image contains a disk of radius r. Applying Schwarz's lemma to $(\hat{h}^a)^{-1} \colon \Delta_r \to \Delta_{r(a)}$ we get $r(a) \geq r$. But then

$$\limsup_{k\to+\infty} |h_k(a_2)|^{1/k} \leq \max_{|a|=R} \limsup_{k\to+\infty} |h_k(a)|^{1/k} = \frac{1}{r(a)} \leq \frac{1}{r} < +\infty \ ;$$

hence $\hat{h}^{a_2}$ is convergent, and we are done. □

Finally, we would like to describe the connection between condition (27) and linearization. From the function theoretical side, given $\theta \in [0,1)$ set

$$r(\theta) = \inf\{r(f) \mid f \in \mathrm{End}(\mathbb{C},0) \text{ has multiplier } e^{2\pi i\theta} \text{ and it is defined and injective in } \Delta\}$$

where $r(f) \geq 0$ is the radius of convergence of the unique formal linearization of f tangent to the identity.

From the number theoretical side, given an irrational number $\theta \in [0,1)$ let $\{p_k/q_k\}$ be the sequence of rational numbers converging to θ given by the expansion in continued fractions, and put

$$\alpha_n = -\frac{q_n\theta - p_n}{q_{n-1}\theta - p_{n-1}}, \qquad \alpha_0 = \theta,$$
$$\beta_n = (-1)^n(q_n\theta - p_n), \qquad \beta_{-1} = 1.$$

Definition 4.18. The *Brjuno function* $B\colon [0,1)\setminus\mathbb{Q} \to (0,+\infty]$ is defined by

$$B(\theta) = \sum_{n=0}^{\infty} \beta_{n-1} \log\frac{1}{\alpha_n}.$$

Then Theorem 4.10 is consequence of what we have seen and the following

Theorem 4.19 (Yoccoz, 1988 [Y2]).
(i) $B(\theta) < +\infty$ *if and only if* $\lambda = e^{2\pi i\theta}$ *satisfies Brjuno's condition* (27)*;*
(ii) *there exists a universal constant* $C > 0$ *such that*

$$|\log r(\theta) + B(\theta)| \leq C$$

for all $\theta \in [0,1)\setminus\mathbb{Q}$ *such that* $B(\theta) < +\infty$*;*
(iii) *if* $B(\theta) = +\infty$ *then there exists a non-linearizable* $f \in \mathrm{End}(\mathbb{C},0)$ *with multiplier* $e^{2\pi i\theta}$.

If 0 is a Siegel point for $f \in \mathrm{End}(\mathbb{C},0)$, the local dynamics of f is completely clear, and simple enough. On the other hand, if 0 is a Cremer point of f, then the local dynamics of f is very complicated and not yet completely understood. Pérez-Marco (in [P3, 5–7]) and Biswas ([B1, 2]) have studied the topology and the dynamics of the stable set in this case. Some of their results are summarized in the following

Theorem 4.20 (Pérez-Marco, 1995 [P6, 7]). *Assume that* 0 *is a Cremer point for an elliptic holomorphic local dynamical system* $f \in \mathrm{End}(\mathbb{C},0)$. *Then:*
(i) *The stable set* K_f *is compact, connected, full (i.e.,* $\mathbb{C}\setminus K_f$ *is connected), it is not reduced to* $\{0\}$*, and it is not locally connected at any point distinct from the origin.*
(ii) *Any point of* $K_f\setminus\{0\}$ *is recurrent (that is, a limit point of its orbit).*
(iii) *There is an orbit in* K_f *which accumulates at the origin, but no non-trivial orbit converges to the origin.*

Theorem 4.21 (Biswas, 2007 [B2]). *The rotation number and the conformal class of* K_f *are a complete set of holomorphic invariants for Cremer points. In other words, two elliptic non-linearizable holomorphic local dynamical systems* f *and* g

are holomorphically locally conjugated if and only if they have the same rotation number and there is a biholomorphism of a neighbourhood of K_f with a neighbourhood of K_g.

Remark 4.22. So, if $\lambda \in S^1$ is not a root of unity and does not satisfy Brjuno's condition (27), we can find f_1, $f_2 \in \mathrm{End}(\mathbb{C},0)$ with multiplier λ such that f_1 is holomorphically linearizable while f_2 is not. Then f_1 and f_2 are formally conjugated without being neither holomorphically nor topologically locally conjugated.

Remark 4.23. Yoccoz [Y2] has proved that if $\lambda \in S^1$ is not a root of unity and does not satisfy Brjuno's condition (27) then there is an uncountable family of germs in $\mathrm{End}(\mathbb{C},O)$ with multiplier λ which are not holomorphically conjugated to each other nor holomorphically conjugated to any entire function.

See also [P2, 4] for other results on the dynamics about a Cremer point. We end this section recalling the somewhat surprising fact that in the elliptic case the multiplier is a topological invariant (in the parabolic case this follows from Proposition 3.20):

Theorem 4.24 (Naishul, 1983 [N]). *Let f, $g \in \mathrm{End}(\mathbb{C},O)$ be two holomorphic local dynamical systems with an elliptic fixed point at the origin. If f and g are topologically locally conjugated then $f'(0) = g'(0)$.*

See [P7] for another proof of this result.

5 Several complex variables: the hyperbolic case

Now we start the discussion of local dynamics in several complex variables. In this setting the theory is much less complete than its one-variable counterpart.

Definition 5.1. Let $f \in \mathrm{End}(\mathbb{C}^n,O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. The *homogeneous expansion* of f is

$$f(z) = P_1(z) + P_2(z) + \cdots \in \mathbb{C}_0\{z_1,\ldots,z_n\}^n ,$$

where P_j is an n-uple of homogeneous polynomials of degree j. In particular, P_1 is the differential df_O of f at the origin, and f is locally invertible if and only if P_1 is invertible.

We have seen that in dimension one the multiplier (i.e., the derivative at the origin) plays a main rôle. When $n > 1$, a similar rôle is played by the eigenvalues of the differential.

Definition 5.2. Let $f \in \mathrm{End}(\mathbb{C}^n,O)$ be a holomorphic local dynamical system at $O \in \mathbb{C}^n$, with $n \geq 2$. Then:

– if all eigenvalues of df_O have modulus less than 1, we say that the fixed point O is *attracting;*
– if all eigenvalues of df_O have modulus greater than 1, we say that the fixed point O is *repelling;*
– if all eigenvalues of df_O have modulus different from 1, we say that the fixed point O is *hyperbolic* (notice that we allow the eigenvalue zero);
– if O is attracting or repelling, and df_O is invertible, we say that f is in the *Poincaré domain;*
– if O is hyperbolic, df_O is invertible, and f is not in the Poincaré domain (and thus df_O has both eigenvalues inside the unit disk and outside the unit disk) we say that f is in the *Siegel domain;*
– if all eigenvalues of df_O are roots of unity, we say that the fixed point O is *parabolic;* in particular, if $df_O = \mathrm{id}$ we say that f is *tangent to the identity;*
– if all eigenvalues of df_O have modulus 1 but none is a root of unity, we say that the fixed point O is *elliptic;*
– if $df_O = O$, we say that the fixed point O is *superattracting.*

Other cases are clearly possible, but for our aims this list is enough. In this survey we shall be mainly concerned with hyperbolic and parabolic fixed points; however, in the last section we shall also present some results valid in other cases.

Remark 5.3. There are situations where one can use more or less directly the one-dimensional theory. For example, it is possible to study the so-called *semi-direct product* of germs, namely germs $f \in \mathrm{End}(\mathbb{C}^2, O)$ of the form

$$f(z_1, z_2) = \big(f_1(z_1), f_2(z_1, z_2)\big) ,$$

or the so-called *unfoldings*, i.e., germs $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z_1, \dots, z_n) = \big(f_1(z_1, \dots, z_n), z_2, \dots, z_n\big) .$$

We refer to [J2] for the study of a particular class of semi-direct products, and to [Ri1–2] for interesting results on unfoldings.

Let us begin assuming that the origin is a hyperbolic fixed point for an $f \in \mathrm{End}(\mathbb{C}^n, O)$ not necessarily invertible. We then have a canonical splitting

$$\mathbb{C}^n = E^s \oplus E^u ,$$

where E^s (respectively, E^u) is the direct sum of the generalized eigenspaces associated to the eigenvalues of df_O with modulus less (respectively, greater) than 1. Then the first main result in this subject is the famous *stable manifold theorem* (originally due to Perron [Pe] and Hadamard [H]; see [FHY], [HK], [HPS], [Pes], [Sh], [AM] for proofs in the C^∞ category, Wu [Wu] for a proof in the holomorphic category, and [A3] for a proof in the non-invertible case):

Theorem 5.4 (Stable manifold theorem). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system with a hyperbolic fixed point at the origin. Then:*

(i) *the stable set K_f is an embedded complex submanifold of (a neighbourhood of the origin in) $\mathbb{C}^n$, tangent to E^s at the origin;*

(ii) *there is an embedded complex submanifold W_f of (a neighbourhood of the origin in) $\mathbb{C}^n$, called the* unstable set *of f, tangent to E^u at the origin, such that $f|_{W_f}$ is invertible, $f^{-1}(W_f) \subseteq W_f$, and $z \in W_f$ if and only if there is a sequence $\{z_{-k}\}_{k\in\mathbb{N}}$ in the domain of f such that $z_0 = z$ and $f(z_{-k}) = z_{-k+1}$ for all $k \geq 1$. Furthermore, if f is invertible then W_f is the stable set of f^{-1}.*

The proof is too involved to be summarized here; it suffices to say that both K_f and W_f can be recovered, for instance, as fixed points of a suitable contracting operator in an infinite dimensional space (see the references quoted above for details).

Remark 5.5. If the origin is an attracting fixed point, then $E^s = \mathbb{C}^n$, and K_f is an open neighbourhood of the origin, its *basin of attraction*. However, as we shall discuss below, this does not imply that f is holomorphically linearizable, not even when it is invertible. Conversely, if the origin is a repelling fixed point, then $E^u = \mathbb{C}^n$, and $K_f = \{O\}$. Again, not all holomorphic local dynamical systems with a repelling fixed point are holomorphically linearizable.

If a point in the domain U of a holomorphic local dynamical system with a hyperbolic fixed point does not belong either to the stable set or to the unstable set, it escapes both in forward time (that is, its orbit escapes) and in backward time (that is, it is not the end point of an infinite orbit contained in U). In some sense, we can think of the stable and unstable sets (or, as they are usually called in this setting, stable and unstable *manifolds*) as skewed coordinate planes at the origin, and the orbits outside these coordinate planes follow some sort of hyperbolic path, entering and leaving any neighbourhood of the origin in finite time.

Actually, this idea of straightening stable and unstable manifolds can be brought to fruition (at least in the invertible case), and it yields one of the possible proofs (see [HK], [Sh], [A3] and references therein) of the *Grobman-Hartman theorem:*

Theorem 5.6 (Grobman, 1959 [G1–2]; Hartman, 1960 [Hr]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then f is topologically locally conjugated to its differential df_O.*

Thus, at least from a topological point of view, the local dynamics about an invertible hyperbolic fixed point is completely clear. This is definitely not the case if the local dynamical system is not invertible in a neighbourhood of the fixed point. For instance, already Hubbard and Papadopol [HP] noticed that a Böttcher-type theorem for superattracting points in several complex variables is just not true: there are holomorphic local dynamical systems with a superattracting fixed point which are not even topologically locally conjugated to the first non-vanishing term of their homogeneous expansion. Recently, Favre and Jonsson (see, e.g., [Fa] and [FJ1, 2]) have begun a very detailed study of superattracting fixed points in $\mathbb{C}^2$, study that might lead to their topological classification. We shall limit ourselves to quote one result.

Definition 5.7. Given $f \in \mathrm{End}(\mathbb{C}^2, O)$, we shall denote by $\mathrm{Crit}(f)$ the set of critical points of f. Put

$$\mathrm{Crit}^\infty(f) = \bigcup_{k \geq 0} f^{-k}\bigl(\mathrm{Crit}(f)\bigr) ;$$

we shall say that f is *rigid* if (as germ in the origin) $\mathrm{Crit}^\infty(f)$ is either empty, a smooth curve, or the union of two smooth curves crossing transversally at the origin. Finally, we shall say that f is *dominant* if $\det(df) \not\equiv 0$.

Rigid germs have been classified by Favre [Fa], which is the reason why next theorem can be useful for classifying superattracting dynamical systems:

Theorem 5.8 (Favre-Jonsson, 2007 [FJ2]). *Let $f \in \mathrm{End}(\mathbb{C}^2, O)$ be superattracting and dominant. Then there exist:*

(a) *a 2-dimensional complex manifold M (obtained by blowing-up a finite number of points; see next section);*
(b) *a surjective holomorphic map $\pi\colon M \to \mathbb{C}^2$ such that the restriction $\pi|_{M \setminus E}\colon M \setminus E \to \mathbb{C}^2 \setminus \{O\}$ is a biholomorphism, where $E = \pi^{-1}(O)$;*
(c) *a point $p \in E$; and*
(d) *a rigid holomorphic germ $\tilde{f} \in \mathrm{End}(M, p)$*

so that $\pi \circ \tilde{f} = f \circ \pi$.

See also Ruggiero [Ru] for similar results for semi-superattracting (one eigenvalue zero, one eigenvalue different from zero) germs in $\mathbb{C}^2$.

Coming back to hyperbolic dynamical systems, the holomorphic and even the formal classification are not as simple as the topological one. The main problem is caused by resonances.

Definition 5.9. Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system, and let denote by $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ the eigenvalues of df_O. A *resonance* for f is a relation of the form

$$\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j = 0 \tag{35}$$

for some $1 \leq j \leq n$ and some $k_1, \ldots, k_n \in \mathbb{N}$ with $k_1 + \cdots + k_n \geq 2$. When $n = 1$ there is a resonance if and only if the multiplier is a root of unity, or zero; but if $n > 1$ resonances may occur in the hyperbolic case too.

Resonances are the obstruction to formal linearization. Indeed, a computation completely analogous to the one yielding Proposition 4.4 shows that the coefficients of a formal linearization have in the denominators quantities of the form $\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j$; hence

Proposition 5.10. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point and no resonances. Then f is formally conjugated to its differential df_O.*

In presence of resonances, even the formal classification is not that easy. Let us assume, for simplicity, that df_O is in Jordan form, that is

$$P_1(z) = (\lambda_1 z, \varepsilon_2 z_1 + \lambda_2 z_2, \dots, \varepsilon_n z_{n-1} + \lambda_n z_n) ,$$

with $\varepsilon_1, \dots, \varepsilon_{n-1} \in \{0,1\}$.

Definition 5.11. We shall say that a monomial $z_1^{k_1} \cdots z_n^{k_n}$ in the j-th coordinate of f is *resonant* if $k_1 + \cdots + k_n \geq 2$ and $\lambda_1^{k_1} \cdots \lambda_n^{k_n} = \lambda_j$.

Then Proposition 5.10 can be generalized to (see [Ar, p. 194] or [IY, p. 53] for a proof):

Proposition 5.12 (Poincaré, 1893 [Po]; Dulac, 1904 [Du]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system with a hyperbolic fixed point. Then it is formally conjugated to a $g \in \mathbb{C}_0[[z_1, \dots, z_n]]^n$ such that dg_O is in Jordan normal form, and g has only resonant monomials.*

Definition 5.13. The formal series g is called a *Poincaré-Dulac normal form* of f.

The problem with Poincaré-Dulac normal forms is that they are not unique. In particular, one may wonder whether it could be possible to have such a normal form including *finitely many* resonant monomials only (as happened, for instance, in Proposition 3.10).

This is indeed the case (see, e.g., Reich [Re1]) when f belongs to the Poincaré domain, that is when df_O is invertible and O is either attracting or repelling. As far as I know, the problem of finding canonical formal normal forms when f belongs to the Siegel domain is still open (see [J2] for some partial results).

It should be remarked that, in the hyperbolic case, the problem of formal linearization is equivalent to the problem of smooth linearization. This has been proved by Sternberg [St1–2] and Chaperon [Ch]:

Theorem 5.14 (Sternberg, 1957 [St1–2]; Chaperon, 1986 [Ch]). *Assume we have f, $g \in \mathrm{End}(\mathbb{C}^n, O)$ two holomorphic local dynamical systems, with f locally invertible and with a hyperbolic fixed point at the origin. Then f and g are formally conjugated if and only if they are smoothly locally conjugated. In particular, f is smoothly linearizable if and only if it is formally linearizable. Thus if there are no resonances then f is smoothly linearizable.*

Even without resonances, the holomorphic linearizability is not guaranteed. The easiest positive result is due to Poincaré [Po] who, using majorant series, proved the following

Theorem 5.15 (Poincaré, 1893 [Po]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a locally invertible holomorphic local dynamical system in the Poincaré domain. Then f is holomorphically linearizable if and only if it is formally linearizable. In particular, if there are no resonances then f is holomorphically linearizable.*

Reich [Re2] describes holomorphic normal forms when df_O belongs to the Poincaré domain and there are resonances (see also [ÉV]); Pérez-Marco [P8] discusses the problem of holomorphic linearization in the presence of resonances (see also Raissy [R1]).

When df_O belongs to the Siegel domain, even without resonances, the formal linearization might diverge. To describe the known results, let us introduce the following definition:

Definition 5.16. For $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ and $m \geq 2$ set

$$\Omega_{\lambda_1,\ldots,\lambda_n}(m) = \min\left\{ |\lambda_1^{k_1} \cdots \lambda_n^{k_n} - \lambda_j| \;\middle|\; k_1, \ldots, k_n \in \mathbb{N},\, 2 \leq k_1 + \cdots + k_n \leq m,\, 1 \leq j \leq n \right\} . \tag{36}$$

If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of df_O, we shall write $\Omega_f(m)$ for $\Omega_{\lambda_1,\ldots,\lambda_n}(m)$.

It is clear that $\Omega_f(m) \neq 0$ for all $m \geq 2$ if and only if there are no resonances. It is also not difficult to prove that if f belongs to the Siegel domain then

$$\lim_{m \to +\infty} \Omega_f(m) = 0 ,$$

which is the reason why, even without resonances, the formal linearization might be diverging, exactly as in the one-dimensional case. As far as I know, the best positive and negative results in this setting are due to Brjuno [Brj2–3] (see also [Rü] and [R4]), and are a natural generalization of their one-dimensional counterparts, whose proofs are obtained adapting the proofs of Theorems 4.13 and 4.6 respectively:

Theorem 5.17 (Brjuno, 1971 [Brj2–3]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that f belongs to the Siegel domain, has no resonances, and df_O is diagonalizable. Assume moreover that*

$$\sum_{k=0}^{+\infty} \frac{1}{2^k} \log \frac{1}{\Omega_f(2^{k+1})} < +\infty . \tag{37}$$

Then f is holomorphically linearizable.

Theorem 5.18. *Let $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ be without resonances and such that*

$$\limsup_{m \to +\infty} \frac{1}{m} \log \frac{1}{\Omega_{\lambda_1,\ldots,\lambda_n}(m)} = +\infty .$$

Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$, with $df_O = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, not holomorphically linearizable.

Remark 5.19. These theorems hold even without hyperbolicity assumptions.

Remark 5.20. It should be remarked that, contrarily to the one-dimensional case, it is not yet known whether condition (37) is necessary for the holomorphic linearizability of all holomorphic local dynamical systems with a given linear part belonging to the Siegel domain. However, it is easy to check that if $\lambda \in S^1$ does not satisfy the one-dimensional Brjuno condition then any $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = \big(\lambda z_1 + z_1^2, g(z)\big)$$

is not holomorphically linearizable: indeed, if $\varphi \in \mathrm{End}(\mathbb{C}^n, O)$ is a holomorphic linearization of f, then $\psi(\zeta) = \varphi(\zeta, O)$ is a holomorphic linearization of the quadratic polynomial $\lambda z + z^2$, against Theorem 4.10.

Pöschel [Pö] shows how to modify (36) and (37) to get partial linearization results along submanifolds, and Raissy [R1] (see also [Ro1] and [R2]) explains when it is possible to pass from a partial linearization to a complete linearization even in presence of resonances. Another kind of partial linearization results, namely "linearization modulo an ideal", can be found in [Sto]. Russmann [Rü] and Raissy [R4] proved that in Theorem 5.17 one can replace the hypothesis "no resonances" by the hypothesis "formally linearizable", up to define $\Omega_f(m)$ by taking the minimum only over the non-resonant multiindeces. See also and Il'yachenko [I1] for an important result related to Theorem 5.18. Raissy, in [R3], describes a completely different way of proving the convergence of Poincaré-Dulac normal forms, based on torus actions and allowing a detailed study of torsion phenomena. Finally, in [DG] results in the spirit of Theorem 5.17 are discussed without assuming that the differential is diagonalizable.

6 Several complex variables: the parabolic case

A first natural question in the several complex variables parabolic case is whether a result like the Leau-Fatou flower theorem holds, and, if so, in which form. To present what is known on this subject in this section we shall restrict our attention to holomorphic local dynamical systems tangent to the identity; consequences on dynamical systems with a more general parabolic fixed point can be deduced taking a suitable iterate (but see also the end of this section for results valid when the differential at the fixed point is not diagonalizable).

So we are interested in the local dynamics of a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form

$$f(z) = z + P_\nu(z) + P_{\nu+1}(z) + \cdots \in \mathbb{C}_0\{z_1, \ldots, z_n\}^n , \tag{38}$$

where P_ν is the first non-zero term in the homogeneous expansion of f.

Definition 6.1. If $f \in \mathrm{End}(\mathbb{C}^n, O)$ is of the form (38), the number $\nu \geq 2$ is the *order* of f.

The two main ingredients in the statement of the Leau-Fatou flower theorem were the attracting directions and the petals. Let us first describe a several variables analogue of attracting directions.

Definition 6.2. Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be tangent at the identity and of order ν. A *characteristic direction* for f is a non-zero vector $v \in \mathbb{C}^n \setminus \{O\}$ such that $P_\nu(v) = \lambda v$

for some $\lambda \in \mathbb{C}$. If $P_\nu(v) = O$ (that is, $\lambda = 0$) we shall say that v is a *degenerate* characteristic direction; otherwise, (that is, if $\lambda \neq 0$) we shall say that v is *non-degenerate*. We shall say that f is *dicritical* if all directions are characteristic; *non-dicritical* otherwise.

Remark 6.3. It is easy to check that $f \in \mathrm{End}(\mathbb{C}^n, O)$ of the form (38) is dicritical if and only if $P_\nu \equiv \lambda\,\mathrm{id}$, where $\lambda\colon \mathbb{C}^n \to \mathbb{C}$ is a homogeneous polynomial of degree $\nu - 1$. In particular, generic germs tangent to the identity are non-dicritical.

Remark 6.4. There is an equivalent definition of characteristic directions that shall be useful later on. The n-uple of ν-homogeneous polynomials P_ν induces a meromorphic self-map of $\mathbb{P}^{n-1}(\mathbb{C})$, still denoted by P_ν. Then, under the canonical projection $\mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ non-degenerate characteristic directions correspond exactly to fixed points of P_ν, and degenerate characteristic directions correspond exactly to indeterminacy points of P_ν. In generic cases, there is only a finite number of characteristic directions, and using Bezout's theorem it is easy to prove (see, e.g., [AT1]) that this number, counting according to a suitable multiplicity, is given by $(\nu^n - 1)/(\nu - 1)$.

Remark 6.5. The characteristic directions are *complex* directions; in particular, it is easy to check that f and f^{-1} have the same characteristic directions. Later on we shall see how to associate to (most) characteristic directions $\nu - 1$ petals, each one in some sense centered about a *real* attracting direction corresponding to the same complex characteristic direction.

The notion of characteristic directions has a dynamical origin.

Definition 6.6. We shall say that an orbit $\{f^k(z_0)\}$ converges to the origin *tangentially* to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ if $f^k(z_0) \to O$ in $\mathbb{C}^n$ and $[f^k(z_0)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$, where $[\cdot]\colon \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ denotes the canonical projection.

Then

Proposition 6.7. *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic dynamical system tangent to the identity. If there is an orbit of f converging to the origin tangentially to a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, then v is a characteristic direction of f.*

Sketch of proof. ([Ha2]) For simplicity let us assume $\nu = 2$; a similar argument works for $\nu > 2$.

If v is a degenerate characteristic direction, there is nothing to prove. If not, up to a linear change of coordinates we can assume $[v] = [1 : v']$ and write

$$\begin{cases} f_1(z) = z_1 + p_2^1(z_1, z') + p_3^1(z_1, z') + \cdots\,, \\ f'(z) = z' + p_2'(z_1, z') + p_3'(z_1, z') + \cdots\,, \end{cases}$$

where $z' = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$, $f = (f_1, f')$, $P_j = (p_j^1, p_j')$ and so on, with $p_2^1(1, v') \neq 0$. Making the substitution

$$\begin{cases} w_1 = z_1 , \\ z' = w' z_1 , \end{cases} \tag{39}$$

which is a change of variable off the hyperplane $z_1 = 0$, the map f becomes

$$\begin{cases} \tilde{f}_1(w) = w_1 + p_2^1(1,w')w_1^2 + p_3^1(1,w')w_1^3 + \cdots , \\ \tilde{f}'(w) = w' + r(w')w_1 + O(w_1^2) , \end{cases} \tag{40}$$

where $r(w')$ is a polynomial such that $r(v') = O$ if and only if $[1 : v']$ is a characteristic direction of f with $p_2^1(1,v') \neq 0$.

Now, the hypothesis is that there exists an orbit $\{f^k(z_0)\}$ converging to the origin and such that $[f^k(z_0)] \to [v]$. Writing $\tilde{f}^k(w_0) = \big(w_1^k, (w')^k\big)$, this implies that $w_1^k \to 0$ and $(w')^k \to v'$. Then, arguing as in the proof of (8), it is not difficult to prove that

$$\lim_{k\to+\infty} \frac{1}{k w_1^k} = -p_2^1(1,v') ,$$

and then that $(w')^{k+1} - (w')^k$ is of order $r(v')/k$. This implies $r(v') = O$, as claimed, because otherwise the telescopic series

$$\sum_k \big((w')^{k+1} - (w')^k\big)$$

would not be convergent. □

Remark 6.8. There are examples of germs $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity with orbits converging to the origin without being tangent to any direction: for instance

$$f(z,w) = \big(z + \alpha z w, w + \beta w^2 + o(w^2)\big)$$

with α, $\beta \in \mathbb{C}^*$, $\alpha \neq \beta$ and $\mathrm{Re}(\alpha/\beta) = 1$ (see [Riv1] and [AT3]).

The several variables analogue of a petal is given by the notion of parabolic curve.

Definition 6.9. A *parabolic curve* for $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity is an injective holomorphic map $\varphi \colon \Delta \to \mathbb{C}^n \setminus \{O\}$ satisfying the following properties:

(a) Δ is a simply connected domain in $\mathbb{C}$ with $0 \in \partial\Delta$;
(b) φ is continuous at the origin, and $\varphi(0) = O$;
(c) $\varphi(\Delta)$ is f-invariant, and $(f|_{\varphi(\Delta)})^k \to O$ uniformly on compact subsets as $k \to +\infty$.

Furthermore, if $[\varphi(\zeta)] \to [v]$ in $\mathbb{P}^{n-1}(\mathbb{C})$ as $\zeta \to 0$ in Δ, we shall say that the parabolic curve φ is *tangent* to the direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$.

Then the first main generalization of the Leau-Fatou flower theorem to several complex variables is due to Écalle and Hakim (see also [W]):

Theorem 6.10 (Écalle, 1985 [É4]; Hakim, 1998 [Ha2]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Then* *for any non-degenerate characteristic direction* $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ *there exist (at least)* $\nu - 1$ *parabolic curves for* f *tangent to* $[v]$.

Sketch of proof. Écalle proof is based on his theory of resurgence of divergent series; we shall describe here the ideas behind Hakim's proof, which depends on more standard arguments.

For the sake of simplicity, let us assume $n = 2$; without loss of generality we can also assume $[v] = [1:0]$. Then after a linear change of variables and a transformation of the kind (39) we are reduced to prove the existence of a parabolic curve at the origin for a map of the form

$$\begin{cases} f_1(z) = z_1 - z_1^\nu + O(z_1^{\nu+1}, z_1^\nu z_2) , \\ f_2(z) = z_2\bigl(1 - \lambda z_1^{\nu-1} + O(z_1^\nu, z_1^{\nu-1} z_2)\bigr) + z_1^\nu \psi(z) , \end{cases}$$

where ψ is holomorphic with $\psi(O) = 0$, and $\lambda \in \mathbb{C}$. Given $\delta > 0$, set $D_{\delta,\nu} = \{\zeta \in \mathbb{C} \mid |\zeta^{\nu-1} - \delta| < \delta\}$; this open set has $\nu - 1$ connected components, all of them satisfying condition (a) on the domain of a parabolic curve. Furthermore, if u is a holomorphic function defined on one of these connected components, of the form $u(\zeta) = \zeta^2 u_o(\zeta)$ for some bounded holomorphic function u_o, and such that

$$u\bigl(f_1\bigl(\zeta, u(\zeta)\bigr)\bigr) = f_2\bigl(\zeta, u(\zeta)\bigr) , \tag{41}$$

then it is not difficult to verify that $\varphi(\zeta) = \bigl(\zeta, u(\zeta)\bigr)$ is a parabolic curve for f tangent to $[v]$.

So we are reduced to finding a solution of (41) in each connected component of $D_{\delta,\nu}$, with δ small enough. For any holomorphic $u = \zeta^2 u_o$ defined in such a connected component, let $f_u(\zeta) = f_1\bigl(\zeta, u(\zeta)\bigr)$, put

$$H(z) = z_2 - \frac{z_1^\lambda}{f_1(z)^\lambda} f_2(z) ,$$

and define the operator T by setting

$$(Tu)(\zeta) = \zeta^\lambda \sum_{k=0}^{\infty} \frac{H\bigl(f_u^k(\zeta), u\bigl(f_u^k(\zeta)\bigr)\bigr)}{f_u^k(\zeta)^\lambda} .$$

Then, if $\delta > 0$ is small enough, it is possible to prove that T is well-defined, that u is a fixed point of T if and only if it satisfies (41), and that T is a contraction of a closed convex set of a suitable complex Banach space — and thus it has a fixed point. In this way if $\delta > 0$ is small enough we get a unique solution of (41) for each connected component of $D_{\delta,\nu}$, and hence $\nu - 1$ parabolic curves tangent to $[v]$. □

Definition 6.11. A set of $\nu - 1$ parabolic curves obtained in this way is a *Fatou flower* for f tangent to $[v]$.

Remark 6.12. When there is a one-dimensional f-invariant complex submanifold passing through the origin tangent to a characteristic direction $[v]$, the previous theorem is just a consequence of the usual one-dimensional theory. But it turns out that in most cases such an f-invariant complex submanifold does not exist: see [Ha2] for a concrete example, and [É4] for a general discussion.

We can also have f-invariant complex submanifolds of dimension strictly greater than one attracted by the origin.

Definition 6.13. Given a holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity of order $\nu \geq 2$, and a non-degenerate characteristic direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$, the eigenvalues $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ of the linear operator

$$\frac{1}{\nu - 1}\left(d(P_\nu)_{[v]} - \mathrm{id}\right) : T_{[v]}\mathbb{P}^{n-1}(\mathbb{C}) \to T_{[v]}\mathbb{P}^{n-1}(\mathbb{C})$$

are the *directors* of $[v]$.

Then, using a more elaborate version of her proof of Theorem 6.10, Hakim has been able to prove the following:

Theorem 6.14 (Hakim, 1997 [Ha3]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, with directors $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$. Furthermore, assume that $\mathrm{Re}\,\alpha_1, \ldots, \mathrm{Re}\,\alpha_d > 0$ and $\mathrm{Re}\,\alpha_{d+1}, \ldots, \mathrm{Re}\,\alpha_{n-1} \leq 0$ for a suitable $d \geq 0$. Then:*

(i) *There exists an f-invariant $(d+1)$-dimensional complex submanifold M of $\mathbb{C}^n$, with the origin in its boundary, such that the orbit of every point of M converges to the origin tangentially to $[v]$;*
(ii) *$f|_M$ is holomorphically conjugated to the translation $\tau(w_0, w_1, \ldots, w_d) = (w_0 + 1, w_1, \ldots, w_d)$ defined on a suitable right half-space in $\mathbb{C}^{d+1}$.*

Remark 6.15. In particular, if all the directors of $[v]$ have positive real part, there is an open domain attracted by the origin. However, the condition given by Theorem 6.14 is not necessary for the existence of such an open domain; see Rivi [Riv1] for an easy example, or Ushiki [Us], Vivas [V] and [AT3] for more elaborate examples.

In his monumental work [É4] Écalle has given a complete set of formal invariants for holomorphic local dynamical systems tangent to the identity with at least one non-degenerate characteristic direction. For instance, he has proved the following

Theorem 6.16 (Écalle, 1985 [É4]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system tangent to the identity of order $\nu \geq 2$. Assume that*

(a) *f has exactly $(\nu^n - 1)/(\nu - 1)$ distinct non-degenerate characteristic directions and no degenerate characteristic directions;*
(b) *the directors of any non-degenerate characteristic direction are irrational and mutually independent over $\mathbb{Z}$.*

Let $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ be a non-degenerate characteristic direction, and denote by $\alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ its directors. Then there exist a unique $\rho \in \mathbb{C}$ and unique (up to dilations) formal series $R_1, \ldots, R_n \in \mathbb{C}[\![z_1, \ldots, z_n]\!]$, where each R_j contains only monomial of total degree at least $\nu+1$ and of partial degree in z_j at most $\nu-2$, such that f is formally conjugated to the time-1 map of the formal vector field

$$X = \frac{1}{(\nu-1)(1+\rho z_n^{\nu-1})}\left\{[-z_n^{\nu}+R_n(z)]\frac{\partial}{\partial z_n}+\sum_{j=1}^{n-1}[-\alpha_j z_n^{\nu-1}z_j+R_j(z)]\frac{\partial}{\partial z_j}\right\}.$$

Other approaches to the formal classification, at least in dimension 2, are described in [BM] and in [AT2].

Using his theory of resurgence, and always assuming the existence of at least one non-degenerate characteristic direction, Écalle has also provided a set of holomorphic invariants for holomorphic local dynamical systems tangent to the identity, in terms of differential operators with formal power series as coefficients. Moreover, if the directors of all non-degenerate characteristic directions are irrational and satisfy a suitable diophantine condition, then these invariants become a complete set of invariants. See [É5] for a description of his results, and [É4] for the details.

Now, all these results beg the question: what happens when there are no non-degenerate characteristic directions? For instance, this is the case for

$$\begin{cases} f_1(z) = z_1 + bz_1z_2 + z_2^2\,, \\ f_2(z) = z_2 - b^2z_1z_2 - bz_2^2 + z_1^3\,, \end{cases}$$

for any $b \in \mathbb{C}^*$, and it is easy to build similar examples of any order. At present, the theory in this case is satisfactorily developed for $n = 2$ only. In particular, in [A2] is proved the following

Theorem 6.17 (Abate, 2001 [A2]). *Every holomorphic local dynamical system $f \in$ $\mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity, with an isolated fixed point, admits at least one Fatou flower tangent to some direction.*

Remark 6.18. Bracci and Suwa have proved a version of Theorem 6.17 for $f \in \mathrm{End}(M, p)$ where M is a *singular* variety with not too bad a singularity at p; see [BrS] for details.

Let us describe the main ideas in the proof of Theorem 6.17, because they provide some insight on the dynamical structure of holomorphic local dynamical systems tangent to the identity, and on how to deal with it. A shorter proof, deriving this theorem directly from Camacho-Sad theorem [CS] on the existence of separatrices for holomorphic vector fields in $\mathbb{C}^2$, is presented in [BCL] (see also [D2]); however, such an approach provides fewer informations on the dynamical and geometrical structures of local dynamical systems tangent to the identity.

The first idea is to exploit in a systematic way the transformation (39), following a procedure borrowed from algebraic geometry.

Definition 6.19. If p is a point in a complex manifold M, there is a canonical way (see, e.g., [GH] or [A1]) to build a complex manifold $\tilde{M}$, called the *blow-up* of M at p, provided with a holomorphic projection $\pi\colon \tilde{M} \to M$, so that $E = \pi^{-1}(p)$, the *exceptional divisor* of the blow-up, is canonically biholomorphic to $\mathbb{P}(T_pM)$, and $\pi|_{\tilde{M}\setminus E}\colon \tilde{M} \setminus E \to M \setminus \{p\}$ is a biholomorphism. In suitable local coordinates, the map π is exactly given by (39). Furthermore, if $f \in \mathrm{End}(M,p)$ is tangent to the identity, there is a unique way to lift f to a map $\tilde{f} \in \mathrm{End}(\tilde{M},E)$ such that $\pi \circ \tilde{f} = f \circ \pi$, where $\mathrm{End}(\tilde{M},E)$ is the set of holomorphic maps defined in a neighbourhood of E with values in $\tilde{M}$ and which are the identity on E.

In particular, the characteristic directions of f become points in the domain of the lifted map $\tilde{f}$; and we shall see that this approach allows to determine which characteristic directions are dynamically meaningful.

The blow-up procedure reduces the study of the dynamics of local holomorphic dynamical systems tangent to the identity to the study of the dynamics of germs $f \in \mathrm{End}(M,E)$, where M is a complex n-dimensional manifold, and $E \subset M$ is a compact smooth complex hypersurface pointwise fixed by f. In [A2], [BrT] and [ABT1] we discovered a rich geometrical structure associated to this situation.

Let $f \in \mathrm{End}(M,E)$ and take $p \in E$. Then for every $h \in \mathscr{O}_{M,p}$ (where $\mathscr{O}_M$ is the structure sheaf of M) the germ $h \circ f$ is well-defined, and we have $h \circ f - h \in \mathscr{I}_{E,p}$, where $\mathscr{I}_E$ is the ideal sheaf of E.

Definition 6.20. The *f-order of vanishing* at p of $h \in \mathscr{O}_{M,p}$ is

$$\nu_f(h;p) = \max\{\mu \in \mathbb{N} \mid h \circ f - h \in \mathscr{I}_{E,p}^{\mu}\} ,$$

and the *order of contact* ν_f of f with E is

$$\nu_f = \min\{\nu_f(h;p) \mid h \in \mathscr{O}_{M,p}\} .$$

In [ABT1] we proved that ν_f does not depend on p, and that

$$\nu_f = \min_{j=1,\dots,n} \nu_f(z_j;p) ,$$

where (U,z) is any local chart centered at $p \in E$ and $z = (z_1,\dots,z_n)$. In particular, if the local chart (U,z) is such that $E \cap U = \{z_1 = 0\}$ (and we shall say that the local chart is *adapted* to E) then setting $f_j = z_j \circ f$ we can write

$$f_j(z) = z_j + (z_1)^{\nu_f} g_j(z) , \tag{42}$$

where at least one among $g_1,\dots,g_n$ does not vanish identically on $U \cap E$.

Definition 6.21. A map $f \in \mathrm{End}(M,E)$ is *tangential* to E if

$$\min\bigl\{\nu_f(h;p) \mid h \in \mathscr{I}_{E,p}\bigr\} > \nu_f$$

for some (and hence any) point $p \in E$.

Choosing a local chart (U,z) adapted to E so that we can express the coordinates of f in the form (42), it turns out that f is tangential if and only if $g^1|_{U\cap E}\equiv 0$.

The g^j's in (42) depend in general on the chosen chart; however, in [ABT1] we proved that setting

$$\mathscr{X}_f=\sum_{j=1}^n g_j\frac{\partial}{\partial z_j}\otimes(dz_1)^{\otimes\nu_f} \tag{43}$$

then $\mathscr{X}_f|_{U\cap E}$ defines a *global* section X_f of the bundle $TM|_E\otimes(N_E^*)^{\otimes\nu_f}$, where N_E^* is the conormal bundle of E into M. The bundle $TM|_E\otimes(N_E^*)^{\otimes\nu_f}$ is canonically isomorphic to the bundle $\mathrm{Hom}(N_E^{\otimes\nu_f},TM|_E)$. Therefore the section X_f induces a morphism still denoted by $X_f\colon N_E^{\otimes\nu_f}\to TM|_E$.

Definition 6.22. The morphism $X_f\colon N_E^{\otimes\nu_f}\to TM|_E$ just defined is the *canonical morphism* associated to $f\in\mathrm{End}(M,E)$.

Remark 6.23. It is easy to check that f is tangential if and only if the image of X_f is contained in TE. Furthermore, if f is the lifting of a germ $f_o\in\mathrm{End}(\mathbb{C}^n,O)$ tangent to the identity, then (see [ABT1]) f is tangential if and only if f_o is non-dicritical; so in this case tangentiality is generic. Finally, in [A2] we used the term “non degenerate” instead of ”tangential”.

Definition 6.24. Assume that $f\in\mathrm{End}(M,E)$ is tangential. We shall say that $p\in E$ is a *singular point for* f if X_f vanishes at p.

By definition, $p\in E$ is a singular point for f if and only if

$$g_1(p)=\cdots=g_n(p)=0$$

for any local chart adapted to E; so singular points are generically isolated.

In the tangential case, only singular points are dynamically meaningful. Indeed, a not too difficult computation (see [A2], [AT1] and [ABT1]) yields the following

Proposition 6.25. *Let $f\in\mathrm{End}(M,E)$ be tangential, and take $p\in E$. If p is not singular, then the stable set of the germ of f at p coincides with E.*

The notion of singular point allows us to identify the dynamically meaningful characteristic directions.

Definition 6.26. Let M be the blow-up of $\mathbb{C}^n$ at the origin, and f the lift of a non-dicritical holomorphic local dynamical system $f_o\in\mathrm{End}(\mathbb{C}^n,O)$ tangent to the identity. We shall say that $[v]\in\mathbb{P}^{n-1}(\mathbb{C})=E$ is a *singular direction* of f_o if it is a singular point of $\tilde{f}$.

It turns out that non-degenerate characteristic directions are always singular (but the converse does not necessarily hold), and that singular directions are always characteristic (but the converse does not necessarily hold). Furthermore, the singular directions are the dynamically interesting characteristic directions, because Propositions 6.7 and 6.25 imply that if f_o has a non-trivial orbit converging to the origin tangentially to $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ then $[v]$ must be a singular direction.

The important feature of the blow-up procedure is that, even though the underlying manifold becomes more complex, the lifted maps become simpler. Indeed, using an argument similar to one (described, for instance, in [MM]) used in the study of singular holomorphic foliations of 2-dimensional complex manifolds, in [A2] it is shown that after a finite number of blow-ups our original holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^2, O)$ tangent to the identity can be lifted to a map $\tilde{f}$ whose singular points (are finitely many and) satisfy one of the following conditions:

(o) they are dicritical; or,
($\star$) in suitable local coordinates centered at the singular point we can write

$$\begin{cases} \tilde{f}_1(z) = z_1 + \ell(z)\big(\lambda_1 z_1 + O(\|z\|^2)\big), \\ \tilde{f}_2(z) = z_2 + \ell(z)\big(\lambda_2 z_2 + O(\|z\|^2)\big), \end{cases}$$

with

- ($\star_1$) λ_1, $\lambda_2 \neq 0$ and λ_1/λ_2, $\lambda_2/\lambda_1 \notin \mathbb{N}$, or
- ($\star_2$) $\lambda_1 \neq 0$, $\lambda_2 = 0$.

Remark 6.27. This "reduction of the singularities" statement holds only in dimension 2, and it is not clear how to replace it in higher dimensions.

It is not too difficult to prove that Theorem 6.10 can be applied to both dicritical and ($\star_1$) singularities; therefore as soon as this blow-up procedure produces such a singularity, we get a Fatou flower for the original dynamical system f.

So to end the proof of Theorem 6.17 it suffices to prove that any such blow-up procedure *must* produce at least one dicritical or ($\star_1$) singularity. To get such a result, we need another ingredient.

Let again $f \in \mathrm{End}(M, E)$, where E is a smooth complex hypersurface in a complex manifold M, and assume that f is tangential; let E^o denote the complement in E of the singular points of f. For simplicity of exposition we shall assume $\dim M = 2$ and $\dim E = 1$; but this part of the argument works for any $n \geq 2$ (even when E has singularities, and it can also be adapted to non-tangential germs).

Since $\dim E = 1 = \mathrm{rk} N_E$, the restriction of the canonical morphism X_f to $N_{E^o}^{\otimes \nu_f}$ is an isomorphism between $N_{E^o}^{\otimes \nu_f}$ and TE^o. Then in [ABT1] we showed that it is possible to define a holomorphic connection ∇ on N_{E^o} by setting

$$\nabla_u(s) = \pi([\mathscr{X}_f(\tilde{u}), \tilde{s}]|_S) \;, \tag{44}$$

where: s is a local section of N_{E^o}; $u \in TE^o$; $\pi \colon TM|_{E^o} \to N_{E^o}$ is the canonical projection; $\tilde{s}$ is any local section of $TM|_{E^o}$ such that $\pi(\tilde{s}|_{S^o}) = s$; $\tilde{u}$ is any local section of $TM^{\otimes \nu_f}$ such that $X_f\big(\pi(\tilde{u}|_{E^o})\big) = u$; and $\mathscr{X}_f$ is locally given by (43). In a chart (U, z) adapted to E, a local generator of N_{E^o} is $\partial_1 = \pi(\partial/\partial z_1)$, a local generator of $N_{E^o}^{\otimes \nu_f}$ is $\partial_1^{\otimes \nu_f} = \partial_1 \otimes \cdots \otimes \partial_1$, and we have

$$X_f(\partial_1^{\otimes \nu_f}) = g_2|_{U\cap E}\frac{\partial}{\partial z_2}\ ;$$

therefore

$$\nabla_{\partial/\partial z_2}\partial_1 = -\left.\frac{1}{g_2}\frac{\partial g_1}{\partial z_1}\right|_{U\cap E}\partial_1\ .$$

In particular, ∇ is a meromorphic connection on N_E, with poles in the singular points of f.

Definition 6.28. The *index* $\iota_p(f,E)$ of f along E at a point $p\in E$ is by definition the opposite of the residue at p of the connection ∇ divided by ν_f:

$$\iota_p(f,E) = -\frac{1}{\nu_f}\mathrm{Res}_p(\nabla)\ .$$

In particular, $\iota_p(f,E)=0$ if p is not a singular point of f.

Remark 6.29. If $[v]$ is a non-degenerate characteristic direction of a non-dicritical $f_o\in\mathrm{End}(\mathbb{C}^2,O)$ with non-zero director $\alpha\in\mathbb{C}^*$, then it is not difficult to check that

$$\iota_{[v]}(f,E) = \frac{1}{\alpha}\ ,$$

where f is the lift of f_o to the blow-up of the origin.

Then in [A2] we proved the following *index theorem* (see [Br1], [BrT] and [ABT1, 2] for multidimensional versions and far reaching generalizations):

Theorem 6.30 (Abate, Bracci, Tovena, 2004 [A2], [ABT1]). *Let E be a compact Riemann surface inside a 2-dimensional complex manifold M. Take $f\in\mathrm{End}(M,E)$, and assume that f is tangential to E. Then*

$$\sum_{q\in E}\iota_q(f,E) = c_1(N_E)\ ,$$

where $c_1(N_E)$ is the first Chern class of the normal bundle N_E of E in M.

Now, a combinatorial argument (inspired by Camacho and Sad [CS]; see also [Ca] and [T]) shows that if we have $f\in\mathrm{End}(\mathbb{C}^2,O)$ tangent to the identity with an isolated fixed point, and such that applying the reduction of singularities to the lifted map $\tilde{f}$ starting from a singular direction $[v]\in\mathbb{P}^1(\mathbb{C})=E$ we end up only with $(\star_2)$ singularities, then the index of $\tilde{f}$ at $[v]$ along E must be a non-negative rational number. But the first Chern class of N_E is -1; so there must be at least one singular directions whose index is not a non-negative rational number. Therefore the reduction of singularities must yield at least one dicritical or $(\star_1)$ singularity, and hence a Fatou flower for our map f, completing the proof of Theorem 6.17.

Actually, we have proved the following slightly more precise result:

Theorem 6.31 (Abate, 2001 [A2]). *Let E be a (not necessarily compact) Riemann surface inside a 2-dimensional complex manifold M, and take $f\in\mathrm{End}(M,E)$*

tangential to E. Let $p \in E$ be a singular point of f such that $\iota_p(f,E) \notin \mathbb{Q}^+$. Then there exist a Fatou flower for f at p. In particular, if $f_o \in \mathrm{End}(\mathbb{C}^2,O)$ is a non-dicritical holomorphic local dynamical system tangent to the identity with an isolated fixed point at the origin, and $[v] \in \mathbb{P}^1(\mathbb{C})$ is a singular direction such that $\iota_{[v]}\big(f,\mathbb{P}^1(\mathbb{C})\big) \notin \mathbb{Q}^+$, where f is the lift of f_o to the blow-up of the origin, then f_o has a Fatou flower tangent to $[v]$.

Remark 6.32. This latter statement has been generalized in two ways. Degli Innocenti [D1] has proved that we can allow E to be singular at p (but irreducible; in the reducible case one has to impose conditions on the indeces of f along all irreducible components of E passing through p). Molino [Mo], on the other hand, has proved that the statement still holds assuming only $\iota_p(f,E) \neq 0$, at least for f of order 2 (and E smooth at p); it is natural to conjecture that this should be true for f of any order.

As already remarked, the reduction of singularities via blow-ups seem to work only in dimension 2. This leaves open the problem of the validity of something like Theorem 6.17 in dimension $n \geq 3$; see [AT1] and [Ro2] for some partial results.

As far as I know, it is widely open, even in dimension 2, the problem of describing the stable set of a holomorphic local dynamical system tangent to the identity, as well as the more general problem of the topological classification of such dynamical systems. Some results in the case of a dicritical singularity are presented in [BM]; for non-dicritical singularities a promising approach in dimension 2 is described in [AT3].

Let $f \in \mathrm{End}(M,E)$, where E is a smooth Riemann surface in a 2-dimensional complex manifold M, and assume that f is tangential; let E^o denote the complement in E of the singular points of f. The connection ∇ on N_{E^o} described above induces a connection (still denoted by ∇) on $N_{E^o}^{\otimes \nu_f}$.

Definition 6.33. In this setting, a *geodesic* is a curve $\sigma\colon I \to E^o$ such that

$$\nabla_{\sigma'} X_f^{-1}(\sigma') \equiv O\,.$$

It turns out that σ is a geodesic if and only if the curve $X_f^{-1}(\sigma')$ is an integral curves of a global holomorphic vector field G on the total space of $N_{E^o}^{\otimes \nu_f}$; furthermore, G extends holomorphically to the total space of $N_E^{\otimes \nu_f}$.

Now, assume that M is the blow-up of the origin in $\mathbb{C}^2$, and E is the exceptional divisor. Then there exists a canonical ν_f-to-1 holomorphic covering map $\chi_{\nu_f}\colon \mathbb{C}^2 \setminus \{O\} \to N_E^{\otimes \nu_f} \setminus E$. Moreover, if f is the lift of a non-dicritical $f_o \in \mathrm{End}(\mathbb{C}^2,O)$ of the form (38) with $P_\nu = (P_\nu^1, P_\nu^2)$, then $\nu_f = \nu - 1$ and it turns out that χ_{ν_f} maps integral curves of the homogeneous vector field

$$Q_\nu = P_\nu^1 \frac{\partial}{\partial z_1} + P_\nu^2 \frac{\partial}{\partial z_2}$$

onto integral curves of G.

Now, the time-1 map of Q_ν is tangent to the identity and of the form (38); the previous argument shows that to study its dynamics it suffices to study the dynamics of such a geodesic vector field G. This is done in [AT3], where we get: a complete formal classification of homogeneous vector fields nearby their characteristic directions; a complete holomorphic classification nearby generic characteristic directions (including, but not limited to, non-degenerate characteristic directions); and powerful tools for the study of the geodesic flow for meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, yielding deep results on the global dynamics of real integral curves of homogeneous vector fields. For instance, we get the following Poincaré-Bendixson theorem:

Theorem 6.34 (Abate, Tovena, 2009 [AT3]). *] Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$ of degree $\nu+1\geq 2$, and let $\gamma\colon [0,\varepsilon_0)\to\mathbb{C}^2$ be a recurrent maximal integral curve of Q. Then γ is periodic or $[\gamma]\colon [0,\varepsilon_0)\to\mathbb{P}^1(\mathbb{C})$ intersects itself infinitely many times, where $[\cdot]\colon \mathbb{C}^2\to\mathbb{P}^1(\mathbb{C})$ is the canonical projection.*

Furthermore, we also get examples of maps tangent to the identity with small cycles, that is periodic orbits of arbitrarily high period accumulating at the origin; and a complete description of the local dynamics in a full neighbourhood of the origin for a large class of holomorphic local dynamical systems tangent to the identity.

Since results like Theorem 3.8 seems to suggest that generic holomorphic local dynamical systems tangent to the identity might be topologically conjugated to the time-1 map of a homogeneous vector field, this approach might eventually lead to a complete topological description of the dynamics for generic holomorphic local dynamical systems tangent to the identity in dimension 2.

We end this section with a couple of words on holomorphic local dynamical systems with a parabolic fixed point where the differential is not diagonalizable. Particular examples are studied in detail in [CD], [A4] and [GS]. In [A1] it is described a canonical procedure for lifting an $f\in\mathrm{End}(\mathbb{C}^n,O)$ whose differential at the origin is not diagonalizable to a map defined in a suitable iterated blow-up of the origin (obtained blowing-up not only points but more general submanifolds) with a canonical fixed point where the differential is diagonalizable. Using this procedure it is for instance possible to prove the following

Theorem 6.35 ([A2]). *Let $f\in\mathrm{End}(\mathbb{C}^2,O)$ be a holomorphic local dynamical system with $df_O=J_2$, the canonical Jordan matrix associated to the eigenvalue 1, and assume that the origin is an isolated fixed point. Then f admits at least one parabolic curve tangent to $[1:0]$ at the origin.*

7 Several complex variables: other cases

Outside the hyperbolic and parabolic cases, there are not that many general results. Theorems 5.17 and 5.18 apply to the elliptic case too, but, as already remarked, it is not known whether the Brjuno condition is still necessary for holomorphic linearizability. However, another result in the spirit of Theorem 5.18 is the following:

Theorem 7.1 (Yoccoz, 1995 [Y2]). *Let $A \in GL(n,\mathbb{C})$ be an invertible matrix such that its eigenvalues have no resonances and such that its Jordan normal form contains a non-trivial block associated to an eigenvalue of modulus one. Then there exists $f \in \mathrm{End}(\mathbb{C}^n, O)$ with $df_O = A$ which is not holomorphically linearizable.*

A case that has received some attention is the so-called semi-attractive case

Definition 7.2. A holomorphic local dynamical system $f \in \mathrm{End}(\mathbb{C}^n, O)$ is said *semi-attractive* if the eigenvalues of df_O are either equal to 1 or have modulus strictly less than 1.

The dynamics of semi-attractive dynamical systems has been studied by Fatou [F4], Nishimura [Ni], Ueda [U1–2], Hakim [H1] and Rivi [Riv1–2]. Their results more or less say that the eigenvalue 1 yields the existence of a "parabolic manifold" M — in the sense of Theorem 6.14.(ii) — of a suitable dimension, while the eigenvalues with modulus less than one ensure, roughly speaking, that the orbits of points in the normal bundle of M close enough to M are attracted to it. For instance, Rivi proved the following

Theorem 7.3 (Rivi, 1999 [Riv1–2]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system. Assume that 1 is an eigenvalue of (algebraic and geometric) multiplicity $q \geq 1$ of df_O, and that the other eigenvalues of df_O have modulus less than 1. Then:*

(i) *We can choose local coordinates $(z,w) \in \mathbb{C}^q \times \mathbb{C}^{n-q}$ such that f expressed in these coordinates becomes*

$$\begin{cases} f_1(z,w) &= A(w)z + P_{2,w}(z) + P_{3,w}(z) + \cdots, \\ f_2(z,w) &= G(w) + B(z,w)z, \end{cases}$$

where: $A(w)$ is a $q \times q$ matrix with entries holomorphic in w and $A(O) = I_q$; the $P_{j,w}$ are q-uples of homogeneous polynomials in z of degree j whose coefficients are holomorphic in w; G is a holomorphic self-map of $\mathbb{C}^{n-q}$ such that $G(O) = O$ and the eigenvalues of dG_O are the eigenvalues of df_O with modulus strictly less than 1; and $B(z,w)$ is an $(n-q) \times q$ matrix of holomorphic functions vanishing at the origin. In particular, $f_1(z,O)$ is tangent to the identity.

(ii) *If $v \in \mathbb{C}^q \subset \mathbb{C}^m$ is a non-degenerate characteristic direction for $f_1(z,O)$, and the latter map has order ν, then there exist $\nu - 1$ disjoint f-invariant $(n-q+1)$-dimensional complex submanifolds M_j of $\mathbb{C}^n$, with the origin in their boundary, such that the orbit of every point of M_j converges to the origin tangentially to $\mathbb{C}v \oplus E$, where $E \subset \mathbb{C}^n$ is the subspace generated by the generalized eigenspaces associated to the eigenvalues of df_O with modulus less than one.*

Rivi also has results in the spirit of Theorem 6.14, and results when the algebraic and geometric multiplicities of the eigenvalue 1 differ; see [Riv1, 2] for details.

Building on work done by Canille Martins [CM] in dimension 2, and using Theorem 3.8 and general results on normally hyperbolic dynamical systems due to Palis and Takens [PT], Di Giuseppe has obtained the topological classification when the eigenvalue 1 has multiplicity 1 and the other eigenvalues are not resonant:

Theorem 7.4 (Di Giuseppe, 2004 [Di]). *Let $f \in \mathrm{End}(\mathbb{C}^n, O)$ be a holomorphic local dynamical system such that df_O has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$, where λ_1 is a primitive q-root of unity, and $|\lambda_j| \neq 0,\ 1$ for $j = 2, \ldots, n$. Assume moreover that $\lambda_2^{r_2} \cdots \lambda_n^{r^n} \neq 1$ for all multi-indeces $(r_2, \ldots, r_n) \in \mathbb{N}^{n-1}$ such that $r_2 + \cdots + r_n \geq 2$. Then f is topologically locally conjugated either to df_O or to the map*

$$z \mapsto (\lambda_1 z_1 + z_1^{kq+1}, \lambda_2 z_2, \ldots, \lambda_n z_n)$$

for a suitable $k \in \mathbb{N}^*$.

We end this survey by recalling results by Bracci and Molino, and by Rong. Assume that $f \in \mathrm{End}(\mathbb{C}^2, O)$ is a holomorphic local dynamical system such that the eigenvalues of df_O are 1 and $e^{2\pi i\theta} \neq 1$. If $e^{2\pi i\theta}$ satisfies the Brjuno condition, Pöschel [Pö] proved the existence of a 1-dimensional f-invariant holomorphic disk containing the origin where f is conjugated to the irrational rotation of angle θ. On the other hand, Bracci and Molino give sufficient conditions (depending on f but not on $e^{2\pi i\theta}$, expressed in terms of two new holomorphic invariants, and satisfied by generic maps) for the existence of parabolic curves tangent to the eigenspace of the eigenvalue 1; see [BrM] for details, and [Ro3] for generalizations to $n \geq 3$.

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