# INDEX THEOREMS FOR HOLOMORPHIC MAPS AND FOLIATIONS

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Abstract. We describe a general construction providing index theorems localizing the Chern classes of the normal bundle of a subvariety inside a complex manifold. As particular instances of our construction we recover both Lehmann-Suwa's generalization of the classical Camacho-Sad index theorem for holomorphic foliations and our index theorem for holomorphic maps with positive dimensional fixed point set. Furthermore, we also obtain generalizations of recent index theorems of Camacho-Movasati-Sad and Camacho-Lehmann for holomorphic foliations transversal to a subvariety.

## 1. Introduction

In 1982, C. Camacho and P. Sad [11] proved the existence of separatrices for singular holomorphic foliations in dimension 2. One of the main tools in their proof was the following index theorem:

**Theorem 1.1** (Camacho-Sad). *Let $S$ be a compact Riemann surface embedded in a smooth complex surface $M$. Let $\mathcal{F}$ be a one-dimensional singular holomorphic foliation defined in a neighbourhood of $S$ and such that $S$ is a leaf of $\mathcal{F}$, that is such that $\mathcal{F}$ is tangent to $S$. Then it is possible to associate to any singular point $q \in S$ of $\mathcal{F}$ a complex number $\iota_q(\mathcal{F}, S) \in \mathbb{C}$, the* index *of the foliation along $S$ at $q$, depending only on the local behavior of $\mathcal{F}$ near $q$, such that*

$$\sum_{q \in \mathrm{Sing}(\mathcal{F})} \iota_q(\mathcal{F}, S) = \int_S c_1(N_S),$$

*where $c_1(N_S)$ is the first Chern class of the normal bundle $N_S$ of $S$ in $M$.*

The index $\iota_q(\mathcal{F}, S)$ can be explicitely computed using a local vector field generating $\mathcal{F}$ in a neighbourhood of $q$ (see Example 7.23).

Thus Theorem 1.1 gives a quantitative connection between the way $S$ sits in $M$ (the integral of the first Chern class of $N_S$ is equal to the self-intersection number $S \cdot S$ of $S$) and the behavior of singular foliations tangent to $S$.

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Due to its importance (see, e.g., [22] and [9] for applications), in the next twenty years this theorem has been generalized in several ways; see, e.g., Lins Neto [22], Suwa [27], Lehmann [19], Lehmann-Suwa [20], [21], and references therein (see also [6], [7], where also the ambient space $M$ is allowed to be singular). In particular, using Čech-de Rham cohomology, Lehmann and Suwa (see [28] for a systematic exposition) proved what we are going to consider as a model index theorem: if the possibly singular holomorphic foliation $\mathcal{F}$ of dimension $\ell$ is tangent to a possibly singular (but not too wild: see Definition 6.3 and Example 6.4) subvariety $S$ of dimension $d$ in the ambient manifold $M$, then the Chern classes of the normal bundle $N_S$ of degree higher than $d-\ell$ can be localized at the singularities — that is, obtained as sum of local residues depending only on the behavior of $S$ and $\mathcal{F}$ nearby the singularities (of $S$ and of $\mathcal{F}$ in $S$). We explicitely remark that one of the main ingredients in their proof is the construction of a partial holomorphic connection on the the normal bundle $N_S$ outside a suitable analytic subset of $S$ (where "partial" here means that we are differentiating only along some tangent directions, the ones contained in $\mathcal{F}$).

The results obtained in those papers are apparently strictly inside the theory of holomorphic foliations; the arguments used needed the existence of the foliation $\mathcal{F}$ in a neighbourhood of the subvariety $S$, and the tangency of $\mathcal{F}$ to $S$. In the last five years, however, a number of results have appeared strongly suggesting that these might be unnecessary limitations: the tangency of $\mathcal{F}$ to $S$ might be replaced by hypotheses on the embedding of $S$ into the ambient space $M$ ([10], [12], [13]), and, perhaps more strikingly, the foliation can be replaced by a holomorphic self-map of the ambient manifold fixing pointwise the subvariety $S$ ([1], [8], [2]). Furthermore, there was the tantalizing fact that the statements of all these new index theorems were clearly similar, and yet they all needed slightly different proofs; none of them was consequence of any of the others.

The main goal of this paper is to show how it is possible to recover all these index theorems (and a couple of new ones) using a universal construction having a priori nothing to do with either foliations or self-maps. More precisely, we shall reduce the proof of such an index theorem to the construction of an $\mathcal{O}_{S^o}$-morphism $\psi\colon \mathcal{F} \to \mathcal{A}$ satisfying a splitting condition (see Theorem 5.9.(ii) for the exact condition), where: $S^o$ is the complement in $S$ of the singular points of $S$ and of the singular points of the object (foliation or self-map) we are interested in; $\mathcal{F}$ is the sheaf of germs of holomorphic sections of a suitable subbundle of the tangent bundle $TS^o$; and $\mathcal{A}$ is an universal $\mathcal{O}_{S^o}$-locally free sheaf depending only on the embedding of $S^o$ into the ambient space $M$. The details of the construction of $\psi$ will of course depend on the particular situation we are dealing with (foliation or self-map, tangential or transversal); but as soon as such a $\psi$ exists then an index theorem analogous to Lehmann-Suwa's model one follows (see Theorem 6.8). We shall also show (in Sections 7 and 8) how to construct such a $\psi$ in several cases, and we shall be able to recover all the index theorems of this kind known up to now (for $M$ smooth), together with two new ones: the first one (Theorem 7.21) on foliations transverse to $S$ generalizes both Camacho-Movasati-Sad's ([13]) and Camacho-Lehmann's ([10], [12]) results, while the second one (Theorem 8.10) extends the results of [2]. We also explicitely compute the index at isolated singularities in a simple but important case using a Grothendieck residue; see Remark 6.10 and Example 7.23.

Very briefly, the reason why the existence of a morphism $\psi\colon \mathcal{F} \to \mathcal{A}$ implies an index theorem can be explained as follows. The sheaf $\mathcal{A}$ comes provided with an important additional structure: an universal holomorphic $\theta_1$-connection (see Sections 4 and 5 for precise definitions) on the normal bundle $N_{S^o}$ of $S^o$ in $M$. Then, following ideas due to Atiyah [4], we shall be able to prove that the existence of such a morphism $\psi$ is equivalent to the existence of a partial holomorphic connection on $\mathcal{N}_{S^o}$ along $\mathcal{F}$. Having this, an argument essentially due to Baum and Bott ([5]; see Theorem 6.1) yields the vanishing on $S^o$ of suitable Chern classes of $N_{S^o}$; and then a general cohomological argument (developed by Lehmann and Suwa [19], [28]) allows one to infer from this vanishing the localization at the singularities of the corresponding Chern classes — that is, an index theorem.

Let us finally describe the plan of the paper. In Sections 2 and 3 we collect a number of definitions and properties concerning infinitesimal neighbourhoods of subvarieties that we shall need in the rest of the paper, and we describe the conditions we shall impose on the embedding of the subvariety into the ambient manifold to deal with the transversal cases (we also refer to [3] for more details on these conditions). In Sections 4 and 5 we introduce the sheaf $\mathcal{A}$ and its additional structures, while in Section 6 we prove the vanishing Theorem 6.1 and our general index Theorem 6.8. In Section 7 we show how to build the morphism $\psi$ for holomorphic foliations (and, in particular, we get the new Theorem 7.21), and finally in Section 8 we do the same for holomorphic self-maps (and, in particular, we get the new Theorem 8.10).

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## 2. Splitting submanifolds

Let us begin by recalling a few general facts on sequences of sheaves. We say that an exact sequence of sheaves (of abelian groups, rings, modules…)

$$O \longrightarrow \mathcal{R} \stackrel{\iota}{\longrightarrow} \mathcal{S} \stackrel{p}{\longrightarrow} \mathcal{T} \longrightarrow O$$

on a variety $S$ *splits* if there is a morphism $\sigma\colon \mathcal{T} \to \mathcal{S}$ of sheaves (of abelian groups, rings, modules…) such that $p \circ \sigma = \mathrm{id}$. Any such morphism is called a *splitting morphism.* A morphism of sheaves of abelian groups $\tau\colon \mathcal{S} \to \mathcal{R}$ such that $\tau \circ \iota = \mathrm{id}$ is called a *left splitting morphism.*

The following facts are well-known, and easy to prove:

**Lemma 2.1.** *Let*

$$O \longrightarrow \mathcal{R} \xrightarrow{\iota} \mathcal{S} \xrightarrow{p} \mathcal{T} \longrightarrow O \tag{1}$$

*be an exact sequence of sheaves of abelian groups over a variety $S$. Then:*

(i) *the sequence* (1) *splits if and only if there exists a left splitting morphism $\tau\colon \mathcal{S} \to \mathcal{R}$;*

(ii) *if* (1) *splits, for any splitting morphism $\sigma\colon \mathcal{T} \to \mathcal{S}$ there exists a unique left splitting morphism $\tau\colon \mathcal{S} \to \mathcal{R}$ such that $\tau \circ \sigma = O$ and*

$$\iota \circ \tau + \sigma \circ p = \mathrm{id};$$

(iii) *if* (1) *splits, then there is a* 1*-to-*1 *correspondance between splitting morphisms and elements of $H^0\big(S, \mathrm{Hom}(\mathcal{T}, \mathcal{R})\big)$. More precisely, if $\sigma_0\colon \mathcal{T} \to \mathcal{S}$ is a splitting morphism, all the other splitting morphisms are of the form $\sigma_0 - \iota \circ \varphi$ with $\varphi \in H^0\big(S, \mathrm{Hom}(\mathcal{T}, \mathcal{R})\big)$, while if $\tau_0\colon \mathcal{T} \to \mathcal{S}$ is a left splitting morphism, all others left splitting morphisms are of the form $\tau_0 + \varphi \circ p$ with $\varphi \in H^0\big(S, \mathrm{Hom}(\mathcal{T}, \mathcal{R})\big)$.*

Following Grothendieck and Atiyah, we can give a useful cohomological characterization of splitting for sequences of locally free $\mathcal{O}_S$-modules. Let

$$O \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow O \tag{2}$$

be an exact sequence of sheaves of locally free $\mathcal{O}_S$-modules. Applying the functor $\mathrm{Hom}(\mathcal{E}'', \cdot)$ to this sequence we get the exact sequence

$$O \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}') \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}) \longrightarrow \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'') \longrightarrow O. \tag{3}$$

Let $\delta\colon H^0\big(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}'')\big) \to H^1\big(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}')\big)$ be the connecting homomorphism in the long exact cohomology sequence of (3). Then we can associate to the exact sequence (2) the cohomology class

$$\delta(\mathrm{id}_{\mathcal{E}''}) \in H^1\big(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}')\big).$$

This procedure gives a 1-to-1 correspondance between the group $H^1\big(S, \mathrm{Hom}(\mathcal{E}'', \mathcal{E}')\big)$ and isomorphism classes of exact sequences of locally free $\mathcal{O}_S$-modules starting with $\mathcal{E}'$ and ending with $\mathcal{E}''$. Indeed, we have (see [4], Proposition 1.2):

**Proposition 2.2.** *Let $S$ be a complex manifold. Then two exact sequences of locally free $\mathcal{O}_S$-modules are isomorphic if and only if they correspond to the same cohomology class. In particular, an exact sequence* (2) *of locally free $\mathcal{O}_S$-modules splits if and only if it corresponds to the zero cohomology class.*

Let us now introduce the sheaves (and sequences of sheaves) we are interested in. Let $M$ be a complex manifold of dimension $n$, and let $S$ be a reduced, globally irreducible subvariety of $M$ of codimension $m \geq 1$. We denote: by $\mathcal{O}_M$ the sheaf of germs of holomorphic functions on $M$; by $\mathcal{I}_S$ the subsheaf of $\mathcal{O}_M$ of germs vanishing on $S$; and by $\mathcal{O}_S$ the quotient sheaf $\mathcal{O}_M/\mathcal{I}_S$ of germs of holomorphic functions on $S$. Furthermore, let $\mathcal{T}_M$ denote the sheaf of germs of holomorphic sections of the holomorphic tangent bundle $TM$

of $M$, and $\Omega_M$ the sheaf of germs of holomorphic 1-forms on $M$. Finally, we shall denote by $\mathcal{T}_{M,S}$ the sheaf of germs of holomorphic sections along $S$ of the restriction $TM|_S$ of $TM$ to $S$, and by $\Omega_{M,S}$ the sheaf of germs of holomorphic sections along $S$ of $T^*M|_S$. It is easy to check that $\mathcal{T}_{M,S} = \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$ and $\Omega_{M,S} = \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_S$.

For $k \geq 1$ we shall denote by $f \mapsto [f]_k$ the canonical projection of $\mathcal{O}_M$ onto $\mathcal{O}_M/\mathcal{I}_S^k$. The *cotangent sheaf* $\Omega_S$ of $S$ is defined by

$$\Omega_S = \Omega_{M,S}\big/d_2(\mathcal{I}_S/\mathcal{I}_S^2),$$

where $d_2\colon \mathcal{O}_M/\mathcal{I}_S^2 \to \Omega_{M,S}$ is given by $d_2[f]_2 = df \otimes [1]_1$. In particular, we have the *conormal sequence* of sheaves of $\mathcal{O}_S$-modules associated to $S$:

$$\mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow O.$$

Applying the functor $\mathrm{Hom}_{\mathcal{O}_S}(\cdot, \mathcal{O}_S)$ to the conormal sequence we get the *normal sequence* of sheaves of $\mathcal{O}_S$-modules associated to $S$:

$$O \longrightarrow \mathcal{T}_S \hookrightarrow \mathcal{T}_{M,S} \xrightarrow{p_2} \mathcal{N}_S,$$

where $\mathcal{T}_S = \mathrm{Hom}_{\mathcal{O}_S}(\Omega_S, \mathcal{O}_S)$ is the *tangent sheaf* of $S$, $p_2$ is the morphism dual to $d_2$, and $\mathcal{N}_S = \mathrm{Hom}_{\mathcal{O}_S}(\mathcal{I}_S/\mathcal{I}_S^2, \mathcal{O}_S)$ is the *normal sheaf* of $S$.

As mentioned in the introduction, to get index theorems in the transversal case we shall need hypotheses on the embedding of $S$ into $M$. The first such hypothesis is:

**Definition 2.3.** Let $S$ be a reduced, globally irreducible subvariety of a complex manifold $M$. We say that $S$ *splits* into $M$ if there exists a morphism of sheaves of $\mathcal{O}_S$-modules $\sigma\colon \Omega_S \to \Omega_{M,S}$ such that $p \circ \sigma = \mathrm{id}$, where $p\colon \Omega_{M,S} \to \Omega_S$ is the canonical projection.

*Remark* 2.4. It is not difficult to prove that if $S$ splits into $M$ then it is necessarily non-singular, and the morphism $d_2\colon \mathcal{I}_S/\mathcal{I}_S^2 \to \Omega_{M,S}$ is injective. In particular, when $S$ splits into $M$ the sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^2 \xrightarrow{d_2} \Omega_{M,S} \xrightarrow{p} \Omega_S \longrightarrow O \tag{4}$$

is a splitting exact sequence of locally free $\mathcal{O}_S$-modules, and we also have a left splitting morphism $\tau\colon \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$.

We shall now describe several equivalent characterizations of splitting subvarieties. In doing so, we shall introduce notations and terminologies that shall be useful in the rest of the paper.

**Definition 2.5.** Let $S$ be a reduced, globally irreducible subvariety of a complex manifold $M$. For any $k \geq 1$ let $\theta_k\colon \mathcal{O}_M/\mathcal{I}_S^{k+1} \to \mathcal{O}_M/\mathcal{I}_S$ be the canonical projection given by $\theta_k([f]_{k+1}) = [f]_1$. The *k-th infinitesimal neighbourhood* of $S$ in $M$ is the ringed space $S(k) = (S, \mathcal{O}_M/\mathcal{I}_S^{k+1})$ together with the canonical inclusion of ringed spaces $\iota_k\colon S =$

$S(0) \to S(k)$ given by $\iota_k = (\mathrm{id}_S, \theta_k)$. We also put $\mathcal{O}_{S(k)} = \mathcal{O}_M/\mathcal{I}_S^{k+1}$. A *k-th order lifting* is a splitting morphism $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(k)}$ for the exact sequence of sheaves of rings

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^{k+1} \hookrightarrow \mathcal{O}_{S(k)} \xrightarrow{\theta_k} \mathcal{O}_S \longrightarrow O.$$

**Definition 2.6.** Let $\mathcal{O}$ and $\mathcal{R}$ be sheaves of rings, $\theta\colon \mathcal{R} \to \mathcal{O}$ a morphism of sheaves of rings, and $\mathcal{M}$ a sheaf of $\mathcal{O}$-modules. A *$\theta$-derivation* of $\mathcal{R}$ in $\mathcal{M}$ is a morphism of sheaves of abelian groups $D\colon \mathcal{R} \to \mathcal{M}$ such that

$$D(r_1 r_2) = \theta(r_1) \cdot D(r_2) + \theta(r_2) \cdot D(r_1)$$

for any $r_1$, $r_2 \in \mathcal{R}$. In other words, $D$ is a derivation with respect to the $\mathcal{R}$-module structure induced via restriction of scalars by $\theta$.

We can now give a first list of conditions equivalent to splitting (see [17], p. 373, [25], Lemma 1.1, and [15], Proposition 16.12 for proofs):

**Proposition 2.7.** *Let $S$ be a reduced, globally irreducible subvariety of a complex manifold $M$. Then there is a* 1*-to-*1 *correspondance among the following classes of morphisms:*

(a) *morphisms $\sigma\colon \Omega_S \to \Omega_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p \circ \sigma = \mathrm{id}$;*
(b) *morphisms $\tau\colon \Omega_{M,S} \to \mathcal{I}_S/\mathcal{I}_S^2$ of sheaves of $\mathcal{O}_S$-modules such that $\tau \circ d_2 = \mathrm{id}$;*
(c) *$\theta_1$-derivations $\tilde{\rho}\colon \mathcal{O}_{S(1)} \to \mathcal{I}_S/\mathcal{I}_S^2$ such that $\tilde{\rho} \circ i_1 = \mathrm{id}$, where $i_1\colon \mathcal{I}_S/\mathcal{I}_S^2 \hookrightarrow \mathcal{O}_{S(1)}$ is the canonical inclusion;*
(d) *morphisms $\rho\colon \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^2$ of sheaves of rings such that $\theta_1 \circ \rho = \mathrm{id}$.*

*In particular, $S$ splits into $M$ if and only if it has a first order lifting. Finally, if any (and hence all) of the classes* (a)–(d) *is not empty, then it is in* 1*-to-*1 *correspondance with the following classes of morphisms:*

(e) *morphisms $\tau^*\colon \mathcal{N}_S \to \mathcal{T}_{M,S}$ of sheaves of $\mathcal{O}_S$-modules such that $p_2 \circ \tau^* = \mathrm{id}$;*
(f) *morphisms $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$ of sheaves of $\mathcal{O}_S$-modules such that $\iota \circ \sigma^* = \mathrm{id}$, where $\iota\colon \mathcal{T}_S \to \mathcal{T}_{M,S}$ is the canonical inclusion.*

We have already noticed (Remark 2.4) that a splitting subvariety is necessarily nonsingular; therefore we can use differential geometric techniques to get another couple of characterizations of splitting submanifolds.

**Definition 2.8.** Let $S$ be a (not necessarily closed) complex submanifold of codimension $m \geq 1$ in a complex manifold $M$ of dimension $n \geq 2$, and let $(U_\alpha, z_\alpha)$ be a chart of $M$. We shall systematically write $z_\alpha = (z_\alpha^1, \ldots, z_\alpha^n) = (z'_\alpha, z''_\alpha)$, with $z'_\alpha = (z_\alpha^1, \ldots, z_\alpha^m)$ and $z''_\alpha = (z_\alpha^{m+1}, \ldots, z_\alpha^n)$. We shall say that $(U_\alpha, z_\alpha)$ is *adapted* to $S$ if either $U_\alpha \cap S = \varnothing$ or

$$U_\alpha \cap S = \{z_\alpha^1 = \cdots = z_\alpha^m = 0\}.$$

In particular, if $(U_\alpha, z_\alpha)$ is adapted to $S$ then $\{z_\alpha^1, \ldots, z_\alpha^m\}$ is a set of generators of $\mathcal{I}_{S,x}$ for all $x \in U_\alpha \cap S$. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ of $M$ is *adapted* to $S$ if all its charts are; then $\mathfrak{U}_S = \{(U_\alpha \cap S, z''_\alpha) \mid U_\alpha \cap S \neq \varnothing\}$ is an atlas for $S$. The *normal bundle* $N_S$ of $S$ in $M$ is the quotient bundle $TM|_S/TS$; its dual is the *conormal bundle* $N_S^*$. If $(U_\alpha, z_\alpha)$ is a chart adapted to $S$, for $r = 1, \ldots, m$ we shall denote by $\partial_{r,\alpha}$ the projection of $\partial/\partial z^r_\alpha|_{U_\alpha \cap S}$ in $N_S$, and by $\omega^r_\alpha$ the local section of $N^*_S$ induced by $dz^r_\alpha|_{U_\alpha \cap S}$. Then $\{\partial_{1,\alpha}, \ldots, \partial_{m,\alpha}\}$ and $\{\omega^1_\alpha, \ldots, \omega^m_\alpha\}$ are local frames over $U_\alpha \cap S$ for $N_S$ and $N^*_S$ respectively, dual to each other.

*Remark* 2.9. From now on, every chart and atlas we consider on $M$ will be adapted to $S$. We shall use Einstein convention on the sum over repeated indices. Indices like $j$, $h$, $k$ will run from 1 to $n$; indices like $r$, $s$, $t$, $u$, $v$ will run from 1 to $m$; and indices like $p$, $q$ will run from $m + 1$ to $n$.

*Remark* 2.10. If $(U_\alpha, z_\alpha)$ and $(U_\beta, z_\beta)$ are two adapted charts with $U_\alpha \cap U_\beta \cap S \neq \varnothing$, then it is easy to check that

$$\left.\frac{\partial z^r_\beta}{\partial z^p_\alpha}\right|_S \equiv O$$

for all $r = 1, \ldots, m$ and $p = m + 1, \ldots, n$.

Then computing the cohomology class associated to the conormal sequence (4) and recalling Proposition 2.2 we get:

**Proposition 2.11.** *Let $S$ be a complex submanifold of codimension $m$ of a complex manifold $M$, and let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an adapted atlas. Then the cohomology class $\mathfrak{s} \in H^1\big(S, \mathrm{Hom}(\Omega_S, \mathcal{N}^*_S)\big)$ associated to the conormal exact sequence of $S$ is represented by the 1-cocycle $\{\mathfrak{s}_{\beta\alpha}\} \in H^1\big(\mathfrak{U}_S, \mathrm{Hom}(\Omega_S, \mathcal{N}^*_S)\big)$ given by*

$$\mathfrak{s}_{\beta\alpha} = -\left.\frac{\partial z^r_\beta}{\partial z^s_\alpha}\frac{\partial z^p_\alpha}{\partial z^r_\beta}\right|_S \omega^s_\alpha \otimes \frac{\partial}{\partial z^p_\alpha} \in H^0(U_\alpha \cap U_\beta \cap S, \mathcal{N}^*_S \otimes \mathcal{T}_S).$$

*In particular, $S$ splits into $M$ if and only if $\mathfrak{s} = O$.*

We can rewrite this characterization in a more useful form using the notion of splitting atlas, originally introduced in [2].

**Definition 2.12.** Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an adapted atlas for a complex submanifold $S$ of codimension $m \geq 1$ of a complex $n$-dimensional manifold $M$. We say that $\mathfrak{U}$ is a *splitting atlas* if

$$\left.\frac{\partial z^p_\beta}{\partial z^r_\alpha}\right|_S \equiv O$$

for all $r = 1, \ldots, m$, $p = m + 1, \ldots, n$ and indices $\alpha$, $\beta$ so that $U_\alpha \cap U_\beta \cap S \neq \varnothing$.

**Definition 2.13.** Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an atlas adapted to $S$. If $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ is a first order lifting for $S$, we say $\mathfrak{U}$ is *adapted* to $\rho$ if

$$\rho([f]_1) = [f]_2 - \left[\frac{\partial f}{\partial z^r_\alpha} z^r_\alpha\right]_2 \tag{5}$$

for all $f \in \mathcal{O}(U_\alpha)$ and all indices $\alpha$ such that $U_\alpha \cap S \neq \varnothing$.

*Remark* 2.14. In [3] it is shown that $\mathfrak{U}$ is adapted to $\rho$ if and only if

$$\rho(g)(z_\alpha) = g_\alpha(O', z''_\alpha) + \mathcal{I}_S^2$$

for all $g \in \mathcal{O}(U_\alpha \cap S)$ and all indices $\alpha$ such that $U_\alpha \cap S \neq \varnothing$, where we are assuming (without loss of generality) that $z_\alpha \in U_\alpha$ implies $(O', z''_\alpha) \in U_\alpha \cap S$.

**Proposition 2.15.** *Let $S$ be a complex submanifold of codimension $m \geq 1$ of an $n$-dimensional complex manifold $M$. Then:*

(i) *$S$ splits into $M$ if and only if there exists a splitting atlas for $S$ in $M$;*
(ii) *an atlas adapted to $S$ is splitting if and only if it is adapted to a first order lifting;*
(iii) *if $S$ splits into $M$, then for any first order lifting there exists an atlas adapted to it.*

*Proof.* (i) By Propositions 2.2 and 2.11, the existence of a splitting atlas clearly implies that $S$ splits into $M$. Conversely, assume that $S$ splits into $M$. Then by Propositions 2.2 and 2.11 we can find an adapted atlas $\mathfrak{U}$ and a 0-cochain $\mathfrak{c} = \{\mathfrak{c}_\alpha\} \in H^0(\mathfrak{U}_S, \mathcal{N}_S^* \otimes \mathcal{T}_S)$ such that $\mathfrak{s}_{\beta\alpha} = \mathfrak{c}_\beta - \mathfrak{c}_\alpha$ on $U_\alpha \cap\ U_\beta \cap S$, that is

$$- \left. \frac{\partial z^r_\alpha}{\partial z^s_\beta} \frac{\partial z^q_\beta}{\partial z^r_\alpha} \right|_S = (c_\alpha)^p_r \left. \frac{\partial z^r_\alpha}{\partial z^s_\beta} \frac{\partial z^q_\beta}{\partial z^p_\alpha} \right|_S - (c_\beta)^q_s \tag{6}$$

on $U_\alpha \cap U_\beta \cap S$ for all $s = 1, \ldots, m$ and $p = m+1, \ldots, n$, where we have written

$$\mathfrak{c}_\alpha = (c_\alpha)^p_s \, \omega^s_\alpha \otimes \frac{\partial}{\partial z^p_\alpha}.$$

Then using (6) it is easy to check that the coordinates

$$\begin{cases} \hat{z}^s_\alpha = z^s_\alpha, \\ \hat{z}^p_\alpha = z^p_\alpha - (c_\alpha)^p_r z^r_\alpha, \end{cases} \tag{7}$$

restricted to suitable open subsets $\hat{U}_\alpha \subseteq U_\alpha$, give a splitting atlas $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$.

(ii) Let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be an atlas adapted to $S$. Setting

$$\rho_\alpha([f]_1) = [f]_2 - \left[ \frac{\partial f}{\partial z^r_\alpha} z^r_\alpha \right]_2$$

we define local first order liftings $\rho_\alpha \colon \mathcal{O}_S|_{U_\alpha \cap S} \to \mathcal{O}_{S(1)}|_{U_\alpha \cap S}$; we claim that in this way we get a global first order lifting $\rho$ if and only if $\mathfrak{U}$ is a splitting atlas (necessarily adapted to $\rho$). But indeed

$$\begin{aligned}\rho_\beta([f]_1)-\rho_\alpha([f]_1) &= \left[\frac{\partial f}{\partial z^r_\alpha}z^r_\alpha\right]_2-\left[\frac{\partial f}{\partial z^s_\beta}z^s_\beta\right]_2\\ &= \left[\frac{\partial f}{\partial z^s_\beta}\left(\frac{\partial z^s_\beta}{\partial z^r_\alpha}z^r_\alpha-z^s_\beta\right)\right]_2+\left[\frac{\partial f}{\partial z^p_\beta}\frac{\partial z^p_\beta}{\partial z^r_\alpha}z^r_\alpha\right]_2\\ &= \left[\frac{\partial f}{\partial z^p_\beta}\frac{\partial z^p_\beta}{\partial z^r_\alpha}z^r_\alpha\right]_2,\end{aligned}$$

and so $\rho_\alpha\equiv\rho_\beta$ on $U_\alpha\cap U_\beta\cap S$ for all $\alpha$ and $\beta$ if and only if $\mathfrak{U}$ is a splitting atlas.

(iii) Let $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ be a splitting atlas, adapted to the first order lifting $\rho_0$, and choose another first order lifting $\rho$. Lemma 2.1.(ii) implies that $\rho=\rho_0-\iota\circ\varphi$ for a suitable $\varphi\in H^0\big(S,\mathrm{Hom}(\mathcal{O}_S,\mathcal{I}_S/\mathcal{I}_S^2)\big)$, and it is easy to check that $\varphi$ is a derivation. Therefore there is a 0-cocycle $\mathfrak{c}=\{\mathfrak{c}_\alpha\}\in H^0(\mathfrak{U}_S,\mathcal{N}_S^*\otimes\mathcal{T}_S)$ such that

$$\rho(g)=\rho_0(g)-(c_\alpha)^p_r\left[\frac{\partial g}{\partial z^p_\alpha}z^r_\alpha\right]_2$$

for all $g\in\mathcal{O}_S|_{U_\alpha\cap S}$. Then defining new coordinates as in (7) we still get a splitting atlas, easily seen adapted to $\rho$. □

*Remark* 2.16. Given a first order lifting $\rho\colon\mathcal{O}_S\to\mathcal{O}_{S(1)}$, let $\tilde\rho$, $\tau^*$ and $\sigma^*$ be the morphisms associated to $\rho$ by Proposition 2.7, and let $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ be an atlas adapted to $S$. Then it is easy to check that the following assertions are equivalent:

(i) $\mathfrak{U}$ is adapted to $\rho$;

(ii) for every $(U_\alpha,z_\alpha)\in\mathfrak{U}$ with $U_\alpha\cap S\neq\varnothing$ and every $f\in\mathcal{O}_M|_{U_\alpha}$ one has

$$\tilde\rho([f]_2)=\left[\frac{\partial f}{\partial z^r_\alpha}z^r_\alpha\right]_2;$$

(iii) for every $(U_\alpha,z_\alpha)\in\mathfrak{U}$ with $U_\alpha\cap S\neq\varnothing$ and every $r=1,\dots,m$ one has

$$\tau^*(\partial_{r,\alpha})=\frac{\partial}{\partial z^r_\alpha};$$

(iv) for every $(U_\alpha,z_\alpha)\in\mathfrak{U}$ with $U_\alpha\cap S\neq\varnothing$ and every $f^j_\alpha\,\partial/\partial z^j_\alpha\in\mathcal{T}_{M,S}|_{U_\alpha\cap S}$ one has

$$\sigma^*\left(f^j_\alpha\frac{\partial}{\partial z^j_\alpha}\right)=f^p_\alpha\frac{\partial}{\partial z^p_\alpha}.$$

*Remark* 2.17. It is also possible to prove (see, e.g., [3]) that a submanifold $S$ splits into $M$ if and only if its first infinitesimal neighbourhood $S(1)$ in $M$ is isomorphic to the first infinitesimal neighbourhood $S_N(1)$ of the zero section in $N_S$.

We end this section with a list of examples of splitting submanifolds.

*Example* 2.18. A local holomorphic retract always splits into the ambient manifold (and thus it is necessarily non-singular). In particular, the zero section of a vector bundle always splits, as well as any slice $S \times \{x\}$ in a product $M = S \times X$ (with both $S$ and $X$ non-singular, of course).

*Example* 2.19. If $S$ is a Stein submanifold of a complex manifold $M$ (e.g., if $S$ is an open Riemann surface), then $S$ splits into $M$. Indeed, we have $H^1(S, \mathcal{T}_S \otimes \mathcal{N}_S^*) = (O)$ by Cartan's Theorem B, and the assertion follows from Proposition 2.11. In particular, if $S$ is a *singular* curve in $M$ then the non-singular part of $S$ always splits into $M$.

*Example* 2.20. Let $\tilde{M}$ be the blow-up of a point in a complex manifold $M$. Then the exceptional divisor splits into $\tilde{M}$: indeed, it is easy to check that the atlas of $\tilde{M}$ induced by the atlas of $M$ is splitting.

*Example* 2.21. A smooth closed irreducible subvariety of $\mathbb{P}^n$ splits into $\mathbb{P}^n$ if and only if it is a linear subspace (see [29], [25], [24]).

*Example* 2.22. Let $S$ be a non-singular, compact, irreducible curve of genus $g$ on a surface $M$. If $S \cdot S < 4 - 4g$ then $S$ splits into $M$. In fact, the Serre duality for Riemann surfaces implies that

$$H^1\big(S, \mathrm{Hom}(\Omega_S, \mathcal{N}_S^*)\big) \cong H^0(S, \Omega_S \otimes \Omega_S \otimes \mathcal{N}_S),$$

and the latter group vanishes because the line bundle $T^*S \otimes T^*S \otimes N_S$ has negative degree by assumption. The bound $S \cdot S < 4 - 4g$ is sharp: for instance, a non-singular compact projective plane conic $S$ has genus $g = 0$ and self-intersection $S \cdot S = 4$, but it does not split in the projective plane (see Example 2.21).

## 3. Comfortably embedded submanifolds

In this section we introduce two other, more stringent, conditions on the embedding of $S$ into $M$. From the definitions it will be clear that they are just the beginning of two infinite lists of progressively more restrictive conditions; we shall however limit ourselves to present only the properties we need in this paper, referring to [3] for a more complete discussion.

We start with a natural generalization of splitting:

**Definition 3.1.** Let $S$ be a submanifold of a complex manifold $M$. We say that $S$ 2-*splits* into $M$ if there exists a second order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(3)}$ or, in other words, if the exact sequence

$$O \longrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \hookrightarrow \mathcal{O}_{S(2)} \xrightarrow{\theta_2} \mathcal{O}_S \longrightarrow O$$

splits as sequence of sheaves of rings. Notice that a 2-splitting submanifold is necessarily splitting.

We have the following analogue of Proposition 2.15:

**Proposition 3.2.** *Let $S$ be an $m$-codimensional submanifold of a complex manifold $M$ of dimension $n$. Then $S$ 2-splits into $M$ if and only if there is an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ such that*

$$\frac{\partial z_\beta^p}{\partial z_\alpha^r} \in \mathcal{I}_S^2 \tag{8}$$

*for all $r = 1, \dots, m$, $p = m+1, \dots, n$ and indices $\alpha$, $\beta$ so that $U_\alpha \cap U_\beta \cap S \neq \varnothing$.*

*Proof.* Let us first assume that we have an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ and such that (8) holds. Let us then define $\rho_\alpha \colon \mathcal{O}_S|_{U_\alpha} \to \mathcal{O}_{S(3)}|_{U_\alpha}$ by

$$\rho_\alpha([f]_1) = [f]_3 - \left[\frac{\partial f}{\partial z_\alpha^r} z_\alpha^r\right]_3 + \frac{1}{2}\left[\frac{\partial^2 f}{\partial z_\alpha^{r_1} \partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 \tag{9}$$

for all $f \in \mathcal{O}(U_\alpha)$. It is easy to check that the right-hand side depends only on $[f]_1$, and that $\rho_\alpha$ is a ring morphism such that $\theta_2 \circ \rho = \mathrm{id}$. So to prove that $S$ 2-splits into $M$ it suffices to show that $\rho_\alpha$ does not depend on $\alpha$. But indeed, since we have

$$[z_\beta^s]_3 = \left[\frac{\partial z_\beta^s}{\partial z_\alpha^r} z_\alpha^r - \frac{1}{2} \frac{\partial^2 z_\beta^s}{\partial z_\alpha^{r_1} \partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 \tag{10}$$

for all $s = 1, \dots, m$, we find

$$\begin{aligned}
\rho_\beta(\quad & [f]_1) - \rho_\alpha([f]_1) \\
= \quad & \left[\frac{\partial f}{\partial z_\alpha^r} z_\alpha^r\right]_3 - \frac{1}{2}\left[\frac{\partial^2 f}{\partial z_\alpha^{r_1} \partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 - \left[\frac{\partial f}{\partial z_\beta^s} z_\beta^s\right]_3 + \frac{1}{2}\left[\frac{\partial^2 f}{\partial z_\beta^{s_1} \partial z_\beta^{s_2}} z_\beta^{s_1} z_\beta^{s_2}\right]_3 \\
= \quad & \left[\frac{\partial f}{\partial z_\beta^s}\left(\frac{\partial z_\beta^s}{\partial z_\alpha^r} z_\alpha^r - z_\beta^s - \frac{1}{2} \frac{\partial^2 z_\beta^s}{\partial z_\alpha^{r_1} \partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right)\right]_3 \\
& -\frac{1}{2}\left[\frac{\partial^2 f}{\partial z_\beta^{s_1} \partial z_\beta^{s_2}}\left(\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^{s_2}}{\partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2} - z_\beta^{s_1} z_\beta^{s_2}\right)\right]_3 \\
& + \left[\frac{\partial f}{\partial z_\beta^p} \frac{\partial z_\beta^p}{\partial z_\alpha^r} z_\alpha^r\right]_3 - \left[\frac{\partial^2 f}{\partial z_\beta^s \partial z_\beta^p} \frac{\partial z_\beta^s}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^p}{\partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 \\
& -\frac{1}{2}\left[\frac{\partial^2 f}{\partial z_\beta^{p_1} \partial z_\beta^{p_2}} \frac{\partial z_\beta^{p_1}}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^{p_2}}{\partial z_\alpha^{r_2}} z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 \\
= \quad & 0,
\end{aligned} \tag{11}$$

because of (10) and (8).

Conversely, let us assume that we have a second order lifting $\rho \colon \mathcal{O}_S \to \mathcal{O}_{S(3)}$; we must build an atlas adapted to $S$ such that (8) holds.

If $\theta_{2,1}\colon \mathcal{O}_{S(2)} \to \mathcal{O}_{S(1)}$ is the canonical projection, then $\rho_1 = \theta_{2,1} \circ \rho$ is a first order lifting; let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a splitting atlas adapted to $\rho_1$. Define then local second order liftings $\rho_\alpha$ as in (9), and set $\sigma_\alpha = \rho - \rho_\alpha$. Now

$$\theta_{2,1} \circ \sigma_\alpha = \rho_1 - \theta_{2,1} \circ \rho_\alpha \equiv O,$$

because the atlas is adapted to $\rho_1$; therefore the image of $\sigma_\alpha$ is contained in $\mathcal{I}_S^2/\mathcal{I}_S^3$, which is an $\mathcal{O}_S$-module. Furthermore, $\sigma_\alpha$ is a derivation; therefore we can find $(s_\alpha)^p_{r_1 r_2} \in \mathcal{O}(U_\alpha \cap S)$, symmetric in the lower indices, such that

$$\sigma_\alpha = (s_\alpha)^p_{r_1 r_2} [z_\alpha^{r_1} z_\alpha^{r_2}]_3 \otimes \frac{\partial}{\partial z_\alpha^p}.$$

Now, since we are using a splitting atlas, the computations in (11) yield

$$\begin{aligned} \left[\frac{\partial z_\beta^p}{\partial z_\alpha^r} z_\alpha^r\right]_3 \otimes \frac{\partial}{\partial z_\beta^p} &= \rho_\beta - \rho_\alpha = \sigma_\alpha - \sigma_\beta \\ &= \left[\left(\frac{\partial z_\beta^p}{\partial z_\alpha^q}(s_\alpha)^q_{r_1 r_2} - (s_\beta)^p_{s_1 s_2} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\right) z_\alpha^{r_1} z_\alpha^{r_2}\right]_3 \otimes \frac{\partial}{\partial z_\beta^p}. \end{aligned} \tag{12}$$

Furthermore, always because we are using a splitting atlas, we can write $z_\beta^p = \phi_{\beta\alpha}(z''_\alpha) + (h_{\beta\alpha})^p_{r_1 r_2} z_\alpha^{r_1} z_\alpha^{r_2}$ with $(h_{\beta\alpha})^p_{r_1 r_2} \in \mathcal{O}(U_\alpha \cap U_\beta)$ symmetric in the lower indices. Putting this in (12) we get

$$2(h_{\beta\alpha})^p_{r_1 r_2} - \frac{\partial z_\beta^p}{\partial z_\alpha^q}(s_\alpha)^q_{r_1 r_2} + (s_\beta)^p_{s_1 s_2} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \in \mathcal{I}_S$$

and hence

$$\left[\frac{\partial z_\beta^p}{\partial z_\alpha^r} - \frac{\partial z_\beta^p}{\partial z_\alpha^q}(s_\alpha)^q_{r_1 r} z_\alpha^{r_1} + (s_\beta)^p_{s_1 s_2} z_\beta^{s_1} \frac{\partial z_\beta^{s_2}}{\partial z_\alpha^r}\right]_2 = 0. \tag{13}$$

Let us then consider the change of coordinates

$$\begin{cases} \hat{z}_\alpha^r = z_\alpha^r, \\ \hat{z}_\alpha^p = z_\alpha^p + \frac{1}{2}(s_\alpha)^p_{r_1 r_2}(z''_\alpha) z_\alpha^{r_1} z_\alpha^{r_2}, \end{cases}$$

defined in suitable open sets $\hat{U}_\alpha \subseteq U_\alpha$; using (13) it is easy to check that $\{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ is the atlas we are looking for. □

**Definition 3.3.** An atlas adapted to $S$ satisfying (8) will be said 2-*splitting.*

*Remark* 3.4. There is a cohomological characterization of splitting submanifolds which are 2-splitting. Indeed, assume that $S$ splits into $M$, and let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a splitting atlas. If we define local second order liftings $\{\rho_\alpha\}$ as in (9), the computations made in the proof of the previous proposition show that setting $\rho_{\beta\alpha} = \rho_\beta - \rho_\alpha$ the cocycle $\{\rho_{\beta\alpha}\}$ defines a cohomology class

$$\mathfrak{g} \in H^1(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes \mathcal{T}_S),$$

and it is easy to check that $S$ 2-splits if and only if this cohomology class vanishes.

It turns out that for our aims it will be much more useful a different, though related, condition on the embedding of $S$ into $M$.

Let $S$ be a splitting submanifold of codimension $m$ of a complex manifold $M$, and let $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ be a first order lifting. The sheaf $\mathcal{I}_S/\mathcal{I}_S^3$ has a natural structure of $\mathcal{O}_{S(1)}$-module; by restriction of scalars via $\rho$, we get a structure of $\mathcal{O}_S$-module, and it is easy to check that with this structure the sequence

$$O \longrightarrow \mathcal{I}_S^2/\mathcal{I}_S^3 \hookrightarrow \mathcal{I}_S/\mathcal{I}_S^3 \xrightarrow{\theta_{2,1}} \mathcal{I}_S/\mathcal{I}_S^2 \longrightarrow O \tag{14}$$

becomes an exact sequence of locally free $\mathcal{O}_S$-modules. In particular, it is clear that if $\{(U_\alpha, z_\alpha)\}$ is an atlas adapted to $\rho$ then $\{[z^r_\alpha]_3, [z^{r_1}_\alpha z^{r_2}_\alpha]_3\}$ is a free set of local generators of $\mathcal{I}_S/\mathcal{I}_S^3$ over $\mathcal{O}_S$.

*Remark* 3.5. The cohomology class $\mathfrak{h} \in H^1\big(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes \mathcal{N}_S\big)$ associated to the sequence (14) by the procedure described at the beginning of the previous section is represented by the cocycle $\{\mathfrak{h}_{\beta\alpha}\}$ given by

$$\mathfrak{h}_{\beta\alpha} = \frac{1}{2}\left[\frac{\partial z^{s_1}_\beta}{\partial z^{r_1}_\alpha}\frac{\partial z^{s_2}_\beta}{\partial z^{r_2}_\alpha}\frac{\partial^2 z^r_\alpha}{\partial z^{s_1}_\beta \partial z^{s_2}_\beta}\right]_1 [z^{r_1}_\alpha z^{r_2}_\alpha]_3 \otimes \partial_{r,\alpha},$$

where $\{(U_\alpha, z_\alpha)\}$ is a splitting atlas associated to the first order lifting $\rho$.

We are thus led to the following

**Definition 3.6.** Let $S$ be a (not necessarily closed) submanifold of a complex manifold $M$. We say that $S$ is *comfortably embedded* in $M$ if there exists a first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ such that the sequence (14) splits as sequence of $\mathcal{O}_S$-modules. We shall sometimes say that $S$ is comfortably embedded *with respect to* $\rho$.

We can characterize comfortably embedded submanifolds using adapted atlases, recovering in particular the original definition of comfortably embedded submanifolds introduced in [2]:

**Proposition 3.7.** *Let $S$ be an $m$-codimensional submanifold of a complex manifold $M$ of dimension $n$. Then $S$ is comfortably embedded into $M$ if and only if there exists an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ such that*

$$\frac{\partial z^p_\beta}{\partial z^r_\alpha} \in \mathcal{I}_S \qquad \text{and} \qquad \frac{\partial^2 z^r_\beta}{\partial z^{s_1}_\alpha \partial z^{s_2}_\alpha} \in \mathcal{I}_S \tag{15}$$

*for all $r$, $s_1$, $s_2 = 1, \dots, m$, $p = m+1, \dots, n$ and indices $\alpha$, $\beta$ such that $U_\alpha \cap U_\beta \cap S \neq \varnothing$.*

*Proof.* If we have an atlas satisfying (15) then, by Proposition 2.15, $S$ splits into $M$, and $\mathfrak{U}$ is adapted to a first order lifting $\rho$. Furthermore, Proposition 2.2 and Remark 3.5 imply that $S$ is comfortably embedded with respect to $\rho$.

Conversely, assume that $S$ is comfortably embedded with respect to a first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$, let $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ be a splitting atlas adapted to $\rho$, and let $\nu\colon \mathcal{I}_S/\mathcal{I}_S^2 \to$

$\mathcal{I}_S/\mathcal{I}_S^3$ be a splitting morphism for the sequence (14). For every index $\alpha$ such that $U_\alpha \cap S \neq \varnothing$ define $\nu_\alpha \colon \mathcal{I}_S/\mathcal{I}_S^2|_{U_\alpha} \to \mathcal{I}_S/\mathcal{I}_S^3|_{U_\alpha}$ by setting $\nu_\alpha([z_\alpha^r]_2) = [z_\alpha^r]_3$ and then extending by $\mathcal{O}_S$-linearity. Notice that (5) and (10) yield

$$\begin{aligned} \nu_\beta([z_\alpha^r]_2) \quad &- \quad \nu_\alpha([z_\alpha^r]_2) \\ &= \quad \rho\left(\left[\frac{\partial z_\alpha^r}{\partial z_\beta^s}\right]_1\right)[z_\beta^s]_3 - [z_\alpha^r]_3 = -\frac{1}{2}\left[\frac{\partial^2 z_\alpha^r}{\partial z_\beta^{s_1}\partial z_\beta^{s_2}} z_\beta^{s_1} z_\beta^{s_2}\right]_3 . \end{aligned} \tag{16}$$

Now set $\sigma_\alpha = \nu - \nu_\alpha$; since $\theta_{2,1} \circ \sigma_\alpha \equiv O$, it follows that $\mathrm{Im}\sigma_\alpha \subseteq \mathcal{I}_S^2/\mathcal{I}_S^3$. In particular, there are $(c_\alpha)_{r_1 r_2}^r \in \mathcal{O}(U_\alpha \cap S)$, symmetric in the lower indices, such that

$$\sigma_\alpha([z_\alpha^r]_2) = (c_\alpha)_{r_1 r_2}^r [z_\alpha^{r_1} z_\alpha^{r_2}]_3.$$

But $\sigma_\alpha - \sigma_\beta = \nu_\beta - \nu_\alpha$; therefore (16) yields

$$(c_\beta)_{s_1 s_2}^s \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}} \frac{\partial z_\beta^{s_2}}{\partial z_\alpha^{r_2}} - \frac{\partial z_\beta^s}{\partial z_\alpha^r}(c_\alpha)_{r_1 r_2}^r + \frac{1}{2}\frac{\partial^2 z_\beta^s}{\partial z_\alpha^{r_1}\partial z_\alpha^{r_2}} \in \mathcal{I}_S. \tag{17}$$

We can finally define new coordinates $\hat{z}_\alpha$ by setting

$$\begin{cases} \hat{z}_\alpha^r = z_\alpha^r + (c_\alpha)_{r_1 r_2}^r z_\alpha^{r_1} z_\alpha^{r_2}, \\ \hat{z}_\alpha^p = z_\alpha^p, \end{cases}$$

on suitable open subsets $\hat{U}_\alpha \subseteq U_\alpha$. It is easy to check that $\hat{\mathfrak{U}} = \{(\hat{U}_\alpha, \hat{z}_\alpha)\}$ still is a splitting atlas adapted to $\rho$. Moreover,

$$\frac{\partial^2 \hat{z}_\beta^s}{\partial \hat{z}_\alpha^{r_1} \hat{z}_\alpha^{r_2}} = \frac{\partial^2 z_\beta^s}{\partial z_\alpha^{r_1}\partial z_\alpha^{r_2}} + 2\left((c_\beta)_{s_1 s_2}^s \frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\frac{\partial z_\beta^{s_2}}{\partial z_\alpha^{r_2}} - \frac{\partial z_\beta^s}{\partial z_\alpha^r}(c_\alpha)_{r_1 r_2}^r\right),$$

and (17) concludes the proof. □

**Definition 3.8.** An atlas satisfying (15) will be said a *comfortable atlas*.

We end this section with a last definition and some examples.

**Definition 3.9.** Let $S$ be a complex submanifold of a complex manifold $M$. We shall say that $S$ is 2-*linearizable* if it is 2-splitting and comfortably embedded (with respect to the first order lifting induced by the 2-splitting).

*Remark* 3.10. In [3] we prove that $S$ is 2-linearizable if and only if its second infinitesimal neighbourhood $S(2)$ in $M$ is isomorphic to the second infinitesimal neighbourhood $S_N(2)$ of the zero section in $N_S$; compare with Remark 2.17.

*Example* 3.11. The zero section of a vector bundle is always 2-linearizable in the total space of the bundle.

*Example* 3.12. A local holomorphic retract is always 2-split in the ambient manifold. Indeed, if $p \colon U \to S$ is a local holomorphic retraction, then a second order lifting $\rho \colon \mathcal{O}_S \to \mathcal{O}_{S(2)}$ is given by $\rho(f) = [f \circ p]_3$.

*Example* 3.13. Let $\tilde{M}$ be the blow-up of a submanifold $X$ in a complex manifold $M$. Then the exceptional divisor $E \subset \tilde{M}$ is 2-linearizable in $\tilde{M}$.

*Example* 3.14. If $S$ is a Stein submanifold of a complex manifold $M$ (e.g., if $S$ is an open Riemann surface), then $S$ is 2-linearizable in $M$. Indeed, by Cartan's Theorem B the first cohomology group of $S$ with coefficients in any coherent sheaf vanishes, and the assertion follows from Proposition 2.2 and Remarks 3.4 and 3.5. In particular, if $S$ is a *singular* curve in $M$ then the non-singular part of $S$ is always comfortably embedded in $M$.

*Example* 3.15. Let $S$ be a non-singular, compact, irreducible curve of genus $g$ in a surface $M$. The Serre duality for Riemann surfaces implies that

$$H^1\big(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes \mathcal{T}_S\big) \cong H^0\big(S, \Omega_S \otimes \Omega_S \otimes \mathcal{N}_S^{\otimes 2}\big).$$

Therefore if $2(S \cdot S) < 4 - 4g$ then $H^1\big(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes \mathcal{T}_S\big) = (O)$. Analogously, we have

$$H^1(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes N_S) \cong H^0(S, \Omega_S \otimes \mathcal{N}_S),$$

and so $S \cdot S < 2 - 2g$ implies $H^1(S, \mathcal{I}_S^2/\mathcal{I}_S^3 \otimes N_S) = (O)$. It follows that if $g \geq 1$ and $S \cdot S < 4 - 4g$, or $g = 0$ and $S \cdot S < 2$, then $S$ is 2-linearizable.

## 4. Partial connections

As explained in the introduction, to get index theorems we need partial holomorphic connections. Atiyah in [4] showed that a complex vector bundle admits a holomorphic connection if and only if a particular exact sequence of locally free sheaves splits. In this section we shall adapt Atiyah's construction to the case of partial holomorphic connections; in the next section we shall describe a more concrete realization of Atiyah's exact sequence that will allow us to explicitely construct splitting morphisms (the morphisms $\psi$ of the introduction).

*Remark* 4.1. From now on, we shall denote the locally free sheaf of germs of holomorphic sections of a vector bundle (e.g., $E$) by the corresponding calligraphic letter (e.g., $\mathcal{E}$).

Let us start briefly recalling Atiyah's construction [4]. Let $E$ be a complex vector bundle of rank $d$ over a complex manifold $S$; we shall denote by $P_E$ the principal bundle associated to $E$, with structure group $GL(d)$. The group $GL(d)$ acts on the tangent vector bundle $T_P$ of the total space of $P_E$, and the quotient $A_E = T_P/GL(d)$ can be identified with the vector bundle on $S$ of rank $d^2 + \dim S$ composed by the fields of tangent vectors to $P_E$ defined along one of its fibres and invariant under the action of $GL(d)$. Since the action of $GL(d)$ on $P_E$ preserves the fibers of the canonical projection $\pi_0 \colon P_E \to S$, the differential of $\pi_0$ defines a vector bundle morphism, still denoted by $\pi_0$, from $A_E$ onto $TS$. Atiyah has shown ([4], Theorem 1 and Proposition 9) that there is a canonical exact sequence

$$O \longrightarrow \mathrm{Hom}(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{A}_E \xrightarrow{\pi_0} \mathcal{T}_S \longrightarrow O, \tag{18}$$

of locally free $\mathcal{O}_S$-modules, where $\mathrm{Hom}(\mathcal{E},\mathcal{E})$ is canonically identified with the sheaf of germs of holomorphic sections of the quotient, under the action of $GL(d)$, of the sub-bundle of $T_P$ formed by vectors tangential to the fibres of $P_E$. Furthermore, this sequence splits if and only if there is a holomorphic connection on $E$ ([4], Theorem 2). See also [16], where part of this theory is extended to subvarieties $S$ having normal crossing singularities.

*Remark* 4.2. Atiyah ([4], p. 190 and 195) also computed the cohomology class associated to the sequence (18). In particular, if $E$ is the normal bundle $N_S$ of a submanifold $S$ of a complex manifold $M$ and $\{(U_\alpha, z_\alpha)\}$ is an atlas adapted to $S$, then the cohomology class is represented by the cocycle $\{\mathfrak{g}_{\alpha\beta}\}$ where

$$\mathfrak{g}_{\alpha\beta} = \left.\frac{\partial z^t_\beta}{\partial z^s_\alpha}\frac{\partial^2 z^r_\alpha}{\partial z^p_\beta \partial z^t_\beta}\right|_S dz^p_\beta \otimes \omega^s_\alpha \otimes \partial_{r,\alpha}. \tag{19}$$

It is easy to adapt Atiyah's construction to the case of partial holomorphic connections.

**Definition 4.3.** Let $F$ be a sub-bundle of the tangent bundle $TS$ of a complex manifold $S$. A *partial holomorphic connection along* $F$ on a complex vector bundle $E$ on $S$ is a $\mathbb{C}$-linear morphism $\nabla\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\nabla(gs) = dg|_{\mathcal{F}} \otimes s + g\nabla s$$

for all $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$.

If $F$ is a sub-bundle of $TS$, we can consider the restriction to $F$ of the sequence (18)

$$O \longrightarrow \mathrm{Hom}(\mathcal{E},\mathcal{E}) \longrightarrow \mathcal{A}_{E,F} \xrightarrow{\pi_0} \mathcal{F} \longrightarrow O, \tag{20}$$

where $\mathcal{A}_{E,F} = \pi_0^{-1}(\mathcal{F})$. Then arguing as in [4] it is easy to prove the following

**Proposition 4.4.** *Let $F$ be a sub-bundle of the tangent bundle $TS$ of a complex manifold $S$, and let $E$ be a complex vector bundle over $S$. Then there is a partial holomorphic connection on $E$ along $F$ if and only if the sequence* (20) *splits, that is if and only if there is an $\mathcal{O}_S$-morphism $\psi_0\colon \mathcal{F} \to \mathcal{A}_E$ such that $\pi_0 \circ \psi_0 = \mathrm{id}$.*

In the next section we shall give a more concrete realization of the sheaf $\mathcal{A}_E$ when $E$ is the normal bundle of a submanifold $S$ into a manifold $M$, allowing us to present an alternative explicit description of the partial holomorphic connection given by a splitting of the sequence (20), and later on to build the morphisms $\psi_0\colon \mathcal{F} \to \mathcal{A}_E$. But we conclude this section with a few general definitions, useful to put in the right perspective what we are going to do.

**Definition 4.5.** Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves of $\mathcal{O}_S$-modules over a complex manifold $S$. Given a section $X \in H^0(S, \mathcal{T}_S \otimes \mathcal{F}^*)$, a *holomorphic $X$-connection on $\mathcal{E}$* (also called a *holomorphic action of $\mathcal{F}$ on $\mathcal{E}$ along $X$*) is a $\mathbb{C}$-linear map $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\tilde{X}(gs) = X^*(dg) \otimes s + g\tilde{X}(s)$$

for all $g \in \mathcal{O}_S$ and $s \in \mathcal{E}$, where $X^* \colon \Omega_S \to \mathcal{F}^*$ is the dual map of $X$. We shall often write $\tilde{X}_v(s)$ instead of $\tilde{X}(s)(v)$, for $s \in \mathcal{E}$ and $v \in \mathcal{F}$, where as usual we have identified $\mathcal{F}^* \otimes \mathcal{E}$ with $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$. Clearly, if $\mathcal{F}$ is an $\mathcal{O}_S$-submodule of $\mathcal{T}_S$ and $X$ is the inclusion, then $\tilde{X}$ is just a partial holomorphic connection on $E$ along $F$.

*Remark* 4.6. Let $\mathcal{E}, \mathcal{F}$ be locally free $\mathcal{O}_S$-modules and $X \colon \mathcal{F} \to \mathcal{T}_S$ be an $\mathcal{O}_S$-morphism. The *sheaf of first jets $J^1_X\mathcal{E}$ of $\mathcal{E}$ along* $X$ is the sheaf of abelian groups $(\mathcal{F}^* \otimes \mathcal{E}) \oplus \mathcal{E}$, with the structure of $\mathcal{O}_S$-module given as follows: for $f \in \mathcal{O}_S$ and $(\omega \otimes e) \oplus e' \in J^1_X\mathcal{E}$ we define

$$f(\omega \otimes e) \oplus e' = (X^*(df) \otimes e' + \omega \otimes fe) \oplus fe'.$$

The natural projection $J^1_X\mathcal{E} \to \mathcal{E}$ given by $(\omega \otimes e) \oplus e' \to e'$ is a surjective $\mathcal{O}_S$-morphism whose kernel is $\mathcal{F}^* \otimes \mathcal{E}$. Thus we obtain the exact sequence

$$O \longrightarrow \mathcal{F}^* \otimes \mathcal{E} \longrightarrow J^1_X\mathcal{E} \longrightarrow \mathcal{E} \to O. \tag{21}$$

Notice that $J^1_X\mathcal{E}$ is locally $\mathcal{O}_S$-free and the sequence (21) is functorial on $\mathcal{E}$. It is easy to see that this sequence splits if and only if there is a holomorphic $X$-connection on $E$. If we denote by $c_X(\mathcal{E}) \in H^1\big(S, \mathcal{F}^* \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{E})\big)$ the class associated to the sequence (21), and by $\hat{c}(\mathcal{E}) \in H^1\big(S, \Omega_S \otimes \mathrm{Hom}(\mathcal{E}, \mathcal{E})\big)$ the class associated to the same sequence when $\mathcal{F} = \mathcal{T}_S$ and $X$ is the identity, then it is not difficult to see that

$$c_X(\mathcal{E}) = (X^* \otimes \mathrm{id})_* \hat{c}(\mathcal{E}).$$

Furthermore, Atiyah ([4], Theorem 5) has shown that $\hat{c}(\mathcal{E})$ is the opposite of the cohomology class associated to the sequence (18).

We shall need a notion of flatness for a holomorphic $X$-connection. To state it in full generality, we need a new definition and a lemma.

**Definition 4.7.** Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_S$-modules over a complex manifold $S$, equipped with an $\mathcal{O}_S$-morphism $X \colon \mathcal{F} \to \mathcal{T}_S$. We say that $\mathcal{F}$ is a *Lie algebroid* of *anchor* $X$ if there is a $\mathbb{C}$-bilinear map $\{\cdot\,, \cdot\} \colon \mathcal{F} \oplus \mathcal{F} \to \mathcal{F}$ such that

(a) $\{v, u\} = -\{u, v\}$;
(b) $\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = O$;
(c) $\{g \cdot u, v\} = g \cdot \{u, v\} - X(v)(g) \cdot u$ for all $g \in \mathcal{O}_S$ and $u, v \in \mathcal{F}$.

*Example* 4.8. Assume that $X \colon \mathcal{F} \to \mathcal{T}_S$ is injective, and that $X(\mathcal{F})$ is an *involutive* subsheaf of $\mathcal{T}_S$ (we recall that a subsheaf $\mathcal{F}$ of $\mathcal{T}_S$ is involutive if it is locally $\mathcal{O}_S$-free and, for each $x \in S$, the fiber $\mathcal{F}_x$ is closed under the bracket operation for vector fields). Then we can easily provide $\mathcal{F}$ with a Lie algebroid structure of anchor $X$ by setting

$$\{u, v\} = X^{-1}([X(u), X(v)]).$$

In particular in this case we have $X(\{u, v\}) = [X(u), X(v)]$.

We refer to [23] and references therein for the general theory of Lie algebroids; here we shall use the definition only.

Let $\mathcal{F}$ be a Lie algebroid of anchor $X$ over a complex manifold $S$, and assume we also have a holomorphic $X$-connection $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ over a locally free $\mathcal{O}_S$-module $\mathcal{E}$. Then it is easy to see that setting

$$R_{uv}s = \tilde{X}_u \circ \tilde{X}_v(s) - \tilde{X}_v \circ \tilde{X}_u(s) - \tilde{X}_{\{u,v\}}(s) \tag{22}$$

we define a $\mathbb{C}$-linear map $R\colon \mathcal{E} \to \bigwedge^2 \mathcal{F}^* \otimes \mathcal{E}$ such that

$$R_{uv}(g \cdot s) = g \cdot R_{uv}(s) + \big([X(u), X(v)] - X(\{u,v\})\big)(g) \cdot s \tag{23}$$

for all $g \in \mathcal{O}_S$, $u$, $v \in \mathcal{F}$ and $s \in \mathcal{E}$.

**Definition 4.9.** Let $\mathcal{F}$ be a Lie algebroid of anchor $X$ over a complex manifold $S$, and $\tilde{X}\colon \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ a holomorphic $X$-connection over a locally free $\mathcal{O}_S$-module $\mathcal{E}$. The *curvature* of $\tilde{X}$ (with respect to the given Lie algebroid structure) is the $\mathbb{C}$-linear map $R\colon \mathcal{E} \to \bigwedge^2 \mathcal{F}^* \otimes \mathcal{E}$ defined in (22). We shall say that $\tilde{X}$ is *flat* if $R \equiv O$.

*Remark* 4.10. Note that if $\tilde{X}$ is flat then from (23) it follows that $[X(u), X(v)] = X(\{u,v\})$ for all $u, v \in \mathcal{F}$.

*Example* 4.11. Let $X\colon \mathcal{F} \to \mathcal{T}_S$ be the inclusion of an involutive locally free $\mathcal{O}_S$-submodule $\mathcal{F}$ of $\mathcal{T}_S$ with locally free quotient $\mathcal{Q} = \mathcal{T}_S/\mathcal{F}$ and consider the associated exact sequence

$$O \longrightarrow \mathcal{F} \xhookrightarrow{X} \mathcal{T}_S \xrightarrow{\phi} \mathcal{Q} \longrightarrow O.$$

Then we can define a partial holomorphic connection $\tilde{X}\colon \mathcal{Q} \to \mathcal{F}^* \otimes \mathcal{Q}$ on $\mathcal{Q}$ along $\mathcal{F}$ by setting

$$\tilde{X}_u(q) = \phi([X(u), \tilde{q}]),$$

where $\tilde{q}$ is any local section of $\mathcal{T}_S$ such that $\phi(\tilde{q}) = q$. Putting on $\mathcal{F}$ the natural Lie algebroid structure of anchor $X$ given by the bracket of vector fields, it is easy to check that $\tilde{X}$ is a flat partial holomorphic connection. The existence of this natural flat partial holomorphic connection is one of the main ingredients in the proof of Baum-Bott index theorem for singular holomorphic foliations; see [5], [12], and [28], Chapter III.

## 5. The universal partial connection on the normal sheaf

In this section we shall describe a concrete incarnation of the sheaf $\mathcal{A}_E$ when $E$ is the normal bundle of a submanifold $S$ of a complex manifold $M$.

**Definition 5.1.** Let $S$ be a (not necessarily closed) complex submanifold of a manifold $M$. Set $\mathcal{T}_{M,S(1)} = \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{O}_{S(1)}$; with a slight abuse of notation we shall denote by $\theta_1$ the $\mathcal{O}_M$-morphism $\mathrm{id} \otimes \theta_1\colon \mathcal{T}_{M,S(1)} \to \mathcal{T}_{M,S}$. Let $\mathcal{T}^S_{M,S(1)}$ be the $\mathcal{O}_M$-submodule of $\mathcal{T}_{M,S(1)}$ given by

$$\mathcal{T}^S_{M,S(1)} = \ker(p_2 \circ \theta_1) \subset \mathcal{T}_{M,S(1)},$$

where $p_2\colon \mathcal{T}_{M,S} \to \mathcal{N}_S$ is the natural projection. Notice that $\theta_1(\mathcal{T}^S_{M,S(1)}) = \mathcal{T}_S$.

In local adapted coordinates, an element $v = [a^j]_2 \frac{\partial}{\partial z^j} \in \mathcal{T}_{M,S(1)}$ belongs to $\mathcal{T}^S_{M,S(1)}$ if and only if $[a^r]_1 = 0$ for $r = 1, \dots, m$. In other words, $v \in \mathcal{T}_{M,S(1)}$ belongs to $\mathcal{T}^S_{M,S(1)}$ if and only if when restricted to $S$ it is tangent to it.

In general, $\mathcal{T}^S_{M,S(1)}$ is not an $\mathcal{O}_S$-module (it is if $S$ splits into $M$, but we do not want to assume this yet). However, we can almost define on it a Lie algebroid structure of anchor $\theta_1$.

If $v \in \mathcal{T}_{M,S}$ and $f \in \mathcal{O}_M$, then $v(f)$ is a well-defined element of $\mathcal{O}_S$. Analogously, if $v \in \mathcal{T}_{M,S(1)}$ and $f \in \mathcal{O}_M$, then $v(f)$ is a well-defined element of $\mathcal{O}_{S(1)}$; on the other hand, if $g \in \mathcal{O}_{S(1)}$ then $v(g)$ is well defined as an element of $\mathcal{O}_S$, but not of $\mathcal{O}_{S(1)}$. This means that we can define a bracket operation $[\cdot\,,\cdot]\colon \mathcal{T}_{M,S(1)} \oplus \mathcal{T}_{M,S(1)} \to \mathcal{T}_{M,S}$ by setting

$$[u,v](f) = u\bigl(v(f)\bigr) - v\bigl(u(f)\bigr) \in \mathcal{O}_S$$

for all $f \in \mathcal{O}_M$. In particular, for every $g \in \mathcal{O}_{S(1)}$ and $u$, $v \in \mathcal{T}_{M,S(1)}$ we have

$$[gu,v] = \theta_1(g)[u,v] - v(g)\cdot\theta_1(u). \tag{24}$$

*Remark* 5.2. In general, if $u$, $v$, $w \in \mathcal{T}_{M,S(1)}$ then $[v,w] \in \mathcal{T}_{M,S}$, and so $[u,[v,w]]$ is *not* defined. But we shall see exceptions to this rule.

**Lemma 5.3.** *Let $S$ be a complex submanifold of a manifold $M$. Then*

(i) *every $v \in \mathcal{T}^S_{M,S(1)}$ induces a derivation $g \mapsto v(g)$ of $\mathcal{O}_{S(1)}$;*

(ii) *there exists a natural $\mathbb{C}$-bilinear map $\{\cdot\,,\cdot\}\colon \mathcal{T}^S_{M,S(1)} \oplus \mathcal{T}^S_{M,S(1)} \to \mathcal{T}^S_{M,S(1)}$ such that*

  (a) $\{v,u\} = -\{u,v\}$,

  (b) $\{u,\{v,w\}\} + \{v,\{w,u\}\} + \{w,\{u,v\}\} = O$,

  (c) $\{gu,v\} = g\{u,v\} - v(g)\cdot u$ *for all* $g \in \mathcal{O}_{S(1)}$, *and*

  (d) $\theta_1\{u,v\} = [\theta_1(u),\theta_1(v)] = [u,v]$.

*Proof.* (i) The relevant fact here is that if we have an adapted chart $(U,z)$ and germs $[f]_2 \in \mathcal{O}_{S(1)}$ and $[h]_2 \in \mathcal{I}_S/\mathcal{I}_S^2$, then we get well-defined elements of $\mathcal{O}_{S(1)}$ by setting

$$\frac{\partial[f]_2}{\partial z^p} = \left[\frac{\partial f}{\partial z^p}\right]_2 \qquad \text{and} \qquad [h]_2\frac{\partial[f]_2}{\partial z^r} = \left[h\frac{\partial f}{\partial z^r}\right]_2$$

for $p = m+1, \dots, n$ and $r = 1, \dots, m$. This implies that if $[f]_2 \in \mathcal{O}_{S(1)}$ and $v = [a^j]_2 \frac{\partial}{\partial z^j} \in \mathcal{T}^S_{M,S(1)}$ then

$$v([f]_2) = [a^r]_2\frac{\partial[f]_2}{\partial z^r} + [a^p]_2\frac{\partial[f]_2}{\partial z^p}$$

is a well-defined (and independent of the local coordinates) element of $\mathcal{O}_{S(1)}$, and not just of $\mathcal{O}_S$, and in this way we clearly get a derivation of $\mathcal{O}_{S(1)}$.

(ii) We define $\{\cdot\,,\cdot\}$ by setting

$$\{u,v\}(g) = u\bigl(v(g)\bigr) - v\bigl(u(g)\bigr),$$

for all $g\in\mathcal{O}_{S(1)}$. It is easy to check (working for instance in local coordinates adapted to $S$) that $\{u,v\}\in\mathcal{T}^S_{M,S(1)}$, and that properties (a)–(d) are satisfied. □

It turns out that a quotient of $\mathcal{T}^S_{M,S(1)}$ has a natural $\mathcal{O}_S$-module structure, and inherites the Lie algebroid structure of anchor $\theta_1$. To prove this, we need the following

**Lemma 5.4.** *Let $S$ be an $m$-codimensional complex submanifold of a manifold $M$ of dimension $n$. Then:*

- (i) *$u\in\mathcal{T}_{M,S(1)}$ is such that $p_2([u,s])=O$ for all $s\in\mathcal{T}_{M,S(1)}$ if and only if $u\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$;*
- (ii) *if $u\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$ and $v\in\mathcal{T}^S_{M,S(1)}$ then $\{u,v\}\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$;*
- (iii) *the quotient sheaf*
$$\mathcal{A}=\mathcal{T}^S_{M,S(1)}/\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$$
*admits a natural structure of $\mathcal{O}_S$-locally free sheaf such that $\theta_1\colon\mathcal{A}\to\mathcal{T}_S$ is an $\mathcal{O}_S$-morphism.*

*Proof.* (i) Let us work in local coordinates adapted to $S$. Writing $u=[a^j]_2\frac{\partial}{\partial z^j}$, $s=[b^h]_2\frac{\partial}{\partial z^h}\in\mathcal{T}_{M,S(1)}$ we have
$$p_2([u,s])=\left[a^j\frac{\partial b^r}{\partial z^j}-b^j\frac{\partial a^r}{\partial z^j}\right]_1\frac{\partial}{\partial z^r}.$$
Now $u\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$ if and only if $a^r\in\mathcal{I}_S^2$ and $a^p\in\mathcal{I}_S$ for $r=1,\dots,m$ and $p=m+1,\dots,n$; in particular it is clear that $u\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$ implies $p_2([u,s])=O$ for all $s\in\mathcal{T}_{M,S(1)}$.

Conversely, assume that $u=[a^j]_2\frac{\partial}{\partial z^j}\in\mathcal{T}_{M,S(1)}$ is such that $p_2([u,s])=O$ for all $s\in\mathcal{T}_{M,S(1)}$. From $p_2([u,\partial/\partial z^p])=O$ for all $p=m+1,\dots,n$ we get that $[a^r]_1$ is a constant $\alpha^r\in\mathbb{C}$ for $r=1,\dots,m$. But then from $p_2([u,\partial/\partial z^s])=O$ for all $s=1,\dots,m$ we get $[a^r]_2=\alpha^r$ for $r=1,\dots,m$. Now from $p_2([u,[z^{s_0}]_2\partial/\partial z^{s_0}])=O$ for all $s_0=1,\dots,m$ (no sum on $s_0$ here) we get $\alpha_r=0$ for $r=1,\dots,m$. Finally, from $p_2([u,[z^{p_0}]_2\partial/\partial z^1])=O$ for all $p_0=m+1,\dots,n$ we get $[a^p]_1=0$ for all $p=m+1,\dots,n$, and so $u\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$, as claimed.

(ii) Working again in local coordinates adapted to $S$, if $u=[a^j]_2\frac{\partial}{\partial z^j}\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$ and $v=[b^h]_2\frac{\partial}{\partial z^h}\in\mathcal{T}^S_{M,S(1)}$ then we have
$$\{u,v\}=\left[a^j\frac{\partial b^r}{\partial z^j}-b^j\frac{\partial a^r}{\partial z^j}\right]_2\frac{\partial}{\partial z^r}+\left[a^j\frac{\partial b^p}{\partial z^j}-b^j\frac{\partial a^p}{\partial z^j}\right]_2\frac{\partial}{\partial z^p}\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$$
because $a^r\in\mathcal{I}_S^2$ and $a^p$, $b^r\in\mathcal{I}_S$ for all $r=1,\dots,m$ and $p=m+1,\dots,n$.

(iii) The sheaf $\mathcal{T}^S_{M,S(1)}$ is an $\mathcal{O}_{S(1)}$-submodule of $\mathcal{T}_{M,S(1)}$ such that (by definition) $g\cdot v\in\mathcal{I}_S\cdot\mathcal{T}^S_{M,S(1)}$ for all $g\in\mathcal{I}_S/\mathcal{I}_S^2$ and $v\in\mathcal{T}^S_{M,S(1)}$. Therefore the $\mathcal{O}_{S(1)}$-module structure induces a natural $\mathcal{O}_S$-module structure on $\mathcal{A}$. It is easy to check that, in terms of local coordinates adapted to $S$, the sheaf $\mathcal{A}$ is a locally free $\mathcal{O}_S$-module freely generated by $\pi(\frac{\partial}{\partial z^p})$ and $\pi([z^s]_2\frac{\partial}{\partial z^r})$ (with $p = m+1, \dots n$, and $r$, $s = 1, \dots m$), where $\pi\colon \mathcal{T}^S_{M,S(1)} \to \mathcal{A}$ is the quotient map.

Finally, since $\mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)} \subset \ker\theta_1$, the morphism $\theta_1$ defines a map, still denoted by the same symbol, $\theta_1\colon \mathcal{A} \to \mathcal{T}_S$, and it is clear that $\theta_1$ is an $\mathcal{O}_S$-morphism for the structure we just defined.

□

**Definition 5.5.** Let $S$ be a complex submanifold of a complex manifold $M$. The *Atiyah sheaf* of $S$ in $M$ is the locally free $\mathcal{O}_S$-module

$$\mathcal{A} = \mathcal{T}^S_{M,S(1)}/\mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)}.$$

The sheaf $\mathcal{A}$ appears also in [26], where it is denoted by $\mathcal{N}^1_{M,C}$, and in [16], where it is denoted by $T^1_M\langle Y\rangle \otimes \mathcal{O}_Y$. We now show that $\mathcal{A}$ is isomorphic to the sheaf $\mathcal{A}_{N_S}$ described in the previous section.

**Theorem 5.6.** *Let $S$ be an $m$-codimensional complex submanifold of a manifold $M$ of dimension $n$. Then there is a natural exact sequence of locally free $\mathcal{O}_S$-modules*

$$O \longrightarrow \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S) \longrightarrow \mathcal{A} \xrightarrow{\theta_1} \mathcal{T}_S \longrightarrow O \tag{25}$$

*which is isomorphic to the sequence* (18) *with $E = N_S$. In particular, the sheaf $\mathcal{A}$ only depends on the normal bundle $N_S$ of the embedding of $S$ into $M$, and $N_S$ admits a holomorphic connection if and only if the sequence* (25) *splits.*

*Proof.* As usual, we work in local coordinates $\{(U_\alpha, z_\alpha)\}$ adapted to $S$. The kernel of $\theta_1$ is freely generated by the images under the canonical projection $\pi\colon \mathcal{T}^S_{M,S(1)} \to \mathcal{A}$ of $[z^s_\alpha]_2\frac{\partial}{\partial z^r_\alpha}$ for $r$, $s = 1, \dots, m$. Now we have

$$\pi\left([z^s_\alpha]_2\frac{\partial}{\partial z^r_\alpha}\right) = \left.\frac{\partial z^s_\alpha}{\partial z^{s_1}_\beta}\right|_S \left.\frac{\partial z^{r_1}_\alpha}{\partial z^r_\beta}\right|_S \pi\left([z^{s_1}_\beta]_2\frac{\partial}{\partial z^{r_1}_\beta}\right),$$

and hence the kernel of $\theta_1$ is naturally isomorphic to $\mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S)$.

To prove that (25) is isomorphic to (18), by Proposition 2.2 it suffices to prove that the cohomology class associated to both sequences is the same.

Define local splittings $\sigma_\alpha$ of (25) by setting $\sigma_\alpha(\frac{\partial}{\partial z^p_\alpha}) = \pi\left(\frac{\partial}{\partial z^p_\alpha}\right)$ and then extending by $\mathcal{O}_S$-linearity. The class of the sequence (25) is then represented by the cocycle given by

$$\begin{aligned}(\sigma_\beta - \sigma_\alpha)\left(\frac{\partial}{\partial z^p_\beta}\right) &= \pi\left(\frac{\partial}{\partial z^p_\beta}\right) - \left.\frac{\partial z^q_\alpha}{\partial z^p_\beta}\right|_S \pi\left(\frac{\partial}{\partial z^q_\alpha}\right) \\ &= \pi\left(\left[\frac{\partial z^r_\alpha}{\partial z^p_\beta}\right]_2 \frac{\partial}{\partial z^r_\alpha}\right) = \pi\left(\left[\frac{\partial^2 z^r_\alpha}{\partial z^p_\beta \partial z^t_\beta} z^t_\beta\right]_2 \frac{\partial}{\partial z^r_\alpha}\right) \\ &= \left.\frac{\partial z^t_\beta}{\partial z^s_\alpha}\frac{\partial^2 z^r_\alpha}{\partial z^p_\beta \partial z^t_\beta}\right|_S \pi\left([z^s_\alpha]_2 \frac{\partial}{\partial z^r_\alpha}\right),\end{aligned}$$

and our claim follows from (19). The last assertion now is an immediate consequence of [4], Theorem 2. □

The main advantage of our $\mathcal{A}$ over Atiyah's $\mathcal{A}_{N_S}$ is that we have an explicit way of going from a splitting of the sequence (25) to a partial holomorphic connection on $N_S$. Indeed, $\mathcal{A}$ comes equipped with both a natural structure of Lie algebroid of anchor $\theta_1$ and a holomorphic $\theta_1$-connection on $N_S$:

**Proposition 5.7.** *Let $S$ be a complex submanifold of a complex manifold $M$. Then:*

(i) *the Atiyah sheaf $\mathcal{A}$ has a natural structure $\{\cdot\,,\cdot\}$ of Lie algebroid of anchor $\theta_1$ such that*

$$\theta_1\{q_1, q_2\} = [\theta_1(q_1), \theta_1(q_2)] \tag{26}$$

*for all $q_1$, $q_2 \in \mathcal{A}$;*

(ii) *there is a natural holomorphic $\theta_1$-connection $\tilde{X}\colon \mathcal{N}_S \to \mathcal{A}^* \otimes \mathcal{N}_S$ on $\mathcal{N}_S$ given by*

$$\tilde{X}_q(s) = p_2([v, \tilde{s}])$$

*for all $q \in \mathcal{A}$ and $s \in \mathcal{N}_S$, where $v \in \mathcal{T}^S_{M,S(1)}$ and $\tilde{s} \in \mathcal{T}_{M,S(1)}$ are such that $\pi(v) = q$ and $p_2 \circ \theta_1(\tilde{s}) = s$;*

(iii) *this holomorphic $\theta_1$-connection $\tilde{X}$ is flat.*

*Proof.* (i) Lemmas 5.3 and 5.4.(ii) imply that setting

$$\{q_1, q_2\} = \pi(\{v_1, v_2\})$$

for all $q_1$, $q_2 \in \mathcal{A}$, where $v_j \in \mathcal{T}^S_{M,S(1)}$ is such that $q_j = \pi(v_j)$, we get a well-defined $\theta_1$-Lie algebroid structure satisying (26).

(ii) Lemma 5.4.(i) implies that if $\pi(v) = \pi(v')$ then $p_2([v, \tilde{s}]) = p_2([v', \tilde{s}])$ for all $\tilde{s} \in \mathcal{T}_{M,S(1)}$. Analogously, Lemma 5.3.(ii).(d) implies that if $p_2 \circ \theta_1(\tilde{s}) = p_2 \circ \theta_1(\tilde{s}')$ then $p_2([v, \tilde{s}]) = p_2([v, \tilde{s}'])$ for all $v \in \mathcal{T}^S_{M,S(1)}$; therefore $\tilde{X}_q(s)$ is a well-defined element of $\mathcal{N}_S$.

Now, (24) yields

$$\tilde{X}_{[f]_1\cdot q}(s) = p_2\big(\big[[f]_2 v, \tilde{s}\big]\big) = p_2\big([f]_1[v,\tilde{s}] - \tilde{s}([f]_2)\cdot\theta_1(v)\big) = [f]_1\cdot\tilde{X}_q(v)$$

because $v\in\ker(p_2\circ\theta_1)$, and so $\tilde{X}(s)\in\mathcal{A}^*\otimes\mathcal{N}_S$, as claimed. Finally,

$$\begin{aligned}\tilde{X}_q([f]_1\cdot s) &= p_2\big(\big[v,[f]_2\tilde{s}\big]\big) = p_2\big([f]_1[v,\tilde{s}] + v([f]_2)\cdot\theta_1(\tilde{s})\big)\\ &= [f]_1\cdot\tilde{X}_q(v) + \theta_1(q)([f]_1)\cdot s,\end{aligned}$$

because

$$\big[v([f]_2)\big]_1 = \theta_1(v)([f]_1) = \theta_1\big(\pi(v)\big)([f]_1)$$

for all $[f]_2\in\mathcal{O}_{S(1)}$ and $v\in\mathcal{T}^S_{M,S(1)}$, and thus $\tilde{X}$ is a holomorphic $\theta_1$-connection.

(iii) We must prove that

$$\big[v_1,\widetilde{[v_2,\tilde{s}]}\big] + \big[v_2,\widetilde{[\tilde{s},v_1]}\big] + [\tilde{s},\{v_1,v_2\}]\in\mathcal{T}_S \tag{27}$$

for all $v_1,\ v_2\in\mathcal{T}^S_{M,S(1)}$ and $\tilde{s}\in\mathcal{T}_{M,S(1)}$, where $\widetilde{[v_j,\tilde{s}]}$ is any element of $\mathcal{T}_{M,S(1)}$ such that its $\theta_1$-image is equal to $[v_j,\tilde{s}]\in\mathcal{T}_{M,S}$. But using local coordinates adapted to $S$ it is easy to see that (27) is a consequence of the usual Jacobi rule for brackets of vector fields. □

**Definition 5.8.** Let $S$ be a complex submanifold of a complex manifold $M$. The holomorphic $\theta_1$-connection $\tilde{X}\colon\mathcal{N}_S\to\mathcal{A}^*\otimes\mathcal{N}_S$ on $\mathcal{N}_S$ defined in Proposition 5.7.(ii) is called the *universal holomorphic connection* on $\mathcal{N}_S$.

We can now summarize what we have done up to now in the following

**Theorem 5.9.** *Let $S$ be a submanifold of a complex manifold $M$, and $F$ a sub-bundle of the tangent bundle $TS$. Then:*

- (i) *if $\psi\colon\mathcal{F}\to\mathcal{A}$ is an $\mathcal{O}_S$-morphism such that $\theta_1\circ\psi=\mathrm{id}$ then the map $\tilde{X}^\psi\colon\mathcal{N}_S\to\mathcal{F}^*\otimes\mathcal{N}_S$ given by*
$$\tilde{X}^\psi_v(s) = \tilde{X}_{\psi(v)}(s)$$
*for all $v\in\mathcal{F}$ and $s\in\mathcal{N}_S$, where $\tilde{X}$ is the universal holomorphic connection on $N_S$, is a partial holomorphic connection on $N_S$ along $F$;*
- (ii) *there exists a partial holomorphic connection on $N_S$ along $F$ if and only if there exists an $\mathcal{O}_S$-morphism $\psi\colon\mathcal{F}\to\mathcal{A}$ such that $\theta_1\circ\psi=\mathrm{id}$;*
- (iii) *if $F$ is involutive, then the partial holomorphic connection $\tilde{X}^\psi$ is flat if and only if $\psi\colon\mathcal{F}\to\mathcal{A}$ is a Lie algebroid morphism.*

*Proof.* (i) The only not completely trivial property is Leibniz's rule. But indeed

$$\tilde{X}^\psi_v(g\cdot s) = \tilde{X}_{\psi(v)}(g\cdot s) = g\cdot\tilde{X}_{\psi(v)}(s) + \theta_1\big(\psi(v)\big)(g)\cdot s = g\cdot\tilde{X}^\psi_v(s) + v(g)\cdot s$$

for all $g\in\mathcal{O}_S$, and we are done.

(ii) In one direction is (i). Conversely, assume that we have a partial holomorphic connection on $N_S$ along $F$. Then Proposition 4.4 yields an $\mathcal{O}_S$-morphism $\psi_0$ from $\mathcal{F}$

to $\mathcal{A}_{N_S}$ such that $\pi_0 \circ \psi_0 = \mathrm{id}$, and hence Theorem 5.6 yields an $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F} \to \mathcal{A}$ such that $\theta_1 \circ \psi = \mathrm{id}$.

(iii) If we denote by $R^\psi$ the curvature of $\tilde{X}^\psi$, recalling that the universal holomorphic connection $\tilde{X}$ is flat, we get

$$R^\psi_{uv} = \tilde{X}_{\{\psi(u),\psi(v)\}-\psi[u,v]}$$

for all $u$, $v \in \mathcal{F}$. One direction is then clear; conversely, assume that $R^\psi \equiv O$. Now, Proposition 5.7.(i) implies that $\{\psi(u), \psi(v)\} - \psi[u, v] \in \ker\theta_1 \subset \mathcal{A}$ for all $u$, $v \in \mathcal{F}$; therefore it suffices to prove that if $q \in \ker\theta_1 \subset \mathcal{A}$ is such that $\tilde{X}_q \equiv O$ then $q = O$. But indeed, if $q \in \ker\theta_1$ then $q = \pi(v)$ with $v = [a^r]_2\partial/\partial z^r$ for suitable $a^r \in \mathcal{I}_S$ (and we are using local coordinates adapted to $S$, as usual). Then from $\tilde{X}_q(\partial_s) = O$ for $s = 1, \ldots, m$ it easily follows that $a^r \in \mathcal{I}_S^2$ for $r = 1, \ldots, m$, and hence $q = O$.

□

*Remark* 5.10. As suggested by [4], the sequence of the first jets sheaves can be interpreted as a subsequence of the extension obtained dualizing the sequence (25) and tensorizing with $\mathcal{N}_S$:

$$\begin{array}{ccccccccc} O & \longrightarrow & \Omega_S \otimes \mathcal{N}_S & \longrightarrow & J^1(\mathcal{N}_S) & \longrightarrow & \mathcal{N}_S & \longrightarrow & 0 \\ & & \| & & & & \downarrow & & \\ O & \longrightarrow & \Omega_S \otimes \mathcal{N}_S & \longrightarrow & \mathcal{A}^* \otimes \mathcal{N}_S & \longrightarrow & \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S) \otimes \mathcal{N}_S & \longrightarrow & O \end{array},$$

where the last vertical map is the injection locally given by $s \mapsto \mathrm{id} \otimes s$, where $s$ is a local section of $\mathcal{N}_S$; it is obtained tensorizing by $\mathcal{N}_S$ the inclusion $\mathcal{O}_S \hookrightarrow \mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S)$ that corresponds to the identity map on $\mathcal{N}_S$. In particular, $J^1(\mathcal{N}_S)$ is a subsheaf of $\mathrm{Hom}_{\mathcal{O}_S}(\mathcal{A}, \mathcal{N}_S)$.

When $S$ has codimension 1 in $M$, we have $\mathrm{Hom}(\mathcal{N}_S, \mathcal{N}_S) \cong \mathcal{O}_S$ and hence the previous remark yields

**Corollary 5.11.** *If $S$ is a codimension 1 submanifold of a complex manifold $M$, then the sequence (25) becomes $O \to \mathcal{O}_S \to \mathcal{A} \to \mathcal{T}_S \to O$, so that $\mathcal{A} \cong J^1(\mathcal{N}_S^*) \otimes \mathcal{N}_S$ and $\mathcal{A}$ admits a nowhere zero holomorphic section.*

We end this section with a remark that will be useful in Section 7:

**Proposition 5.12.** *Let $S$ be a submanifold of a complex manifold $M$, comfortably embedded with respect to a first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$. Then there exists an $\mathcal{O}_S$-morphism $\tilde{\pi}\colon \mathcal{T}_{M,S(1)} \to \mathcal{A}$ such that $\tilde{\pi}|_{\mathcal{T}^S_{M,S(1)}} = \pi$, where $\mathcal{T}_{M,S(1)}$ is endowed with the structure of $\mathcal{O}_S$-module given by restriction of scalars via $\rho$.*

*Proof.* Fix a comfortable atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $\rho$. If $v \in \mathcal{T}_{M,S(1)}$, we can write

$$v = [x^j_\alpha]_2 \frac{\partial}{\partial z^j_\alpha}$$

for suitable $[x^j_\alpha]_2 \in \mathcal{O}_{S(1)}$; then we set

$$\tilde\pi(v) = \pi\left(\tilde\rho([x^r_\alpha]_2)\frac{\partial}{\partial z^r_\alpha} + [x^p_\alpha]_2\frac{\partial}{\partial z^p_\alpha}\right) = \pi\left(v - \rho([x^r_\alpha]_1)\frac{\partial}{\partial z^r_\alpha}\right).$$

We claim that $\tilde\pi$ is well-defined, that is it does not depend on the particular chart chosen to express $v$. Indeed, if we also write $v = [x^k_\beta]_2 \frac{\partial}{\partial z^k_\beta}$ then we have

$$[x^k_\beta]_2 = [x^j_\alpha]_2 \left[\frac{\partial z^k_\beta}{\partial z^j_\alpha}\right]_2 .$$

Applying $\rho \circ \theta_1$ to both sides and recalling that we are working with an atlas adapted to $S$ we get

$$\rho([x^s_\beta]_1) = \rho([x^r_\alpha]_1)\,\rho\left(\left[\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right]_1\right).$$

Then

$$\rho([x^s_\beta]_1)\frac{\partial}{\partial z^s_\beta} = \rho([x^r_\alpha]_1)\,\rho\left(\left[\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right]_1\right)\frac{\partial}{\partial z^s_\beta} = \rho([x^r_\alpha]_1)\,\left[\frac{\partial z^s_\beta}{\partial z^r_\alpha}\right]_2\frac{\partial}{\partial z^s_\beta}$$

where we used (5) and the fact that $\mathfrak{U}$ is a comfortable atlas adapted to $\rho$. So

$$\left(v - \rho([x^s_\beta]_1)\frac{\partial}{\partial z^s_\beta}\right) - \left(v - \rho([x^r_\alpha]_1)\frac{\partial}{\partial z^r_\alpha}\right) \in \mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)}$$

because we are using a splitting atlas, and thus $\tilde\pi$ is well-defined. Finally, it is easy to check that $\tilde\pi$ is an $\mathcal{O}_S$-morphism extending $\pi$, and we are done. □

## 6. The general index theorem

In this section we shall prove our general index theorem following the strategy indicated in the introduction; in the next two sections we shall show how all the known (and a couple of new ones) index theorems of this kind (with $M$ smooth) for both holomorphic maps and holomorphic foliations are just particular instances of our general statement.

In the previous sections we have shown how to get a partial holomorphic connection on the normal bundle from a splitting morphism $\psi$. The next step is showing how the existence of a partial holomorphic connection forces the vanishing of some Chern classes. This has been proved, for instance, by Baum and Bott ([5]; see also [14]); we report here a statement (and a proof) adapted to our situation.

**Theorem 6.1.** *Let $S$ be a complex manifold, $F$ a sub-bundle of $TS$ of rank $\ell$, and $E$ a complex vector bundle on $S$. Assume we have a partial holomorphic connection on $E$ along $F$. Then:*

(i) *every symmetric polynomial in the Chern classes of $E$ of degree larger than* $\dim S - \ell + \lfloor \ell/2 \rfloor$ *vanishes.*

(ii) *Furthermore, if $F$ is involutive and the partial holomorphic connection is flat then every symmetric polynomial in the Chern classes of $E$ of degree larger than* $\dim S-\ell$ *vanishes.*

*Proof.* Write

$$T^{\mathbb{R}}S\otimes\mathbb{C}=F\oplus F_1\oplus T^{(0,1)}S, \tag{28}$$

where $F_1$ is any $C^\infty$-complement of $F$ in $TS=T^{(1,0)}S$. Define a (real) connection $\nabla$ on $E$ using the given partial holomorphic connection on $F$, any connection on $F_1$, and $\bar\partial$ on $T^{(0,1)}S$.

Let $\omega$ be the curvature form of $\nabla$. We claim that

$$\omega(v,\bar w)=\omega(\bar u,\bar w)=O \tag{29}$$

for all $v\in F$ and $\bar u$, $\bar w\in T^{(0,1)}S$. It is enough to prove that they vanish when applied to holomorphic sections of $E$, since these generate $\Gamma(E)$ as a $C^\infty$-module, and the curvature is a tensor. But if $\sigma$ is a holomorphic section of $E$ we have

$$\omega(v,\bar w)(\sigma)=\nabla_v(\nabla_{\bar w}\sigma)-\nabla_{\bar w}(\nabla_v\sigma)-\nabla_{[v,\bar w]}\sigma=O,$$

because $\nabla_{\bar w}$ kills every holomorphic section, $\nabla_v\sigma$ is holomorphic because $\nabla$ is holomorphic along $F$, and $[v,\bar w]=O$. Analogously, since $[\bar u,\bar w]\in T^{(0,1)}S$, one shows that $\omega(\bar u,\bar w)=O$.

Choose local coordinates and local forms $\eta^1,\ldots,\eta^n$ (where $n=\dim S$) so that

$$\{\eta^1,\ldots,\eta^\ell,\eta^{\ell+1},\ldots,\eta^n,d\overline{z^1},\ldots,d\overline{z^n}\}$$

is a local frame for the dual of $T^{\mathbb{R}}S\otimes\mathbb{C}$ respecting (28); in particular, $\{\eta^1|_F,\ldots,\eta^\ell|_F\}$ is a local frame for the dual of $F$, and $\{\eta^{\ell+1}|_{F_1},\ldots,\eta^n|_{F_1}\}$ is a local frame for the dual of $F_1$. Then (29) implies that in this local frame the curvature matrix is composed by forms which are linear combinations of

$$\eta^{p'}\wedge\eta^{q'},\ \eta^{p'}\wedge\eta^{q''},\ \eta^{p''}\wedge\eta^{q''},\ d\bar{z^j}\wedge\eta^{q''},$$

where $1\le p'<q'\le\ell$, $\ell+1\le p''<q''\le n$ and $1\le j\le n$. Since any product of more than $n-\ell+\lfloor\ell/2\rfloor$ of these forms vanishes, (i) follows.

If $F$ is involutive and the partial holomorphic connection along $F$ is flat, we moreover have $\omega(v,w)=O$ for all $v$, $w\in F$. This means that we can drop the forms $\eta^{p'}\wedge\eta^{q'}$ from the previous list, and then any product of more than $n-\ell$ of the remaining forms vanishes, giving part (ii). □

*Remark* 6.2. The previous proof shows not only that Chern classes of suitable degree vanish, but that the standard differential forms representing them (the one obtained starting from the curvature matrix of a connection) vanish too.

We have now all the ingredients needed to apply the general cohomological argument devised by Lehmann and Suwa. Let us first introduce a couple of definitions to simplify the statements of our theorems.

**Definition 6.3.** Let $S$ be a (possibly singular) subvariety of a complex manifold $M$, and let $S^{\mathrm{reg}} \subseteq S$ be the regular part of $S$. We shall say that $S$ has an *extendable normal bundle* if there exists a coherent sheaf of $\mathcal{C}^\infty_M$-modules $\mathcal{N}$ defined on an open neighborhood of $S$ in $M$ such that $\mathcal{N} \otimes_{\mathcal{O}_M} \mathcal{O}_{S^{\mathrm{reg}}} = \mathcal{N}_{S^{\mathrm{reg}}}$. We say that $\mathcal{N}$ is an *extension* of $\mathcal{N}_{S^{\mathrm{reg}}}$.

*Example* 6.4. Any nonsingular submanifold has an extendable normal bundle: an extension of $\mathcal{N}_S$ is given by the pull-back (under the retraction) to a tubular neighbourhood. If $S$ is singular but has codimension one in $M$ then the line bundle $\mathcal{O}([S])$ associated to the divisor $[S]$ provides an extension of $\mathcal{N}_{S^{\mathrm{reg}}}$, and hence $S$ has an extendable normal bundle. More generally, if $S$ is a *locally complete intersection defined by a section,* or a *strongly locally complete intersection,* then it has an extendable normal bundle (see [20] and [21]).

*Remark* 6.5. The extension of the normal bundle might be, in general, not unique. However, in all cases described in the previous example there is a natural extension to consider.

The next definition will considerably shorten several statements.

**Definition 6.6.** Let $S$ be a compact, complex, reduced, irreducible, possibly singular, subvariety of dimension $d$ of an $n$-dimensional complex manifold $M$. Assume that $S$ has extendable normal bundle. Let $\Sigma$ be an analytic subset of $S$, containing the singular part $S^{\mathrm{sing}}$ of $S$, so that $S^o = S \setminus \Sigma \subseteq S^{\mathrm{reg}}$, and let furthermore $\mathfrak{F}$ denote another analytic object involved in the problem (for instance, in our applications $\mathfrak{F}$ will be either a holomorphic foliation or a holomorphic self-map, and $\Sigma$ the union of the singular set of $S$ with the singular set of $\mathfrak{F}$). We shall say that $S$ *has the Lehmann-Suwa index property of level* $\ell \geq 1$ *on* $\Sigma$ *with respect to* $\mathfrak{F}$ if given an extension $\mathcal{N}$ of $\mathcal{N}_{S^{\mathrm{reg}}}$ we can associate to every homogeneous symmetric polynomial $\varphi$ of degree $k > d - \ell$ and every connected component $\Sigma_\lambda$ of $\Sigma$ a homology class

$$\mathrm{Res}_\varphi(\mathfrak{F}, \mathcal{N}; \Sigma_\lambda) \in H_{2(d-k)}(\Sigma_\lambda; \mathbb{C}),$$

depending only on $\mathcal{N}$ and on the local behavior of $\mathfrak{F}$ near $\Sigma_\lambda$, so that

$$\sum (i_\lambda)_* \mathrm{Res}_\varphi(\mathfrak{F}, \mathcal{N}; \Sigma_\lambda) = [S] \frown \varphi(\mathcal{N}) \quad \text{in } H_{2(d-k)}(S; \mathbb{C}),$$

where the sum ranges over all the connected components of $\Sigma$, the map $i_\lambda \colon \Sigma_\lambda \hookrightarrow S$ is the inclusion, and $\varphi(\mathcal{N})$ denotes the class obtained evaluating $\varphi$ in the Chern classes of $\mathcal{N}$.

*Remark* 6.7. When $k = d$ then $[S] \frown \varphi(\mathcal{N}) = \int_S \varphi(\mathcal{N}) \in \mathbb{C}$.

And now, our general index theorem:

**Theorem 6.8.** *Let $S$ be a compact, complex, reduced, irreducible, possibly singular, subvariety of dimension $d$ of an $n$-dimensional complex manifold $M$, and assume that $S$ has extendable normal bundle. Let $\Sigma$ be an analytic subset of $S$ containing $S^{\mathrm{sing}}$ such that there exist a sub-bundle $F$ of rank $\ell$ of $TS^o$ (where $S^o = S \setminus \Sigma \subseteq S^{\mathrm{reg}}$) and an $\mathcal{O}_{S^o}$-morphism $\psi \colon \mathcal{F} \to \mathcal{A}$ with $\theta_1 \circ \psi = \mathrm{id}$, where $\mathcal{A}$ is the Atiyah sheaf of $S^o$. Then:*

(i) *$S$ has the Lehmann-Suwa index property of level $\ell - \lfloor \ell/2 \rfloor$ on $\Sigma$ with respect to $\psi$.*
(ii) *If furthermore $F$ is involutive and $\psi$ is a Lie algebroid morphism, then $S$ has the Lehmann-Suwa index property of level $\ell$ on $\Sigma$ with respect to $\psi$.*

*Proof.* Theorem 5.9 yields a partial holomorphic connection on $N_{S^o}$ along $F$, which is flat in case (ii). Theorem 6.1 then implies that every symmetric polynomial in the Chern classes of $N_{S^o}$ of degree larger than $d-\ell+\lfloor \ell/2 \rfloor$ (or, in case (ii), larger than $d-\ell$) vanishes. The theorem then follows from the general Lehmann-Suwa theory (see, e.g., Chapter VI in [28], or [21]). □

*Remark* 6.9. For the sake of completeness, let us summarize here the gist of Lehmann-Suwa's argument, warning the reader that the complete proof is a bit technical and requires Čech-de Rham cohomology (see, e.g., [28]). Let

$$H^*(S, S^o; \mathbb{C}) \longrightarrow H^*(S; \mathbb{C}) \longrightarrow H^*(S^o; \mathbb{C})$$

be the long exact cohomology sequence of the pair $(S, S^o)$. The vanishing Theorem 6.1 says that the cohomology class $\varphi(\mathcal{N})$ vanishes when restricted to $S^o$; hence it must be the image of some cohomology class $\eta \in H^*(S, S^o; \mathbb{C})$ which is, by definition of cohomology of a pair, concentrated in an arbitrary neighbourhood of $S \setminus S^o = \Sigma$. Such a class is not unique in general, and it should be chosen in a suitable way depending on the partial holomorphic connection given by $\psi$. Now, since $S$ is compact, the Poincaré homomorphism (consisting exactly in taking the cap product with $[S]$) gives a natural map from $H^*(S; \mathbb{C})$ to $H_{2d-*}(S; \mathbb{C})$. On the other hand, since $\Sigma$ is an analytic subset of $S$, the Alexander homomorphism $A$ gives a natural map from $H^*(S, S^o; \mathbb{C})$ to $H_{2d-*}(\Sigma; \mathbb{C})$. Furthermore, if we denote by $i \colon \Sigma \to S$ the inclusion, we have the equality $i_* A(\eta) = [S] \frown \varphi(\mathcal{N})$. Now, if $\Sigma = \bigcup_\lambda \Sigma_\lambda$ is the decomposition in connected components of $\Sigma$, we have $H_{2d-*}(\Sigma; \mathbb{C}) = \bigoplus_\lambda H_{2d-*}(\Sigma_\lambda; \mathbb{C})$; therefore if we denote by $\mathrm{Res}_\varphi(\psi, \mathcal{N}; \Sigma_\lambda)$ the component of $A(\eta)$ belonging to $H_{2d-*}(\Sigma_\lambda; \mathbb{C})$, we obtain

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\psi, \mathcal{N}; \Sigma_\lambda) = [S] \frown \varphi(\mathcal{N}),$$

that is the index theorem.

*Remark* 6.10. Let us now describe how to compute $\mathrm{Res}_\varphi(\psi; \Sigma_\lambda)$ in a simple (but useful) case. Assume that: $S$ is a locally complete intersection defined by a section; the connected component $\Sigma_\lambda$ reduces to an isolated point $p \in S$; the sub-bundle $F$ has rank $\ell = 1$; and there exists a local vector field $v \in (\mathcal{T}_S)_p \subset (\mathcal{T}_{M,S})_p$ vanishing at $p$ and generating $F$ in a pointed neighbourhood of $p$. Let $l^1, \dots, l^m$ be a local system of defininig functions for $S$ near $p$, so that $\{[l^1]_2, \dots, [l^m]_2\}$ is a local frame for $\mathcal{I}_S/\mathcal{I}_S^2 = \mathcal{N}_S^*$, and denote by $\{\zeta_1, \dots, \zeta_m\}$ the corresponding dual local frame of $\mathcal{N}_S$. If $\tilde{X}^\psi$ is the partial holomorphic connection induced by $\psi$, then writing

$$\tilde{X}^\psi_v(\zeta_r) = c_r^s \zeta_s$$

we get an $m \times m$ matrix $C = (c^r_s)$ of holomorphic functions defined in a pointed neighbourhood of $p$. Finally (see [20]), it is possible to choose a local chart $(U, z)$ at $p$ so that if we write $v = [a^j]_1 \partial/\partial z^j$ then

$$\{l^1 = \cdots = l^m = a^{m+1} = \cdots = a^n = 0\} = \{p\}$$

(if $p$ is a regular point of $S$ it suffices to take any chart adapted to $S$). Take now a homogeneous symmetric polynomial $\varphi$ of degree $\dim S$; then $\mathrm{Res}_\varphi(\psi; \{p\})$ is given by the Grothendieck residue

$$\mathrm{Res}_\varphi(\psi, \mathcal{N}; \{p\}) = \frac{1}{(2\pi i)^{n-m}} \int_\Gamma \frac{\varphi(C)}{a^{m+1} \cdots a^n} \, dz^{m+1} \wedge \cdots \wedge dz^n, \tag{30}$$

where $\Gamma = \{q \in S \mid |a^{m+1}(q)| = \cdots = |a^n(q)| = \varepsilon\}$ for $0 < \varepsilon << 1$, oriented so that $d \arg a^{m+1} \wedge \cdots \wedge d \arg a^n$ is positive, $\mathcal{N}$ is the natural extension of $\mathcal{N}_{S^{\mathrm{reg}}}$ mentioned in Remark 6.5, and $\varphi(C)$ denotes $\varphi$ evaluated on the eigenvalues of the matrix $C$. This formula can be obtained by observing that if $\tilde{v} \in \mathcal{T}^S_{M,S(1)}$ is such that $\pi(\tilde{v}) = \psi(v)$ outside $p$, then the local partial holomorphic connection on $\mathcal{N}_{S^{\mathrm{reg}}}$ induced by $\tilde{v}$ coincides with $\tilde{X}^\psi_v$, and the residue $\mathrm{Res}_\varphi(\psi, \mathcal{N}; \{p\})$ coincides with the residue associated to $\tilde{v}$ and obtained in [20]. By the way, an explicit algorithm for computing the Grothendieck residue (30) when $p$ is a regular point of $S$ is described in [5], p. 280.

We end this section by describing the general strategy we are going to use to build the morphism $\psi \colon \mathcal{F} \to \mathcal{A}$. Such a morphism exists if and only if the sequence

$$O \longrightarrow \mathrm{Hom}(\mathcal{N}_{S^o}, \mathcal{N}_{S^o}) \longrightarrow \theta_1^{-1}(\mathcal{F}) \xrightarrow{\theta_1} \mathcal{F} \longrightarrow O$$

splits, that is if and only if the associated cohomology class in $H^1\big(S^o, \mathcal{F}^* \otimes \mathrm{Hom}(\mathcal{N}_{S^o}, \mathcal{N}_{S^o})\big)$ vanishes. The latter class is represented by a cocycle of the form $\{\psi_\beta - \psi_\alpha\}$, where the $\psi_\alpha$ are local splitting morphisms. Therefore the morphism $\psi$ exists if and only if we can find local morphisms $x_\alpha$ from $\mathcal{F}$ to $\mathrm{Hom}(\mathcal{N}_{S^o}, \mathcal{N}_{S^o})$ such that $\psi_\beta - \psi_\alpha = x_\beta - x_\alpha$.

Our strategy then will be to use the additional data involved (foliation or self-map) to build local splitting morphisms; in this way we shall be able to express the cohomological problem in terms of the geometry of the additional data, and then to give sufficient conditions for the problem to be solvable.

Notice in particular that if $S^o$ is Stein then this cohomological problem is *always* solvable, and thus we have

**Corollary 6.11.** *Let $S$ be a compact, complex, reduced, irreducible, possibly singular, subvariety of dimension $d$ of an $n$-dimensional complex manifold $M$, and assume that $S$ has extendable normal bundle. Let $\Sigma$ be an analytic subset containing $S^{\mathrm{sing}}$ such that $S^o = S \setminus \Sigma$ is Stein. Then $S$ has the Lehmann-Suwa index property of level $d - \lfloor d/2 \rfloor$ on $\Sigma$ with respect to anything providing local splitting morphisms for the sequence* (25) *over $S^o$.*

## 7. Holomorphic foliations

The aim of this section is to show how to use a holomorphic foliation on the ambient manifold to implement the strategy just discussed. We recall that a (possibly singular) *holomorphic foliation* $\mathcal{F}$ on a complex $n$-dimensional manifold $M$ is, by definition, a coherent involutive subsheaf of $\mathcal{T}_M$. The *singular locus* $\mathrm{Sing}(\mathcal{F})$ is the set of points $x \in M$ such that the quotient $\mathcal{T}_M/\mathcal{F}$ is not free at $x$. In particular, $\mathcal{F}$ is a locally free $\mathcal{O}_M$-module of some rank $1 \le \ell \le n$ outside $\mathrm{Sing}(\mathcal{F})$; the number $\ell$ is the *dimension* of the holomorphic folitaion. The foliation is called *non-singular* if the singular locus is empty. We refer to [5] and [28], chapter VI, for more details on holomorphic foliations.

**Definition 7.1.** Let $S$ be a complex (not necessarily closed) $m$-codimensional submanifold of an $n$-dimensional complex manifold $M$, and let $\mathcal{F}$ be a (possibly singular) holomorphic foliation $\mathcal{F}$ on $M$, of dimension $\ell \le n-m = \dim S$. We shall denote by $\mathcal{F}_{S(1)}$ the $\mathcal{O}_{S(1)}$-submodule $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_{S(1)}$ of $\mathcal{T}_{M,S(1)}$, and by $\mathcal{F}_S$ the $\mathcal{O}_S$-submodule $\mathcal{F} \otimes_{\mathcal{O}_M} \mathcal{O}_S$ of $\mathcal{T}_{M,S}$. If $\mathcal{F}_S \subseteq \mathcal{T}_S \subset \mathcal{T}_{M,S}$, then $\mathcal{F}$ is *tangent* to $S$; otherwise, the foliation $\mathcal{F}$ is *tranverse* to $S$.

*Remark* 7.2. We shall always assume that $S$ is not contained in the singular locus of $\mathcal{F}$.

In the tangential case, we clearly have $\mathcal{F}_{S(1)} \subseteq \mathcal{T}^S_{M,S(1)}$. Furthermore, $\mathcal{F}_S$ is a (possibly singular) holomorphic foliation of $S$ of dimension $\ell$. The singular locus of $\mathcal{F}_S$ (which is the intersection of $\mathrm{Sing}(\mathcal{F})$ with $S$) is an analytic subset of $S$; therefore since our aim is to build a splitting morphism $\psi$ outside the singularities, we shall assume that

*Case* 1. $\mathcal{F}_S$ is a non-singular holomorphic foliation of $S$ of dimension $\ell \le \dim S$ (and thus, in particular, it is the sheaf of germs of holomorphic sections of an involutive subbundle $F$ of $TS$ of rank $\ell$). To be consistent with the non-tangential case, we shall also set $\mathcal{F}^\sigma = \mathcal{F}_S$ and $\sigma^* = \mathrm{id}_{\mathcal{F}_S}$.

If $\mathcal{F}$ is not tangent to $S$ then $\mathcal{F}_S$ is not a subsheaf of $\mathcal{T}_S$, but only of $\mathcal{T}_{M,S}$. To get a subsheaf of $\mathcal{T}_S$, we must project $\mathcal{F}_S$ into it.

**Definition 7.3.** Let $S$ be a splitting submanifold of a complex manifold $M$. Given a first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^2$, let $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$ be the left splitting morphism associated to $\rho$ by Proposition 2.7. If $\mathcal{F}$ is a holomorphic foliation on $M$ of dimension $\ell \le \dim S$, we shall denote by $\mathcal{F}^\sigma$ the coherent sheaf of $\mathcal{O}_S$-modules given by

$$\mathcal{F}^\sigma = \sigma^*(\mathcal{F}_S) \subseteq \mathcal{T}_S.$$

We shall say that $\rho$ is *$\mathcal{F}$-faithful outside an analytic subset* $\Sigma \subset S$ if $\mathcal{F}^\sigma$ is a non-singular holomorphic foliation of dimension $\ell$ on $S \setminus \Sigma$. If $\Sigma = \varnothing$ we shall simply say that $\rho$ is *$\mathcal{F}$-faithful.*

It might happen that a first order lifting is not $\mathcal{F}$-faithful while another one is. Furthermore, $\mathcal{F}^\sigma$ might be as well as not be involutive depending on the choice of $\rho$.

*Example* 7.4. Let $M = \mathbb{C}^4$, take $S = \{z^1 = 0\}$ and let $\mathcal{F}$ be the non-singular foliation generated over $\mathcal{O}_M$ by the global vector fields $(z^2 - z^1)\frac{\partial}{\partial z^3} + \frac{\partial}{\partial z^4}$ and $\frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^2}$, so that $\mathcal{F}_S$ is generated over $\mathcal{O}_S$ by $z^2\frac{\partial}{\partial z^3} + \frac{\partial}{\partial z^4}$ and $\frac{\partial}{\partial z^1} + \frac{\partial}{\partial z^2}$. The submanifold $S$ clearly splits into $M$, and a natural choice of first order lifting is

$$\rho([f]_1) = [f]_2 - \left[\frac{\partial f}{\partial z^1} z^1\right]_2 .$$

The corresponding left splitting morphism $\sigma^*$ is the identity on $\mathcal{T}_S$ and kills $\frac{\partial}{\partial z^1}$; therefore $\mathcal{F}^\sigma$ is generated over $\mathcal{O}_S$ by $z^2\frac{\partial}{\partial z^3} + \frac{\partial}{\partial z^4}$ and $\frac{\partial}{\partial z^2}$, and thus $\mathcal{F}^\sigma$ is not involutive.

If we choose as first order lifting the less standard $\rho_1$ given by

$$\rho_1([f]_1) = [f]_2 - \left[\left(\frac{\partial f}{\partial z^1} + \frac{\partial f}{\partial z^2}\right) z^1\right]_2 ,$$

then the corresponding left splitting morphism $\sigma_1^*$ sends $\frac{\partial}{\partial z^1}$ in $-\frac{\partial}{\partial z^2}$, and $\mathcal{F}^{\sigma_1}$ turns out to be generated by $z^2\frac{\partial}{\partial z^3} + \frac{\partial}{\partial z^4}$ only, and so it is involutive, but of the wrong dimension. Finally, if we take as first order lifting

$$\rho_2([f]_1) = [f]_2 - \left[\left(\frac{\partial f}{\partial z^1} + \frac{\partial f}{\partial z^2} - \frac{\partial f}{\partial z^3}\right) z^1\right]_2$$

then $\sigma_2^*$ sends $\frac{\partial}{\partial z^1}$ in $\frac{\partial}{\partial z^3} - \frac{\partial}{\partial z^2}$, so that $\mathcal{F}^{\sigma_2}$ is generated by $z^2\frac{\partial}{\partial z^3} + \frac{\partial}{\partial z^4}$ and $\frac{\partial}{\partial z^3}$, and thus $\rho_2$ is $\mathcal{F}$-faithful.

If $\mathcal{F}^\sigma$ has dimension equal to 1 or to the dimension of $S$, then it is automatically involutive. In this case it is easy to have faithfulness:

**Lemma 7.5.** *Let $S$ be a splitting submanifold of a complex manifold $M$, and let $\mathcal{F}$ be a holomorphic foliation on $M$ of dimension equal to 1 or to the dimension of $S$. If there exists $x_0 \in S \setminus \mathrm{Sing}(\mathcal{F})$ such that $\mathcal{F}$ is tangent to $S$ at $x_0$, i.e., $(\mathcal{F}_S)_{x_0} \subseteq \mathcal{T}_{S,x_0}$, then any first order lifting is $\mathcal{F}$-faithful outside a suitable analytic subset of $S$.*

*Proof.* Let $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$ be the left-splitting morphism associated to a first order lifting $\rho$. By assumption, $\ker \sigma^*_{x_0} \cap (F_S)_{x_0} = (O)$; therefore $\ker \sigma^*_x \cap (F_S)_x = (O)$ for all $x \in S$ outside an analytic subset $\Sigma_0$ of $S$. Furthermore, $\mathcal{F}^\sigma$ has dimension equal to 1 or to $\dim S$; therefore it is involutive, and hence $\rho$ is $\mathcal{F}$-faithful outside $\Sigma_0 \cup \mathrm{Sing}(\mathcal{F}^\sigma)$. □

Notice that there are topological obstructions for a foliation to be everywhere non-tangential to $S$. For instance, in [9], [18] it is proved that if $S$ is a curve in a surface $M$, the number of points of tangency between $S$ and an one-dimensional holomorphic reduced foliation $\mathcal{F}$ of $M$, counted with multiplicity, is $S \cdot S - S \cdot \mathcal{F}$. Therefore if $S \cdot S \neq S \cdot \mathcal{F}$ then every first order lifting is $\mathcal{F}$-faithful outside a suitable analytic subset.

Another result of this kind shows that for one-dimensional foliations most first order liftings are faithful:

**Lemma 7.6.** *Let $S$ be a non-singular hypersurface splitting in a complex manifold $M$, and let $\mathcal{F}$ be a one dimensional holomorphic foliation on $M$. Assume that $S$ is not contained in* $\mathrm{Sing}(\mathcal{F})$*. Then there is at most one first order lifting $\rho$ which is not $\mathcal{F}$-faithful outside a suitable analytic subset of $S$.*

*Proof.* Suppose $\rho$ is a first order lifting of $S$ which is *not* $\mathcal{F}$-faithful; since $\mathcal{F}$ is one-dimensional, this means that $(\mathcal{F}_S)_x \subseteq \ker \sigma^*_x$ for all $x \in S \setminus \mathrm{Sing}(\mathcal{F})$, where $\sigma^*$ is the left splitting morphism associated to $\rho$. By Lemma 2.1.(iii) any other left splitting morphism is of the form $\sigma^*_1 = \sigma^* + \varphi \circ p_2$ with $\varphi \in H^0\big(S, \mathrm{Hom}(\mathcal{N}_S, \mathcal{T}_S)\big)$; in particular, $\sigma^*_1(v) = \varphi\big(p_2(v)\big)$ for all $v \in \mathcal{F}_S$. Now, since $\sigma^*$ is a left splitting morphism, we have $\ker \sigma^*_x \cap \ker(p_2)_x = (O)$ for all $x \in S$; therefore $p_2|_{\mathcal{F}_S}$ is injective. Furthermore, since $\mathcal{N}_S$ has rank one, $\varphi_x$ is either injective or identically zero; hence $(\sigma^*_1)_x$ restricted to $(\mathcal{F}_S)_x$ is either injective or identically zero. Now, if $\varphi \neq O$ then $\varphi_x \neq O$ for $x$ outside an analytic subset $\Sigma_0$ of $S$; therefore it follows that if $\varphi \neq O$ then the first order lifting associated to $\sigma^*_1$ is $\mathcal{F}$-faithful outside $\Sigma_0 \cup \mathrm{Sing}(\mathcal{F}^{\sigma_1})$. □

**Corollary 7.7.** *Let $S$ be a non-singular hypersurface splitting in a complex manifold $M$, and let $\mathcal{F}$ be a one dimensional holomorphic foliation on $M$. Assume that $S$ is not contained in* $\mathrm{Sing}(\mathcal{F})$*. If $H^0(S, \mathcal{T}_S \otimes \mathcal{N}^*_S) \neq (O)$ then there exists at least one first order lifting $\mathcal{F}$-faithful outside a suitable anlytic subset of $S$.*

*Proof.* If $H^0(S, \mathcal{T}_S \otimes \mathcal{N}^*_S) \neq (O)$ then by Lemma 2.1.(iii) there exist at least two different splitting morphisms. Then the assertion follows from the previous lemma. □

Coming back to our main concern, in the non-tangential case we shall momentarily make the following assumption:

*Case* 2. There exists an $\mathcal{F}$-faithful first order lifting $\rho$, with associated left-splitting morphism $\sigma^*$; in particular, $\mathcal{F}^\sigma$ is a non-singular holomorphic foliation of $S$ of dimension $\ell \leq \dim S$, and $\sigma^*|_{\mathcal{F}_S} \colon \mathcal{F}_S \to \mathcal{F}^\sigma$ is an isomorphism of $\mathcal{O}_S$-modules.

If $\mathcal{G} \subseteq \mathcal{T}_S$ is a non-singular holomorphic foliation of $S$ of dimension $\ell \leq \dim S$, Frobenius' theorem implies that we can always find an atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ such that the $\{\partial/\partial z^{m+1}_\alpha, \ldots, \partial/\partial z^{m+\ell}_\alpha\}$ are local frames for $\mathcal{G}$. Furthermore, it is easy to check that if $S$ is split (2-split, comfortably embedded) in $M$ we can also assume that $\mathfrak{U}$ is a splitting (2-splitting, comfortable) atlas.

**Definition 7.8.** Assume we are either in Case 1 or in Case 2. An atlas $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ adapted to $S$ such that the $\{\partial/\partial z^{m+1}_\alpha, \ldots, \partial/\partial z^{m+\ell}_\alpha\}$ are local frames for $\mathcal{F}^\sigma$ shall be said *adapted to $S$ and $\mathcal{F}$.* We explicitly notice that if $\mathfrak{U}$ is adapted to $S$ and $\mathcal{F}$ then

$$\frac{\partial z^{q''}_\beta}{\partial z^{p'}_\alpha} \in \mathcal{I}_S$$

for all $p' = m+1, \ldots, m+\ell$, $q'' = m+\ell+1, \ldots, n$ and indices $\alpha$, $\beta$ such that $U_\alpha \cap U_\beta \cap S \neq \varnothing$.

*Remark* 7.9. From now on, indices like $p'$, $q'$, $\tilde{p}'$ and $\tilde{q}'$ will run from $m+1$ to $m+\ell$, while indices like $p''$ and $q''$ will run from $m+\ell+1$ to $n$.

Using adapted atlas we can find special local frames for the foliation:

**Lemma 7.10.** *Assume we are in Case 1 or in Case 2, and let $\{(U_\alpha, z_\alpha)\}$ be an atlas adapted to $S$ and $\mathcal{F}$ (and to $\rho$ too in Case 2).*

(i) *For each index $\alpha$ there exists a unique $\ell$-uple $(v_{\alpha,m+1}, \dots, v_{\alpha,m+\ell})$ of elements of $\mathcal{F}$ of the form*

$$v_{\alpha,p'} = \frac{\partial}{\partial z_\alpha^{p'}} + (a_\alpha)_{p'}^r \frac{\partial}{\partial z_\alpha^r} + (a_\alpha)_{p'}^{p''} \frac{\partial}{\partial z_\alpha^{p''}} \tag{31}$$

*with $(a_\alpha)_{p'}^{p''} \in \mathcal{I}_S$ for $p' = m+1, \dots, m+\ell$ and $p'' = m+\ell+1, \dots, n$, (and, in Case 1, $(a_\alpha)_{p'}^r \in \mathcal{I}_S$ for $r = 1, \dots, m$), so that $\sigma^*(v_{\alpha,p'} \otimes [1]_1) = \partial/\partial z_\alpha^{p'}$.*

(ii) *The set $\{v_{\alpha,m+1}, \dots, v_{\alpha,m+\ell}\}$ is a local frame for the sheaf $\mathcal{F}$ in a neighbourhood of $S \cap U_\alpha$. Furthermore, writing*

$$v_{\beta,q'} = (c_{\beta\alpha})_{q'}^{p'} v_{\alpha,p'}$$

*the $(c_{\beta\alpha})_{q'}^{p'} \in \mathcal{O}_M$ define a cocycle $(c_{\beta\alpha})$ representing the vector bundle associated to $\mathcal{F}$ in a neighbourhood of $S$ and satisfy the following relations:*

$$\begin{cases} (c_{\beta\alpha})_{q'}^{p'} = \dfrac{\partial z_\alpha^{p'}}{\partial z_\beta^{q'}} + (a_\beta)_{q'}^s \dfrac{\partial z_\alpha^{p'}}{\partial z_\beta^s} + (a_\beta)_{q'}^{q''} \dfrac{\partial z_\alpha^{p'}}{\partial z_\beta^{q''}}, \\ (c_{\beta\alpha})_{q'}^{p'} (a_\alpha)_{p'}^r = \dfrac{\partial z_\alpha^r}{\partial z_\beta^{q'}} + (a_\beta)_{q'}^s \dfrac{\partial z_\alpha^r}{\partial z_\beta^s} + (a_\beta)_{q'}^{q''} \dfrac{\partial z_\alpha^r}{\partial z_\beta^{q''}}, \\ (c_{\beta\alpha})_{q'}^{p'} (a_\alpha)_{p'}^{p''} = \dfrac{\partial z_\alpha^{p''}}{\partial z_\beta^{q'}} + (a_\beta)_{q'}^s \dfrac{\partial z_\alpha^{p''}}{\partial z_\beta^s} + (a_\beta)_{q'}^{q''} \dfrac{\partial z_\alpha^{p''}}{\partial z_\beta^{q''}}. \end{cases} \tag{32}$$

*Proof.* (i) Since $\{(U_\alpha, z_\alpha)\}$ is adapted to $S$ and $\mathcal{F}$, the $\partial/\partial z_\alpha^{p'}$'s form local frames for $\mathcal{F}^\sigma$. Hence we can find $\tilde{v}_{\alpha,p'} \in \mathcal{F}$ such that $\sigma^*(\tilde{v}_{\alpha,p'} \otimes [1]_1) = \partial/\partial z_\alpha^{p'}$. Write

$$\tilde{v}_{\alpha,p'} = (b_\alpha)_{p'}^{q'} \frac{\partial}{\partial z_\alpha^{q'}} + (b_\alpha)_{p'}^r \frac{\partial}{\partial z_\alpha^r} + (b_\alpha)_{p'}^{q''} \frac{\partial}{\partial z_\alpha^{q''}}$$

for suitable $(b_\alpha)_{p'}^j \in \mathcal{O}_M$; we must have $[(b_\alpha)_{p'}^q]_1 = \delta_{p'}^q$, and $[(b_\alpha)_{p'}^r]_1 = 0$ in Case 1. In particular, $([(b_\alpha)_{p'}^{q'}]_1)$ is the identity matrix; hence $\big((b_\alpha)_{p'}^{q'}\big)$ is invertible as matrix of germs. Multiplying then the $\ell$-uple $(\tilde{v}_{\alpha,m+1}, \dots, \tilde{v}_{\alpha,m+\ell})$ by the inverse of this matrix we get an $\ell$-uple $(v_{\alpha,m+1}, \dots, v_{\alpha,m+\ell})$ of elements of $\mathcal{F}$ of the desired form. Furthermore, since $\mathrm{rk}_{\mathcal{O}_M} \mathcal{F} = \ell$, the $v_{\alpha,p'}$'s form a local frame for $\mathcal{F}$; an exercise in linear algebra then shows that they are uniquely determined.

(ii) The elements $v_{\alpha,m+1},\dots,v_{\alpha,m+\ell}$ form a local frame for the sheaf $\mathcal{F}$ in a neighbourhood of $S\cap U_\alpha$ since their restriction to $S$ form a local frame for $\mathcal{F}_S$ on $U_\alpha\cap S$. The relations (32) then follows directly from (31). □

Restricting to $S$ the first equation in (32) we get:

**Corollary 7.11.** *Assume we are in Case 1 or in Case 2, let $\{(U_\alpha,z_\alpha)\}$ be an atlas adapted to $S$ and $\mathcal{F}$ (and to $\rho$ too in Case 2), and let $v_{\alpha,m+1},\dots,v_{\alpha,m+\ell}$ be given by the previous lemma. Then the vector bundles associated to $\mathcal{F}_S$ and $\mathcal{F}^\sigma$ are represented by the same cocycle*

$$[(c_{\beta\alpha})^{p'}_{q'}]_1=\left[\frac{\partial z^{p'}_\alpha}{\partial z^{q'}_\beta}\right]_1. \tag{33}$$

*in the frames $\{v_{\alpha,m+1}\otimes[1]_1,\dots,v_{\alpha,m+\ell}\otimes[1]_1\}$ and $\{\partial/\partial z^{m+1}_\alpha,\dots,\partial/\partial z^{m+\ell}_\alpha\}$, respectively. In particular, the isomorphism $\sigma^*|_{\mathcal{F}_S}\colon\mathcal{F}_S\to\mathcal{F}^\sigma$ is represented with respect to these frames by the identity matrix.*

The restriction to $S$ of the second equation in (32) gives

$$[(c_{\beta\alpha})^{p'}_{q'}]_1[(a_\alpha)^r_{p'}]_1=[(a_\beta)^s_{q'}]_1\left[\frac{\partial z^r_\alpha}{\partial z^s_\beta}\right]_1,$$

and thus we get a global section $T\in H^0(S,(\mathcal{F}^\sigma)^*\otimes\mathcal{N}_S)$ by setting

$$T|_{U_\alpha}=[(a_\alpha)^r_{p'}]_1\,\omega^{p'}_\alpha\otimes\partial_{r,\alpha},$$

where $\omega^{p'}_\alpha$ is the local section of $(\mathcal{F}^\sigma)^*$ induced by $dz^{p'}_\alpha$. It is easy to check that the corresponding morphism $T\colon\mathcal{F}^\sigma\to\mathcal{N}_S$ is given by the composition

$$T\colon\mathcal{F}^\sigma\xrightarrow{(\sigma^*|_{\mathcal{F}_S})^{-1}}\mathcal{F}_S\hookrightarrow\mathcal{T}_{M,S}\xrightarrow{p_2}\mathcal{N}_S.$$

*Remark* 7.12. The morphism $T$ is non-zero if and only if the foliation $\mathcal{F}$ is transversal to $S$.

Now we are ready to characterize the existence of morphisms $\psi\colon\mathcal{F}^\sigma\to\mathcal{A}$ such that $\theta_1\circ\psi=\mathrm{id}$:

**Proposition 7.13.** *Assume we are in Case 1, or in Case 2 with $S$ comfortably embedded in $M$. Given a (comfortable in Case 2) atlas $\mathfrak{U}=\{(U_\alpha,z_\alpha)\}$ adapted to $S$ and $\mathcal{F}$ (and to $\rho$ too in Case 2), let $\{f_{\beta\alpha}\}$ be the cocycle defined by*

$$\begin{aligned} f_{\beta\alpha} &= \left[(c_{\alpha\beta})^{q'}_{p'}\right]_1\tilde\rho\left(\left[(c_{\beta\alpha})^{\tilde p'}_{q'}\right]_2\right)\otimes\frac{\partial}{\partial z^{\tilde p'}_\alpha}\otimes\omega^{p'}_\alpha\\ &= \left.\frac{\partial z^{q'}_\beta}{\partial z^{p'}_\alpha}\frac{\partial(c_{\beta\alpha})^{\tilde p'}_{q'}}{\partial z^t_\alpha}\right|_S\left[z^t_\alpha\right]_2\otimes\frac{\partial}{\partial z^{\tilde p'}_\alpha}\otimes\omega^{p'}_\alpha, \end{aligned}$$

*and denote by* $\mathfrak{f} \in H^1\big(S, \mathcal{N}_S^* \otimes \mathcal{F}^\sigma \otimes (\mathcal{F}^\sigma)^*\big)$ *the corresponding cohomology class. Then there exists a morphism* $\psi\colon \mathcal{F}^\sigma \to \mathcal{A}$ *such that* $\theta_1 \circ \psi = \mathrm{id}$ *if and only if*

$$T_*(\mathfrak{f}) = O$$

*in* $H^1\big(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes (\mathcal{F}^\sigma)^*\big)$*, where* $T_*$ *is the map induced in cohomology by the morphism* $\mathrm{id} \otimes T \otimes \mathrm{id}$*.*

*Proof.* It is easy to check that $\mathfrak{f}$ is a well-defined cohomology class indepedent of the particular atlas $\mathfrak{U}$ chosen. Thus to prove the assertion it suffices to find local splittings $\psi_\alpha\colon \mathcal{F}^\sigma|_{U_\alpha \cap S} \to \mathcal{A}|_{U_\alpha \cap S}$ so that $\{\psi_\beta - \psi_\alpha\}$ represents the cohomology class $T_*(\mathfrak{f})$.

A local frame for $\mathcal{F}^\sigma$ is $\{\partial/\partial z_\alpha^{m+1}, \ldots, \partial/\partial z_\alpha^{m+\ell}\}$. We then define $\psi_\alpha$ by setting

$$\psi_\alpha\left(\frac{\partial}{\partial z_\alpha^{p'}}\right) = \tilde{\pi}(v_{\alpha,p'} \otimes [1]_2) = \pi\left(\frac{\partial}{\partial z_\alpha^{p'}} + \tilde{\rho}([(a_\alpha)^r_{p'}]_2)\frac{\partial}{\partial z_\alpha^r}\right) \tag{34}$$

and then extending by $\mathcal{O}_S$-linearity, where $\pi\colon \mathcal{T}^S_{M,S(1)} \to \mathcal{A}$ is the canonical projection, and $\tilde{\pi}\colon \mathcal{T}_{M,S(1)} \to \mathcal{A}$ is the morphism introduced in Proposition 5.12. Notice that, in Case 1, $v_{\alpha,p'} \otimes [1]_2 \in \mathcal{T}^S_{M,S(1)}$, and so $\psi_\alpha$ is defined without assuming anything on the embedding of $S$ into $M$.

Now, we have:

$$\begin{aligned}
\psi_\beta\left(\frac{\partial}{\partial z_\alpha^{p'}}\right) &= \psi_\beta\left(\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \frac{\partial}{\partial z_\beta^{q'}}\right) = \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \pi\left(\frac{\partial}{\partial z_\beta^{q'}} + \tilde{\rho}\big(\big[(a_\beta)^s_{q'}\big]_2\big)\frac{\partial}{\partial z_\beta^s}\right) \\
&= \pi\left(\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_2 \frac{\partial}{\partial z_\beta^{q'}} + \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_2 \tilde{\rho}([(a_\beta)^s_{q'}]_2)\frac{\partial}{\partial z_\beta^s}\right) \\
&= \pi\left(\frac{\partial}{\partial z_\alpha^{p'}} + \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_2 \left[\frac{\partial z_\alpha^r}{\partial z_\beta^{q'}}\right]_2 \frac{\partial}{\partial z_\alpha^r}\right) \\
&\quad + \pi\left(\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_2 \tilde{\rho}([(a_\beta)^s_{q'}]_2) \left[\frac{\partial z_\alpha^r}{\partial z_\beta^s}\right]_2 \frac{\partial}{\partial z_\alpha^r}\right)
\end{aligned}$$

Hence

$$\begin{aligned}
(\psi_\beta &- \psi_\alpha)\left(\frac{\partial}{\partial z_\alpha^{p'}}\right) \\
&= \pi\left(\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \left[\frac{\partial z_\alpha^r}{\partial z_\beta^{q'}}\right]_2 \frac{\partial}{\partial z_\alpha^r}\right) \\
&\quad + \pi\left(\left\{\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \tilde{\rho}([(a_\beta)^s_{q'}]_2) \left[\frac{\partial z_\alpha^r}{\partial z_\beta^s}\right]_1 - \tilde{\rho}([(a_\alpha)^r_{p'}]_2)\right\} \frac{\partial}{\partial z_\alpha^r}\right)
\end{aligned}$$

In Case 1 the second line of (32) yields $\psi_\beta - \psi_\alpha \equiv O$. In Case 2, applying $\tilde\rho$ to the second line of (32), and recalling that we are using a comfortable atlas, we get

$$\left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \tilde\rho([(a_\beta)^s_{q'}]_2)\left[\frac{\partial z_\alpha^r}{\partial z_\beta^s}\right]_1 - \tilde\rho([(a_\alpha)^r_{p'}]_2) + \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \left[\frac{\partial z_\alpha^r}{\partial z_\beta^{q'}}\right]_2 = \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \tilde\rho([(c_{\beta\alpha})^{\tilde p'}_{q'}]_2)[(a_\alpha)^r_{\tilde p'}]_1.$$

Hence

$$(\psi_\beta - \psi_\alpha)\left(\frac{\partial}{\partial z_\alpha^{p'}}\right) = [(a_\alpha)^r_{\tilde p'}]_1 \left[\frac{\partial z_\beta^{q'}}{\partial z_\alpha^{p'}}\right]_1 \left[\frac{\partial (c_{\beta\alpha})^{\tilde p'}_{q'}}{\partial z_\alpha^t}\right]_1 \pi\left([z_\alpha^t]_2 \frac{\partial}{\partial z_\alpha^r}\right), \tag{35}$$

and we are done. □

**Corollary 7.14.** *Let $S$ be a complex $m$-codimensional submanifold of an $n$-dimensional complex manifold $M$, and let $\mathcal{F}$ be a holomorphic foliation $\mathcal{F}$ on $M$, of dimension $\ell \le n - m = \dim S$ tangent to $S$. Assume that $\mathcal{F}_S$ is a non-singular holomorphic foliation of $S$. Then we can always find an $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F}_S \to \mathcal{A}$ such that $\theta_1 \circ \psi = \mathrm{id}$ which is furthermore a Lie algebroid morphism.*

*Proof.* In this case (35) shows that (34) defines a global morphism $\psi$. To prove that it is a Lie algebroid morphism, it suffices to show that $\{\tilde\pi(v_{\alpha,p'} \otimes [1]_2), \tilde\pi(v_{\alpha,q'} \otimes [1]_2)\} = O$ for all $p', q' = m+1, \ldots, m+\ell$. But since $\mathcal{F}$ is tangent to $S$, we have $\tilde\pi(v_{\alpha,p'} \otimes [1]_2) = \pi(v_{\alpha,p'} \otimes [1]_2)$; hence it suffices to show that

$$\{v_{\alpha,p'} \otimes [1]_2, v_{\alpha,q'} \otimes [1]_2\} \in \mathcal{I}_S \cdot \mathcal{T}^S_{M,S(1)}. \tag{36}$$

Now, $\{v_{\alpha,m+1}, \ldots, v_{\alpha,m+\ell}\}$ is a local frame for $\mathcal{F}$, which is an involutive sheaf; it follows that $[v_{\alpha,p'}, v_{\alpha,q'}] = O$, and (36) is an immediate consequence. □

**Definition 7.15.** Let $S$ be a complex submanifold of a complex manifold $M$, and $\mathcal{F}$ a holomorphic foliation of $M$ of dimension $d \le \dim S$. Assume that $S$ splits into $M$, with first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated projection $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$. We shall say that $\mathcal{F}$ *splits along* $\rho$ if $\mathfrak{f} = O$ in $H^1\big(S, \mathrm{Hom}(\mathcal{F}^\sigma, \mathcal{N}_S^* \otimes \mathcal{F}^\sigma)\big)$.

**Corollary 7.16.** *Let $S$ be a complex $m$-codimensional submanifold of an $n$-dimensional complex manifold $M$, and let $\mathcal{F}$ be a holomorphic foliation $\mathcal{F}$ on $M$, of dimension $\ell \le n - m = \dim S$ tangent to $S$. Assume that $S$ is comfortably embedded in $M$ with respect to an $\mathcal{F}$-faithful first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$. If $\mathcal{F}$ splits along $\rho$ then there is an $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F}^\sigma \to \mathcal{A}$ such that $\theta_1 \circ \psi = \mathrm{id}$.*

*Remark* 7.17. In general, the morphism $\psi$ provided by the previous corollary might not be a Lie algebroid morphism, unless $\ell = 1$.

*Remark* 7.18. As a consequence of (33), we know that

$$\rho([(c_{\beta\alpha})^{p'}_{q'}]_1) = \rho\left(\left[\frac{\partial z^{p'}_\alpha}{\partial z^{q'}_\beta}\right]_1\right) = \left[\frac{\partial z^{p'}_\alpha}{\partial z^{q'}_\beta}\right]_2 ;$$

therefore $\tilde\rho\big([(c_{\beta\alpha})^{p'}_{q'}]_2\big) = O$ is equivalent to

$$[(c_{\beta\alpha})^{p'}_{q'}]_2 = \left[\frac{\partial z^{p'}_\alpha}{\partial z^{q'}_\beta}\right]_2 ;$$

compare with Corollary 7.11.

Since we are using an adapted atlas, the first line of (32) yields

$$\tilde\rho\big([(c_{\beta\alpha})^{p'}_{q'}]_2\big) = [(a_\beta)^s_{q'}]_1 \left[\frac{\partial z^{p'}_\alpha}{\partial z^s_\beta}\right]_2 + \left[\frac{\partial z^{p'}_\alpha}{\partial z^{q''}_\beta}\right]_1 [(a_\beta)^{q''}_{q'}]_2. \tag{37}$$

This suggests a couple of sufficient conditions for the splitting of $\mathcal{F}$ along $\rho$:

**Corollary 7.19.** *Let $S$ be a comfortably embedded submanifold of a complex manifold $M$, with first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated left splitting morphism $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$. Let $\mathcal{F}$ be a holomorphic foliation of $M$ of dimension $\ell \le \dim S$ such that $\rho$ is $\mathcal{F}$-faithful. Assume moreover that one of the following conditions is satisfied:*

(a) *$S$ is 2-linearizable, and $\ell = \dim S$;*
(b) *$S$ is 2-linearizable, and there exists a nonsingular holomorphic foliation of $S$ transversal to $\mathcal{F}^\sigma$.*

*Then $\mathcal{F}$ splits along $\rho$.*

*Proof.* In both cases we can find a comfortable 2-splitting atlas $\{(U_\alpha, z_\alpha)\}$ adapted to $S$, $\mathcal{F}^\sigma$ and $\rho$ such that $\tilde\rho([(c_{\beta\alpha})^{p'}_{q'}]_2) = 0$ always. In case (a) this follows directly from (37), because $m + \ell = n$; in case (b) the hypothesis implies the existence of a comfortable 2-splitting atlas adapted to $S$, $\mathcal{F}$ and $\rho$ and such that $[\partial z^{p'}_\alpha / \partial z^{q''}_\beta]_1 = 0$, and we are done. □

In Case 2 there is another condition ensuring the existence of the morphism $\psi$:

**Lemma 7.20.** *Let $S$ be a comfortably embedded submanifold of a complex manifold $M$, with first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_{S(1)}$ and associated left splitting morphism $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$. Let $\mathcal{F}$ be a holomorphic foliation of $M$ of dimension $\ell \le \dim S$ such that $\rho$ is $\mathcal{F}$-faithful. Assume moreover that $\mathcal{F}_{S(1)}$ is (isomorphic to) the trivial sheaf $\mathcal{O}^{\oplus\ell}_{S(1)}$ of dimension $\ell$ (this happens, for instance, if $\mathcal{F}$ is globally generated by $\ell$ global vector fields). Then there exists an $\mathcal{O}_S$-morphism $\psi\colon \mathcal{F}^\sigma \to \mathcal{A}$ such that $\theta_1 \circ \psi = \mathrm{id}$.*

*Proof.* Let $v_1, \ldots, v_d$ be global generators of $\mathcal{F}_{S(1)}$; by assumption, the $\hat{v}_j = \sigma^*(v_j \otimes [1]_1)$ for $j = 1, \ldots, \ell$ are global generators of $\mathcal{F}^\sigma$. We then define $\psi\colon \mathcal{F}^\sigma \to \mathcal{A}$ by setting

$$\psi(\hat{v}_j) = \tilde{\pi}(v_j),$$

where $\tilde{\pi}$ is the $\mathcal{O}_S$-morphism defined in Proposition 5.12, and then extending by $\mathcal{O}_S$-linearity. It is then easy to check that $\theta_1 \circ \psi = \mathrm{id}$, and we are done. □

We finally have all the ingredients needed to prove our most general index theorem for holomorphic foliations:

**Theorem 7.21.** *Let $S$ be a compact, complex, reduced, irreducible, possibly singular, subvariety of dimension $d$ of an $n$-dimensional complex manifold $M$, and assume that $S$ has extendable normal bundle. Let $\mathcal{F}$ be a (possibly singular) holomorphic foliation $\mathcal{F}$ on $M$, of dimension $\ell \le d$. Assume that there exists an analytic subset $\Sigma$ of $S$ containing $\big(\mathrm{Sing}(\mathcal{F}) \cap S\big) \cup S^{\mathrm{sing}}$ such that, setting $S^o = S \setminus \Sigma$, we have either*

(1) *$\mathcal{F}$ is tangent to $S^o$ and $\mathcal{F}_{S^o}$ is a non-singular holomorphic foliation of $S^o$ (and in this case we can take $\Sigma = \big(\mathrm{Sing}(\mathcal{F}) \cap S\big) \cup S^{\mathrm{sing}}$); or*
(2) *$S^o$ is comfortably embedded in $M$ with respect to a first order lifting $\rho$ which is $\mathcal{F}$-faithful outside of $\Sigma$, and*
  (2.a) *$S^o$ is 2-linearizable, and $\ell = \dim S$, or*
  (2.b) *$S^o$ is 2-linearizable, and there exists a nonsingular holomorphic foliation of $S^o$ transversal to $\mathcal{F}^\sigma$, or*
  (2.c) *$\mathcal{F}_{S^o(1)}$ is (isomorphic to) the trivial sheaf $\mathcal{O}_{S^o(1)}^{\oplus \ell}$ of dimension $\ell$, or, more generally,*
  (2.d) *$T_*(\mathfrak{f}) = O$ in $H^1\big(S^o, \mathcal{N}_{S^o}^* \otimes \mathcal{N}_{S^o} \otimes (\mathcal{F}^\sigma)^*\big)$.*

*Then $S$ has the Lehmann-Suwa index property of level $\ell$ in case* (1)*, and of level $\ell - \lfloor \ell/2 \rfloor$ in case* (2)*, on $\Sigma$ with respect to $\mathcal{F}$.*

*Proof.* It follows from Theorem 6.8, Proposition 7.13, Corollaries 7.14, 7.19, and Lemma 7.20. □

*Remark* 7.22. Theorem 7.21.(1) is Lehmann-Suwa's theorem (see [19], [20] and [28]); Theorem 7.21.(2.a) generalizes both Camacho-Movasati-Sad theorem (Appendix of [13]) and Camacho's ([10]) and Camacho-Lehmann's results ([12]); Theorems 7.21.(2.b), (2.c) and (2.d) are, as far as we know, new.

*Example* 7.23. We would like to compute the residue in the situation studied in [13], to show that we recover their theorem exactly. So let $S$ be a Riemann surface 2-linearizable in a complex manifold $M$, and let $\mathcal{F}$ be a one-dimensional foliation of $M$ generated by a local vector field $v \in \mathcal{T}_{M,S}$ at a regular point $p \in S$ which is an isolated singular point for $\mathcal{F}^\sigma$. If $(U_\alpha, z_\alpha)$ is a local chart at $p$ adapted to $S$ and to the first order lifting $\rho$, this means that we can write $v = a^1 \partial/\partial z_\alpha^1 + a^2 \partial/\partial z_\alpha^2$ and $p$ is an isolated zero of $a_2$ on $S$. In a pointed neighbourhood of $p$ the element $v_{\alpha,2}$ defined in (31) is given by

$$v_{\alpha,2} = \frac{\partial}{\partial z_\alpha^2} + \frac{a^1}{a^2}\frac{\partial}{\partial z_\alpha^1}.$$

Therefore

$$\psi\bigl(\sigma^*(v)\bigr) = \pi\left([a^2]_2\frac{\partial}{\partial z_\alpha^2} + [a^2]_2\tilde{\rho}\left(\left[\frac{a^1}{a^2}\right]_2\right)\frac{\partial}{\partial z_\alpha^1}\right),$$

and so

$$\tilde{X}_v^\psi\left(\frac{\partial}{\partial z_\alpha^1}\right) = a^2\frac{\partial(a^1/a^2)}{\partial z_\alpha^1}\bigg|_S\frac{\partial}{\partial z_\alpha^1}.$$

The unique (up to a constant) homogeneous symmetric polynomial of degree 1 in one variable is the identity id; hence (30) yields

$$\mathrm{Res}_{\mathrm{id}}(\mathcal{F};p) = \frac{1}{2\pi i}\int_{|a^2|=\varepsilon}\frac{\partial(a^1/a^2)}{\partial z_\alpha^1}\bigg|_S dz_\alpha^2 = \mathrm{Res}_p\left(\frac{\partial(a^1/a^2)}{\partial z_\alpha^1}\bigg|_S dz_\alpha^2\right),$$

which is exactly the formula given in [13] (and used in Theorem 1.1).

## 8. Holomorphic maps

In this final section we shall describe how to apply the strategy discussed in Section 6 using holomorphic maps instead of foliations.

Let $S$ be an irreducible subvariety of a complex manifold $M$, and let us denote by $\mathrm{End}(M,S)$ the space of holomorphic self-maps of $M$ fixing $S$ pointwise. We recall a few definitions and facts from [2].

**Definition 8.1.** Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. The *order of contact* $\nu_f$ of $f$ with $S$ is defined by

$$\nu_f = \min_{h\in\mathcal{O}_{M,p}} \max\bigl\{\mu \in \mathbb{N} \mid h\circ f - h \in \mathcal{I}_{S,p}^\mu\bigr\} \in \mathbb{N}^*,$$

where $p$ is any point of $S$.

In [2], Section 1, we proved that the order of contact is well-defined (i.e., it does not depend on the point $p \in S$), and that it can be computed by the formula

$$\nu_f = \min_{j=1,\dots,n} \max\bigl\{\mu \in \mathbb{N} \mid f_\alpha^j - z_\alpha^j \in \mathcal{I}_{S,p}^\mu\bigr\},$$

where $(U_\alpha, z_\alpha)$ is any local chart at $p$, and $f_\alpha^j = z_\alpha^j \circ f$. In particular, $[f_\alpha^j - z_\alpha^j]_{\nu_f+1} \otimes \partial/\partial z_\alpha^j$ defines a local section of $\mathcal{I}_S^{\nu_f}/\mathcal{I}_S^{\nu_f+1} \otimes \mathcal{T}_{M,S} = \mathrm{Sym}^{\nu_f}(\mathcal{N}_S^*) \otimes \mathcal{T}_{M,S}$, that (see [2], Section 3) turns out to be independent of the particular chart chosen:

**Definition 8.2.** Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. The *canonical section* $X_f \in H^0\bigl(S, \mathrm{Sym}^{\nu_f}(\mathcal{N}_S^*)\otimes \mathcal{T}_{M,S}\bigr)$ is given by

$$X_f = [f_\alpha^j - z_\alpha^j]_{\nu_f+1} \otimes \partial/\partial z_\alpha^j$$

for any local chart $(U_\alpha, z_\alpha)$ at a point $p \in S$.

Since we have $\mathrm{Sym}^{\nu_f}(\mathcal{N}_S^*) = \big(\mathrm{Sym}^{\nu_f}(\mathcal{N}_S)\big)^*$, the canonical section $X_f$ can be thought of as an $\mathcal{O}_S$-morphism $X_f\colon \mathrm{Sym}^{\nu_f}(\mathcal{N}_S) \to \mathcal{T}_{M,S}$.

**Definition 8.3.** Let $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$. The *canonical distribution* $\mathcal{F}_f$ associated to $f$ (it was denoted by $\Xi_f$ in [2]) is the subsheaf of $\mathcal{T}_{M,S}$ defined by

$$\mathcal{F}_f = X_f\big(\mathrm{Sym}^{\nu_f}(\mathcal{N}_S)\big) \subseteq \mathcal{T}_{M,S}.$$

We shall say that $f$ is *tangential* if $\mathcal{F}_f \subseteq \mathcal{T}_S$.

*Remark* 8.4. If $S$ is smooth, in [2], Corollary 3.2, we proved that $f$ is tangential if and only if

$$f_\alpha^r - z_\alpha^r \in \mathcal{I}_S^{\nu_f+1}$$

for all $r = 1, \dots, m$ and all local charts $(U_\alpha, z_\alpha)$ adapted to $S$. We also refer to [2] for a discussion of the relevance of this notion.

Until further notice, we shall assume that $S$ is a smooth complex submanifold of $M$. We shall also assume that

$$\mathrm{rk}_{\mathcal{O}_S}\mathrm{Sym}^{\nu_f}(\mathcal{N}_S) = \binom{m+\nu_f-1}{\nu_f} \le \dim S, \tag{38}$$

where $m$ is the codimension of $S$.

If $(U_\alpha, z_\alpha)$ is a local chart adapted to $S$, we can find $(g_\alpha)^j_{r_1\dots r_{\nu_f}} \in \mathcal{O}_M$ symmetric in the lower indices such that

$$f_\alpha^j - z_\alpha^j = (g_\alpha)^j_{r_1\dots r_{\nu_f}}\, z_\alpha^{r_1}\cdots z_\alpha^{r_{\nu_f}}.$$

The $(g_\alpha)^j_{r_1\dots r_{\nu_f}}$ are not uniquely defined as elements of $\mathcal{O}_M$, but it is not difficult to check that the $[(g_\alpha)^j_{r_1\dots r_{\nu_f}}]_1 \in \mathcal{O}_S$ *are* uniquely defined. Furthermore, the sheaf $\mathcal{F}_f$ is locally generated by the elements

$$v_{r_1\dots r_{\nu_f},\alpha} = [(g_\alpha)^j_{r_1\dots r_{\nu_f}}]_1\, \frac{\partial}{\partial z_\alpha^j}.$$

Finally, the fact that $X_f$ is well-defined is equivalent to the formula

$$[(g_\alpha)^j_{r_1,\dots,r_{\nu_f}}]_1 \left[\frac{\partial z_\beta^h}{\partial z_\alpha^j}\right]_1 = \left[\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{\nu_f}}}{\partial z_\alpha^{r_{\nu_f}}}\right]_1 [(g_\beta)^h_{s_1,\dots,s_{\nu_f}}]_1,$$

so that

$$v_{r_1\dots r_{\nu_f},\alpha} = \left[\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{\nu_f}}}{\partial z_\alpha^{r_{\nu_f}}}\right]_1 v_{s_1\dots s_{\nu_f},\beta}. \tag{39}$$

We would like to build our morphism $\psi$ outside singularities. So in the tangential case we shall momentarily assume that

*Case* 3. The sheaf $\mathcal{F}_f$ is the sheaf of germs of holomorphic sections of a sub-bundle of $TS$ of rank

$$\ell = \binom{m+\nu_f-1}{\nu_f}.$$

To be consistent with the non-tangential case, we shall also set $\mathcal{F}_f^\sigma = \mathcal{F}_f$, $\sigma^* = \mathrm{id}_{\mathcal{F}_f}$ and $v^\sigma_{r_1\ldots r_{\nu_f},\alpha} = v_{r_1\ldots r_{\nu_f},\alpha}$.

*Remark* 8.5. In other words, we are removing from $S$ the analytic subset of the points of $S$ where $X$ is not injective, together with the analytic subset of the points of $S$ where $\mathcal{T}_S/\mathcal{F}_f$ is not locally free.

In the non-tangential case, we project $\mathcal{F}_f$ into $\mathcal{T}_S$, as usual.

**Definition 8.6.** Let $S$ be a splitting submanifold of a complex manifold $M$. Given a first order lifting $\rho\colon \mathcal{O}_S \to \mathcal{O}_M/\mathcal{I}_S^2$, let $\sigma^*\colon \mathcal{T}_{M,S} \to \mathcal{T}_S$ be the left splitting morphism associated to $\rho$ by Proposition 2.7. If $f \in \mathrm{End}(M,S)$, $f \not\equiv \mathrm{id}_M$, has order of contact $\nu_f$, and (38) holds, we shall denote by $\mathcal{F}_f^\sigma$ the coherent sheaf of $\mathcal{O}_S$-modules given by

$$\mathcal{F}_f^\sigma = \sigma^* \circ X \circ (df)^{\otimes \nu_f}\big(\mathrm{Sym}^{\nu_f}(\mathcal{N}_S)\big) \subseteq \mathcal{T}_S,$$

where $(df)^{\otimes\nu_f}$ is the endomorphism of $\mathrm{Sym}^{\nu_f}(\mathcal{N}_S)$ induced by the action of $df$ on $N_S$. Notice that if $\nu_f > 1$ (or $\nu_f = 1$ and $f$ is tangential) we have $df|_{N_S} = \mathrm{id}$, and hence the presence of $df$ is meaningful only for $\nu_f = 1$ and $f$ not tangential. We shall say that $\rho$ is *$f$-faithful outside an analytic subset* $\Sigma \subseteq S$ if $\mathcal{F}_f^\sigma$ is the sheaf of germs of holomorphic sections of a sub-bundle of rank $\ell = \binom{m+\nu_f-1}{\nu_f}$ of $TS$ on $S \setminus \Sigma$. If $\Sigma = \varnothing$ we shall simply say that $\rho$ is *$f$-faithful.*

*Remark* 8.7. The assumption of faithfulness amounts to saying that $\sigma^* \circ X \circ (df)^{\otimes\nu_f}$ is injective and $\mathcal{T}_S/\mathcal{F}_f^\sigma$ is locally free outside $\Sigma$. In particular, if $m = 1$ then either $\sigma^* \circ X \circ (df)^{\otimes\nu_f}$ is identically zero or $\rho$ is $f$-faithful outside a suitable analytic subset.

So in the non-tangential case we shall assume that

*Case* 4. There exists an $f$-faithful first order lifting $\rho$, with associated left splitting morphism $\sigma^*$. If $\{(U_\alpha, z_\alpha)\}$ is an atlas adapted to $\rho$, we shall also set

$$v^\sigma_{r_1\ldots r_{\nu_f},\alpha} = \sigma^*(v_{r_1\ldots r_{\nu_f},\alpha}) = [(g_\alpha)^p_{r_1\ldots r_{\nu_f}}]_1 \frac{\partial}{\partial z^p_\alpha}$$

when $\nu_f > 1$, and

$$v^\sigma_{r,\alpha} = \sigma^* \circ X \circ df(\partial_{r,\alpha}) = \big[\big(\delta^s_r + (g_\alpha)^s_r\big)(g_\alpha)^p_s\big]_1 \frac{\partial}{\partial z^p_\alpha}$$

when $\nu_f = 1$, so that the $v^\sigma_{r_1\ldots r_{\nu_f},\alpha}$ form a local frame for $\mathcal{F}_f^\sigma$ and (39) still holds.

We are now ready to compute the obstruction to the existence of the morphism $\psi$.

**Proposition 8.8.** *Assume we are in Case 3 or in Case 4. Given an atlas* $\mathfrak{U} = \{(U_\alpha, z_\alpha)\}$ *adapted to* $S$ *(and to* $\rho$ *in Case 4), let* $\{m_{\beta\alpha}\}$ *be the cocycle defined by*

$$m_{\beta\alpha} = \frac{\partial z_\beta^q}{\partial z_\alpha^p} \frac{\partial^2 z_\alpha^r}{\partial z_\beta^q \partial z_\beta^t} \frac{\partial z_\beta^t}{\partial z_\alpha^s} (g_\alpha)^p_{r_1 \ldots r_{\nu_f}} \bigg|_S \omega_\alpha^s \otimes \partial_{r,\alpha} \otimes v_\alpha^{\sigma, r_1 \ldots r_{\nu_f}},$$

*if* $\nu_f > 1$ *or* $f$ *is tangential, or by*

$$m_{\beta\alpha} = \frac{\partial z_\beta^q}{\partial z_\alpha^p} \frac{\partial^2 z_\alpha^r}{\partial z_\beta^q \partial z_\beta^t} \frac{\partial z_\beta^t}{\partial z_\alpha^s} \big(\delta^u_{r_1} + (g_\alpha)^u_{r_1}\big)(g_\alpha)^p_u \bigg|_S \omega_\alpha^s \otimes \partial_{r,\alpha} \otimes v_\alpha^{\sigma, r_1},$$

*if* $\nu_f = 1$ *and* $f$ *is not tangential, where the* $v_\alpha^{\sigma, r_1 \ldots r_{\nu_f}}$ *form the frame of* $(\mathcal{F}_f^\sigma)^*$ *dual to the frame* $\{v^\sigma_{r_1 \ldots r_{\nu_f}, \alpha}\}$ *of* $\mathcal{F}_f^\sigma$*, and denote by* $\mathfrak{m} \in H^1\big(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes (\mathcal{F}_f^\sigma)^*\big)$ *the corresponding cohomology class. Then there exists a morphism* $\psi \colon \mathcal{F}_f^\sigma \to \mathcal{A}$ *such that* $\theta_1 \circ \psi = \mathrm{id}$ *if and only if*

$$\mathfrak{m} = O$$

*in* $H^1\big(S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes (\mathcal{F}_f^\sigma)^*\big)$*.*

*Proof.* Let us first assume $\nu_f > 1$ or $f$ tangential. We then define local $\mathcal{O}_S$-morphism $\psi_\alpha \colon \mathcal{F}_f^\sigma|_{U_\alpha} \to \mathcal{A}$ by setting

$$\psi_\alpha(v^\sigma_{r_1 \ldots r_{\nu_f}, \alpha}) = \pi\left( [(g_\alpha)^p_{r_1 \ldots r_{\nu_f}}]_2 \frac{\partial}{\partial z_\alpha^p} \right) \tag{40}$$

where $\pi \colon \mathcal{T}^S_{M,S(1)} \to \mathcal{A}$ is the canonical projection, and then extending by $\mathcal{O}_S$-linearity. Notice that the argument of $\pi$ in (40) is not well-defined, but its image under $\pi$ is. Since $\theta_1 \circ \psi_\alpha = \mathrm{id}$, it suffices to show that the cocycle $\{m_{\beta\alpha}\}$ is represented by $\psi_\beta - \psi_\alpha$. But indeed, recalling (39) and using either that $f$ is tangential or that we are working with an atlas adapted to $\rho$, we have

$$\begin{aligned}\psi_\beta \quad & (v^\sigma_{r_1\dots r_{\nu_f},\alpha}) - \psi_\beta(v^\sigma_{r_1\dots r_{\nu_f},\alpha})\\
&= \left[\frac{\partial z_\beta^{s_1}}{\partial z_\alpha^{r_1}}\cdots\frac{\partial z_\beta^{s_{\nu_f}}}{\partial z_\alpha^{r_{\nu_f}}}\right]_1 \psi_\beta(v^\sigma_{s_1\dots s_{\nu_f},\beta}) - [(g_\alpha)^p_{r_1\dots r_{\nu_f}}]_1\,\pi\left(\frac{\partial}{\partial z_\alpha^j}\right)\\
&= [(g_\alpha)^j_{r_1\dots r_{\nu_f}}]_1 \left[\frac{\partial z_\beta^q}{\partial z_\alpha^j}\right]_1 \pi\left(\left[\frac{\partial z_\alpha^k}{\partial z_\beta^q}\right]_2 \frac{\partial}{\partial z_\alpha^k}\right) - [(g_\alpha)^p_{r_1\dots r_{\nu_f}}]_1\,\pi\left(\frac{\partial}{\partial z_\alpha^j}\right)\\
&= [(g_\alpha)^j_{r_1\dots r_{\nu_f}}]_1 \left[\frac{\partial z_\beta^q}{\partial z_\alpha^j}\right]_1 \left\{\left[\frac{\partial z_\alpha^p}{\partial z_\beta^q}\right]_1 \pi\left(\frac{\partial}{\partial z_\alpha^p}\right) + \pi\left(\left[\frac{\partial z_\alpha^r}{\partial z_\beta^q}\right]_2 \frac{\partial}{\partial z_\alpha^r}\right)\right\}\\
&\quad -[(g_\alpha)^p_{r_1\dots r_{\nu_f}}]_1\,\pi\left(\frac{\partial}{\partial z_\alpha^j}\right)\\
&= [(g_\alpha)^p_{r_1\dots r_{\nu_f}}]_1 \left[\frac{\partial z_\beta^q}{\partial z_\alpha^p}\right]_1 \left[\frac{\partial^2 z_\alpha^r}{\partial z_\beta^q \partial z_\beta^t}\right]_1 \left[\frac{\partial z_\beta^t}{\partial z_\alpha^s}\right]_1 \pi\left([z_\alpha^s]_2 \frac{\partial}{\partial z_\alpha^r}\right),\end{aligned}$$

and we are done in this case.

If $\nu_f = 1$ and $f$ is not tangential we define $\psi_\alpha$ by

$$\psi_\alpha(v^\sigma_{r,\alpha}) = \pi\left(\left[\left(\delta^s_r + (g_\alpha)^s_r\right)(g_\alpha)^p_s\right]_2 \frac{\partial}{\partial z_\alpha^p}\right),$$

and the assertion follows as before. □

It turns out that in codimension 1 (assuming $S$ comfortably embedded in Case 4) we have $\mathfrak{m} = O$ always:

**Proposition 8.9.** *Assume we are in Case 3 or in Case 4 with $S$ comfortably embedded, and that $S$ has codimension 1. Then $\mathfrak{m} = O$.*

*Proof.* Let $\{(U_\alpha, z_\alpha)\}$ be an atlas adapted to $S$, and also comfortable and adapted to $\rho$ in Case 4. We define a 0-cochain $\{x_\alpha\} \in H^0\big(\mathfrak{U}_S, \mathcal{N}_S^* \otimes \mathcal{N}_S \otimes (\mathcal{F}_f^\sigma)^*\big)$ by setting

$$x_\alpha(v^\sigma_{1\dots 1,\alpha}) = \pi\left(\tilde\rho([(g_\alpha)^1_{1\dots 1}]_2)\frac{\partial}{\partial z_\alpha^1}\right) = \pi\left(\left[\frac{\partial (g_\alpha)^1_{1\dots 1}}{\partial z_\alpha^1}\right]_1 [z_\alpha^1]_2 \frac{\partial}{\partial z_\alpha^1}\right), \tag{41}$$

where in Case 3 we have $\tilde\rho([(g_\alpha)^1_{1\dots 1}]_2) = [(g_\alpha)^1_{1\dots 1}]_2$, and thus we do not need to assume anything on the embedding of $S$ into $M$. Notice furthermore that, since the codimension of $S$ is 1, the germ $(g_\alpha)^1_{1\dots 1}$ is well-defined *as germ in $\mathcal{O}_M$*, and not only as germ in $\mathcal{O}_S$, and so (41) is well-defined.

To prove the assertion it suffices then to show that $x_\alpha - x_\beta = m_{\beta\alpha}$. Now we have

$$\begin{aligned}(x_\alpha \quad - \quad x_\beta)(v^\sigma_{1\dots1,\alpha}) &= \pi\left(\tilde\rho([(g_\alpha)^1_{1\dots1}]_2)\frac{\partial}{\partial z^1_\alpha}\right) - \left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_1^{\nu_f} x_\beta(v^\sigma_{1\dots1,\beta})\\ &= \pi\left(\left\{\tilde\rho([(g_\alpha)^1_{1\dots1}]_2) - \left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2^{\nu_f}\tilde\rho([(g_\beta)^1_{1\dots1}]_2)\left[\frac{\partial z^1_\alpha}{\partial z^1_\beta}\right]_2\right\}\frac{\partial}{\partial z^1_\alpha}\right).\end{aligned}$$

If $\nu_f > 1$ from

$$\begin{aligned}(g_\beta)^1_{1\dots1}(z^1_\beta)^{\nu_f} &= f^1_\beta - z^1_\beta = \frac{\partial z^1_\beta}{\partial z^j_\alpha}(f^j_\alpha - z^j_\alpha) + R_{2\nu_f}\\ &= \frac{\partial z^1_\beta}{\partial z^j_\alpha}(g_\alpha)^j_{1\dots1}(z^1_\alpha)^{\nu_f} + R_{2\nu_f},\end{aligned}$$

where $R_{2\nu_f} \in \mathcal{I}_S^{2\nu_f}$, we get

$$\left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2^{\nu_f}[(g_\beta)^1_{1\dots1}]_2 = \left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2[(g_\alpha)^1_{1\dots1}]_2 + \left[\frac{\partial z^1_\beta}{\partial z^p_\alpha}\right]_2[(g_\alpha)^p_{1\dots1}]_2.$$

Since we are working with a comfortable atlas, $\tilde\rho([\partial z^1_\beta/\partial z^1_\alpha]_2) = O$; therefore applying (in Case 4) $\tilde\rho$ to the previous formula we get

$$\left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2^{\nu_f}\tilde\rho([(g_\beta)^1_{1\dots1}]_2) = \left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2\tilde\rho([(g_\alpha)^1_{1\dots1}]_2) + \left[\frac{\partial z^1_\beta}{\partial z^p_\alpha}\right]_2[(g_\alpha)^p_{1\dots1}]_2.$$

Remarking that

$$\left[\frac{\partial^2 z^1_\beta}{\partial z^p_\alpha\partial z^1_\alpha}\right]_1\left[\frac{\partial z^1_\alpha}{\partial z^1_\beta}\right]_1 = -\left[\frac{\partial z^1_\alpha}{\partial z^1_\beta}\right]_1\left[\frac{\partial z^q_\beta}{\partial z^p_\alpha}\right]_1\left[\frac{\partial^2 z^1_\alpha}{\partial z^q_\beta\partial z^1_\beta}\right]_1,$$

we obtain the assertion if $\nu_f > 1$ or $f$ is tangential.

If instead $\nu_f = 1$, recalling that we are using a comfortable atlas, we have

$$\begin{aligned}(g_\beta)^1_1 z^1_\beta &= f^1_\beta - z^1_\beta = \frac{\partial z^1_\beta}{\partial z^j_\alpha}(f^j_\alpha - z^j_\alpha) + \frac{1}{2}\frac{\partial^2 z^1_\beta}{\partial z^h_\alpha\partial z^k_\alpha}(f^h_\alpha - z^h_\alpha)(f^k_\alpha - z^k_\alpha) + R_3\\ &= \frac{\partial z^1_\beta}{\partial z^j_\alpha}(g_\alpha)^j_1 z^1_\alpha + \frac{\partial^2 z^1_\beta}{\partial z^p_\alpha\partial z^1_\alpha}(g_\alpha)^p_1(g_\alpha)^1_1(z^1_\alpha)^2 + R_3,\end{aligned}$$

where $R_3 \in \mathcal{I}_S^3$, and so

$$\left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2[(g_\beta)^1_1]_2 = \left[\frac{\partial z^1_\beta}{\partial z^1_\alpha}\right]_2[(g_\alpha)^1_1]_2 + \left[\frac{\partial z^1_\beta}{\partial z^p_\alpha}\right]_2\big(1 + [(g_\alpha)^1_1]_2\big)[(g_\alpha)^p_1]_2.$$

Applying $\tilde\rho$ and arguing as before we obtain the assertion in this case too. □

We are now ready for our most general index theorem for holomorphic maps:

**Theorem 8.10.** *Let $S$ be a compact, complex, reduced, irreducible, possibly singular, subvariety of codimension $m$ of an $n$-dimensional complex manifold $M$, and assume that $S$ has extendable normal bundle. Let $f \in \mathrm{End}(M, S)$, $f \not\equiv \mathrm{id}_M$, have order of contact $\nu_f$ with $S$, and such that $\ell = \binom{m+\nu_f-1}{\nu_f} \leq \dim S$. Assume that there exists an analytic subset $\Sigma$ of $S$ containing $S^{\rm sing}$ such that, setting $S^o = S \setminus \Sigma$, we have either*

(1) *$f$ is tangential to $S^o$, $X|_{S^o} \colon \mathrm{Sym}^{\nu_f}(\mathcal{N}^*_{S^o}) \to \mathcal{T}_{S^o}$ is injective, $\mathcal{T}_{S^o}/(\mathcal{F}_f|_{S^o})$ is locally free, and*
  (1.a) *$S$ has codimension $m = 1$, or, more generally,*
  (1.b) *$\mathfrak{m} = \mathcal{O}$;*

(2) *$S^o$ is comfortably embedded in $M$ with respect to a first order lifting $\rho$ which is $f$-faithful outside of $\Sigma$, and*
  (2.a) *$S$ has codimension $m = 1$, or, more generally,*
  (2.b) *$\mathfrak{m} = \mathcal{O}$.*

*Then $S$ has the Lehmann-Suwa index property of level $\ell - \lfloor \ell/2 \rfloor$ on $\Sigma$ with respect to $f$.*

*Proof.* It follows from Theorem 6.8 and Propositions 8.8 and 8.9. □

*Remark* 8.11. Theorems 8.10.(1.a) and (2.a) are contained in [2]. Theorems 8.10.(1.b) and (2.b) are, as far as we know, new.

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