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Poincaré–Bendixson theorems for meromorphic connections and holomorphic homogeneous vector fields

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ABSTRACT

We first study the dynamics of the geodesic flow of a meromorphic connection on a Riemann surface, and prove a Poincaré–Bendixson theorem describing recurrence properties and ω-limit sets of geodesics for a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$. We then show how to associate to a homogeneous vector field Q in $\mathbb{C}^n$ a rank 1 singular holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^{n-1}(\mathbb{C})$ and a (partial) meromorphic connection ∇^o along $\mathcal{F}$ so that integral curves of Q are described by the geodesic flow of ∇^o along the leaves of $\mathcal{F}$, which are Riemann surfaces. The combination of these results yields powerful tools for a detailed study of the dynamics of homogeneous vector fields. For instance, in dimension two we obtain a description of recurrence properties of integral curves of Q, and of the behavior of the geodesic flow in a neighborhood of a singularity, classifying the possible singularities both from a formal point of view and (for generic singularities) from a holomorphic point of view. We also get examples of unexpected new phenomena, we put in a coherent context scattered results previously known, and we obtain (as far as we know for the first time) a complete description of the dynamics in a full neighborhood of the origin for a substantial class of holomorphic maps tangent to the identity. Finally, as an example of application of our methods we study in detail the dynamics of quadratic homogeneous vector fields in $\mathbb{C}^2$.

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0. Introduction

In this paper we shall study the dynamics of two apparently unrelated objects: geodesics for meromorphic connections on a Riemann surfaces, and integral curves of homogeneous vector fields in $\mathbb{C}^n$.

Meromorphic connections on Riemann surfaces have been well studied, particularly from an algebraic geometrical point of view (see, e.g., [20]); however, to our knowledge the dynamical properties of the *real* geodesic curves associated to a meromorphic connection have not been investigated before. Here, by a *geodesic* for a meromorphic connection ∇ on a Riemann surface S we mean a real smooth curve $\sigma : I \to S^o$, where $I \subseteq \mathbb{R}$ is an interval and S^o is the complement in S of the poles of ∇, satisfying the geodesic equation $\nabla_{\sigma'}\sigma' \equiv 0$.

One of the main results of this paper is (as the title suggests) a type of Poincaré–Bendixson theorem describing the recurrence properties and ω-limit sets of geodesics for a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$. The classical Poincaré–Bendixson theorem (see, e.g., [17, Theorem 14.1.1]) deals with integral curves of vector fields defined on open subsets of the sphere S^2 (notice that, as a differentiable manifold, $\mathbb{P}^1(\mathbb{C})$ is diffeomorphic to S^2): recurrent integral curves are necessarily periodic, and the ω-limit set of an integral curve either contains singular points or is a periodic integral curve. In our Poincaré–Bendixson theorem (see Theorem 4.6) the poles of the connection replace the singular points of the vector field:

Theorem 0.1. *Let $\sigma : [0, \varepsilon_0) \to S^o$ be a maximal geodesic for a meromorphic connection ∇^o on $\mathbb{P}^1(\mathbb{C})$, where $S^o = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\}$ and $p_0, \dots, p_r$ are the poles of ∇^o. Then either*

(i) *$\sigma(t)$ tends to a pole of ∇^o as $t \to \varepsilon_0$; or*
(ii) *σ is closed, and then surrounds poles $p_1, \dots, p_g$ with $\sum_{j=1}^g \operatorname{Re}\operatorname{Res}_{p_j}(\nabla^o) = -1$; or*
(iii) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is given by the support of a closed geodesic surrounding poles $p_1, \dots, p_g$ with $\sum_{j=1}^g \operatorname{Re}\operatorname{Res}_{p_j}(\nabla^o) = -1$; or*
(iv) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is a cycle of saddle connections (see below) surrounding poles $p_1, \dots, p_g$ with $\sum_{j=1}^g \operatorname{Re}\operatorname{Res}_{p_j}(\nabla^o) = -1$; or*
(v) *σ intersects itself infinitely many times, and in this case every simple loop of σ surrounds a set of poles whose sum of residues has real part belonging to $(-3/2, -1) \cup (-1, -1/2)$.*

In particular, a recurrent geodesic either intersects itself infinitely many times or is closed.

Here, a *saddle connection* is a geodesic connecting two (not necessarily distinct) poles of ∇^o; a *cycle of saddle connections* is a closed curve composed of saddle connections. Furthermore, the residue of the connection at a pole p is defined as the residue at p of the meromorphic 1-form representing the connection with respect to any holomorphic local coordinate in p.

We have examples (see Examples 6.1, 8.1 and 8.2) of cases (i), (ii), (iii) and (v); we do not know yet whether case (iv) can actually be realized (but we are able to exclude it in several situations; see Remark 8.3). Notice furthermore that (see, e.g., [18, Theorem III.17.33]) the only limitation on the residues of a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$ is that their sum should be -2; more precisely, given any finite set of pairs $\{(p_1, r_1), \dots, (p_g, r_g)\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{C}$ with $r_1 + \cdots + r_g = -2$ there exists a meromorphic connection ∇^o with poles $\{p_1, \dots, p_g\}$ and $\operatorname{Res}_{p_j}(\nabla^o) = r_j$ for $j = 1, \dots, g$. Since cases (ii)–(v) of Theorem 0.1 impose additional conditions on the residues (in particular, in the last case the condition should be satisfied by each of the infinitely many simple loops of the geodesic; see also Proposition 4.7), it follows that maximal geodesics of a meromorphic connection often display no recurrence phenomena at all, being simply saddle connections.

It is important to notice that in general a meromorphic connection ∇^o on a Riemann surface is not the Chern connection of a Hermitian metric (unless all residues are real: see Proposition 1.2 and Corollary 3.7). Furthermore, even when it is, the associated Hermitian metric is never complete (except in trivial cases: see Corollary 2.5); so the behavior of our geodesics is subtly different from the behavior of the usual geodesics in Riemannian geometry (for instance, we can have closed geodesics which are not periodic: geodesics for meromorphic connections are not necessarily of constant speed).

Nevertheless, we shall be able to associate to ∇^o a conformal family of local Hermitian flat metrics that shall be very useful.

The proof of Theorem 0.1 depends on three main ingredients, developed in the first four sections of this paper. The first ingredient is a detailed study of geodesics for holomorphic connections on a simply connected Riemann surface (the universal cover of the complement of the poles); since a holomorphic connection is necessarily flat, it turns out that its geodesics behave locally as Euclidean segments. The second ingredient is Theorem 4.1, relating the external angles of a geodesic polygon to the residues of the poles inside the polygon; the proof depends on the crucial observation that a global Gauss–Bonnet formula still holds for conformal families of local flat metrics. It also turns out that the residues control the monodromy of the connection (Proposition 3.6) and the speed of the geodesics (Lemma 4.4). Finally, the proof of Theorem 0.1 is completed by a delicate argument introducing a sort of Poincaré return map on a transversal defined at a point in the ω-limit set of the geodesic.

The second half of the paper is devoted to the dynamics of homogeneous vector fields. A homogeneous vector field in $\mathbb{C}^n$ is a vector field of the form

$$Q = Q^1 \frac{\partial}{\partial z^1} + \cdots + Q^n \frac{\partial}{\partial z^n}, \tag{0.1}$$

where $Q^1, \dots, Q^n$ are homogeneous polynomials of degree $\nu + 1 \geqslant 2$. The complex foliation generated by Q is not that difficult to study (see, e.g., Theorem 6.2); but here we are mostly interested in the dynamics of the *real* integral curves of Q. The reason is that we arrived to this problem because we wanted to study the dynamics of holomorphic maps tangent to the identity (that is, of germs of holomorphic self-maps of $\mathbb{C}^n$ fixing the origin whose differential at the origin is the identity), and homogeneous vector fields provide good examples of those. Indeed, it is easy to see that the time-1 map of a vector field of the form (0.1) has a homogeneous expansion of the form

$$f(z) = z + Q_{\nu+1}(z) + \cdots,$$

where $Q_{\nu+1} = (Q^1, \dots, Q^n)$, and thus f is tangent to the identity. In dimension one, the classical Leau–Fatou theorem (see, e.g., [21] and [4]) gives a complete description of the dynamics of a holomorphic function tangent to the identity in a full neighborhood of the origin. Using this, in 1978 Camacho [9] (see also [24]) proved that, from a topological point of view, time-1 maps of homogeneous vector fields provide a complete list of models for the dynamics:

Theorem 0.2. *(See Camacho, 1978 [9].) Let $f(z) = z + a_{\nu+1} z^{\nu+1} + \cdots$ with $a_{\nu+1} \neq 0$, be a germ of holomorphic function tangent to the identity. Then f is locally topologically conjugated to the time-1 map of the homogeneous vector field*

$$Q = z^{\nu+1} \frac{\partial}{\partial z}.$$

In dimension greater than one nothing of the sort is (as yet) known. More precisely, understanding the (topological) dynamics of holomorphic germs tangent to the identity in a full neighborhood of the origin is one of the main open problems in local dynamics of several complex variables. There are versions of the Leau–Fatou flower theorem in several variables, obtained by Écalle (see [11–14]) and Hakim (see [15,16]) in any dimension but for generic germs, and for all germs in dimension 2 by the first author (see [2]). But these theorems mostly give the existence of 1-dimensional invariant sets only, and are quite far from providing a description of the dynamics in a full neighborhood of the origin. In fact, as far as we know, before the present paper such a description was available for a handful of examples only.

On the other hand, along the lines of Camacho's Theorem 0.2, it is conjectured that in any dimension a *generic* (e.g., with only non-degenerate characteristic directions; see below) germ tangent to the identity is locally topologically conjugated to the time-1 map of a homogeneous vector field. To build such a conjugation, one usually needs a precise description of the dynamics of the model; so we decided to study in detail the dynamics of time-1 maps of homogeneous vector fields. Since the orbit of a point p under the action of such a map is contained in the *real* integral curve issuing from p, we were led to the study of the dynamics of real integral curves.

To state the (somewhat unexpected) link between integral curves of a homogeneous vector field and geodesics of a meromorphic connection we need a few definitions. Let Q be a homogeneous vector field of the form (0.1), and denote by $[\cdot]:\mathbb{C}^n\setminus\{O\}\to\mathbb{P}^{n-1}(\mathbb{C})$ the canonical projection. Following Écalle (see [11–14]) and Hakim (see [15,16]) we say that a direction $[v]\in\mathbb{P}^1(\mathbb{C})$ is a *characteristic direction* of Q if the line $L_v=\mathbb{C}v$ is Q-invariant (and in that case we say that L_v is a *characteristic leaf*). If Q is identically zero along L_v we say that $[v]$ is *degenerate;* otherwise, it is *non-degenerate.* It turns out (see, e.g., [5]) that either Q has $\frac{1}{\nu}[(\nu+1)^n-1]$ characteristic directions, counting multiplicities, or infinitely many directions are characteristic. In particular, if all directions are characteristic we shall say that Q is *dicritical.* When $n=2$, it turns out that either Q is dicritical or it has $\nu+2$ characteristic directions, counting multiplicities.

The dynamical meaning of characteristic directions is expressed by the following fact (see [15]): if γ is an integral curve of Q converging to the origin tangentially to some direction $[v]\in\mathbb{P}^{n-1}(\mathbb{C})$ then $[v]$ is a characteristic direction. Notice however that (as noted by Rivi [22]) there might exist integral curves converging to the origin without being tangent to any direction; see Example 6.1 (and Corollary 8.5, giving an explanation of this phenomenon).

The dynamics inside a characteristic leaf is 1-dimensional, and easy to study (see Lemma 5.4). So we are interested in the dynamics outside characteristic leaves of (necessarily) non-dicritical vector fields. Our second main result says that integral curves outside characteristic leaves are given by geodesics for a suitable meromorphic connection on suitable Riemann surfaces, foliating a projective space. Let $\mathcal{F}$ be a rank 1 singular holomorphic foliation of a complex manifold M. Let $\mathrm{Sing}(\mathcal{F})$ be the singular set of $\mathcal{F}$, and set $M^o=M\setminus\mathrm{Sing}(\mathcal{F})$. A (*partial*) *meromorphic connection along* $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla^o:\mathcal{F}|_{M^o}\to\mathcal{F}|^*_{M^o}\otimes\mathcal{F}|_{M^o}$ satisfying the usual Leibniz condition (see Section 5 for details); roughly speaking, ∇^o allows to differentiate sections of $\mathcal{F}$ along directions tangent to the foliation. In particular, ∇^o induces a (classical) meromorphic connection on each (1-dimensional) leaf of the foliation.

If $\sigma:I\to M^o$ is a curve contained in a leaf of the foliation (that is, $\sigma'(t)\in\mathcal{F}_{\sigma(t)}$ for all $t\in I$), and ∇^o is a meromorphic connection along $\mathcal{F}$, then we can consider $\nabla^o_{\sigma'}\sigma'$; we shall say that σ is a ∇^o-*geodesic* if $\nabla^o_{\sigma'}\sigma'\equiv O$. In other words, a curve σ contained in a leaf L is a ∇^o-geodesic if and only if it is a geodesic for the meromorphic connection induced on L.

The link between integral curves and geodesics is then provided by the following result (see Theorem 5.3):

Theorem 0.3. *Let Q be a non-dicritical homogeneous vector field in $\mathbb{C}^n$ of degree $\nu+1\geqslant 2$, and let $\hat{S}_Q$ be the complement in $\mathbb{C}^n$ of the characteristic leaves of Q. Then there exists a rank* 1 *singular holomorphic foliation $\mathcal{F}$ on $\mathbb{P}^{n-1}(\mathbb{C})$, whose singular points are characteristic directions of Q, and a meromorphic connection ∇^o along $\mathcal{F}$ such that*:

(i) *if $\gamma:I\to\hat{S}_Q$ is an integral curve of Q then its direction $[\gamma]:I\to\mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^o-geodesic*; *conversely,*
(ii) *if $\sigma:I\to\mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^o-geodesic then there exists exactly ν integral curves $\gamma_1,\dots,\gamma_\nu:I\to\hat{S}_Q$ of Q, differing only by the multiplication by a ν-th root of unity, whose direction is given by σ, that is such that $\sigma=[\gamma_j]$.*

This result then reduces the study of integral curves of a homogeneous vector field to the study of the foliation $\mathcal{F}$ and to the study of geodesics for a meromorphic connection on Riemann surfaces.

The proof of Theorem 0.3 depends on another couple of ingredients. Let $\pi:M\to\mathbb{C}^n$ be the blow-up of the origin (see [1] for a description of the blow-up construction adapted to dynamical purposes); the exceptional divisor (that is, the preimage of the origin under π) is canonically identified with $\mathbb{P}^{n-1}(\mathbb{C})$. Let $p : N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})} \to \mathbb{P}^{n-1}(\mathbb{C})$ be the ν-th tensor power of the normal bundle $N_{\mathbb{P}^{n-1}(\mathbb{C})}$ of the exceptional divisor in M. Then it is possible to define (see Proposition 5.2) a ν-to-1 holomorphic covering map $\chi_\nu : \mathbb{C}^n \setminus \{O\} \to N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})} \setminus \mathbb{P}^{n-1}(\mathbb{C})$ (where we are identifying $\mathbb{P}^{n-1}(\mathbb{C})$ with the zero section of $N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})}$) such that $p \circ \chi_\nu(z) = [z]$ for all $z \in \mathbb{C}^n \setminus \{O\}$.

Usually, the push-forward of a vector field is not a vector field. However, the homogeneity of Q implies (see Theorem 5.3) that $d\chi_\nu(Q)$ is a holomorphic vector field G globally defined on the total space of $N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})}$; so using χ_ν we transform the study of integral curves of Q in the study of integral curves of G.

In [7] we showed (in a more general setting) how to associate to the (non-dicritical) homogeneous vector field Q (or, more precisely, to its time-1 map) a canonical morphism $X_Q : N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})} \to T\mathbb{P}^{n-1}(\mathbb{C})$. The zeroes of X_Q are exactly the characteristic directions of Q; so X_Q is an isomorphism outside the characteristic directions, and thus it defines a rank 1 singular holomorphic foliation $\mathcal{F}$ of $\mathbb{P}^{n-1}(\mathbb{C})$. Furthermore, again in [7] we showed how to use Q to define (in an essentially unique way; see [8]) a partial holomorphic connection ∇ along $\mathcal{F}$ on $N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})}$.

In Section 5 we shall describe this construction in our context, adding a few new ideas. In particular, we shall remark that using X_Q we can push ∇ to $T\mathbb{P}^{n-1}(\mathbb{C})$ obtaining a meromorphic connection ∇^o along $\mathcal{F}$ in the sense mentioned above, and we shall show (Proposition 5.1) that a curve σ in $\mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^o-geodesic if and only if the image of σ is contained in a leaf of $\mathcal{F}$ and the curve $X_Q^{-1}(\sigma')$ in the total space of $N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})}$ is an integral curve of the global vector field $G = d\chi_\nu(Q)$. So everything fits together, and we can use meromorphic connections on Riemann surfaces to study the dynamics of homogeneous vector fields.

To exemplify the strength of this method, in the rest of the paper we specialize to the case of $n = 2$, where the foliation $\mathcal{F}$ has only one leaf, the complement of the characteristic directions. Then, as a corollary of Theorem 0.1 we immediately get (see Theorem 6.3) a description of recurrent integral curves:

Theorem 0.4. *Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$, and let γ be a recurrent maximal integral curve of Q. Then either γ is periodic or $[\gamma]$ intersects itself infinitely many times.*

More can be said along these lines (see, e.g., Section 9); but to fully understand the dynamics we need to know what happens to integral curves nearby the characteristic leaves, that is to the geodesics nearby the poles. So in Sections 7 and 8 we turn to a detailed study of the geodesic field G and its singularities in dimension 2, showing that we must distinguish between three types of singularities: apparent, Fuchsian (which is the generic case) and irregular. We shall be able to give a complete formal classification of all cases (see Proposition 7.1 and Theorem 8.1), and a complete holomorphic classification of the first two cases (Proposition 7.1 and Theorem 8.3); in particular, it is worthwhile to remark that in the Fuchsian case resonances appear. More precisely, we shall prove the following (see Theorem 8.3):

Theorem 0.5. *Let $p_0 \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian pole of G, that is assume that in local coordinates (U_α, z_α) centered at p_0, denoting by v_α the induced coordinate along the fibers of $N^{\otimes\nu}_{\mathbb{P}^1(\mathbb{C})}$, we can write*

$$G = z_\alpha^\mu (a_0 + a_1 z_\alpha + \cdots) v_\alpha \frac{\partial}{\partial z_\alpha} - z_\alpha^{\mu-1}(b_0 + b_1 z_\alpha + \cdots) v_\alpha^2 \frac{\partial}{\partial v_\alpha},$$

with $\mu \geqslant 1$ and a_0, $b_0 \neq 0$. Put $\rho = b_0/a_0 = \mathrm{Res}_{p_0}(\nabla)$. Then we can find a chart (U, z) centered at p_0 in which G is given by

$$z^{\mu-1}\left(zv\frac{\partial}{\partial z} - \rho v^2 \frac{\partial}{\partial v}\right)$$

if $\mu - 1 - \rho \notin \mathbb{N}^*$*, or by*

$$z^{\mu-1}\left(zv\frac{\partial}{\partial z} - \rho\left(1 + az^n\right)v^2\frac{\partial}{\partial v}\right)$$

for a suitable $a \in \mathbb{C}$ *if* $n = \mu - 1 - \rho \in \mathbb{N}^*$.

It is easy to check that non-degenerate characteristic directions with non-zero residue are Fuchsian (with $\mu = 1$). Using Theorem 0.5 we then get a complete description of the dynamics in a neighborhood of Fuchsian characteristic directions (see Proposition 8.4 and Corollary 8.5); we also have a complete description of the dynamics in a neighborhood of apparent singularities (see Corollaries 7.2 and 7.3). Putting this together with our Poincaré–Bendixson theorems we get a complete description of the dynamics for a substantial class of 2-dimensional homogeneous vector fields (see Corollary 8.6):

Theorem 0.6. *Let* Q *be a non-dicritical homogeneous vector field on* $\mathbb{C}^2$*. Assume that all characteristic directions of* Q *are non-degenerate with non-zero residue. Assume moreover that for no set of characteristic directions the real part of the sum of the residues of* ∇^o *is equal to* -1.

Let $\gamma : [0, \varepsilon_0) \to \mathbb{C}^2$ *be a maximal integral curve of* Q*. Then*:

(a) *If* $\gamma(0)$ *belongs to a characteristic leaf* L_{v_0}*, then the whole image of* γ *is contained in* L_{v_0}*. Moreover, either* $\gamma(t) \to O$ (*and this happens for an open Zariski dense set of initial conditions in* L_{v_0}), *or* $\|\gamma(t)\| \to +\infty$.

(b) *If* $\gamma(0)$ *does not belong to a characteristic leaf, then either*
  - (i) γ *converges to the origin tangentially to a characteristic direction* $[v_0]$ *whose residue has negative real part*; *or*
  - (ii) $\|\gamma(t)\| \to +\infty$ *tangentially to a characteristic direction* $[v_0]$ *whose residue has positive real part*; *or*
  - (iii) $[\gamma] : [0, \varepsilon_0) \to \mathbb{P}^1(\mathbb{C})$ *intersects itself infinitely many times.*

  Furthermore, if (iii) *never occurs then* (i) *holds for an open Zariski dense set of initial conditions.*

In particular, if the residues of Q do not satisfy the condition corresponding to the one described in Theorem 0.1(v) – and we already remarked that this is often the case – then case (b)(iii) of Theorem 0.6 cannot occur, and we get a description of the dynamics of the time-1 map of Q in a full neighborhood of the origin; as mentioned before, as far as we know this is the first such description for a non-trivial class of maps tangent to the identity. We also remark that we do have a description of the dynamics even when the real part of the sum of the residues might be -1, or for some classes of degenerate characteristic directions, and so the scope of our results is larger than Theorem 0.6; see Section 8 for details and Section 9 for examples.

In our opinion, this approach not only offers effective tools for studying the dynamics of homogeneous vector fields (and thus hopefully of maps tangent to the identity), but it also gives a better understanding of what is going on. For instance, Hakim's theorem [16] on the existence of parabolic basins in this context is explained by the fact that non-degenerate characteristic directions whose residue has negative real part are attractors (see Corollary 8.5, that actually extends Hakim's theorem to some degenerate characteristic directions in dimension two). Or, Rivi's [22] example of orbits going to the origin without being tangent to any direction turns out to be related to the existence of characteristic directions with purely imaginary residue (see again Corollary 8.5). We are also able to give examples of unexpected phenomena. In dimension one, the Leau–Fatou flower theorem implies that a map tangent to the identity has no small cycles: there is a neighborhood of the origin containing no periodic points beside the origin itself. It was expected that even in several complex variables maps tangent to the identity could not have small cycles; surprisingly, this turns out to be false, and in Corollaries 7.3 and 8.5 we shall give examples having periodic points of arbitrarily high period accumulating at the origin (see also Corollary 6.4).

This paper is organized as follows. In Section 1 we shall introduce the local metrics and the global (metric) foliation associated to a holomorphic connection on a line bundle over a Riemann surface S. Along the way, we shall characterize the holomorphic connections which are the Chern connection of a Hermitian metric on a line bundle over S. In Section 2 we shall study in depth the geodesic flow of a holomorphic connection over a simply connected Riemann surface. In Section 3 we shall use the monodromy representation of a holomorphic connection over a multiply connected Riemann surface to study the geodesic flow there. In Section 4 we shall consider meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, and use the previous material to prove Theorem 0.1. In Section 5 we shall clarify the relations between maps tangent to the identity, homogeneous vector fields and meromorphic connections in any dimension, proving in particular Theorem 0.3. In Section 6 we shall specialize to dimension 2, proving Theorem 0.4. In Section 7 we shall begin the study of the geodesic flow nearby the singularities in dimension two; the formal and holomorphic classifications, as well as their dynamical consequences and the proofs of Theorems 0.5 and 0.6, are contained in Section 8. Finally, in Section 9 we shall discuss in detail 2-dimensional quadratic homogeneous vector fields.

Developing the ideas leading to this paper has been a long process, carried out not only in our home institutions but (mostly) in several other places. We would like to thank the Department of Mathematics of Niigata, Kyoto and Barcelona Universities, the IMPA (Rio de Janeiro, Brazil) and, in particular, the Institut Mittag-Leffler (Djursholm, Sweden) for their warm hospitality and productive environment.

1. The metric, horizontal and geodesic foliations

Let us begin recalling a few standard facts about holomorphic connections on line bundles over Riemann surfaces (see, e.g., [19]).

Definition 1.1. Let E be a complex line bundle on a Riemann surface S. A *holomorphic connection* on E is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \Omega_S^1 \otimes \mathcal{E}$ satisfying the Leibniz rule

$$\nabla(se) = ds \otimes e + s\nabla e$$

for all $s \in \mathcal{O}_S$ and $e \in \mathcal{E}$, where $\mathcal{E}$ denotes the sheaf of germs of holomorphic sections of E, while $\mathcal{O}_S$ is the structure sheaf of S and Ω_S^1 is the sheaf of holomorphic 1-forms on S. A *horizontal section* of ∇ is a section $e \in \mathcal{E}$ such that $\nabla e \equiv O$.

Let $\{(U_\alpha, z_\alpha, e_\alpha)\}$ be an atlas of S trivializing E, where (U_α, z_α) are local charts of S and e_α is a holomorphic generator of $E|_{U_\alpha}$. Over U_α, a holomorphic connection ∇ is represented by a holomorphic 1-form $\eta_\alpha \in \Omega_S^1(U_\alpha)$ such that

$$\nabla e_\alpha = \eta_\alpha \otimes e_\alpha.$$

If $\{\xi_{\alpha\beta}\}$ is the cocycle representing the cohomology class $\xi \in H^1(S, \mathcal{O}^*)$ of E, over $U_\alpha \cap U_\beta$ we have

$$e_\beta = e_\alpha \xi_{\alpha\beta}$$

and

$$\eta_\beta = \eta_\alpha + \frac{1}{\xi_{\alpha\beta}} \partial \xi_{\alpha\beta}. \tag{1.1}$$

Recalling the short exact sequence of sheaves

$$O \to \mathbb{C}^* \to \mathcal{O}^* \xrightarrow{\partial \log} \Omega_S^1 \to O,$$

we see that equality (1.1) shows that the existence of a holomorphic connection ∇ is equivalent to the vanishing of the image of ξ under the map $\partial \log : H^1(S, \mathcal{O}^*) \to H^1(S, \Omega_S^1)$ induced on cohomology. So, the class ξ is the image of a class $\hat{\xi} \in H^1(S, \mathbb{C}^*)$: we now recall how to find a representative $\hat{\xi}_{\alpha\beta}$

of $\hat{\xi}$. Up to shrinking the U_α's, we can find holomorphic functions $K_\alpha \in \mathcal{O}(K_\alpha)$ such that $\eta_\alpha = \partial K_\alpha$ on U_α. Then (1.1) implies that on $U_\alpha \cap U_\beta$

$$\hat{\xi}_{\alpha\beta} = \frac{\exp(K_\alpha)}{\exp(K_\beta)} \xi_{\alpha\beta} \tag{1.2}$$

is a complex non-zero constant defining a cocycle representing ξ. We furthermore remark that

$$\nabla\big(\exp(-K_\alpha)e_\alpha\big) \equiv O, \tag{1.3}$$

that is $\exp(-K_\alpha)e_\alpha$ is a horizontal section on U_α.

Definition 1.2. The homomorphism $\rho : \pi_1(S) \to \mathbb{C}^*$ corresponding to the class $\hat{\xi}$ under the canonical isomorphism $H^1(S, \mathbb{C}^*) \cong \mathrm{Hom}(H_1(S, \mathbb{Z}), \mathbb{C}^*) = \mathrm{Hom}(\pi_1(S), \mathbb{C}^*)$ is the *monodromy representation* of the holomorphic connection ∇. We shall say that ∇ has *real periods* if the image of ρ is contained in S^1, that is if $\hat{\xi}$ is the image of a class in $H^1(S, S^1)$ under the natural inclusion $S^1 \hookrightarrow \mathbb{C}^*$.

In Proposition 3.5 we shall explicitly compute the monodromy representation when $S \subseteq \mathbb{C}$, explaining the rationale behind the terminology.

Now, it is well known that to a Hermitian metric g on a complex vector bundle over a complex manifold M can be associated a unique $(1,0)$-connection ∇ (not necessarily holomorphic) such that $\nabla g \equiv O$, the *Chern connection* of g. We would like to study the converse problem: given a holomorphic connection ∇, does there exist a Hermitian metric g so that $\nabla g \equiv O$?

Definition 1.3. Let E be a complex line bundle on a Riemann surface S, and $\nabla : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$ a holomorphic connection on E. We say that a Hermitian metric g on E is *adapted* to ∇ if $\nabla g \equiv O$, that is if

$$X\big(g(R,T)\big) = g(\nabla_X R, T) + g(R, \nabla_{\overline{X}} T)$$

and

$$\overline{X}\big(g(R,T)\big) = g(\nabla_{\overline{X}} R, T) + g(R, \nabla_X T)$$

for all (not necessarily holomorphic) sections R, T of E, and all vector fields X on S.

Let us see what this condition means in local coordinates. With respect to an atlas $\{(U_\alpha, z_\alpha, e_\alpha)\}$ trivializing E, a Hermitian metric g on E is locally represented by a positive C^∞ function $n_\alpha \in C^\infty(U_\alpha, \mathbb{R}^+)$ given by

$$n_\alpha = g(e_\alpha, e_\alpha).$$

Then it is easy to see that g is adapted to ∇ over U_α if and only if

$$\partial n_\alpha = n_\alpha \eta_\alpha, \tag{1.4}$$

where η_α is the holomorphic 1-form representing ∇. A standard argument shows how to solve this equation; for later reference we formally state here the result, whose proof is elementary.

Proposition 1.1. *Let E be a complex line bundle on a Riemann surface S, and $\nabla : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$ a holomorphic connection on E. Let $(U_\alpha, z_\alpha, e_\alpha)$ be a local chart trivializing E, and define $\eta_\alpha \in \Omega^1_S(U_\alpha)$ by setting $\nabla e_\alpha = \eta_\alpha \otimes e_\alpha$. Assume that we have a holomorphic primitive K_α of η_α on U_α. Then*

$$n_\alpha = \exp(2 \operatorname{Re} K_\alpha) = \exp(K_\alpha + \overline{K_\alpha}) \tag{1.5}$$

is a positive solution of (1.4)*. Conversely, if n_α is a positive solution of* (1.4) *then for any $z_0 \in U_\alpha$ and simply connected neighborhood $U \subseteq U_\alpha$ of z_0 there is a holomorphic primitive $K_\alpha \in \mathcal{O}(U)$ of η_α over U such that $n_\alpha = \exp(2 \operatorname{Re} K_\alpha)$ in U. Furthermore, K_α is unique up to a purely imaginary additive constant. Finally, two* (*local*) *solutions of* (1.4) *differ* (*locally*) *by a positive multiplicative constant.*

Remark 1.1. It is well known that a holomorphic 1-form $k\,dz$ defined in an open set $U \subseteq S$ has a (necessarily holomorphic) primitive if and only if

$$\int_\gamma k\,dz = 0$$

for all closed loops γ in U. However, the obstructions to the existence of n_α are slightly weaker, because we just need the exponential of the real part of a primitive. To be more explicit, let assume that $U = \Delta^*$ is a pointed disk, and use the Laurent expansion to write

$$k(z) = k^*(z) + \frac{\rho}{z},$$

where ρ is the residue of k at the origin. Then k^* has vanishing residue at the origin, and thus it admits a primitive K^* on U. Locally, a primitive of ρ/z is of the form $\rho \log z$; therefore setting

$$K(z) = K^*(z) + \rho \log z$$

we have a locally defined (multivalued) primitive of k. We are interested in the exponential of the real part of K, given by

$$\exp\bigl(2 \operatorname{Re} K(z)\bigr) = \exp\bigl(2 \operatorname{Re} K^*(z)\bigr)|z|^{\operatorname{Re} \rho} \exp\bigl(-(\operatorname{Im} \rho) \arg(z)\bigr).$$

But $\arg(z)$ is defined up to an integer multiple of 2π; therefore, the indeterminacy of $\exp(2 \operatorname{Re} K)$ is a multiplicative factor of the form $e^{-2\pi h \operatorname{Im} \rho}$, with $h \in \mathbb{Z}$. In particular, if the residue ρ is real, then we get a well-defined solution of (1.4) in the whole Δ^*. We shall prove (see Propositions 1.2 and 3.5) that, roughly speaking, this will be the only obstruction to the existence of a global metric adapted to ∇.

Remark 1.2. The Gaussian curvature of a local metric g adapted to ∇ is identically zero. Indeed, g is of the form $\exp(2 \operatorname{Re} K) g_0$, where g_0 is the Euclidean metric. The Gaussian curvature of a metric of the form $h g_0$ is $-\frac{1}{h} \Delta \log h$. In our case $h = \exp(2 \operatorname{Re} K) = |\exp(K)|^2$ is the modulus squared of a holomorphic function; so $\log h$ is harmonic, and hence $\Delta h = 0$.

Having solved the local problem, let us see when we get a global Hermitian metric adapted to ∇. Let $\{(U_\alpha, z_\alpha, e_\alpha)\}$ be an atlas of S trivializing E. Up to shrinking the U_α's, we can take a holomorphic primitive K_α of the holomorphic 1-form η_α representing ∇ on U_α. Taking the logarithm of the modulus of (1.2) on $U_\alpha \cap U_\beta$ we get

$$\operatorname{Re}(K_\alpha - K_\beta) + \log|\xi_{\alpha\beta}| = \log|\hat{\xi}_{\alpha\beta}|. \tag{1.6}$$

Hence:

Proposition 1.2. *Let E be a complex line bundle on a Riemann surface S, and $\nabla : \mathcal{E} \to \Omega^1_S \otimes \mathcal{E}$ a holomorphic connection on E. Then there exists a Hermitian metric adapted to ∇ if and only if ∇ has real periods.*

Proof. Let g be a Hermitian metric on E. If $\{(U_\alpha, z_\alpha, e_\alpha)\}$ is any atlas trivializing E, setting $n_\alpha = g(e_\alpha, e_\alpha)$ over U_α, we must have

$$n_\beta = |\xi_{\alpha\beta}|^2 n_\alpha \tag{1.7}$$

over $U_\alpha \cap U_\beta$. If g is adapted to ∇, up to shrinking the U_α's, we can assume that $n_\alpha = \exp(2\operatorname{Re} K_\alpha)$ over U_α, where K_α is a holomorphic primitive of the form η_α representing ∇ over U_α. Then (1.7) says that

$$\operatorname{Re}(K_\beta - K_\alpha) = \log|\xi_{\alpha\beta}|$$

over $U_\alpha \cap U_\beta$, and hence ∇ has real periods.

Conversely, assume that ∇ has real periods. Then we can find an atlas $\{(U_\alpha, z_\alpha, e_\alpha)\}$ trivializing E, holomorphic primitives K_α, and constants $c_\alpha \in \mathbb{C}^*$ such that $\hat{\xi}_{\alpha\beta} = (c_\beta / c_\alpha)\tilde{\xi}_{\alpha\beta}$ with $\tilde{\xi}_{\alpha\beta} \in S^1$, so that

$$\operatorname{Re}(K_\beta - K_\alpha) - \log|\xi_{\alpha\beta}| = \log|c_\alpha| - \log|c_\beta|.$$

Then $\tilde{K}_\alpha = K_\alpha + \log|c_\alpha|$ is a holomorphic primitive of η_α such that (1.7) is satisfied by $n_\alpha = \exp(2\operatorname{Re}\tilde{K}_\alpha)$, and thus setting $g(e_\alpha, e_\alpha) = n_\alpha$ we get a global Hermitian metric adapted to ∇. $\square$

Actually, we shall be more interested in the case when does not exist a metric adapted to ∇. Indeed, the first main result of this section is the following:

Proposition 1.3. *Let ∇ be a holomorphic connection on a complex line bundle E over a Riemann surface S. Then there exists a real rank 3 non-singular foliation of $E \setminus S$ (where we are identifying S with the zero section of E) whose leaves are the level sets of any local or global Hermitian metric on E adapted to ∇.*

Proof. Choose an atlas $\{(U_\alpha, z_\alpha, e_\alpha)\}$ of S trivializing E (with connected intersections) and such that on each U_α we can find a holomorphic primitive K_α of the holomorphic form η_α representing ∇. Set $n_\alpha = \exp(2\operatorname{Re} K_\alpha)$, and define $g_\alpha : p^{-1}(U_\alpha) \to \mathbb{R}^+$ by setting $g_\alpha(v) = n_\alpha(p(v))|v_\alpha|^2$, where $p : E \to S$ is the canonical projection, and $v_\alpha \in \mathbb{C}$ is so that $v = v_\alpha e_\alpha$. Clearly, g_α is a submersion out of the zero section, and thus its level sets define a real rank 3 non-singular foliation of $p^{-1}(U_\alpha) \setminus U_\alpha$; we must show that the foliation is independent of α. But indeed if $z_0 \in U_\alpha \cap U_\beta$ and $v \in E_{z_0}$ formula (1.6) yields $c_{\alpha\beta} \in \mathbb{R}$ such that

$$\begin{aligned} g_\beta(v) &= n_\beta(z_0)|v_\beta|^2 = n_\alpha(z_0)\exp\bigl(2\operatorname{Re}(K_\beta - K_\alpha)(z_0)\bigr)|v_\beta|^2 = |\hat{\xi}_{\alpha\beta}|^{-2} n_\alpha(z_0)\bigl|\xi_{\alpha\beta}(z_0)\bigr|^2|v_\beta|^2 \\ &= |\hat{\xi}_{\alpha\beta}|^{-2} n_\alpha(z_0)|v_\alpha|^2 = |\hat{\xi}_{\alpha\beta}|^{-2} g_\alpha(v); \end{aligned}$$

therefore g_α and g_β differ by a multiplicative constant, and thus they have the same level sets in $p^{-1}(U_\alpha \cap U_\beta)$.

Finally, if g is a Hermitian metric adapted to ∇ defined on an open set U, Proposition 1.1 says that on each $U \cap U_\alpha$ the function $n = g(e_\alpha, e_\alpha)$ locally is a positive multiple of n_α, and hence the level sets of the norm induced by g coincide with the level sets of g_α. $\square$

Definition 1.4. The foliation just defined induced by a holomorphic connection ∇ on a complex line bundle E over a Riemann surface S is the *metric foliation* of ∇ on E.

Remark 1.3. When g is a globally defined Hermitian metric adapted to ∇, the leaves of the metric foliation are simply the sets $\{v \in E \setminus S \mid g(v, v) = \text{const.}\}$, and thus they are diffeomorphic to $S^1 \times S$ and closed in the total space of E (zero section included). But when ∇ does not admit an adapted metric the leaves might have a more complicated behavior; in particular, they can accumulate the zero section (and thus they are not closed in the total space of E); see Lemma 3.2 and Theorem 3.3.

The choice of a local chart $(U_\alpha, z_\alpha, e_\alpha)$ trivializing the line bundle $p : E \to S$ yields local coordinates (z_α, v_α) on the total space of E, and thus a local frame for TE. Let us denote by $\{\partial_\alpha, \partial/\partial v_\alpha\}$ this local frame, where ∂_α is the tangent vector corresponding to the coordinate z_α, and by $\{p^*(dz_\alpha), dv_\alpha\}$ the dual co-frame. From $e_\beta = e_\alpha \xi_{\alpha\beta}$ in $U_\alpha \cap U_\beta$ we get

$$v_\alpha = (\xi_{\alpha\beta} \circ p) v_\beta, \tag{1.8}$$

and

$$dv_\alpha = v_\beta p^*(\partial \xi_{\alpha\beta}) + (\xi_{\alpha\beta} \circ p)\, dv_\beta = v_\beta \left(\frac{\partial \xi_{\alpha\beta}}{\partial z_\beta} \circ p \right) p^*(dz_\beta) + (\xi_{\alpha\beta} \circ p)\, dv_\beta. \tag{1.9}$$

Furthermore, we have

$$p^*(dz_\alpha) = (\psi_{\alpha\beta} \circ p) p^*(dz_\beta),$$

where $\psi_{\alpha\beta} = \partial z_\alpha / \partial z_\beta$. It follows that

$$\frac{\partial}{\partial v_\beta} = (\xi_{\alpha\beta} \circ p) \frac{\partial}{\partial v_\alpha} \quad \text{and} \quad \partial_\beta = (\psi_{\alpha\beta} \circ p) \partial_\alpha + v_\alpha \left(\frac{1}{\xi_{\alpha\beta}} \frac{\partial \xi_{\alpha\beta}}{\partial z_\beta} \right) \circ p \frac{\partial}{\partial v_\alpha}.$$

In particular, the local sections $v_\alpha \partial/\partial v_\alpha$ give a globally defined section R of TE, whose integral curves are the real lines through the origin in each fiber of E. Analogously, the integral curves of iR are circumferences around the origin in each fiber of E, and gives a rank 1 real foliation of the total space of E, singular along the zero section.

Using these notations, it is easy to see that in a local chart $(U_\alpha, z_\alpha, e_\alpha)$ trivializing E the metric foliation is generated by the real 1-form

$$\varpi_\alpha = \operatorname{Re}\bigl(|v_\alpha|^2 p^* \eta_\alpha + \overline{v_\alpha}\, dv_\alpha \bigr), \tag{1.10}$$

where η_α is the holomorphic 1-form representing ∇. In particular, the tangent space to the foliation is generated by H_α, iH_α and iR, where H_α is the local section of TE defined by

$$H_\alpha = \partial_\alpha - (k_\alpha \circ p) v_\alpha \frac{\partial}{\partial v_\alpha}, \tag{1.11}$$

with $k_\alpha = \eta_\alpha(\partial/\partial z_\alpha)$. In particular, it is clear that the metric foliation is transversal to the fibers of E.

The local fields H_α define a complex rank 1 foliation of the total space of E, because

$$H_\beta = (\psi_{\alpha\beta} \circ p) H_\alpha. \tag{1.12}$$

Furthermore, it is easy to check that a local section s_α of E is an integral curve of H_α if and only if $\nabla s_\alpha \equiv O$, that is if and only if s_α is a horizontal section.

Definition 1.5. The complex rank 1 non-singular foliation on $E \setminus S$ induced by the local fields H_α is the *horizontal foliation* of the holomorphic connection ∇. Clearly, the leaves of the horizontal foliation are transversal to the fibers of E, and are contained in the leaves of the metric foliation.

It is also easy to describe this foliation using a global holomorphic 1-form on the total space of E. Indeed, let as always $\{(U_\alpha, z_\alpha, e_\alpha)\}$ be an atlas trivializing E, and let (z_α, v_α) denote the corresponding local coordinates on $E|_{U_\alpha}$. Denote by $p : E \to S$ the projection. Then (1.9) and (1.1) yield

$$p^*\eta_\beta + \frac{1}{v_\beta}\,dv_\beta = p^*\eta_\alpha + \frac{1}{v_\alpha}\,dv_\alpha.$$

Therefore setting

$$\omega = p^*\eta_\alpha + \frac{1}{v_\alpha}\,dv_\alpha$$

on $E|_{U_\alpha} \setminus U_\alpha$, we get a global holomorphic 1-form on $E \setminus S$. Furthermore, $\omega(H_\alpha) \equiv 0$, and so ω induces the horizontal foliation, as claimed.

Remark 1.4. We clearly have

$$\varpi_\alpha = |v_\alpha|^2 \operatorname{Re}\omega,$$

where ϖ_α is given by (1.10).

Remark 1.5. We can extend the horizontal foliation to a non-singular foliation of the whole total space of E just by adding the zero section as a new leaf. Indeed, we have

$$v_\beta p^*\eta_\beta + dv_\beta = v_\beta\omega = \frac{1}{\xi_{\alpha\beta} \circ p} v_\alpha\omega = \frac{1}{\xi_{\alpha\beta} \circ p}\bigl[v_\alpha p^*\eta_\alpha + dv_\alpha\bigr];$$

therefore the local forms $v_\alpha p^*\eta_\alpha + dv_\alpha$ define a complex rank 1 non-singular foliation on E which coincides with the horizontal foliation off the zero section.

Later on we shall need local parametrizations for the leaves of the horizontal foliation. We need a holomorphic map $\varphi : V \to E$ defined on some open set $V \subseteq \mathbb{C}$ and such that

$$\varphi' = H_\alpha \circ \varphi.$$

Writing in local coordinates $\varphi(\zeta) = (z_\alpha(\zeta), v_\alpha(\zeta))$ we see that we need

$$z'_\alpha \equiv 1 \quad \text{and} \quad v'_\alpha = -k_\alpha v_\alpha.$$

Hence

$$z_\alpha(\zeta) = \zeta + c_0, \qquad v_\alpha(\zeta) = c_1 \exp\bigl(-K_\alpha(\zeta + c_0)\bigr), \tag{1.13}$$

where $c_0, c_1 \in \mathbb{C}$ and K_α is a holomorphic primitive of η_α on $V + c_0$; compare with (1.3).

Since the local fields H_α do not glue together in the intersections, they do not define a *real* rank 1 foliation of $E \setminus S$. As discussed in the introduction, in the cases we shall be interested in we shall have another ingredient available: an isomorphism $X : E \to TS$, allowing us to introduce a real rank 1 foliation of $E \setminus S$ by using X to define ∇-geodesics, and then considering the geodesic flow.

Definition 1.6. Let E be a line bundle on a Riemann surface S, and assume we have a holomorphic connection ∇ on E and an isomorphism $X: E \to TS$. We say that a smooth curve $\sigma: I \to S$, where $I \subseteq \mathbb{R}$ is an interval, is a *geodesic* (with respect to ∇ and X) if $\nabla_{\sigma'} X^{-1}(\sigma') \equiv O$. If σ is a geodesic, then $X^{-1}(\sigma')$ is a curve in the total space of E; we shall momentarily show that it is an integral curve of a vector field on E.

Remark 1.6. The reason we are explicitly using the isomorphism X instead of just considering geodesics for a holomorphic connection on TS is that in the applications we have in mind the line bundle E will be the restriction to S of a line bundle $\hat{E}$ defined on a larger Riemann surface $\hat{S} \supset S$. We shall have a morphism $X: \hat{E} \to T\hat{S}$, but this will be an isomorphism only over S. Furthermore, we shall be interested in the behavior of geodesics in $\hat{S}$, and of the geodesic flow in the total space of $\hat{E}$; and to study those it will be important to work in E using X instead of working in TS. However, in the next three sections we shall deal with $E = TS$ and $X = \mathrm{id}$ only.

If $(U_\alpha, z_\alpha, e_\alpha)$ is a local chart trivializing E, then there is a holomorphic function $X_\alpha \in \mathcal{O}^*(U_\alpha)$ such that

$$X(e_\alpha) = X_\alpha \frac{\partial}{\partial z_\alpha},$$

and it is easy to check that changing coordinates X_α changes according to the rule

$$X_\beta = \frac{\xi_{\alpha\beta}}{\psi_{\alpha\beta}} X_\alpha. \tag{1.14}$$

Then we have

Proposition 1.4. *Let ∇ be a holomorphic connection on a complex line bundle $p: E \to S$ over a Riemann surface S, and $X: E \to TS$ an isomorphism. Let $\{(U_\alpha, z_\alpha, e_\alpha)\}$ be an atlas trivializing E. Then*:

(i) *setting*

$$G|_{p^{-1}(U_\alpha)} = (X_\alpha \circ p) v_\alpha H_\alpha = (X_\alpha \circ p) v_\alpha \partial_\alpha - (X_\alpha k_\alpha) \circ p (v_\alpha)^2 \frac{\partial}{\partial v_\alpha}$$

we define a global holomorphic section G of TE, vanishing only on the zero section;

(ii) *a curve $\sigma: I \to S$ is a geodesic if and only if $X^{-1}(\sigma')$ is an integral curve of G.*

Proof. (i) follows immediately from (1.14), (1.12) and (1.8). Denoting by $z_\alpha(t)$ the expression of the curve $\sigma: I \to S$ in the local chart (U_α, z_α), it is easy to see that σ is a geodesic if and only if

$$\left(\frac{z'_\alpha}{X_\alpha}\right)' + k_\alpha X_\alpha \left(\frac{z'_\alpha}{X_\alpha}\right)^2 \equiv 0. \tag{1.15}$$

On the other hand, a curve $t \mapsto (z_\alpha(t), v_\alpha(t))$ is an integral curve of G if and only if

$$\begin{cases} z'_\alpha = X_\alpha(z_\alpha) v_\alpha, \\ v'_\alpha = -k_\alpha(z_\alpha) X_\alpha(z_\alpha) v_\alpha^2. \end{cases} \tag{1.16}$$

Since $X^{-1}(\sigma')$ is expressed in local coordinates by $(z_\alpha, z'_\alpha / X_\alpha(z_\alpha))$, assertion (ii) follows. $\square$

Definition 1.7. The global holomorphic field G is the *geodesic field* associated to ∇ and X. The rank 1 non-singular real foliation of $E \setminus S$ given by the integral curves of G is the *geodesic foliation* associated to ∇ and X. Clearly, the leaves of the geodesic foliation are contained in the leaves of the horizontal foliation.

Remark 1.7. Since G is a global field, the leaves of the geodesic foliation, being integral curves of G, are equipped with a canonical parametrization. In principle, we can get such a parametrization by quadratures and taking inverses. Indeed, let $t \mapsto (z_\alpha(t), v_\alpha(t))$ be a local integral curve of G. By (1.13) we must have $v_\alpha(t) = c_1 \exp(-K_\alpha(z_\alpha(t)))$; hence the first equation in (1.16) yields

$$\frac{\exp(K_\alpha(z_\alpha))}{X_\alpha(z_\alpha)} z'_\alpha = c_1.$$

If F_α is a primitive of $\exp(K_\alpha)/X_\alpha$ we then get $F_\alpha(z_\alpha(t)) = c_1 t + c_2$; since $F'_\alpha \neq 0$ always we finally get

$$z_\alpha(t) = F_\alpha^{-1}(c_1 t + c_2).$$

We shall use this procedure in the last section of this paper.

Remark 1.8. The leaves of the geodesic foliation are contained in the leaves of the horizontal foliation, and thus in the leaves of the metric foliation. Furthermore, they are transversal to the fibers of E. So we have cut the total space of E off the zero section in three real foliations, of real rank 3, 2 and 1 respectively, one inside the other, and all transversal to the fibers of E.

Remark 1.9. Clearly, the field iG defines another real rank 1 non-singular foliation of $E \setminus S$; but we shall not use it in this paper.

The main goal of this paper will be the study of the dynamics of the geodesic foliation, and then the application of our results to the study of the dynamics of homogeneous vector fields in $\mathbb{C}^n$. Along the way we shall also get a few (usually easier) results on the dynamics of the metric and horizontal foliations.

2. Simply connected Riemann surfaces

Let $\tilde{S}$ be a simply connected Riemann surface, and assume that we have a holomorphic connection $\tilde{\nabla}$ on $T\tilde{S}$. In particular (see, e.g., [18, Theorem III.17.33]), $\tilde{S}$ cannot be $\mathbb{P}^1(\mathbb{C})$, and so $\tilde{S}$ is biholomorphic either to $\mathbb{C}$ or to the unit disk Δ. In both cases, $T\tilde{S} = \tilde{S} \times \mathbb{C}$, and we have a global coordinate z on $\tilde{S}$. We would like to study the metric, horizontal and geodesic foliations associated to $\tilde{\nabla}$ on $T\tilde{S}$.

We use $\partial/\partial z$ as global section of $T\tilde{S}$, giving an explicit isomorphism between $T\tilde{S}$ and $\tilde{S} \times \mathbb{C}$. Let $\tilde{\eta} = \tilde{k}\, dz$ be the (global) holomorphic 1-form associated to $\tilde{\nabla}$, and $\tilde{K} : \tilde{S} \to \mathbb{C}$ a (global) holomorphic primitive of $\tilde{\eta}$. By Proposition 1.1, the function $\tilde{g} : \tilde{S} \times \mathbb{C} \to \mathbb{C}$ given by

$$\tilde{g}(z; v) = \exp\bigl(2 \operatorname{Re} \tilde{K}(z)\bigr) |v|^2 \tag{2.1}$$

is the norm squared of a Hermitian metric (that we shall also denote by $\tilde{g}$) adapted to $\tilde{\nabla}$. In particular, the leaves of the metric foliation are just given by the level sets of $\tilde{g}$:

$$\exp\bigl(2 \operatorname{Re} \tilde{K}(z)\bigr) |v|^2 = \text{const.} \in \mathbb{R}^+.$$

Furthermore, we have a similar description for the leaves of the horizontal foliation; indeed, (1.13) says that they can be expressed by

$$\exp\bigl(\tilde{K}(z)\bigr)v = \text{const.} \in \mathbb{C}^*, \tag{2.2}$$

that is as the level sets of the holomorphic function $(z; v) \mapsto \exp(\tilde{K}(z))v$.

As a consequence, the dynamics of both the metric and the horizontal foliations over a simply connected Riemann surface is pretty trivial. In particular, each leaf of the horizontal foliation intersects each fiber in exactly one point, and we have an explicit biholomorphism between $\tilde{S}$ and the leaf through the point $(z_0; v_0)$ given by

$$z \mapsto \bigl(z; \exp\bigl(-\tilde{K}(z)\bigr)\exp\bigl(\tilde{K}(z_0)\bigr)v_0\bigr).$$

Analogously, each leaf of the metric foliation intersects each fiber in exactly one circumference, and we have an explicit diffeomorphism between $S^1 \times \tilde{S}$ and the leaf through the point $(z_0; v_0)$ given by

$$\bigl(e^{2\pi i\theta}, z\bigr) \mapsto \bigl(z; \exp\bigl(-\operatorname{Re}\tilde{K}(z)\bigr)\exp\bigl(\operatorname{Re}\tilde{K}(z_0)\bigr)|v_0|e^{2\pi i\theta}\bigr).$$

The rest of this section is devoted to the study of the geodesic foliation. As a, somewhat unexpected (at least by us), consequence, we shall see (Corollary 2.5) that the metric $\tilde{g}$ adapted to $\tilde{\nabla}$ is *never* complete (unless $\tilde{S} = \mathbb{C}$ and $\tilde{\nabla}$ is trivial), preventing the use of standard theorems like the Hopf–Rinow theorem, even though in this case our geodesics are the usual Riemannian geodesics of the Riemannian metric $\operatorname{Re}\tilde{g}$.

Our first result is the following:

Proposition 2.1. *Let $\tilde{\nabla}$ be a holomorphic connection on $T\tilde{S} = \tilde{S} \times \mathbb{C}$, where $\tilde{S} \cong \mathbb{C}$ or Δ is a simply connected Riemann surface. Let $\tilde{\eta}$ be the holomorphic 1-form associated to $\tilde{\nabla}$, and $\tilde{K} : \tilde{S} \to \mathbb{C}$ a holomorphic primitive of $\tilde{\eta}$. Finally, let $J : \tilde{S} \to \mathbb{C}$ be a holomorphic primitive of $\exp(\tilde{K})$. Then $J : \tilde{S} \to \mathbb{C}$ is a local isometry, where $\tilde{S}$ is endowed with the metric $\tilde{g}$ adapted to $\tilde{\nabla}$ corresponding to $\tilde{K}$, and $\mathbb{C}$ is endowed with the Euclidean metric.*

Proof. First of all, since $\tilde{S}$ is simply connected, J exists. Now, by (2.1), the $\tilde{g}$-length of $v \in T_zS$ is

$$\exp\bigl(\operatorname{Re}\tilde{K}(z)\bigr)|v| = \bigl|J'(z)v\bigr|,$$

and hence J is a local isometry. □

In particular, and this will be important in the sequel, J sends a geodesic segment contained in any open set $U \subseteq \tilde{S}$ where J is injective onto a line segment contained in $J(U)$. Notice that J is locally invertible because $J' = \exp(\tilde{K})$ is never vanishing.

Using J, we can say a lot more on the geodesics of $\tilde{\nabla}$. We begin with

Proposition 2.2. *Let $\tilde{\nabla}$ be a holomorphic connection on $T\tilde{S} = \tilde{S} \times \mathbb{C}$, where $\tilde{S} \cong \mathbb{C}$ or Δ is a simply connected Riemann surface. Let $\tilde{\eta}$ be the holomorphic 1-form associated to $\tilde{\nabla}$, and $\tilde{K} : \tilde{S} \to \mathbb{C}$ a holomorphic primitive of $\tilde{\eta}$. Finally, let $\tilde{J} : \tilde{S} \to \mathbb{C}$ be a holomorphic primitive of $\exp(\tilde{K})$. Then a smooth curve $\sigma : I \to \tilde{S}$ is a geodesic if and only if there are $c_0, w_0 \in \mathbb{C}$ such that*

$$J\bigl(\sigma(t)\bigr) = c_0 t + w_0.$$

In particular, the geodesic issuing from $z_0 \in \tilde{S}$ along the direction $v_0 \in \mathbb{C}^*$ is given by $\sigma(t) = J^{-1}(c_0 t + J(z_0))$, where $c_0 = \exp(\tilde{K}(z_0))v_0$ and J^{-1} is an analytic continuation of the local inverse of J nearby $J(z_0)$ chosen so that $J^{-1}(J(z_0)) = z_0$.

Proof. The first assertion follows directly from the previous proposition; but let us describe another proof giving a useful formula.

We know that if σ is a geodesic then the support of σ' is contained in a leaf of the horizontal foliation. Recalling (2.2), this means that σ' must satisfy the differential equation

$$\sigma' = c_0 \exp\bigl(-\tilde{K}(\sigma)\bigr) \tag{2.3}$$

for some $c_0 \in \mathbb{C}$.

Conversely, if σ satisfies this equation it is easy to check that it satisfies (1.15) too (remember that $X_\alpha \equiv 1$ here), and thus it is a geodesic. But we have

$$\sigma' = c_0 \exp\bigl(-\tilde{K}(\sigma)\bigr) \quad\Longleftrightarrow\quad \exp\bigl(\tilde{K}(\sigma)\bigr)\sigma' \equiv c_0 \quad\Longleftrightarrow\quad \bigl(J' \circ \sigma\bigr)\sigma' \equiv c_0$$
$$\Longleftrightarrow\quad J\bigl(\sigma(t)\bigr) = c_0 t + w_0,$$

and the first assertion follows. The second is an easy consequence of the fact that

$$c_0 = \exp\bigl(\tilde{K}(z_0)\bigr)v_0. \qquad \square$$

Remark 2.1. In particular, the proof shows that a curve $\sigma : [0, \varepsilon) \to \tilde{S}$ is a geodesic if and only if

$$\sigma'(t) = \exp\bigl(-\tilde{K}\bigl(\sigma(t)\bigr)\bigr)\exp\bigl(\tilde{K}\bigl(\sigma(0)\bigr)\bigr)\sigma'(0)$$

if and only if

$$J\bigl(\sigma(t)\bigr) = \exp\bigl(\tilde{K}\bigl(\sigma(0)\bigr)\bigr)\sigma'(0)t + J\bigl(\sigma(0)\bigr).$$

The first important fact we deduce from this result is that geodesics cannot accumulate points in $\tilde{S}$. To better express this fact let us recall two standard definitions.

Definition 2.1. We say that a curve $\gamma : [0, \varepsilon) \to \tilde{S}$ (with $\varepsilon \in (0, +\infty]$) *tends to the boundary* of $\tilde{S}$ if $\gamma(t)$ eventually leaves every compact subset of $\tilde{S}$. In other words, γ does not tend to the boundary if and only if there is a sequence $t_k \uparrow \varepsilon$ such that $\gamma(t_k) \to \tilde{z}_0 \in \tilde{S}$.

Definition 2.2. An *asymptotic value* of a holomorphic function $J : \tilde{S} \to \mathbb{C}$ is a $w_0 \in \mathbb{C}$ such that there exists a curve $\gamma : [0, 1) \to \tilde{S}$ tending to the boundary of $\tilde{S}$ with $J(\gamma(t)) \to w_0$ as $t \to 1$.

Proposition 2.3. *Let $\tilde{\nabla}$ be a holomorphic connection on $T\tilde{S} = \tilde{S} \times \mathbb{C}$, where $\tilde{S} \cong \mathbb{C}$ or Δ is a simply connected Riemann surface, and let $\sigma_{v_0} : [0, \varepsilon_{v_0}) \to \tilde{S}$ be the maximal geodesic issuing from $z_0 \in \tilde{S}$ in the direction $v_0 \in \mathbb{C}^*$. Then σ_{v_0} tends to the boundary of $\tilde{S}$. Furthermore, if $\varepsilon_{v_0} < +\infty$ then $w_0 = J(z_0) + J'(z_0)v_0\varepsilon_{v_0}$ is an asymptotic value of J.*

Proof. Let us first consider the case $\varepsilon_{v_0} = +\infty$. If σ_{v_0} does not tend to the boundary we can find a sequence $t_k \to +\infty$ so that $\sigma_{v_0}(t_k) \to \tilde{z}_0 \in \tilde{S}$. Hence $J(\sigma_{v_0}(t_k)) \to J(\tilde{z}_0) \in \mathbb{C}$; but $J(\sigma_{v_0}(t_k)) = J(z_0) + J'(z_0)v_0 t_k$ is unbounded, contradiction.

Assume then $\varepsilon_{v_0} < +\infty$, and put $\gamma(t) = \sigma_{v_0}(\varepsilon_{v_0} t)$ and $\gamma_1(t) = J(\gamma(t)) = J(z_0) + J'(z_0)v_0\varepsilon_{v_0} t$. Clearly $J(\gamma(t)) \to w_0$ as $t \to 1$; so to end the proof it suffices to show that γ tends to the boundary of $\tilde{S}$.

If γ does not tend to the boundary, then there is a sequence $t_k \to 1$ with $\gamma(t_k) \to \tilde{z}_0 \in \tilde{S}$; hence $\gamma_1(t_k) = J(\gamma(t_k)) \to J(\tilde{z}_0)$, and thus $w_0 = J(\tilde{z}_0) \in J(\tilde{S})$. Let $F : D \to \tilde{S}$ be the local inverse of J with $F(w_0) = \tilde{z}_0$, where D is a disk centered at z_0. By Proposition 2.2, there is an inverse F_1 of J defined in a neighborhood U of the support of γ_1 with $F_1 \circ \gamma_1 \equiv \gamma$; up to shrinking U we can also assume that

$U \cap D$ is connected. We have $\gamma_1(t_k) \in D \cap U$ eventually. Furthermore, $F(D)$ is an open neighborhood of $\tilde{z}_0$; hence $\gamma(t_k) \in F(D)$ eventually. But

$$J\big(F_1\big(\gamma_1(t_k)\big)\big) = \gamma_1(t_k) = J\big(F\big(\gamma_1(t_k)\big)\big);$$

since J is injective in $F(D)$, it follows that $F_1(\gamma_1(t_k)) = F(\gamma_1(t_k))$ eventually. But then F and F_1 are two branches of the inverse of J defined in the connected open set $U \cap D$ and assuming the same value at $\gamma_1(t_k)$; it follows that $F \equiv F_1$ on $U \cap D$. Therefore the curve $t \mapsto F(J(z_0) + J'(z_0)v_0 t)$ is a geodesic extending σ_{v_0} beyond ε_{v_0}, against the maximality of ε_{v_0}. $\square$

The next step consists in studying the set of points reached by geodesics issuing from a given base point.

Definition 2.3. For $(z_0, v_0) \in T\tilde{S} \setminus \tilde{S}$, let $\sigma_{v_0} : [0, \varepsilon_{v_0}) \to \tilde{S}$ denote the maximal geodesic issuing from z_0 in the direction v_0, with $\varepsilon_{v_0} \in (0, +\infty]$. Put $\mathcal{D}_{z_0} = \{v \in \mathbb{C} \mid \varepsilon_v > 1\}$ and define $\exp_{z_0} : \mathcal{D}_{z_0} \to \tilde{S}$ by setting $\exp_{z_0}(v) = \sigma_v(1)$.

Then:

Proposition 2.4. *Let $\tilde{\nabla}$ be a holomorphic connection on $T\tilde{S} = \tilde{S} \times \mathbb{C}$, where $\tilde{S} \cong \mathbb{C}$ or Δ is a simply connected Riemann surface, and fix $z_0 \in \tilde{S}$. Then*:

(i) $J \circ \exp_{z_0} = J(z_0) + J'(z_0)\mathrm{id}$;
(ii) $\exp_{z_0}$ *is a biholomorphism with its image*;
(iii) *J is globally injective on the open simply connected set $\exp_{z_0}(\mathcal{D}_{z_0})$ of points that can be joined to z_0 by a geodesic, and the inverse $J^{-1} : J(\exp_{z_0}(\mathcal{D}_{z_0})) = J(z_0) + J'(z_0)\mathcal{D}_{z_0} \to \tilde{S}$ is given by*

$$J^{-1}(w) = \exp_{z_0}\left(\frac{w - J(z_0)}{J'(z_0)}\right).$$

Proof. (i) Take $v \in \mathcal{D}_{z_0}$. Then Proposition 2.2 yields

$$J\big(\exp_{z_0}(v)\big) = J\big(\sigma_v(1)\big) = J(z_0) + J'(z_0)v,$$

as stated.

(ii) Part (i) implies that $\exp_{z_0}$ is injective. The holomorphicity follows from the fact that σ_v solves the Cauchy problem

$$\begin{cases} \sigma' = \exp\big(-\tilde{K}(\sigma)\big) J'(z_0)v, \\ \sigma(0) = z_0, \qquad \sigma'(0) = v, \end{cases}$$

and thus $\sigma_v(1)$ depends holomorphically on v. Then part (i) yields

$$\big(J' \circ \exp_{z_0}\big) \exp'_{z_0} \equiv J'(z_0); \tag{2.4}$$

and since $J' = \exp(\tilde{K})$ it follows that $\exp'_{z_0}$ is never vanishing. Being globally injective, $\exp_{z_0}$ is then a biholomorphism with its image.

(iii) It follows immediately from (i) and (ii), noticing that $\exp_{\tilde{p}_0}(\mathcal{D}_{z_0})$ is simply connected because $\mathcal{D}_{z_0}$ is star-shaped with respect to the origin. $\square$

As a consequence, we have an interesting corollary:

Corollary 2.5. *Let $\tilde\nabla$ be a holomorphic connection on $T\tilde S = \tilde S \times \mathbb{C}$, where $\tilde S \cong \mathbb{C}$ or Δ is a simply connected Riemann surface. Then a metric $\tilde g$ adapted to $\tilde\nabla$ is never complete, unless $\tilde S \cong \mathbb{C}$ and $\tilde\nabla$ is the trivial connection.*

Proof. Assume $\tilde g$ complete. Then the geodesics are defined for all times, and hence we have $\mathcal{D}_z = \mathbb{C}$ for all $z \in \tilde S$. But then $\exp_z(\mathcal{D}_z)$ is a copy of $\mathbb{C}$ contained in $\tilde S$; therefore $\tilde S = \mathbb{C} = \exp_z(\mathcal{D}_z)$. In particular, $\exp_z$ must be affine linear, sending the origin to z and with derivative 1 at the origin, by (2.4); thus $\exp_z(v) = v + z$. From Proposition 2.4 it follows that J is affine linear too; therefore $\exp(\tilde K) = J'$ is constant. Then $\tilde K$ is constant, and hence $\tilde\eta = \partial\tilde K \equiv 0$, that is $\tilde\nabla$ is the trivial connection. □

We end this section with a couple of remarks on the case $\tilde S = \mathbb{C}$. In this case, if $\tilde\nabla$ is not trivial, $\exp_z$ cannot be surjective. In fact, if $\exp_z$ is surjective then J is globally injective and thus (being $\tilde S = \mathbb{C}$) J must be affine linear and, as before, we find that $\tilde\nabla$ is trivial. Notice that in general J has an essential singularity at infinity; therefore (open simply connected) sets where J is injective tend to become very thin near infinity.

We have also seen that $\mathcal{D}_z$ cannot be $\mathbb{C}$, unless $\tilde\nabla$ is trivial; so there are geodesics going to the boundary in finite time. However, we cannot give a bound on this time. More precisely, we have

Proposition 2.6. *Let $\tilde\nabla$ be a holomorphic connection on $T\mathbb{C}$. Given $z_0 \in \mathbb{C}$, let $\varepsilon : S^1 \to (0, +\infty]$ be defined by $\varepsilon(e^{2\pi i\theta}) = \varepsilon_{e^{2\pi i\theta}} = \sup\{t > 0 \mid te^{2\pi i\theta} \in \mathcal{D}_{z_0}\}$. Then ε is unbounded on every interval of S^1.*

Proof. Assume, by contradiction, that there are $0 \leqslant \theta_0 < \theta_1 < 2\pi$ and $M > 0$ such that $\varepsilon(e^{2\pi i\theta}) < M$ for all $\theta \in [\theta_0, \theta_1]$. We know that $\exp_{z_0}$ is injective, and that all geodesics tend to the boundary of $\tilde S = \mathbb{C}$ (that is, to infinity); therefore the geodesics issuing from z_0 with direction $e^{2\pi i\theta}$ with $\theta \in [\theta_1, \theta_2]$ swap a wedge-like simply connected region $W \subset \mathbb{C}$ bounded by the geodesics starting with direction $v_0 = e^{2\pi i\theta_0}$ and $v_1 = e^{2\pi i\theta_1}$. If $v = e^{2\pi i\theta}$ we have

$$\bigl|J\bigl(\sigma_v(t)\bigr)\bigr| \leqslant \bigl|J(z_0)\bigr| + \bigl|J'(z_0)\bigr|\varepsilon(v) < \bigl|J(z_0)\bigr| + \bigl|J'(z_0)\bigr|M.$$

So J is bounded on W; by a Phragmen–Lindelöf argument (see [23, Theorem III.3.4]), it follows that $J(\sigma_{v_0}(t))$ and $J(\sigma_{v_1}(t))$ must have the same limit. But this would imply $v_0 = v_1$, contradiction. □

When $\tilde S = \Delta$, this argument just says that we cannot have an interval of geodesics all converging to the same boundary point of Δ in finite time.

3. Multiply connected Riemann surfaces

Now let S be any Riemann surface, and assume we have a holomorphic connection ∇ on TS; in this section we shall study the metric, horizontal and geodesic foliations induced by ∇ on $TS \setminus S$.

Let us begin with a few preliminaries. Let $\pi : \tilde S \to S$ be the universal covering map. Since $S \neq \mathbb{P}^1(\mathbb{C})$, the universal covering space $\tilde S$ is biholomorphic either to $\mathbb{C}$ or to Δ. Let $\tilde\nabla = \pi^*\nabla$ be the holomorphic connection on $T\tilde S$ induced by ∇ via π (it is well defined because π is locally invertible); it satisfies the equations

$$d\pi(\tilde\nabla_{\tilde v}\tilde e) = \nabla_{d\pi(\tilde v)}\, d\pi(\tilde e) \quad\Longleftrightarrow\quad (\mathrm{id} \otimes d\pi) \circ \tilde\nabla = \bigl(\pi^* \otimes \mathrm{id}\bigr) \circ \nabla \circ d\pi. \tag{3.1}$$

Let (U_α, z_α) be a chart of S, and η_α the local holomorphic 1-form representing ∇ on U_α. Denote by w the coordinate on $\tilde S = \mathbb{C}$ or Δ, and by $\tilde\eta$ the global holomorphic 1-form representing $\tilde\nabla$. We define a local derivative $\pi'_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{C}^*$ by

$$d\pi_w\left(\frac{\partial}{\partial w}\right) = \pi'_\alpha(w) \left.\frac{\partial}{\partial z_\alpha}\right|_{\pi(w)}. \tag{3.2}$$

Then using (3.1) it is easy to see that η_α and $\tilde{\eta}$ are related by

$$\tilde{\eta} = \pi^* \eta_\alpha + \frac{1}{\pi'_\alpha} d\pi'_\alpha \tag{3.3}$$

over $\pi^{-1}(U_\alpha)$.

As a first consequence we have

Proposition 3.1. *Let S be a (multiply connected) Riemann surface, and ∇ a holomorphic connection on TS. Let $\tilde{\pi} : \tilde{S} \to S$ be the universal covering map, and $\tilde{\nabla}$ the holomorphic connection on $T\tilde{S}$ induced by ∇ via π. Then:*

(i) *$d\pi$ sends leaves of the metric (respectively, horizontal) foliation in $T\tilde{S}$ onto leaves of the metric (respectively, horizontal) foliation in TS;*
(ii) *a curve $\tilde{\sigma} : I \to \tilde{S}$ is a geodesic for $\tilde{\nabla}$ if and only if $\sigma = \pi \circ \tilde{\sigma}$ is a geodesic for ∇.*

Proof. (i) Let $\omega = p^* \eta_\alpha + v_\alpha^{-1} dv_\alpha$ be the (global) 1-form generating the horizontal foliation on $TS \setminus S$, and $\tilde{\omega} = \tilde{p}^* \tilde{\eta} + \tilde{v}^{-1} d\tilde{v}$ the corresponding form generating the horizontal foliation on $T\tilde{S} \setminus \tilde{S}$, where $p : TS \to S$ and $\tilde{p} : T\tilde{S} \to \tilde{S}$ are the projections. In local coordinates, we can express $d\pi : T\tilde{S} \to TS$ by

$$d\pi(w, \tilde{v}) = \bigl(\pi(w), \pi'_\alpha(w)\tilde{v}\bigr),$$

that is $v_\alpha \circ d\pi = (\pi'_\alpha \circ \tilde{p})\tilde{v}$. Therefore

$$\begin{aligned}(d\pi)^*\omega &= (d\pi)^* p^* \eta_\alpha + (d\pi)^* \left(\frac{1}{v_\alpha} dv_\alpha\right) = \tilde{p}^* \pi^* \eta_\alpha + \frac{1}{v_\alpha \circ d\pi} d(v_\alpha \circ d\pi) \\ &= \tilde{p}^* \pi^* \eta_\alpha + \frac{1}{\pi'_\alpha \circ \tilde{p}} d\bigl(\pi'_\alpha \circ \tilde{p}\bigr) + \frac{1}{\tilde{v}} d\tilde{v} = \tilde{p}^* \tilde{\eta} + \frac{1}{\tilde{v}} d\tilde{v} \\ &= \tilde{\omega},\end{aligned} \tag{3.4}$$

and this means exactly that $d\pi$ sends leaves of the horizontal foliation upstairs onto leaves of the horizontal foliation downstairs.

By Remark 1.5, the metric foliation downstairs (respectively, upstairs) is generated by the local forms $\varpi_\alpha = |v_\alpha|^2 \operatorname{Re} \omega$ (respectively, $\tilde{\varpi} = |\tilde{v}|^2 \operatorname{Re} \tilde{\omega}$). Then

$$(d\pi)^* \varpi_\alpha = |v_\alpha \circ d\pi|^2 \operatorname{Re}(d\pi)^* \omega = \bigl|\pi'_\alpha \circ \tilde{p}\bigr|^2 |\tilde{v}|^2 \operatorname{Re} \tilde{\omega} = \bigl|\pi'_\alpha \circ \tilde{p}\bigr|^2 \tilde{\varpi},$$

and so $d\pi$ also sends leaves of the metric foliation upstairs onto leaves of the metric foliation downstairs.

(ii) By definition we have

$$\nabla_{\sigma'} \sigma' = \nabla_{d\pi(\tilde{\sigma}')} d\pi\,(\tilde{\sigma}') = d\pi\,(\tilde{\nabla}_{\tilde{\sigma}'} \tilde{\sigma}'),$$

and so $\nabla_{\sigma'} \sigma' \equiv O$ if and only if $\tilde{\nabla}_{\tilde{\sigma}'} \tilde{\sigma}' \equiv O$. $\square$

We shall need to know how $\tilde{\eta}$ behaves under the action of the automorphism group $\operatorname{Aut}(\pi)$ of π.

Lemma 3.2. *Let S be a (multiply connected) Riemann surface, and ∇ a holomorphic connection on TS. Let $\tilde{\pi} : \tilde{S} \to S$ be the universal covering map, and $\tilde{\nabla}$ the holomorphic connection on $T\tilde{S}$ induced by ∇ via π. Let $\tilde{\eta}$ be the holomorphic form representing $\tilde{\nabla}$, and $\tilde{K}$ a global primitive of $\tilde{\eta}$. Then*

$$\exp(\tilde{K}\circ\gamma)=\frac{\rho(\gamma)}{\gamma'}\exp(\tilde{K}) \tag{3.5}$$

for all $\gamma\in\mathrm{Aut}(\pi)$*, where* $\rho:\mathrm{Aut}(\pi)\to\mathbb{C}^*$ *is the monodromy representation of* ∇ *(and we are identifying* $\mathrm{Aut}(\pi)$ *with the fundamental group of* S*).*

Proof. Let $\gamma\in\mathrm{Aut}(\pi)$. From $\pi\circ\gamma=\pi$ we get $\gamma(\pi^{-1}(U_\alpha))=\pi^{-1}(U_\alpha)$ and

$$(\pi'_\alpha\circ\gamma)\gamma'=\pi'_\alpha.$$

Therefore

$$\begin{aligned}\gamma^*\tilde{\eta}&=\gamma^*\pi^*\eta_\alpha+\gamma^*\left(\frac{1}{\pi'_\alpha}\,d\pi'_\alpha\right)=\pi^*\eta_\alpha+\frac{1}{\pi'_\alpha\circ\gamma}\,d(\pi'_\alpha\circ\gamma)=\pi^*\eta_\alpha+\frac{\gamma'}{\pi'_\alpha}\,d\left(\frac{\pi'_\alpha}{\gamma'}\right)\\&=\tilde{\eta}-\frac{1}{\gamma'}\,d\gamma'.\end{aligned} \tag{3.6}$$

Let now $\tilde{K}$ be a holomorphic primitive of $\tilde{\eta}$. Then (3.6) becomes

$$d(\tilde{K}\circ\gamma-\tilde{K})=-\frac{1}{\gamma'}\,d\gamma',$$

and thus we can find a $\rho(\gamma)\in\mathbb{C}^*$ such that

$$\exp(\tilde{K}\circ\gamma)=\frac{\rho(\gamma)}{\gamma'}\exp(\tilde{K}).$$

So we are left to proving that $\rho(\gamma)$ is given by the monodromy representation.

Choose an open cover $\{(U_\alpha,z_\alpha)\}$ of S, where the U_α are simply connected (and all non-empty intersections $U_\alpha\cap U_\beta$ are connected). For any α, fix a connected component $\tilde{U}_{\alpha,\mathrm{id}}$ of $\pi^{-1}(U_\alpha)$; setting $\tilde{U}_{\alpha,\gamma}=\gamma(\tilde{U}_{\alpha,\mathrm{id}})$, varying $\gamma\in\mathrm{Aut}(\pi)$ we get all connected components of $\pi^{-1}(U_\alpha)$. In particular, $\{(\tilde{U}_{\alpha,\gamma},z_\alpha\circ\pi,\partial/\partial w)\}$ is an open cover of $\tilde{S}$ trivializing $T\tilde{S}$. By construction, the cocycle representing $T\tilde{S}$ with respect to this cover is trivial.

Choose a holomorphic primitive K_α of η_α on U_α. Then (3.3) yields constants $c_{\alpha,\gamma}\in\mathbb{C}^*$ such that

$$\exp(\tilde{K})|_{\tilde{U}_{\alpha,\gamma}}=c_{\alpha,\gamma}\exp(K_\alpha\circ\pi)\pi'_\alpha\big|_{\tilde{U}_{\alpha,\gamma}}.$$

Assume that $U_\alpha\cap U_\beta\neq\varnothing$. Then for every $\gamma\in\mathrm{Aut}(\pi)$ there is a unique $\gamma'\in\mathrm{Aut}(\pi)$ so that $\tilde{U}_{\alpha,\gamma}\cap\tilde{U}_{\beta,\gamma'}\neq\varnothing$. In this intersection we have

$$1=\frac{\exp(\tilde{K})}{\exp(\tilde{K})}=\frac{c_{\alpha,\gamma}\exp(K_\alpha\circ\pi)\pi'_\alpha}{c_{\beta,\gamma'}\exp(K_\beta\circ\pi)\pi'_\beta}=\frac{c_{\alpha,\gamma}}{c_{\beta,\gamma'}}\frac{\exp(K_\alpha\circ\pi)}{\exp(K_\beta\circ\pi)}\left(\frac{\partial z_\alpha}{\partial z_\beta}\circ\pi\right)=\frac{c_{\alpha,\gamma}}{c_{\beta,\gamma'}}\hat{\psi}_{\alpha\beta}, \tag{3.7}$$

where $\{\hat{\psi}_{\alpha\beta}\}$ is the (locally constant) cocycle representing the monodromy representation of ∇; see (1.2).

Now take γ_0, $\gamma\in\mathrm{Aut}(\pi)$. Then

$$\begin{aligned}\exp(\tilde{K}\circ\gamma)|_{\tilde{U}_{\alpha,\gamma_0}}&=\exp(\tilde{K})|_{\tilde{U}_{\alpha,\gamma\gamma_0}}\circ\gamma=c_{\alpha,\gamma\gamma_0}\exp(K_\alpha\circ\pi\circ\gamma)(\pi'_\alpha\circ\gamma)\\&=\frac{c_{\alpha,\gamma\gamma_0}}{\gamma'}\exp(K_\alpha\circ\pi)\pi'_\alpha=\frac{c_{\alpha,\gamma\gamma_0}}{c_{\alpha,\gamma}}\frac{1}{\gamma'}\exp(\tilde{K})\bigg|_{\tilde{U}_{\alpha,\gamma_0}}.\end{aligned}$$

Therefore $\rho(\gamma) = c_{\alpha,\gamma\gamma_0}/c_{\alpha,\gamma}$, and the assertion follows from (3.7) and the definition of the canonical isomorphism between Čech cohomology and singular cohomology. □

Definition 3.1. We shall denote by $\rho(\pi) \subseteq \mathbb{C}^*$ the image of $\mathrm{Aut}(\pi)$ under ρ, and by $|\rho|(\pi) \subseteq \mathbb{R}^+$ the image of $\mathrm{Aut}(\pi)$ under $|\rho|$; in particular ∇ has real periods if and only if $|\rho|(\pi) = \{1\}$.

Using the monodromy representation we can describe the metric and horizontal foliations:

Theorem 3.3. *Let S be a (multiply connected) Riemann surface, and ∇ a holomorphic connection on TS. Let L be a leaf of the metric foliation, and take $v_0 \in T_{z_0}S \cap L$. Then*

$$L \cap T_{z_0}S = |\rho|(\pi) \cdot \left(S^1 \cdot v_0\right) \quad \text{and} \quad \bar{L} \cap T_{z_0}S = \overline{|\rho|(\pi)} \cdot \left(S^1 \cdot v_0\right). \tag{3.8}$$

In particular, either

(i) *∇ has real periods, and in that case all leaves of the metric foliation are closed in TS; or,*
(ii) *∇ has not real periods, and in that case all leaves of the metric foliation accumulate all points of the zero section of TS.*

Proof. Clearly $S^1 \cdot v_0 \subset L \cap T_{z_0}S$. Let $\pi : \tilde{S} \to S$ be the universal covering map; by Proposition 3.1 we can find a leaf $\tilde{L}$ of the metric foliation upstairs so that $L = d\pi(\tilde{L})$. Fix $\tilde{z}_0 \in \pi^{-1}(z_0)$, and let $\tilde{v}_0 \in T_{\tilde{z}_0}\tilde{S}$ such that $d\pi_{\tilde{z}_0}(\tilde{v}_0) = v_0$. Then $(\tilde{z}, \tilde{v}) \in \tilde{L}$ if and only if

$$\exp\bigl(\operatorname{Re}\tilde{K}(\tilde{z})\bigr)|\tilde{v}| = \exp\bigl(\operatorname{Re}\tilde{K}(\tilde{z}_0)\bigr)|\tilde{v}_0|,$$

where $\tilde{K}$ is a holomorphic primitive of the holomorphic 1-form $\tilde{\eta}$ representing the holomorphic connection $\tilde{\nabla}$ induced by ∇ via π. In particular, if $\tilde{z} = \gamma(\tilde{z}_0)$ for some $\gamma \in \mathrm{Aut}(\pi)$, then (3.5) implies that $(\tilde{z}, \tilde{v}) \in \tilde{L}$ if and only if

$$|\tilde{v}| = \frac{|\gamma'(\tilde{z}_0)|}{|\rho(\gamma)|}|\tilde{v}_0|. \tag{3.9}$$

In other words,

$$\tilde{L}_{\gamma(\tilde{z}_0)} = \frac{|\gamma'(\tilde{z}_0)|}{|\rho(\gamma)|} \cdot \tilde{L}_{\tilde{z}_0},$$

where we put $\tilde{L}_{\tilde{z}} = \tilde{L} \cap T_{\tilde{z}}\tilde{S}$. From $d\pi_{\tilde{z}_0} = \gamma'(\tilde{z}_0)\, d\pi_{\gamma(\tilde{z}_0)}$ we then get

$$d\pi_{\gamma(\tilde{z}_0)}(\tilde{L}_{\gamma(\tilde{z}_0)}) = \frac{1}{|\rho(\gamma)|}\, d\pi_{\tilde{z}_0}(\tilde{L}_{\tilde{z}_0}) = \left|\rho\left(\gamma^{-1}\right)\right| d\pi_{\tilde{z}_0}(\tilde{L}_{\tilde{z}_0}).$$

But $d\pi_{\tilde{z}_0}(\tilde{L}_{\tilde{z}_0}) = S^1 \cdot v_0$; hence

$$L \cap T_{z_0}S = \bigcup_{\gamma \in \mathrm{Aut}(\pi)} d\pi_{\gamma(\tilde{z}_0)}(\tilde{L}_{\gamma(\tilde{z}_0)}) = |\rho|(\pi) \cdot \left(S^1 \cdot v_0\right).$$

Taking the closure we get (3.8). In particular, if ∇ has not real periods then $0 \in \overline{|\rho|(\pi)}$, and (ii) follows.

Finally, assume that ∇ has real periods. Let $L \subset TS \setminus S$ be a leaf of the metric foliation, and $\{(z_k, v_k)\} \subset L$ with $z_k \to z_0 \in S$ and $v_k \to v_0 \in T_{z_0}S$; to get (i) we must prove that $(z_0, v_0) \in L$.

Let $\tilde{L}$ be the leaf of the metric foliation upstairs such that $L = d\pi(\tilde{L})$, and take a sequence $\{(\tilde{z}_k, \tilde{v}_k)\} \subset \tilde{L}$ such that $\pi(\tilde{z}_k) = z_k$ and $d\pi_{\tilde{z}_k}(\tilde{v}_k) = v_k$. Fix a point $\hat{z}_0 \in \pi^{-1}(z_0)$. Since $z_k \to z_0$ in S, we can find a sequence $\hat{z}_k \to \hat{z}_0$ in $\tilde{S}$ and a sequence $\{\gamma_k\} \subset \mathrm{Aut}(\pi)$ such that $\tilde{z}_k = \gamma_k(\hat{z}_k)$. Put $\hat{v}_k = \tilde{v}_k / \gamma_k'(\hat{z}_k)$, so that $d\pi_{\hat{z}_k}(\hat{v}_k) = d\pi_{\tilde{z}_k}(\tilde{v}_k) = v_k$. Furthermore, (3.9) yields $(\hat{z}_k, \hat{v}_k) \in \tilde{L}$.

Now, since $\exp(\operatorname{Re} \tilde{K}(\hat{z}_k))|\hat{v}_k|$ is a non-zero constant and $\exp(\operatorname{Re} \tilde{K}(\hat{z}_k)) \to \exp(\operatorname{Re} \tilde{K}(\hat{z}_0)) \neq 0$, the sequence $\{\hat{v}_k\}$ is bounded; therefore, up to a subsequence, we can assume that $\hat{v}_k \to \hat{v}_0 \in \mathbb{C}$. Clearly, $(\hat{z}_0, \hat{v}_0)$ still belongs to the leaf $\tilde{L}$; hence

$$d\pi_{\hat{z}_0}(\hat{v}_0) = \lim_{k \to +\infty} d\pi_{\hat{w}_k}(\hat{v}_k) = \lim_{k \to \infty} v_k = v_0$$

belongs to L, as claimed. □

In a similar way we can prove the following

Theorem 3.4. *Let S be a (multiply connected) Riemann surface, and ∇ a holomorphic connection on TS, and let L be a leaf of the horizontal foliation. Then $p(L) = S$, where $p : TS \to S$ is the canonical projection. Furthermore, take any $z_0 \in S$ and $v_0 \in T_{z_0}S \cap L$. Then*

$$L \cap T_{z_0}S = \rho(\pi) \cdot v_0 \quad \text{and} \quad \bar{L} \cap T_{z_0}S = \overline{\rho(\pi)} \cdot v_0. \tag{3.10}$$

In particular, either

(i) *∇ has real periods, and in that case either all leaves of the horizontal foliation are closed in TS or any leaf of the horizontal foliation is dense in the leaf of the metric foliation containing it; or,*
(ii) *∇ has not real periods, and in that case all leaves of the horizontal foliation accumulate all points of the zero section of TS.*

Proof. Let $\pi : \tilde{S} \to S$ be the universal covering map; by Proposition 3.1 we can find a leaf $\tilde{L}$ of the horizontal foliation upstairs so that $L = d\pi(\tilde{L})$. Since (2.2) implies $\tilde{p}(\tilde{L}) = \tilde{S}$, it follows immediately that $p(L) = S$.

Fix $\tilde{z}_0 \in \pi^{-1}(z_0)$, and let $\tilde{v}_0 \in T_{\tilde{z}_0}\tilde{S}$ such that $d\pi_{\tilde{z}_0}(\tilde{v}_0) = v_0$. Then $(\tilde{z}, \tilde{v}) \in \tilde{L}$ if and only if

$$\exp\bigl(\tilde{K}(\tilde{z})\bigr)\tilde{v} = \exp\bigl(\tilde{K}(\tilde{z}_0)\bigr)\tilde{v}_0,$$

where $\tilde{K}$ is a holomorphic primitive of the holomorphic 1-form $\tilde{\eta}$ representing the holomorphic connection $\tilde{\nabla}$ induced by ∇ via π. In particular, if $\tilde{z} = \gamma(\tilde{z}_0)$ for some $\gamma \in \mathrm{Aut}(\pi)$, then (3.5) implies that $(\tilde{z}, \tilde{v}) \in \tilde{L}$ if and only if

$$\tilde{v} = \frac{\gamma'(\tilde{z}_0)}{\rho(\gamma)} \tilde{v}_0. \tag{3.11}$$

Notice that $\tilde{L}$ intersects each $T_{\tilde{z}}\tilde{S}$ in just one point. From $d\pi_{\tilde{z}_0} = \gamma'(\tilde{z}_0)\, d\pi_{\gamma(\tilde{z}_0)}$ we then get

$$d\pi_{\gamma(\tilde{z}_0)}(\tilde{v}) = \frac{1}{\rho(\gamma)} d\pi_{\tilde{z}_0}(\tilde{v}_0) = \rho\bigl(\gamma^{-1}\bigr)\, d\pi_{\tilde{z}_0}(\tilde{v}_0).$$

Hence

$$L \cap T_{z_0}S = \bigcup_{\gamma \in \mathrm{Aut}(\pi)} d\pi_{\gamma(\tilde{z}_0)}\bigl(\tilde{L} \cap T_{\gamma(\tilde{z}_0)}\tilde{S}\bigr) = \rho(\pi) \cdot v_0.$$

Taking the closure we get (3.10). In particular, if ∇ has not real periods then $0 \in \overline{\rho(\pi)}$, and (ii) follows.

Finally, if ∇ has real periods then $\rho(\pi)$ is a subgroup of S^1. The subgroups of S^1 are either cyclic or dense; in the first case it is easy to check that L is closed, and in the second case the assertion follows from (3.10) and (3.8). $\square$

The monodromy representation enters in another question: deciding when the automorphisms of π are isometries for a Hermitian metric adapted to $\tilde{\nabla}$.

Proposition 3.5. *Let S be a (multiply connected) Riemann surface, and ∇ a holomorphic connection on TS. Let $\tilde{\pi}:\tilde{S}\to S$ be the universal covering map, $\tilde{\nabla}$ the holomorphic connection on $T\tilde{S}$ induced by ∇ via π, and $\tilde{g}=\exp(\operatorname{Re}\tilde{K})g_0$ a Hermitian metric adapted to $\tilde{\nabla}$ (where g_0 is the Euclidean metric). Let $\rho:\operatorname{Aut}(\pi)\to\mathbb{C}^*$ be the monodromy representation of ∇. Then:*

(i) *$\gamma\in\operatorname{Aut}(\pi)$ is an isometry of $\tilde{g}$ if and only if $|\rho(\gamma)|=1$; in particular, $\operatorname{Aut}(\pi)\subseteq\operatorname{Iso}(\tilde{g})$ if and only if ∇ has real periods;*

(ii) *every $\gamma\in\operatorname{Aut}(\pi)$ sends $\tilde{\nabla}$-geodesics in $\tilde{\nabla}$-geodesics, and we have*

$$\gamma\circ\exp_{\tilde{z}_0}(\tilde{v})=\exp_{\gamma(\tilde{z}_0)}\bigl(\gamma'(\tilde{z}_0)\tilde{v}\bigr)$$

for all $\tilde{z}_0\in\tilde{S}$ and $\tilde{v}\in\mathbb{C}$;

(iii) *if $J:\tilde{S}\to\mathbb{C}$ is a primitive of $\exp(\tilde{K})$ then*

$$J\bigl(\gamma(\tilde{z})\bigr)-J\bigl(\gamma(\tilde{z}_0)\bigr)=\rho(\gamma)\bigl[J(\tilde{z})-J(\tilde{z}_0)\bigr]$$

for all $\tilde{z}_0,\tilde{z}\in\tilde{S}$ and $\gamma\in\operatorname{Aut}(\pi)$.

Proof. (i) Formula (3.5) yields

$$\exp(\operatorname{Re}\tilde{K}\circ\gamma)=\frac{|\rho(\gamma)|}{|\gamma'|}\exp(\operatorname{Re}\tilde{K}).$$

Therefore

$$\begin{aligned}\tilde{g}\bigl(\gamma(\tilde{z});d\gamma_{\tilde{z}}(\tilde{v})\bigr)&=\exp\bigl(2\operatorname{Re}\tilde{K}\bigl(\gamma(\tilde{z})\bigr)\bigr)\bigl|\gamma'(\tilde{z})\bigr|^2|\tilde{v}|^2\\&=\bigl|\rho(\gamma)\bigr|^2\exp\bigl(2\operatorname{Re}\tilde{K}(\tilde{z})\bigr)|\tilde{v}|^2=\bigl|\rho(\gamma)\bigr|^2\tilde{g}(\tilde{z};\tilde{v}),\end{aligned}$$

and $\gamma\in\operatorname{Aut}(\pi)$ is an isometry if and only if $|\rho(\gamma)|=1$.

(ii) Remark 2.1 says that a curve $\sigma:I\to\tilde{S}$ is a geodesic if and only if

$$\sigma'=\exp\bigl(-\tilde{K}(\sigma)\bigr)\exp\bigl(\tilde{K}\bigl(\sigma(t_0)\bigr)\bigr)\sigma'(t_0)$$

for some (and hence all) $t_0\in I$. Now

$$\begin{aligned}&\exp\bigl(-\tilde{K}(\gamma\circ\sigma)\bigr)\exp\bigl(\tilde{K}\bigl(\gamma\circ\sigma(t_0)\bigr)\bigr)(\gamma\circ\sigma)'(t_0)\\&\quad=\frac{\gamma'\circ\sigma}{\rho(\gamma)}\exp(-\tilde{K}\circ\sigma)\frac{\rho(\gamma)}{\gamma'(\sigma(t_0))}\exp\bigl(\tilde{K}\bigl(\sigma(t_0)\bigr)\bigr)\gamma'\bigl(\sigma(t_0)\bigr)\sigma'(t_0)\\&\quad=\bigl(\gamma'\circ\sigma\bigr)\bigl[\exp(-\tilde{K}\circ\sigma)\exp\bigl(\tilde{K}\bigl(\sigma(t_0)\bigr)\bigr)\sigma'(t_0)\bigr],\end{aligned}$$

and thus $\gamma\circ\sigma$ is a geodesic if and only if σ is, even when γ is not an isometry.

(iii) Formula (3.5) yields

$$\exp(\tilde{K} \circ \gamma)\gamma' = \rho(\gamma) \exp(\tilde{K}).$$

Integrating this from $\tilde{z}_0$ to $\tilde{z}$ we get the assertion. □

In the next result we show how to compute the monodromy representation when $S \subseteq \mathbb{C}$, that is when S is covered by a single chart. The most interesting case will be when S is the complement in $\mathbb{P}^1(\mathbb{C})$ of a finite set of points.

Proposition 3.6. *Let $S \subseteq \mathbb{C}$ be a (multiply connected) domain, ∇ a holomorphic connection on TS, and η the holomorphic 1-form representing ∇. Then the monodromy representation $\rho : H_1(S, \mathbb{Z}) \to \mathbb{C}^*$ is given by*

$$\rho(\gamma) = \exp\biggl(\int_\gamma \eta\biggr)$$

for all $\gamma \in H_1(S, \mathbb{Z})$.

Proof. Let $\pi : \tilde{S} \to S$ be the universal covering map of S. Choose a $z_0 \in S$, a $\tilde{z}_0 \in \pi^{-1}(z_0)$, and a loop (still denoted by γ) based at z_0 representing $\gamma \in H_1(S, \mathbb{Z})$. Let $\tilde{\gamma}$ be the lift of γ based at $\tilde{z}_0$; then the action on $\tilde{z}_0$ of the element of $\mathrm{Aut}(\pi)$ corresponding to γ (again denoted by γ) is given by $\tilde{\gamma}(1)$.

Let $\tilde{\nabla}$ be the holomorphic connection on $T\tilde{S}$ induced by ∇ via π, and $\tilde{\eta}$ the holomorphic 1-form representing $\tilde{\nabla}$. Choose a holomorphic primitive $\tilde{K}$ of $\tilde{\eta}$, and a determination of $\log \pi'$. Then

$$\begin{aligned}\tilde{K}\bigl(\gamma(w_0)\bigr) - \tilde{K}(w_0) &= \tilde{K}\bigl(\tilde{\gamma}(1)\bigr) - \tilde{K}\bigl(\tilde{\gamma}(0)\bigr)\\ &= \int_{\tilde{\gamma}} \tilde{\eta} = \int_{\tilde{\gamma}} \bigl(\pi^*\eta + d \log \pi'\bigr) = \int_{\tilde{\gamma}} \pi^*\eta + \bigl(\log \pi'\bigr)\bigl(\gamma(\tilde{z}_0)\bigr) - \log \pi'(\tilde{z}_0)\\ &= \int_\gamma \eta + \bigl(\log \pi'\bigr)\bigl(\gamma(\tilde{z}_0)\bigr) - \log \pi'(\tilde{z}_0).\end{aligned} \tag{3.12}$$

Therefore

$$\exp\bigl[\tilde{K}\bigl(\gamma(\tilde{z}_0)\bigr) - \tilde{K}(\tilde{z}_0)\bigr] = \frac{\pi'(\gamma(\tilde{z}_0))}{\pi'(\tilde{z}_0)} \exp\biggl(\int_\gamma \eta\biggr) = \frac{1}{\gamma'(\tilde{z}_0)} \exp\biggl(\int_\gamma \eta\biggr),$$

and the assertion follows from (3.5). □

We have actually proved something more. Keeping the notations introduced in the previous proof, from $(\pi' \circ \gamma)\gamma' = \pi'$ we deduce that for each $\gamma \in \mathrm{Aut}(\pi)$ there is a unique determination of the logarithm of γ' such that

$$\log \gamma' = \log \pi' - \bigl(\log \pi'\bigr) \circ \gamma.$$

Then (3.12) becomes

$$\tilde{K}\bigl(\gamma(w_0)\bigr) = \tilde{K}(w_0) - \log \gamma'(w_0) + \int_\gamma \eta.$$

Put

$$\rho_0(\gamma) = \frac{1}{2\pi i}\int\limits_\gamma \eta;$$

then it is easy to check that $\rho_0 : H_1(S,\mathbb{Z}) \to \mathbb{C}$ is a homomorphism of abelian groups, and we can write (3.12) as

$$\tilde{K} \circ \gamma = \tilde{K} - \log\gamma' + 2\pi i \rho_0(\gamma). \tag{3.13}$$

Definition 3.2. The homomorphism $\rho_0 : H_1(S,\mathbb{Z}) \to \mathbb{C}$ just introduced is the *period map* associated to ∇.

Since $\rho = \exp(2\pi i \rho_0)$, the connection ∇ has real periods if and only if the image of the period map is contained in $\mathbb{R}$. In particular, Proposition 1.2 yields:

Corollary 3.7. *Let $S \subseteq \mathbb{C}$ be a domain in the plane, and ∇ a holomorphic connection on TS. Then there exists a Hermitian metric adapted to ∇ if and only if the period map is real-valued.*

4. Meromorphic connections

Let us now specialize to the case we are mostly interested in, that is meromorphic connections on $\mathbb{P}^1(\mathbb{C})$. If ∇ is a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, then we can consider it as a holomorphic connection on $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0,\dots,p_r\}$, where $\{p_0,\dots,p_r\}$ are the poles of the meromorphic connection. Without loss of generality, we shall always assume $p_0 = \infty$, so that $S \subseteq \mathbb{C}$, and thus we have the period map $\rho_0 : H_1(S,\mathbb{Z}) \to \mathbb{C}$ associated to ∇. The homology group $H_1(S,\mathbb{Z})$ is generated by the counterclockwise loops $\gamma_1,\dots,\gamma_r$ around, respectively, $p_1,\dots,p_r$; therefore $\rho_0(\gamma_j)$ is, practically by definition, the *residue* $\operatorname{Res}_{p_j}(\nabla)$:

$$\operatorname{Res}_{p_j}(\nabla) = \rho_0(\gamma_j) = \frac{1}{2\pi i}\int\limits_{\gamma_j} \eta,$$

where η is the holomorphic 1-form representing ∇ on S. We also have the residue at ∞, which is given by $\rho(\gamma_0)$, where γ_0 is a clockwise Jordan loop in $\mathbb{C}$ containing $\{p_1,\dots,p_r\}$ in its interior. It is useful to keep in mind that the classical residue theorem for meromorphic connections (see, e.g., [18, Theorem III.17.33]) says that

$$\sum_{j=0}^{r} \operatorname{Res}_{p_j}(\nabla) = \deg T\mathbb{P}^1(\mathbb{C}) = -2. \tag{4.1}$$

The aim of this section is to describe the recurrence properties of the geodesics on S, and of the geodesic flow on TS, recurrence properties that, as we shall see, are strikingly different from the recurrence properties of the metric and horizontal foliations described by Theorems 3.3 and 3.4.

To state the main technical tool for our study we need a definition.

Definition 4.1. Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0 = \infty, p_1,\dots,p_r\}$, and set $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0,\dots,p_r\} \subseteq \mathbb{C}$. An *s-sided geodesic polygon* is a simply connected domain $R_0 \subset \mathbb{P}^1(\mathbb{C})$ whose boundary is composed by $s \geqslant 1$ simple geodesics $\sigma_j : [0,\ell_j] \to S$ with $\sigma_j(\ell_j) = \sigma_{j+1}(0) = z_{j+1}$ for $j = 1,\dots,s-1$, $\sigma_r(\ell_n) = \sigma_1(0) = z_1$ and no other intersections; the geodesics are listed so that ∂R_0 is positively oriented (that is R_0 is the interior of ∂R_0). The points $z_1,\dots,z_s$ are the *vertices* of R_0.

Then:

Theorem 4.1. *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0=\infty, p_1,\ldots,p_r\}$, and set $S=\mathbb{P}^1(\mathbb{C})\setminus\{p_0,\ldots,p_r\}\subseteq\mathbb{C}$. Let $R_0\subset\mathbb{P}^1(\mathbb{C})$ be an s-sided geodesic polygon with vertices $z_1,\ldots,z_s$. For $j=1,\ldots,s$ let $\varepsilon_j\in(-\pi,\pi)$ be the external angle in z_j, and let $\{p_1,\ldots,p_g\}$ be the poles of ∇ contained in R_0. Then*

$$\sum_{j=1}^{s}\varepsilon_j=2\pi\left(1+\sum_{j=1}^{g}\operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\right). \tag{4.2}$$

In particular,

$$\sum_{j=1}^{g}\operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\in\left(-\frac{s+2}{2},\frac{s-2}{2}\right). \tag{4.3}$$

Proof. The idea is to apply the Gauss–Bonnet theorem, even if we do not have a global metric. What we do have is the local Gauss–Bonnet theorem, expressed in terms of a local metric adapted to ∇. We know that the Gaussian curvature of all such metrics is identically zero (see Remark 1.2); furthermore, any such metric is a positive multiple of any other one, and thus the external angles (and the notion of orthogonal parametrizations) are the same for all of them. It follows that also the integral of the geodesic curvature does not depend on the chosen local metric (and this can be verified directly too; see below), and hence the standard proof (see, e.g., [6, Theorem 6.3.9] or [10, p. 274]) of the global Gauss–Bonnet theorem based on the local Gauss–Bonnet theorem still works.

We shall apply the Gauss–Bonnet theorem to the region R obtained removing from R_0 small disks around $p_1,\ldots,p_g$. Denoting by $\tau^1,\ldots,\tau^g:[0,2\pi]\to S$ the small clockwise circles bounding the disks around $p_1,\ldots,p_g$ respectively, and by κ_g^j the geodesic curvature of τ^j, the Gauss–Bonnet theorem says that

$$\sum_{j=1}^{g}\int_{\tau^j}\kappa_g^j\,ds+\sum_{j=1}^{s}\varepsilon_j=2\pi(1-g). \tag{4.4}$$

Since all the local metrics adapted to ∇ are (non-constant) multiples of the Euclidean metric, the standard real coordinates (x,y) on $S\subseteq\mathbb{C}$ are orthogonal, and we can use the formula

$$\int_{\tau^j}\kappa_g^j=\int_0^{2\pi}\left[\frac{d\theta^j}{dt}+\frac{1}{2\sqrt{(EG)\circ\tau^j}}\left(\left(\operatorname{Im}\tau^j\right)'\frac{\partial G}{\partial x}\circ\tau^j-\left(\operatorname{Re}\tau^j\right)'\frac{\partial E}{\partial y}\circ\tau^j\right)\right]dt, \tag{4.5}$$

where $E=G=\exp(2\operatorname{Re}K)$ and θ^j is the angle between $\partial/\partial x$ and $(\tau^j)'$. Here K is any local holomorphic primitive of the form η representing ∇ in S; since two such local primitives differ only by an additive constant, the integrand of (4.5) does not depend on the choice of K.

We have $\tau^j(t)=p_j+re^{-it}$ for $r>0$ small enough; hence $d\theta^j/dt\equiv-1$. The Cauchy–Riemann equations yield

$$\frac{1}{2\sqrt{EG}}\frac{\partial G}{\partial x}=\operatorname{Re}\frac{\partial K}{\partial z}\quad\text{and}\quad-\frac{1}{2\sqrt{EG}}\frac{\partial E}{\partial y}=\operatorname{Im}\frac{\partial K}{\partial z}.$$

Hence (4.5) becomes

$$\int_{\tau^j} \kappa_g^j = -2\pi + \int_0^{2\pi} \operatorname{Im} \frac{d}{dt}\bigl(K \circ \tau^j\bigr)\, dt.$$

Let $\pi : \tilde{S} \to S$ be the universal covering map, and $\tilde{\tau}^j$ a lifting of τ^j. If $\gamma_j \in \operatorname{Aut}(\pi)$ is the generator associated to the counterclockwise loop around p_j, we have $\tilde{\tau}^j(2\pi) = \gamma_j^{-1}(\tilde{\tau}^j(0))$. Let now $\tilde{K}$ be a holomorphic primitive of the holomorphic 1-form $\tilde{\eta}$ representing the holomorphic connection $\tilde{\nabla}$ induced by ∇ via π. Then, choosing a determination for the logarithm of π', we have

$$d\tilde{K} = d\bigl(K \circ \pi + \log \pi'\bigr).$$

Therefore

$$\frac{d}{dt}\bigl(K \circ \tau^j\bigr) = \frac{d}{dt}\bigl(K \circ \pi \circ \tilde{\tau}^j\bigr) = \frac{d}{dt}\bigl((\tilde{K} - \log \pi') \circ \tilde{\tau}^j\bigr).$$

Hence, using (3.13) and $(\pi' \circ \gamma)\gamma' = \pi'$ for all $\gamma \in \operatorname{Aut}(\pi)$, we get

$$\begin{aligned}\int_{\tau^j} \kappa_g^j &= -2\pi + \operatorname{Im}\bigl[\tilde{K}\bigl(\gamma_j^{-1}\bigl(\tilde{\tau}^j(0)\bigr)\bigr) - \tilde{K}\bigl(\tilde{\tau}^j(0)\bigr) + \log \pi'\bigl(\tilde{\tau}^j(0)\bigr) - \log \pi'\bigl(\gamma_j^{-1}\bigl(\tilde{\tau}^j(0)\bigr)\bigr)\bigr] \\ &= -2\pi - \operatorname{Im}\bigl(2\pi i \rho_0(\gamma_j)\bigr) = -2\pi\bigl[1 + \operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\bigr],\end{aligned}$$

and so

$$\sum_{j=1}^{g} \int_{\tau^j} \kappa_g^j = -2\pi g - 2\pi \sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla). \tag{4.6}$$

Putting this into (4.4) we get (4.2). Finally, (4.3) follows from (4.2) and the fact that the sum of the external angles belongs to the interval $(-s\pi, s\pi)$. □

Corollary 4.2. *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0 = \infty, p_1, \dots, p_r\}$, and set $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\} \subseteq \mathbb{C}$. Let $\sigma : [0, \ell] \to S$ be a geodesic with $\sigma(0) = \sigma(\ell)$ and no other self-intersections; in particular, σ is an oriented Jordan curve. Let $\{p_1, \dots, p_g\}$ be the poles of ∇ contained in the interior of σ, and $\varepsilon \in (-\pi, \pi)$ the external angle at $\sigma(0)$. Then*

$$\varepsilon = 2\pi\left(1 + \sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\right), \tag{4.7}$$

and hence

$$\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) \in (-3/2, -1/2). \tag{4.8}$$

Proof. It follows from Theorem 4.1 with $s = 1$. □

Corollary 4.3. *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0=\infty, p_1, \dots, p_r\}$, and set $S=\mathbb{P}^1(\mathbb{C})\setminus\{p_0,\dots,p_r\}\subseteq\mathbb{C}$. Let $\sigma_0:[0,\ell_0]\to S$ and $\sigma_1:[0,\ell_1]\to S$ be two distinct geodesics with $\sigma_0(0)=z_0=\sigma_1(0)$ and $\sigma_0(\ell_0)=z_1=\sigma_1(\ell_1)$ and not intersecting elsewhere. Let $\{p_1,\dots,p_g\}$ be the poles of ∇ contained in the simply connected domain R_0 bounded by σ_0 and σ_1, and $\varepsilon_j\in(-\pi,\pi)$ the external angle at z_j, for $j=1,2$. Then*

$$\varepsilon_0+\varepsilon_1=2\pi\left(1+\sum_{j=1}^{g}\operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\right), \tag{4.9}$$

and hence

$$\sum_{j=1}^{g}\operatorname{Re}\operatorname{Res}_{p_j}(\nabla)\in(-2,0). \tag{4.10}$$

Proof. It follows from Theorem 4.1 with $s=2$. $\square$

To prove the next corollary we need a lemma and a definition.

Lemma 4.4. *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0=\infty,p_1,\dots,p_r\}$, and set $S=\mathbb{P}^1(\mathbb{C})\setminus\{p_0,\dots,p_r\}\subseteq\mathbb{C}$. Let $R_0\subset\mathbb{P}^1(\mathbb{C})$ be an s-sided geodesic polygon with vertices $z_1,\dots,z_s$. Let $\sigma_j:[0,\ell_j]\to S$ be the geodesics composing the boundary of R_0, with $\sigma_j(0)=z_j$ for $j=1,\dots,s$, and let $\{p_1,\dots,p_g\}$ be the poles of ∇ contained in R_0. Then*

$$\prod_{j=1}^{s}\sigma_j'(\ell_j)=\exp\left(-2\pi i\sum_{j=1}^{g}\operatorname{Res}_{p_j}(\nabla)\right)\prod_{j=1}^{s}\sigma_j'(0). \tag{4.11}$$

Proof. Choose a point $\tilde z_1\in\pi^{-1}(z_1)$, and let $\tilde\sigma_1:[0,\ell_1]\to\tilde S$ be the lifting of σ_1 with $\tilde\sigma_1(0)=\tilde z_1$. Then recursively choose the lifting $\tilde\sigma_j:[0,\ell_j]\to\tilde S$ of σ_j with $\tilde\sigma_j(0)=\tilde\sigma_{j-1}(\ell_{j-1})$. In particular, $\tilde\sigma_s(\ell_s)=\gamma(w_1)$, where $\gamma\in\operatorname{Aut}(\pi)$ is the element associated to the class of ∂R_0 in $\pi_1(S,z_1)$.

By Proposition 3.1(ii), the $\tilde\sigma_j$ are geodesics for $\tilde\nabla$; hence (Remark 2.1)

$$\tilde\sigma_j'(\ell_j)=\exp\bigl(-\tilde K\bigl(\tilde\sigma_j(\ell_j)\bigr)\bigr)\exp\bigl(-\tilde K\bigl(\tilde\sigma_j(0)\bigr)\bigr)\tilde\sigma_j'(0).$$

Recalling that $\tilde\sigma_j(0)=\tilde\sigma_{j-1}(\ell_{j-1})$ and using (3.13) we get

$$\begin{aligned}\prod_{j=1}^{s}\tilde\sigma_j'(\ell_j)&=\exp\bigl(-\tilde K\bigl(\gamma(\tilde z_1)\bigr)\bigr)\exp\bigl(\tilde K(\tilde z_1)\bigr)\prod_{j=1}^{s}\tilde\sigma_j'(0)\\&=\exp\bigl(-2\pi i\rho_0(\gamma)\bigr)\gamma'(\tilde z_1)\prod_{j=1}^{s}\tilde\sigma_j'(0).\end{aligned}$$

Now, $\sigma_j'=(\pi'\circ\tilde\sigma_j)\tilde\sigma_j'$; therefore

$$\begin{aligned}\prod_{j=1}^{s}\sigma_j'(\ell_j)&=\prod_{j=1}^{s}\pi'\bigl(\tilde\sigma_j(\ell_j)\bigr)\prod_{j=1}^{s}\tilde\sigma_j'(\ell_j)=\exp\bigl(-2\pi i\rho_0(\gamma)\bigr)\gamma'(\tilde z_1)\prod_{j=1}^{s}\pi'\bigl(\tilde\sigma_j(\ell_j)\bigr)\prod_{j=1}^{s}\tilde\sigma_j'(0)\\&=\exp\bigl(-2\pi i\rho_0(\gamma)\bigr)\gamma'(\tilde z_1)\frac{\prod_{j=1}^{s}\pi'(\tilde\sigma_j(\ell_j))}{\prod_{j=1}^{s}\pi'(\tilde\sigma_j(0))}\prod_{j=1}^{s}\sigma_j'(0)\end{aligned}$$

$$= \exp\bigl(-2\pi i \rho_0(\gamma)\bigr) \frac{\pi'(\gamma(\tilde{z}_1))\gamma'(\tilde{z}_1)}{\pi'(\tilde{z}_1)} \prod_{j=1}^{s} \sigma_j'(0)$$

$$= \exp\bigl(-2\pi i \rho_0(\gamma)\bigr) \prod_{j=1}^{s} \sigma_j'(0).$$

Now by construction, $[\gamma] = [\gamma_1] + \cdots + [\gamma_g]$ in $H_1(S, \mathbb{Z})$, where γ_j is a counterclockwise loop around p_j; therefore

$$\rho_0(\gamma) = \rho_0(\gamma_1) + \cdots + \rho_0(\gamma_g) = \sum_{j=1}^{g} \operatorname{Res}_{p_j}(\nabla), \tag{4.12}$$

and we are done. □

Definition 4.2. A geodesic $\sigma : [0, \ell] \to S$ is *closed* if $\sigma(\ell) = \sigma(0)$ and $\sigma'(\ell)$ is a positive multiple of $\sigma'(0)$; it is *periodic* if $\sigma(\ell) = \sigma(0)$ and $\sigma'(\ell) = \sigma'(0)$.

Remark 4.1. Contrarily to the case of Riemannian geodesics, closed geodesics are not necessarily periodic; see Example 6.1.

Corollary 4.5. *Let ∇ be a meromorphic connection on $\mathbb{P}^1(\mathbb{C})$, with poles $\{p_0 = \infty, p_1, \dots, p_r\}$, and set $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\} \subseteq \mathbb{C}$. Let $\sigma : [0, \ell] \to S$ be a geodesic with $\sigma(0) = \sigma(\ell)$ and no other self-intersections; in particular, σ is an oriented Jordan curve. Let $\{p_1, \dots, p_g\}$ be the poles of ∇ contained in the interior of σ. Then σ is a closed geodesic if and only if*

$$\sum_{j=1}^{g} \operatorname{Re} \operatorname{Res}_{p_j}(\nabla) = -1,$$

and it is a periodic geodesic if and only if

$$\sum_{j=1}^{g} \operatorname{Res}_{p_j}(\nabla) = -1.$$

If σ is closed, it can be extended to an infinite length geodesic $\sigma : J \to S$, where J is a half-line (possibly $J = \mathbb{R}$). Moreover,

(i) *if $\sum_{j=1}^{g} \operatorname{Im} \operatorname{Res}_{p_j}(\nabla) < 0$ then $\sigma'(t) \to O$ as $t \to +\infty$ and $|\sigma'(t)| \to +\infty$ as t tends to the other end of J;*
(ii) *if $\sum_{j=1}^{g} \operatorname{Im} \operatorname{Res}_{p_j}(\nabla) > 0$ then $\sigma'(t) \to O$ as $t \to -\infty$ and $|\sigma'(t)| \to +\infty$ as t tends to the other end of J.*

Remark 4.2. It might actually happen that the a closed geodesic blows up at finite time (that is $J \neq \mathbb{R}$); see Example 6.1.

Proof. A self-intersecting geodesic σ is closed if and only if the external angle at the intersection point is 0; therefore the first assertion follows from Corollary 4.2.

Let $\pi : \tilde{S} \to S$ be the universal covering map, and let $\tilde{\sigma}$ be a lifting of σ. In this case (4.11) becomes

$$\sigma'(\ell) = \exp\left(-2\pi i \sum_{j=1}^{g} \operatorname{Res}_{p_j}(\nabla)\right)\sigma'(0).$$

So σ is periodic if and only if the sum of the residues is an integer contained (by Corollary 4.2) in the interval $(-3/2, -1/2)$, and the second assertion follows.

Finally, when σ is closed at every turn the velocity vector is multiplied by

$$\exp\left(2\pi \sum_{j=1}^{g} \operatorname{Im}\operatorname{Res}_{p_j}(\nabla)\right),$$

and (i) and (ii) follow. □

To state our main theorem we need two more definitions.

Definition 4.3. Let $\sigma : I \to S$ be a curve in $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\}$. A *simple loop in* σ is the restriction of σ to a closed interval $[t_0, t_1] \subseteq I$ such that $\sigma|_{[t_0,t_1]}$ is a simple loop γ. If $p_1, \dots, p_g$ are the poles of ∇ contained in the interior of γ, we shall say that γ *surrounds* $p_1, \dots, p_g$.

Definition 4.4. A *saddle connection* for a meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C})$ with poles $\{p_0, \dots, p_r\}$ is a maximal geodesic $\sigma : (\varepsilon_-, \varepsilon_+) \to S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\}$ (with $\varepsilon_- \in [-\infty, 0)$ and $\varepsilon_+ \in (0, +\infty]$) such that $\sigma(t)$ tends to a pole of ∇ both when $t \uparrow \varepsilon_+$ and when $t \downarrow \varepsilon_-$. A *cycle of saddle connections* is a closed piecewise smooth curve in $\mathbb{P}^1(\mathbb{C})$ made up of saddle connections. Again, we shall say that a cycle of saddle connections *surrounds* the poles of ∇ contained in its interior.

We can now prove a Poincaré–Bendixson theorem for meromorphic connections on $\mathbb{P}^1(\mathbb{C})$:

Theorem 4.6. *Let $\sigma : [0, \varepsilon_0) \to S$ be a maximal geodesic for a meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C})$, where $S = \mathbb{P}^1(\mathbb{C}) \setminus \{p_0, \dots, p_r\}$ and $p_0, \dots, p_r$ are the poles of ∇. Then either*

(i) *$\sigma(t)$ tends to a pole of ∇ as $t \to \varepsilon_0$; or*
(ii) *σ is closed, and then surrounds poles $p_1, \dots, p_g$ with $\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$; or*
(iii) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is given by the support of a closed geodesic surrounding poles $p_1, \dots, p_g$ with $\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$; or*
(iv) *the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$ is a cycle of saddle connections surrounding poles $p_1, \dots, p_g$ with again $\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1$; or*
(v) *σ intersects itself infinitely many times, and in this case every simple loop of σ surrounds a set of poles whose sum of residues has real part belonging to $(-3/2, -1) \cup (-1, -1/2)$.*

In particular, a recurrent geodesic either intersects itself infinitely many times or is closed.

Proof. Assume that σ is not closed, nor intersect itself infinitely many times (the condition on the residues in these cases follows from Corollaries 4.2 and 4.5). Then up to changing the starting point we can assume that σ does not intersect itself. Let W be the ω-limit set of σ in $\mathbb{P}^1(\mathbb{C})$. Since W is connected, to end the proof it suffices to show that if W contains a point $z_0 \in S$ then we are in cases (iii) or (iv).

Let $\pi : \tilde{S} \to S$ be the universal covering map, and $\tilde{\nabla}$ the holomorphic connection on $T\tilde{S}$ induced by ∇ via π. Choose $\tilde{z}_0 \in \pi^{-1}(z_0)$, a simply connected neighborhood $U \subset S$ of z_0, and let $\tilde{U}$ be the connected component of $\pi^{-1}(U)$ containing $\tilde{z}_0$. By Proposition 3.1(ii), the segments of ∇-geodesic contained in U are exactly the images under π of the segments of $\tilde{\nabla}$-geodesic contained in $\tilde{U}$. Furthermore, by Proposition 2.1 up to shrinking U and $\tilde{U}$ we can find an isometry J between $\tilde{U}$ endowed with the Hermitian metric adapted to $\tilde{\nabla}$ and an open set of $\mathbb{C}$ endowed with the Euclidean metric. Therefore $\pi \circ J^{-1}$ sends line segments into segments of ∇-geodesic in U, and conversely every segment of ∇-geodesic in U is image of a line segment via $\pi \circ J^{-1}$. So the geometry of the geodesics in a neighborhood U of z_0 is the same as the geometry of line segments; in particular, we can find in U a (simple) geodesic τ issuing from z_0 and intersecting σ in infinitely many points converging to z_0 in U. Notice that all intersections are transversal because σ does not intersect itself.

Let z_1 be an intersection point between σ and τ. Following σ from z_1, let z_1' be the first intersection point between σ and τ closer to z_0 than z_1; let R be the Jordan domain bounded by the segments of σ and τ between z_1 and z_1'. By Corollary 4.3, this domain must contain at least one pole of ∇.

If the two external angles of R have opposite signs, then we set $R_1 = R$. If not, we follow σ after z_1' until it intersects again τ in a point z_1'' between z_1 and z_0. If z_1'' is between z_1 and z_1', then the Jordan domain bounded by σ and τ between z_1 and z_1'' has external angles with opposite signs, and we call R_1 this domain. If instead z_1'' is between z_1' and z_0 we have two possibilities. If the Jordan domain bounded by σ and τ between z_1' and z_1'' has external angles with opposite signs, we call R_1 this domain. If not, since by construction this domain R' is disjoint from R, the poles inside R' must be disjoint from the poles inside R. Since the number of poles is finite, repeating this construction sooner or later we get a Jordan domain R_1 whose external angles have opposite signs.

Thus in this way we can build a sequence $\{R_j\}$ of disjoint 2-sided geodesic polygons whose external angles have opposite signs bounded by a segment of σ and a segment of τ in such a way that both vertices converge to z_0. Every R_j must contain poles; since there are only finitely many poles, up to a subsequence we can assume they all contain the same poles $p_1, \ldots, p_g$. Since their boundaries are disjoint, they are nested; so up to a subsequence we can also assume that either $R_{j+1} \subset R_j$ for all j or $R_{j+1} \supset R_j$ for all j. Up to a subsequence, we can also assume that the direction of σ at the vertex closest to z_0 along τ is converging to a given direction v_0 in S^1. Since σ does not self-intersect, the local geometry of the geodesics near z_0 implies that the direction of σ at the other vertex must also converge to v_0, and thus the sum of the external angles must converge to 0. But, by Corollary 4.3, the sum of the external angles is constant; so it must be zero. This means that in each R_j the two intersections of σ with τ are parallel, and that

$$\sum_{j=1}^{g} \operatorname{Re}\operatorname{Res}_{p_j}(\nabla) = -1.$$

We have a decreasing or increasing sequence of 2-sided geodesic polygons, with opposite external angles. If $z \in S$ is any point accumulated by this sequence, again using the fact that σ does not self-intersect and the local geometrical structure of the geodesics, we see that this sequence actually accumulates the support of a geodesic issuing from z. Therefore $W \cap S$ is the union of supports of finitely many disjoint geodesics. So if $W \not\subset S$ then we are in case (iv); if instead $W \subset S$, then W is the support of a self-intersecting geodesic σ_0 surrounding $p_1, \ldots, p_g$; hence, by Corollary 4.5, σ_0 is closed, and we are done.

Finally, the ω-limit set of a recurrent geodesic intersects the support of the geodesic, and the last assertion follows. $\square$

Remark 4.3. We have examples (see Examples 6.1, 8.1 and 8.2) of cases (i), (ii), (iii) and (v), but no examples yet of case (iv).

Using these methods we can also say something about self-intersecting geodesics. For instance, we can prove the following:

Proposition 4.7. *Let $\sigma : [0, \varepsilon_0) \to S$ be a geodesic for a meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C})$, where S is the complement in $\mathbb{P}^1(\mathbb{C})$ of the poles of ∇. Assume that σ contains two distinct simple loops $\sigma|_{[t_0,t_1]}$ and $\sigma|_{[t_0',t_1']}$ based on the same point $z_0 = \sigma(t_0) = \sigma(t_1) = \sigma(t_0') = \sigma(t_1')$ and representing the same class $[\gamma_0] \in \pi_1(S, z_0)$. Then σ is closed.*

Proof. Let $\pi : \tilde{S} \to S$ be the universal covering map, and $\tilde{\sigma} : [0, \varepsilon_0) \to \tilde{S}$ a lifting of σ. Since $\sigma|_{[t_0,t_1]}$ is a simple loop representing $[\gamma_0]$, we must have $\tilde{\sigma}(t_1) = \gamma_0(\tilde{\sigma}(t_0))$, where γ_0 is the element of $\mathrm{Aut}(\pi)$ corresponding to $[\gamma_0] \in \pi_1(S, z_0)$. Then Propositions 2.2 and 3.5(iii) yield

$$\begin{aligned}\rho(\gamma_0)c_0t_0 &= \rho(\gamma_0)\big[J\big(\tilde{\sigma}(t_0)\big) - J\big(\tilde{\sigma}(0)\big)\big] = J\big(\gamma_0\big(\tilde{\sigma}(t_0)\big)\big) - J\big(\gamma_0\big(\tilde{\sigma}(0)\big)\big)\\ &= J\big(\tilde{\sigma}(t_1)\big) - J\big(\gamma_0\big(\tilde{\sigma}(0)\big)\big) = c_0t_1 + J\big(\tilde{\sigma}(0)\big) - J\big(\gamma_0\big(\tilde{\sigma}(0)\big)\big),\end{aligned}$$

that is

$$t_1 - \rho(\gamma_0)t_0 = \frac{1}{c_0}\big[J\big(\gamma_0\big(\tilde{\sigma}(0)\big)\big) - J\big(\tilde{\sigma}(0)\big)\big],$$

for a suitable $c_0 \neq 0$. Repeating this argument for $\sigma|_{[t_0',t_1']}$, we get

$$t_1' - \rho(\gamma_0)t_0' = t_1 - \rho(\gamma_0)t_0.$$

So, taking the imaginary part and recalling that $t_0' \neq t_0$, we get $\mathrm{Im}\,\rho(\gamma_0) = 0$. Since $\rho(\gamma_0) = \exp(2\pi i\rho_0(\gamma_0))$, this implies $\sin(2\pi\,\mathrm{Re}\,\rho_0(\gamma_0)) = 0$, that is $2\pi\,\mathrm{Re}\,\rho_0(\gamma_0) = k\pi$ for a suitable $k \in \mathbb{Z}$. By (4.12) and Corollary 4.2 the only possibility is $k = -2$, that is $\mathrm{Re}\,\rho_0(\gamma_0) = -1$. But then, by Corollary 4.5, σ is necessarily closed, and we are done. □

5. Holomorphic self-maps, homogeneous vector fields and meromorphic connections

We start this section adapting concepts introduced in [7] to our situation.

Definition 5.1. Let $f : M \to M$ be a holomorphic self-map of a complex n-dimensional manifold M, and assume that f leaves a smooth hypersurface $S \subset M$ pointwise fixed; we write $f \in \mathrm{End}(M, S)$, and always assume that $f \not\equiv \mathrm{id}_M$. We shall say that a local chart (U, z) of M, with $z = (z^1, \dots, z^n)$, is *adapted* to S if $S \cap U = \{z^1 = 0\}$.

Example 5.1. A particularly interesting example of map $f \in \mathrm{End}(M, S)$ is obtained blowing up a map tangent to the identity. Let f_o be a (germ of) holomorphic self-map of $\mathbb{C}^n$ fixing the origin and *tangent to the identity*, that is such that $d(f_o)_O = \mathrm{id}$. If $\pi : M \to \mathbb{C}^n$ denotes the blow-up of the origin, let $S = \pi^{-1}(O) = \mathbb{P}^1(\mathbb{C})$ be the exceptional divisor; then the lifting f of f_o, that is the unique holomorphic self-map of M such that $f_o \circ \pi = \pi \circ f$ (see, e.g., [1] for details), belongs to $\mathrm{End}(M, S)$.

We denote by $N_S = TM|_S/TS$ the normal bundle of S into M, by $\mathcal{N}_S$ the sheaf of germs of holomorphic sections of N_S, by $\mathcal{T}_M$ the sheaf of germs of holomorphic sections of TM, and we put $\mathcal{T}_{M,S} = \mathcal{T}_M \otimes \mathcal{O}_S$, where $\mathcal{O}_S$ is the structure sheaf of S. More generally, given a complex vector bundle (e.g., E), we shall denote by the corresponding calligraphic letter (e.g., $\mathcal{E}$) the sheaf of germs of its holomorphic sections.

Let $f \in \mathrm{End}(M, S)$ and take $p \in S$. Then for every $h \in \mathcal{O}_{M,p}$ (where $\mathcal{O}_M$ is the structure sheaf of M) the germ $h \circ f$ is well defined, and we have $h \circ f - h \in \mathcal{I}_{S,p}$, where $\mathcal{I}_S$ is the ideal sheaf of S.

Definition 5.2. The f*-order of vanishing* at p of $h \in \mathcal{O}_{M,p}$ is

$$\nu_f(h; p) = \max\big\{\mu \in \mathbb{N} \bigm| h \circ f - h \in \mathcal{I}_{S,p}^{\mu}\big\},$$

and the *order of contact* ν_f of f with S is

$$\nu_f = \min\big\{\nu_f(h; p) \bigm| h \in \mathcal{O}_{M,p}\big\}.$$

In [7] we proved that ν_f does not depend on p, and that

$$\nu_f = \min\bigl\{\nu_f\bigl(z^1; p\bigr), \dots, \nu_f\bigl(z^n; p\bigr)\bigr\},$$

where (U, z) is any local chart centered at $p \in S$ and $z = (z^1, \dots, z^n)$.

Definition 5.3. A map $f \in \mathrm{End}(M, S)$ is *tangential* to S if

$$\min\bigl\{\nu_f(h; p) \bigm| h \in \mathcal{I}_{S,p}\bigr\} > \nu_f$$

for some (and hence any) point $p \in S$.

Let $p \in S$, and take a chart (U, z) adapted to S and centered at p. If $f \in \mathrm{End}(M, S)$ and $f^j = z^j \circ f$, we can then write

$$f^j - z^j = \bigl(z^1\bigr)^{\nu_f} g^j, \tag{5.1}$$

where $g^1, \dots, g^n$ are holomorphic and do not all vanish when restricted to S. They in general depend on the chosen chart; however, in [7] we proved that setting

$$\mathcal{X}_f = \sum_{j=1}^{n} g^j \frac{\partial}{\partial z^j} \otimes \bigl(dz^1\bigr)^{\otimes \nu_f} \tag{5.2}$$

then $\mathcal{X}_f|_{U \cap S}$ defines a *global* section X_f of the bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$, where N_S^* is the conormal bundle of S into M. The bundle $TM|_S \otimes (N_S^*)^{\otimes \nu_f}$ is canonically isomorphic to the bundle $\mathrm{Hom}(N_S^{\otimes \nu_f}, TM|_S)$; therefore the section X_f induces a morphism from $N_S^{\otimes \nu_f}$ to $TM|_S$, still denoted by X_f.

Definition 5.4. The morphism $X_f : N_S^{\otimes \nu_f} \to TM|_S$ just defined is the *canonical morphism* associated to $f \in \mathrm{End}(M, S)$.

It is easy to check (see [7]) that f is tangential if and only if the image of X_f is contained in TS, which amounts to saying that $g^1|_{U \cap S} \equiv O$ for any local chart adapted to S.

Definition 5.5. Assume that $f \in \mathrm{End}(M, S)$ is tangential. We shall say that $p \in S$ is a *singular point for* f if X_f vanishes at p. We shall denote by $\mathrm{Sing}(f)$ the set of singular points for f, and by $S^o = S \setminus \mathrm{Sing}(f)$ the subset of regular points. Since $N_S^{\otimes \nu_f}$ is a line bundle, X_f is injective on $N_{S^o}^{\otimes \nu_f}$. In particular, X_f defines a rank 1 singular holomorphic foliation $\mathcal{F}_f$ of S, regular on S^o.

Definition 5.6. Assume we have a complex vector bundle $\pi_F : F \to S^o$ on a complex manifold S^o, and a morphism $X : F \to TS^o$. A *partial holomorphic X-connection* on a complex vector bundle $\pi_E : E \to S^o$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{F}^* \otimes \mathcal{E}$ such that

$$\nabla_u(gs) = X(u)(g)s + g\nabla_u s$$

for all $s \in \mathcal{E}$, $u \in \mathcal{F}$ and $g \in \mathcal{O}_{S^o}$. Clearly, if X is injective we can identify F with its image in TS^o; in that case we shall talk of a *partial holomorphic connection along* $X(F) \subset TS^o$. Finally, if both E, F and X extend to a larger manifold S, with S^o dense in S and X injective in S^o but not necessarily in S, we shall sometimes say that ∇ is a *partial meromorphic connection along* X on E.

When f is tangential, in [7] we introduced a partial meromorphic connection ∇ along X_f on N_S by setting

$$\nabla_u(s) = \pi\big(\big[\mathcal{X}_f(\tilde{u}), \tilde{s}\big]\big|_S\big), \tag{5.3}$$

where: $s \in \mathcal{N}_{S^o}$; $u \in \mathcal{N}_{S^o}^{\otimes \nu_f}$; $\pi : \mathcal{T}_{M,S^o} \to \mathcal{N}_{S^o}$ is the canonical projection; $\tilde{s}$ is any element in $\mathcal{T}_{M,S^o}$ such that $\pi(\tilde{s}|_{S^o}) = s$; $\tilde{u}$ is any element in $\mathcal{T}_{M,S^o}^{\otimes \nu_f}$ such that $\pi(\tilde{u}|_S) = u$; and $\mathcal{X}_f$ is locally given by (5.2). In a chart (U, z) adapted to S, a local generator of N_{S^o} is $\partial_1 = \pi(\partial/\partial z^1)$, and a local generator of $N_{S^o}^{\otimes \nu_f}$ is $\partial_1^{\otimes \nu_f} = \partial_1 \otimes \cdots \otimes \partial_1$. Therefore using $\tilde{u} = (\partial/\partial z^1)^{\otimes \nu_f}$ as extension of $\partial_1^{\otimes \nu_f}$, and $\tilde{s} = \partial/\partial z^1$ as extension of ∂_1 we get

$$\nabla_{\partial_1^{\otimes \nu_f}} \partial_1 = -\frac{\partial g^1}{\partial z^1}\bigg|_{U \cap S^o} \partial_1.$$

Up to here we limited ourselves to summarize [7]; let us now introduce new ideas.

A partial meromorphic connection ∇ along X_f on N_S canonically induces a partial meromorphic connection (still denoted by ∇) along X_f on $N_S^{\otimes \nu_f}$ by setting

$$\nabla(s_1 \otimes \cdots \otimes s_{\nu_f}) = \sum_{j=1}^{\nu_f} s_1 \otimes \cdots \otimes \nabla s_j \otimes \cdots \otimes s_{\nu_f}.$$

In particular we get

$$\nabla_{\partial_1^{\otimes \nu_f}} \partial_1^{\otimes \nu_f} = -\nu_f \frac{\partial g^1}{\partial z^1}\bigg|_{U \cap S^o} \partial_1^{\otimes \nu_f}.$$

As remarked before, the morphism X_f defines a rank 1 singular holomorphic foliation $\mathcal{F}_f$ on S, locally generated by

$$v_o = X_f\big(\partial_1^{\otimes \nu_f}\big) = \sum_{p=2}^{n} g^p\big|_{U \cap S} \frac{\partial}{\partial z^p}. \tag{5.4}$$

We can then define a partial meromorphic connection $\nabla^o : \mathcal{F}_f \to \mathcal{F}_f^* \otimes \mathcal{F}_f$ along the identity on $\mathcal{F}_f$, holomorphic on S^o, by setting

$$\nabla^o_v s = X_f\big(\nabla_{X_f^{-1}(v)} X_f^{-1}(s)\big).$$

Notice that, by construction, ∇^o *induces a* (*standard*) *holomorphic connection on each leaf of the foliation* $\mathcal{F}_f$; so the geodesics we shall introduce momentarily will be *geodesics for a holomorphic connection on a Riemann surface*, that is exactly of the kind we have studied in the first part of this paper.

Remark 5.1. When $n = 2$, the morphism X_f is an isomorphism between $N_{S^o}^{\otimes \nu_f}$ and TS^o; so ∇^o is a standard meromorphic connection on S. In particular, locally we have $\partial/\partial z^2 = \frac{1}{g^2} v_0$, and thus ∇^o is represented by the 1-form

$$\eta^o = -\left[\nu_f \frac{1}{g^2} \frac{\partial g^1}{\partial z^1} - \frac{(g^2)'}{g^2}\right]\bigg|_{U \cap S^o} dz^2.$$

Definition 5.7. A ∇^o-*geodesic* is a (real) curve $\sigma : I \to S^o$ such that $\sigma'(t) \in (\mathcal{F}_f)_{\sigma(t)}$ for all $t \in I$ (that is, the image of σ is contained in a leaf of the foliation) and $\nabla^o_{\sigma'}\sigma' \equiv O$.

In local coordinates (U, z) adapted to S, saying that the image of σ is contained in a leaf of the foliation $\mathcal{F}_f$ is equivalent to saying that σ' is a multiple of the generator v_0 introduced in (5.4). In other words, writing $\sigma^j = z^j \circ \sigma$ and denoting, with a slight abuse of notation, by σ_o' this multiple, we have that the image of σ is contained in a leaf of the foliation if and only if

$$\left(\sigma^j\right)' = \sigma_o' \cdot \left(g^j \circ \sigma\right) \tag{5.5}$$

for $j = 1, \dots, n$ (in particular, $\sigma^1 \equiv 0$). Furthermore, σ is a ∇^o-geodesic if and only if $X_f^{-1}(\sigma') = \sigma_o' \partial_1^{\otimes \nu_f}$ with

$$\left(\sigma_o'\right)' - \nu_f \left(\frac{\partial g^1}{\partial z^1} \circ \sigma\right) \left(\sigma_o'\right)^2 = 0. \tag{5.6}$$

This suggests to introduce a holomorphic vector field G defined on the total space of $p : N_S^{\otimes \nu_f} \to S$ by setting

$$G|_{p^{-1}(U)} = \sum_{p=2}^{n} g^p|_{U \cap S}\, v \frac{\partial}{\partial z^p} + \nu_f \frac{\partial g^1}{\partial z^1}\bigg|_{U \cap S} v^2 \frac{\partial}{\partial v}, \tag{5.7}$$

where v is the fiber coordinate corresponding to the generator $\partial_1^{\otimes \nu_f}$ (and v^2 is the square of the coordinate v).

Proposition 5.1. *Let $f \in \mathrm{End}(M, S)$ be tangential. Then*:

(i) *the formula* (5.7) *defines a global holomorphic vector field on the total space of $N_S^{\otimes \nu_f}$*;
(ii) *a curve $\sigma : I \to S^o$ is a ∇^o-geodesic if and only if the image of σ is contained in a leaf of $\mathcal{F}_f$ and $X_f^{-1}(\sigma')$ is an integral curve of G.*

Proof. (i) follows from a not too difficult computation (using, e.g., [7, (3.6) and (4.2)]), while (ii) follows from (5.5) and (5.6). □

As mentioned before, our main example is when S is the exceptional divisor of the blow-up of the origin in $\mathbb{C}^n$, and f is the lifting of a germ tangent to the identity. Let us now discuss some peculiar features of this case.

Let $\pi : M \to \mathbb{C}^n$ be the blow-up of the origin in $\mathbb{C}^n$, and $S = \pi^{-1}(O) = \mathbb{P}^{n-1}(\mathbb{C})$ the exceptional divisor. Let $w = (w^1, \dots, w^n)$ denote coordinates in $\mathbb{C}^n$, and set $H_j = \{w \in \mathbb{C}^n \mid w^j \neq 0\} \subset \mathbb{C}^n$ for $j = 1, \dots, n$. We can cover M with n charts (U_j, z_j), where $U_j = \pi^{-1}(H_j)$ for $j = 1, \dots, n$; the chart (U_j, z_j) is centered in $[0 : \cdots : 1 : \cdots : 0] \in \mathbb{P}^{n-1}(\mathbb{C})$, and $U_j \cap S = \{z_j^j = 0\}$, where $z_j = (z_j^1, \dots, z_j^n)$. The projection π on U_j is given by

$$\pi\left(z_j(p)\right) = \left(z_j^1 z_j^j, \dots, z_j^j, \dots, z_j^n z_j^j\right);$$

and the coordinate changes in $U_i \cap U_j = \{z_i^j, z_j^i \neq 0\}$ are given by

$$z_j^h = \begin{cases} z_i^i z_i^j & \text{for } h = j; \\ 1/z_i^j & \text{for } h = i; \\ z_i^h / z_i^j & \text{for } h \neq i, j; \end{cases}$$

see [1] for details. It follows that tangent vectors and covectors change according to the following rules:

$$dz_j^h = \begin{cases} z_i^i \, dz_i^j + z_i^j \, dz_i^i & \text{for } h = j; \\ -\frac{1}{(z_i^j)^2} \, dz_i^j & \text{for } h = i; \\ \frac{1}{z_i^j} \, dz_i^h - \frac{z_i^h}{(z_i^j)^2} \, dz_i^j & \text{for } h \neq i, j; \end{cases}$$

and

$$\frac{\partial}{\partial z_j^h} = \begin{cases} \frac{1}{z_i^j} \frac{\partial}{\partial z_i^i} & \text{for } h = j; \\ z_i^j (2 z_i^i \frac{\partial}{\partial z_i^i} - \sum_{k=1}^n z_i^k \frac{\partial}{\partial z_i^k}) & \text{for } h = i; \\ z_i^j \frac{\partial}{\partial z_i^h} & \text{for } h \neq i, j. \end{cases}$$

We shall denote by (ζ_j, v_j) the induced coordinates on $N_S^{\otimes \nu}$, where

$$\zeta_j = \left(\zeta_j^1, \ldots, \zeta_j^{n-1}\right) = \left(z_j^1, \ldots, \widehat{z_j^j}, \ldots, z_j^n\right) \in \mathbb{C}^{n-1}.$$

The coordinate changes in $N_S^{\otimes \nu}$ are then given by

$$\zeta_j^h = \begin{cases} \zeta_i^h / \zeta_i^j & \text{for } 1 \leqslant h \leqslant j-1 \text{ and } i \leqslant h \leqslant n-1, \\ \zeta_i^{h+1} / \zeta_i^j & \text{for } j \leqslant h \leqslant i-2, \\ 1/\zeta_i^j & \text{for } h = i-1, \end{cases} \quad \text{and} \quad v_j = \left(\zeta_i^j\right)^\nu v_i \tag{5.8}$$

when $j < i$, and by similar formulas when $j > i$.

The first consequence of these formulas is the following:

Proposition 5.2. *Let $\pi : M \to \mathbb{C}^n$ be the blow-up of the origin in $\mathbb{C}^2$, and let $S = \pi^{-1}(O) = \mathbb{P}^{n-1}(\mathbb{C})$ be the exceptional divisor. Then for any $\nu \in \mathbb{N}^*$ we can define a ν-to-1 holomorphic covering map $\chi_\nu : \mathbb{C}^n \setminus \{O\} \to N_S^{\otimes \nu} \setminus S$ by setting*

$$\zeta_j(w) = \left(\frac{w^1}{w^j}, \ldots, \widehat{\frac{w^j}{w^j}}, \ldots, \frac{w^n}{w^j}\right) \quad \textit{and} \quad v_j(w) = (w_j)^\nu$$

for all $w \in H_j$ and $j = 1, \ldots, n$. In particular, $p \circ \chi_\nu : \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ is the canonical projection, where $p : N_S^{\otimes \nu} \to S = \mathbb{P}^{n-1}(\mathbb{C})$ is the projection.

Proof. It suffices to remark that χ_ν is well defined, thanks to (5.8). $\square$

Definition 5.8. We shall call the map $\chi_\nu : \mathbb{C}^n \setminus \{O\} \to N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})} \setminus \mathbb{P}^{n-1}(\mathbb{C})$ just defined the *ν-polar coordinates* of $\mathbb{C}^n$. Furthermore, we shall denote by $[\cdot] : \mathbb{C}^n \setminus \{O\} \to \mathbb{P}^{n-1}(\mathbb{C})$ the canonical projection; so

$$p \circ \chi_\nu(w) = [w]$$

for all $w \in \mathbb{C}^n \setminus \{O\}$, where $p : N^{\otimes\nu}_{\mathbb{P}^{n-1}(\mathbb{C})} \to \mathbb{P}^{n-1}(\mathbb{C})$ is the projection.

Let us now $f \in \mathrm{End}(M, S)$ be obtained blowing up a germ $f_o \in \mathrm{End}(\mathbb{C}^n, O)$ tangent to the identity, as in Example 5.1. Write

$$f_o^j(w) = w^j + \sum_{h \geqslant \nu+1} Q_h^j(w), \tag{5.9}$$

where Q_h^j is a homogeneous polynomial of degree h, and $\nu + 1 \geqslant 2$ is the *order* $\nu(f_o)$ of f_o, chosen so that $(Q^1_{\nu+1}, \ldots, Q^n_{\nu+1}) \not\equiv O$. We associate to f_o the homogeneous vector field of degree $\nu + 1$

$$Q_{f_o} = \sum_{j=1}^n Q^j_{\nu+1} \frac{\partial}{\partial w^j}.$$

Definition 5.9. We say that a homogeneous vector field Q is *dicritical* if it is a multiple of the *radial vector field*

$$\sum_{j=1}^n w^j \frac{\partial}{\partial w^j}.$$

In other words, $Q = \sum_j Q^j \frac{\partial}{\partial w^j}$ is dicritical if and only if

$$w^h Q^k \equiv w^k Q^h$$

for all $h, k = 1, \ldots, n$. A map f_o tangent to the identity is *dicritical* if Q_{f_o} is.

Let $\pi : M \to \mathbb{C}^n$ be the blow-up of the origin, and $f \in \mathrm{End}(M, \mathbb{P}^{n-1}(\mathbb{C}))$ the lift of f_o to the blow-up. Then in the chart (U_1, z_1) introduced before setting $f_1 = z_1 \circ f$ we get

$$f_1^j(z_1) = \begin{cases} z_1^1 + (z_1^1)^\nu \sum_{h \geqslant \nu+1} (z_1^1)^{h-\nu-1} Q_h^1(1, \zeta_1) & \text{for } j = 1, \\ z_1^j + (z_1^1)^\nu \frac{\sum_{h \geqslant \nu+1} (z_1^1)^{h-\nu-1} [Q_h^j(1,\zeta_1) - z_1^j Q_h^1(1,\zeta_1)]}{1 + (z_1^1)^\nu \sum_{h \geqslant \nu+1} (z_1^1)^{h-\nu-1} Q_h^1(1,\zeta_1)} & \text{for } j > 1; \end{cases}$$

similar formulas hold in the other charts. In particular, it follows that

(i) if f_o is non-dicritical then f is tangential to the exceptional divisor $\mathbb{P}^{n-1}(\mathbb{C})$ and $\nu_f = \nu(f_o) - 1$;
(ii) if f_o is dicritical then f is not tangential and $\nu_f = \nu(f_o)$.

Thus *most maps constructed with this procedure are tangential.*

Assume then that f_o is non-dicritical, so that f is tangential and $\nu_f = \nu(f_o) - 1 = \nu$. Then in the canonical chart (U_1, z_1) we have

$$\frac{\partial g_1^1}{\partial z_1^1}\bigg|_{U_1\cap S} = Q_{\nu+1}^1(1,\zeta_1), \qquad g_1^p\big|_{U_1\cap S} = Q_{\nu+1}^p(1,\zeta_1) - \zeta_1^{p-1} Q_{\nu+1}^1(1,\zeta_1) \quad \text{for } p = 2,\dots,n,$$

$$G|_{p^{-1}(U_1\cap S)} = \sum_{h=1}^{n-1}\big[Q_{\nu+1}^{h+1}(1,\zeta_1) - \zeta_1^h Q_{\nu+1}^1(1,\zeta_1)\big]v_1\frac{\partial}{\partial\zeta_1^h} + \nu Q_{\nu+1}^1(1,\zeta_1)v_1^2\frac{\partial}{\partial v_1},$$

and similar formulas hold in the other charts. In particular, it follows that the canonical morphism X_f and the connection ∇ (and hence the connection ∇^o and the ∇^o-geodesics) depend only on the homogeneous vector field Q_{f_o}. Thus *we can use the same formulas to associate to* any *non-dicritical homogeneous vector field Q of degree $\nu+1$ the canonical morphism $X_Q : N_{\mathbb{P}^{n-1}(\mathbb{C})}^{\otimes\nu} \to T\mathbb{P}^{n-1}(\mathbb{C})$, the meromorphic connection ∇ on $N_{\mathbb{P}^{n-1}(\mathbb{C})}^{\otimes\nu}$, and the geodesic field G.* In other words, we associate to $Q = \sum_j Q^j \frac{\partial}{\partial z^j}$ all the objects we would get starting from the time-1 map f_Q of Q, which is of the form

$$f_Q^j(z) = z^1 + Q^j(z) + O\big(\|z\|^{\nu+2}\big)$$

for $j = 1,\dots,n$.

To describe another consequence of these formulas we need another definition.

Definition 5.10. A *characteristic direction* of a homogeneous vector field $Q = \sum_j Q^j \frac{\partial}{\partial w^j}$ is a direction $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ such that the line $L_v = \mathbb{C}v$ is Q-invariant, where $v \in \mathbb{C}^n \setminus \{O\}$ is any representative of $[v]$; in this case we shall say that L_v is a *characteristic leaf* of Q. If moreover $Q|_{L_v} \equiv O$ we say that $[v]$ is *degenerate*; otherwise, it is *non-degenerate.* If $S^o \subset \mathbb{P}^{n-1}(\mathbb{C})$ is the complement of the characteristic directions of Q, we shall write $\hat{S}_Q = \{w \in \mathbb{C}^n \setminus \{O\} \mid [w] \in S^o\}$.

It is easy to see that $[v] \in \mathbb{P}^{n-1}(\mathbb{C})$ is characteristic if and only if $(Q^1(v),\dots,Q^n(v)) = \lambda v$ for some $\lambda \in \mathbb{C}$ (clearly depending on the representative v of $[v]$), and that $[v]$ is non-degenerate if and only if $\lambda \neq 0$. Then it is clear that *the singular points of X_Q are exactly the characteristic directions of Q.*

This is just the first signal that we can relate the dynamics of Q to the geodesic flow of ∇^o. And indeed we have the following:

Theorem 5.3. *Let Q be a non-dicritical homogeneous vector field of degree $\nu+1 \geqslant 2$ in $\mathbb{C}^n$. Then*

$$d\chi_\nu(Q) = G, \tag{5.10}$$

where G is the geodesic field on $N_{\mathbb{P}^{n-1}(\mathbb{C})}^{\otimes\nu}$ associated to Q. In particular a real curve $\gamma : I \to \hat{S}_Q$ is an integral curve of Q if and only if $\chi_\nu \circ \gamma$ is an integral curve of G. Furthermore:

(i) *if $\gamma : I \to \hat{S}_Q$ is an integral curve of Q then its direction $[\gamma] : I \to \mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^o-geodesic*; *conversely,*
(ii) *if $\sigma : I \to \mathbb{P}^{n-1}(\mathbb{C})$ is a ∇^o-geodesic then there exists exactly ν integral curves $\gamma_1,\dots,\gamma_\nu : I \to \hat{S}_Q$ of Q, differing only by the multiplication by a ν-th root of unity, whose direction is given by σ, that is such that $\sigma = [\gamma_j]$.*

Proof. (5.10) follows from the homogeneity of Q and the formula

$$d\chi_\nu\left(\frac{\partial}{\partial w^h}\right) = \begin{cases} \frac{1}{w^j}\frac{\partial}{\partial\zeta_j^h} & \text{if } h < j, \\ \frac{1}{w^j}\frac{\partial}{\partial\zeta_j^{h-1}} & \text{if } h > j, \\ -\frac{1}{(w^j)^2}\big[\sum_{k=1}^{j-1} w^k \frac{\partial}{\partial\zeta_j^k} + \sum_{k=j}^{n-1} w^{k+1}\frac{\partial}{\partial\zeta_j^k}\big] + \nu(w^j)^{\nu-1}\frac{\partial}{\partial v_j} & \text{if } h = j. \end{cases}$$

The assertions (i) and (ii) then follow from Proposition 5.1. $\square$

In particular, if γ is an integral curve of Q then $[\gamma]$ necessarily belongs to a leaf L of the holomorphic singular foliation $\mathcal{F}_Q$ of $\mathbb{P}^{n-1}(\mathbb{C})$ induced by the canonical morphism X_Q, and it is a geodesic for a meromorphic connection on L. Thus the study of the dynamics of Q boils down to the study of the singular foliation $\mathcal{F}_Q$ of $\mathbb{P}^{n-1}(\mathbb{C})$ and of the geodesic flow of meromorphic connections on Riemann surfaces.

To show the power of this approach, from the next section on we shall discuss what happens in dimension 2, where the foliation $\mathcal{F}_Q$ is trivial; but we end this section describing the dynamics of dicritical vector fields, the only case our approach does not work.

Actually, the dynamics of a dicritical vector field is very easy to study, because all directions are characteristic and the dynamics inside a characteristic leaf is 1-dimensional, as shown by the following

Lemma 5.4. *Let $L_v = \mathbb{C}v$ be a characteristic leaf of a homogeneous vector field Q of degree $\nu + 1 \geqslant 2$ in $\mathbb{C}^n$. Then*:

(i) *if $[v] \in \mathbb{P}^1(\mathbb{C})$ is a degenerate characteristic direction then the dynamics of Q on L_v is trivial*;
(ii) *if $[v] \in \mathbb{P}^1(\mathbb{C})$ is a non-degenerate characteristic direction, then the integral curve of Q issuing from $\zeta_0 v \in L_v$ is given by*

$$\gamma_{\zeta_0 v}(t) = \frac{\zeta_0 v}{(1 - \lambda_0 \zeta_0^\nu \nu t)^{1/\nu}}, \tag{5.11}$$

where $\lambda_0 \in \mathbb{C}^*$ is such that $Q^j(v) = \lambda_0 v^j$ for $j = 1, \dots, n$. In particular no (*non-constant*) *integral curve is recurrent, and we have*:
(a) *if $\lambda_0 \zeta_0^\nu \notin \mathbb{R}^+$ then $\lim_{t \to +\infty} \gamma_{\zeta_0 v}(t) = O$*;
(b) *if $\lambda_0 \zeta_0^\nu \in \mathbb{R}^+$ then $\lim_{t \to (\lambda_0 \zeta_0^\nu \nu)^{-1}} \|\gamma_{\zeta v}(t)\| = +\infty$.*

Proof. Part (i) is clear. For part (ii), let $\varphi(\zeta) = \zeta v$ be a parametrization of L_v. Then

$$d\varphi^{-1}(Q|_{L_v}) = \lambda_0 \zeta^{\nu+1} \frac{\partial}{\partial \zeta}.$$

The integral curves of this 1-dimensional vector field are

$$\zeta(t) = \frac{\zeta_0}{(1 - \lambda_0 \nu \zeta_0^\nu t)^{1/\nu}},$$

where the determination of the ν-th root is chosen so that $\zeta(0) = \zeta_0$; therefore the integral curve of Q issuing from $\zeta_0 v$ is given by (5.11). $\square$

6. Global dynamics in dimension 2

Let $f \in \mathrm{End}(M, S)$ be tangential, where M is a 2-dimensional complex manifold and S a 1-dimensional complex submanifold of M. Then the partial connection ∇ introduced in the previous section is a *bona fide* holomorphic connection on $N_{S^o}^{\otimes \nu_f}$, and we have an isomorphism $X_f : N_{S^o}^{\otimes \nu_f} \to TS^o$. We then are in the situation described in Section 1, with $E = N_{S^o}^{\otimes \nu_f}$ and $X = X_f$. In particular, we get

- the metric foliation on $N_{S^o}^{\otimes \nu_f} \setminus S^o$, a real non-singular foliation of real rank 3;
- the horizontal foliation on $N_{S^o}^{\otimes \nu_f}$, a complex non-singular foliation of complex rank 1;
- the geodesic foliation on $N_{S^o}^{\otimes \nu_f}$, a real foliation of real rank 1, singular only on the zero section.

Furthermore, the holomorphic connection ∇^o induced by ∇ via X_f on TS^o is a meromorphic connection on S, and the geodesic field G introduced in the previous section coincides with the geodesic field introduced in Section 1. Locally, if (U_α, z_α) is a chart adapted to S, then z_α^2 is a local coordinate on S, and $\partial/\partial z_\alpha^2$ is a local generator of TS; hence we have

$$X_\alpha = g_\alpha^2\big|_{U_\alpha \cap S^o} \tag{6.1}$$

and

$$G = X_\alpha v_\alpha H_\alpha = g_\alpha^2\big|_{U_\alpha\cap S} v_\alpha \partial_\alpha + \nu_f \frac{\partial g_\alpha^1}{\partial z_\alpha^1}\bigg|_{U_\alpha \cap S} v_\alpha^2 \frac{\partial}{\partial v_\alpha}. \tag{6.2}$$

In particular, G vanishes only on the zero section and on the fibers over those singular points where $\partial g_\alpha^1/\partial z_\alpha^1|_S$ vanishes too.

In particular, G defines a singular extension of the horizontal foliation to the whole of $N_S^{\otimes \nu_f}$. We can use (6.2) to study the associated saturated foliation. Indeed, assume that the chart (U_α, z_α) is centered at a singular point $p \in U_\alpha \cap S$. Then on $U_\alpha \cap S$ we can write

$$X_\alpha v_\alpha H_\alpha = v_\alpha \big(z_\alpha^2\big)^\mu \left[h_\alpha^2 \partial_\alpha + \nu_f h_\alpha^1 v_\alpha \frac{\partial}{\partial v_\alpha}\right],$$

with h_α^1, h_α^2 holomorphic functions on $U_\alpha \cap S$ not both vanishing at p. Then the section in square brackets generates the saturation of the horizontal foliation, and it is vanishing only when $h_\alpha^2(p) = 0$ and $v_\alpha = 0$, that is in the singular points for f in the zero section where g_α^2 vanishes at a higher order than $\partial g_\alpha^1/\partial z_\alpha^1$.

Definition 6.1. Let $f \in \mathrm{End}(M, S)$ be tangential, where M is a 2-dimensional complex manifold and S a 1-dimensional complex submanifold of M. The *order* μ_p of a point $p \in S$ is the order of vanishing of X_f at p. In a local chart (U_α, z_α) adapted to S, we have

$$\mu_p = \mathrm{ord}_p\big(g_\alpha^2\big|_S\big).$$

In particular, $p \in \mathrm{Sing}(f)$ if and only if $\mu_p \geqslant 1$. We say that $p \in \mathrm{Sing}(f)$ is an *apparent singularity* if it is not a pole of ∇, that is if $\mu_p \leqslant \mathrm{ord}_p(\partial g_\alpha^1/\partial z_\alpha^1)$ for a (and hence any) local chart (U_α, z_α) adapted to S. Furthermore, we shall say that p is a *Fuchsian singularity* if $\mu_p = \mathrm{ord}_p(\partial g_\alpha^1/\partial z_\alpha^1) + 1$, and that is an *irregular singularity* if $\mu_p > \mathrm{ord}_p(\partial g_\alpha^1/\partial z_\alpha^1) + 1$. Finally, we say that p is a *strictly fixed point* if it is a pole of ∇ (that is, a Fuchsian or irregular singularity), and that it is a *degenerate singularity* if μ_p, $\mathrm{ord}_p(\partial g_\alpha^1/\partial z_\alpha^1) \geqslant 1$. Finally, the *index* of $p \in S$ is given by

$$\iota_p(f, S) = -\frac{1}{\nu_f}\,\mathrm{Res}_p(\nabla).$$

It is not difficult to check (see [2,7] and Section 7) that these definitions do not depend on the adapted local chart.

Remark 6.1. The index defined here is the same one introduced in [2] and [7]. In particular, there we proved that if S is compact then

$$\sum_{p\in S} \iota_p(f, S) = \int_S c_1(N_S),$$

where $c_1(N_S)$ is the first Chern class of the normal bundle N_S.

Using these definitions we can say that *the geodesic field G extends to a holomorphic section of $T(N_S^{\otimes \nu_f})$, vanishing only on the zero section and on the fibers over degenerate singularities*; *and the horizontal foliation extends to a saturated rank* 1 *complex foliation of $N_S^{\otimes \nu_f}$, whose singular points are the strictly fixed points for f in the zero section.*

Remark 6.2. If we replace the map f by its inverse f^{-1}, it is easy to see that X_α changes sign whereas η_α does not change. Therefore f and f^{-1} induce the same metric and horizontal foliations, but the geodesic field of f is the opposite of the geodesic field of f^{-1} – and thus the orientation of the leaves of the geodesic foliations of f is opposite to the orientation of the leaves of the geodesic foliations of f^{-1}, even though the leaves are the same.

Let $\pi : M \to \mathbb{C}^2$ be the blow-up of the origin in $\mathbb{C}^2$, and $S = \pi^{-1}(O) = \mathbb{P}^1(\mathbb{C})$ the exceptional divisor. In this case we have an atlas of M adapted to S composed by two charts (U_0, z_0) and (U_∞, z_∞), with $z_0(U_0) = z_\infty(U_\infty) = \mathbb{C}^2$. The chart (U_0, z_0) is centered at the point $0 = [1:0] \in \mathbb{P}^1(\mathbb{C})$ of the exceptional divisor, while the chart (U_∞, z_∞) is centered at the point $\infty = [0:1] \in \mathbb{P}^1(\mathbb{C})$ of the exceptional divisor. If we denote by (w^1, w^2) the coordinates of $\mathbb{C}^2$, the projection π from M onto $\mathbb{C}^2$ is given by

$$\begin{cases} w^1 = z_0^1, \\ w^2 = z_0^1 z_0^2, \end{cases} \quad \text{on } U_0, \quad \text{and by} \quad \begin{cases} w^1 = z_\infty^1 z_\infty^2, \\ w^2 = z_\infty^1, \end{cases} \quad \text{on } U_\infty.$$

In particular, $\pi(U_0) = \mathbb{C}^2 \setminus \{(0, w^2) \mid w^2 \neq 0\}$ and $\pi(U_\infty) = \mathbb{C}^2 \setminus \{(w^1, 0) \mid w^1 \neq 0\}$, and the cocycles of TS and N_S are respectively given by

$$\psi_{0\infty}\big(z_0^2\big) = -\big(z_0^2\big)^2 \quad \text{and} \quad \xi_{0\infty}\big(z_0^2\big) = \frac{1}{z_0^2}. \tag{6.3}$$

Furthermore, if $(\zeta_0, v_{0,\nu})$ and $(\zeta_\infty, v_{\infty,\nu})$ are the local coordinates on the total space of $N_S^{\otimes \nu}$ for $\nu \in \mathbb{N}^*$ induced by the canonical charts (U_0, z_0) and (U_∞, z_∞) of M, we have

$$\zeta_\infty = \frac{1}{\zeta_0} \quad \text{and} \quad v_{\infty,\nu} = \frac{1}{\xi_{0\infty}(\zeta_0)^\nu} v_{0,\nu} = {\zeta_0}^\nu v_{0,\nu}. \tag{6.4}$$

Let now $Q = Q^1 \frac{\partial}{\partial w^1} + Q^2 \frac{\partial}{\partial w^2}$ be a non-dicritical homogeneous vector field of degree $\nu + 1 \geqslant 2$. Using as $f \in \mathrm{End}(M, S)$ the blow-up of the time-1 map f_Q of Q, and writing for simplicity v_0 and ζ_0 instead of $v_{0,\nu}$ and z_0^2, and v_∞ and ζ_∞ instead of $v_{\infty,\nu}$ and z_∞^2, we get

$$\left.\frac{\partial g_0^1}{\partial z_0^1}\right|_{U_0 \cap S} = Q^1(1, \zeta_0), \qquad g_0^2\big|_{U_0 \cap S} = Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0),$$

$$X_0 = Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0), \qquad \eta_0 = -\frac{\nu Q^1(1, \zeta_0)}{Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0)} d\zeta_0,$$

$$\omega|_{p^{-1}(U_0 \cap S^o)} = \frac{-\nu Q^1(1, \zeta_0)}{Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0)} d\zeta_0 + \frac{1}{v_0} dv_0,$$

$$H_0 = \frac{\partial}{\partial \zeta_0} + \frac{\nu Q^1(1, \zeta_0)}{Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0)} v_0 \frac{\partial}{\partial v_0},$$

$$G|_{p^{-1}(U_0 \cap S^o)} = \big(Q^2(1, \zeta_0) - \zeta_0 Q^1(1, \zeta_0)\big) v_0 \frac{\partial}{\partial \zeta_0} + \nu Q^1(1, \zeta_0) (v_0)^2 \frac{\partial}{\partial v_0},$$

and

$$\left.\frac{\partial g^1_\infty}{\partial z^1_0}\right|_{U_\infty\cap S} = Q^2(\zeta_\infty,1), \qquad g^2_\infty\big|_{U_\infty\cap S} = Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1),$$

$$X_\infty = Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1), \qquad \eta_\infty = -\frac{\nu Q^2(\zeta_\infty,1)}{Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1)}\,d\zeta_\infty,$$

$$\omega|_{p^{-1}(U_\infty\cap S^o)} = \frac{-\nu Q^2(\zeta_\infty,1)}{Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1)}\,d\zeta_\infty + \frac{1}{v_\infty}\,dv_\infty,$$

$$H_\infty = \frac{\partial}{\partial\zeta_\infty} + \frac{\nu Q^2(\zeta_\infty,1)}{Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1)}\,v_\infty\frac{\partial}{\partial v_\infty},$$

$$G|_{p^{-1}(U_\infty\cap S^o)} = \big(Q^1(\zeta_\infty,1) - \zeta_\infty Q^2(\zeta_\infty,1)\big)v_\infty\frac{\partial}{\partial\zeta_\infty} + \nu Q^2(\zeta_\infty,1)(v_\infty)^2\frac{\partial}{\partial v_\infty}.$$

If $[v]\in\mathbb{P}^1(\mathbb{C})$, the *index* $\iota_{[v]}(Q)$ of Q at $[v]$ is the index introduced in Definition 6.1, that is

$$\iota_{[v]}(Q) = -\frac{1}{\nu}\operatorname{Res}_{[v]}(\nabla).$$

In particular, if $[v]=[1:v_0]$ then

$$\iota_{[v]}(Q) = -\frac{1}{\nu}\operatorname{Res}_{[1:v_0]}(\eta_0) = \operatorname{Res}_{v_0}\left(\frac{Q^1(1,\zeta)}{Q^2(1,\zeta) - \zeta Q^1(1,\zeta)}\right); \tag{6.5}$$

a similar formula using η_∞ yields the index of Q at $[0:1]$. Clearly, the index at $[v]$ can be different from zero only if $[v]$ is a characteristic direction.

Remark 6.3. If $[v]=[1:v_0]$ is a non-degenerate characteristic direction, a related number is the *director* $\delta_{[v]}(Q)$ of $[v]$ given by (see [11–15])

$$\delta_{[v]}(Q) = \frac{1}{Q^1(1,v_0)}\frac{\partial(Q^2(1,\zeta) - \zeta Q^1(1,\zeta))}{\partial\zeta}(v_0).$$

A similar formula yields the director of $\infty=[0:1]$ when the latter is a characteristic direction.

Remark 6.4. A homogeneous vector field of degree $\nu+1$ is dicritical if and only if every direction is characteristic. If Q is non-dicritical then it has $\nu+2$ characteristic directions, counted with multiplicity (which is just the order introduced in Definition 6.1; see [5]).

Remark 6.5. Comparing definitions we immediately see that:

(a) $[v]\in\mathbb{P}^1(\mathbb{C})$ is a characteristic direction for Q if and only if it is a singular point of X_Q;
(b) $[v]\in\mathbb{P}^1(\mathbb{C})$ is a degenerate characteristic direction for Q if and only if it is a degenerate singularity of X_Q;
(c) if $[v]\in\mathbb{P}^1(\mathbb{C})$ is a non-degenerate characteristic direction then it is a strictly fixed point of X_Q;
(d) $[v]\in\mathbb{P}^1(\mathbb{C})$ is a non-degenerate characteristic direction with non-vanishing director if and only if it is a Fuchsian singularity of X_Q of order $\mu_{[v]}=1$, and then

$$\delta_{[v]}(Q) = \frac{-\nu}{\operatorname{Res}_{[v]}(\nabla)} = \frac{1}{\iota_{[v]}(Q)}.$$

We can now study the complex foliation induced by Q in $\mathbb{C}^2$ by means of the horizontal foliation of $N_S^{\otimes\nu}$. The link between the two is provided by the following analogous of Theorem 5.3:

Proposition 6.1. *Let Q be a non-dicritical homogeneous vector field of degree $\nu+1 \geqslant 2$ in $\mathbb{C}^2$, and let $S^o \subset \mathbb{P}^1(\mathbb{C})$ be the complement in $\mathbb{P}^1(\mathbb{C})$ of the characteristic directions of Q. Then*

$$\chi_\nu^*\omega = \frac{\nu}{w^2 Q^1(w) - w^1 Q^2(w)}\big[-Q^2(w)\,dw^1 + Q^1(w)\,dw^2\big]$$

on S^o where ω is the holomorphic 1-form representing the horizontal foliation on the total space of $N_{S^o}^{\otimes\nu}$. In particular a complex curve $\Lambda \subset \hat{S}_Q$ is a (complex) leaf of the holomorphic foliation induced by Q if and only if $\chi_\nu(\Lambda)$ is a leaf of the horizontal foliation of $N_{S^o}^{\otimes\nu}$.

Proof. It is an easy computation. □

So to study the holomorphic foliation induced by Q is equivalent to studying the horizontal foliation of the total space of $N_S^{\otimes\nu}$. For instance, we can get a description of the closure of the leaves completely analogous to the one given in Theorem 3.4. To state it we need a couple of definitions.

Definition 6.2. Let $[v_1], \dots, [v_g] \in \mathbb{P}^1(\mathbb{C})$ be the characteristic directions of a non-dicritical homogeneous vector field Q of degree $\nu+1 \geqslant 2$ in $\mathbb{C}^2$. Then the *monodromy group* associated to Q is the subgroup $G(Q)$ of $\mathbb{C}^*$ given by

$$G(Q) = \exp\left[2\pi i\left(\frac{1}{\nu}\mathbb{Z} \oplus \bigoplus_{h=1}^{g} \mathbb{Z}\iota_{[v_j]}(Q)\right)\right] \subset \mathbb{C}^*.$$

Notice that $G(Q) \subset S^1$ if and only if all indices of Q are real numbers; in this case we shall say that Q has *real periods.*

Remark 6.6. It is not difficult to check that $G(Q)$ is a finite cyclic subgroup of S^1 if and only if there is $\ell \in \mathbb{N}^*$ such that

$$\nu\iota_{[v_h]}(Q) \in \frac{1}{\ell}\mathbb{Z}$$

for all $h = 1, \dots, g$.

Definition 6.3. Let $Q = Q^1\frac{\partial}{\partial z^1} + Q^2\frac{\partial}{\partial z^2}$ be a non-dicritical homogeneous vector field of degree $\nu+1 \geqslant 2$. The *metric foliation* of $\hat{S}_Q$ induced by Q is the real rank 3 foliation given by the form

$$\mathrm{Re}\left(\frac{\nu}{w^2 Q^1(w) - w^1 Q^2(w)}\big[-Q^2(w)\,dw^1 + Q^1(w)\,dw^2\big]\right) = \mathrm{Re}\big(\chi_\nu^*\omega\big).$$

In other words, it is the foliation induced by the metric foliation of $N_S^{\otimes\nu}$ via χ_ν.

Then we have the following

Theorem 6.2. *Let $\Lambda \subset \mathbb{C}^2 \setminus \{O\}$ be a non-characteristic leaf of the foliation induced by a non-dicritical homogeneous vector field Q of degree $\nu+1 \geqslant 2$ in $\mathbb{C}^2$. Then $[\Lambda] \subset \mathbb{P}^1(\mathbb{C})$ is the complement S^o in $\mathbb{P}^1(\mathbb{C})$ of the characteristic directions of Q. Furthermore, take $[v] \in S^o$ and $z_0 \in \Lambda \cap \mathbb{C}^* v$. Then*

$$\Lambda \cap \mathbb{C}^* v = G(Q)\cdot z_0 \quad \textit{and} \quad \overline{\Lambda} \cap \mathbb{C}^* v = \overline{G(Q)}\cdot z_0. \tag{6.6}$$

In particular, either

(i) *Q has real periods, and then either all non-characteristic leaves of Q are closed in $\hat{S}_Q$ (and this happens if and only if $G(Q)$ is a finite cyclic group) or any non-characteristic leaf is dense in the leaf of the metric foliation containing it (which is necessarily closed in $\hat{S}_Q$); or*
(ii) *Q does not have real periods, and then all non-characteristic leaves of Q accumulate both the origin and infinity in all directions.*

Proof. A non-characteristic leaf cannot intersect a characteristic one, and so $[\Lambda] \subseteq S^o$. By Proposition 6.1, $\chi_\nu(\Lambda)$ is then a ν-to-1 cover of a leaf L of the horizontal foliation of $N_{S^o}^{\otimes\nu}$; in particular, since $p(L) = S^o$ we get $[\Lambda] = S^o$.

Theorem 3.4 says that

$$\chi_\nu\bigl(\Lambda \cap \mathbb{C}^* v\bigr) = L \cap N_{[v]}^{\otimes\nu} = \rho(\pi) \cdot \chi_\nu(z_0),$$

where $\rho(\pi) \subset \mathbb{C}^*$ is the image of the monodromy representation of the holomorphic connection ∇ induced by Q on $N_{S^o}^{\otimes\nu}$. Therefore

$$\Lambda \cap \mathbb{C}^* v = \rho(\pi)^{1/\nu} \cdot z_0 \quad \text{and} \quad \overline{\Lambda} \cap \mathbb{C}^* v = \overline{\rho(\pi)^{1/\nu}} \cdot z_0,$$

where $\rho(\pi)^{1/\nu} = \{\zeta \in \mathbb{C} \mid \zeta^\nu \in \rho(\pi)\}$. Hence to prove (6.6) it suffices to show that $\rho(\pi)^{1/\nu} = G(Q)$.

Let $[v_1], \dots, [v_g] \in \mathbb{P}^1(\mathbb{C})$ be the characteristic directions of Q. By Proposition 3.6,

$$\rho(\pi) = \exp\left[2\pi i \bigoplus_{h=1}^{g} \mathbb{Z} r_h\right],$$

where $r_h \in \mathbb{C}$ is the residue in $[v_h]$ of the meromorphic connection ∇^o induced by ∇ on TS^o via X_Q. Therefore

$$\rho(\pi)^{1/\nu} = \exp\left[2\pi i \left(\mathbb{Z}\frac{1}{\nu} \oplus \bigoplus_{h=1}^{g} \mathbb{Z}\frac{r_h}{\nu}\right)\right].$$

Now, it is easy to check that the meromorphic form η_α^o representing ∇^o in a canonical chart is related to the form η_α representing ∇ in the corresponding chart by

$$\eta_\alpha^o = \eta_\alpha - \frac{1}{X_\alpha}\, dX_\alpha;$$

therefore

$$r_h = \mathrm{Res}_{[v_h]}\bigl(\nabla^o\bigr) = \mathrm{Res}_{[v_h]}(\nabla) - \mathrm{ord}_{[v_h]}(X_Q). \tag{6.7}$$

Since $\mathrm{ord}_{[v_h]}(X_Q) \in \mathbb{N}$ we get $\rho(\pi)^{1/\nu} = G(Q)$, as claimed.

If Q does not have real periods, then $G(Q) \subset \mathbb{C}^*$ accumulates both 0 and ∞, and (ii) follows. If Q has real periods, then $G(Q)$ is a subgroup of S^1, and hence it is either cyclic or dense, and (i) follows from Theorem 3.4(i). □

The real flow of Q is instead described by the geodesic foliation of $N_{\mathbb{P}^1(\mathbb{C})}^{\otimes\nu}$, that we studied in the first part of this paper. As a first consequence, we get the following Poincaré–Bendixson theorem for homogeneous vector fields:

Theorem 6.3. *Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$ of degree $\nu+1 \geqslant 2$, and let $\gamma:[0,\varepsilon_0) \to \mathbb{C}^2$ be a recurrent maximal integral curve of Q. Then γ is periodic or $[\gamma]:[0,\varepsilon_0) \to \mathbb{P}^1(\mathbb{C})$ intersects itself infinitely many times.*

Proof. If Q is dicritical, Lemma 5.4 implies that no (non-constant) integral curve is recurrent.

Assume then that Q is not dicritical. Then γ, being a recurrent integral curve, cannot intersect a characteristic leaf, because in that case it would be contained in it and, again by Lemma 5.4, no (non-constant) integral curve in a characteristic leaf is recurrent.

Since γ is recurrent in $\mathbb{C}^2 \setminus \{O\}$, then $\chi_\nu \circ \gamma$ is recurrent in $N_{\mathbb{P}^1(\mathbb{C})}^{\otimes\nu} \setminus \mathbb{P}^1(\mathbb{C})$. More precisely, if we denote by S the complement in $\mathbb{P}^1(\mathbb{C})$ of the characteristic directions of Q, we know that $\chi_\nu \circ \gamma$ is recurrent in $N_S^{\otimes\nu} \setminus S$ because the support of γ is contained in $\hat{S}_Q$. Then $p \circ \chi_\nu \circ \gamma$ is recurrent in S; by Theorem 4.6, this implies that $p \circ \chi_\nu \circ \gamma = [\gamma]$ is closed or intersect itself infinitely many times. In the latter case we are done. In the former case, $[\gamma]$ must satisfy the conditions described in Corollary 4.5; in particular, the only way for $\chi_\nu \circ \gamma$ to be recurrent is to be periodic. Since χ_ν is a finite-to-one map, it follows that γ is periodic, as claimed. $\square$

Corollary 6.4. *Let $\gamma:\mathbb{R} \to \mathbb{C}^2 \setminus \{O\}$ be a non-constant periodic integral curve of a homogeneous vector field Q of degree $\nu+1 \geqslant 2$. Then the characteristic directions $[v_1], \dots, [v_g] \in \mathbb{P}^1(\mathbb{C})$ surrounded by $[\gamma]$ satisfy*

$$\sum_{j=1}^{g} \mathrm{Res}_{[v_j]}(Q) = -1 + \sum_{j=1}^{g} \mathrm{ord}_{[v_j]}(X_Q).$$

Proof. It follows immediately from Corollary 4.5 and (6.7). $\square$

We can clearly say more along this line; but to get a better understanding of the dynamics of Q we need to know something about the behavior of the geodesic field near the singularities. The next two sections are devoted to this task; we instead end this section describing a homogeneous vector field actually having a periodic integral curve and, more generally, examples of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$ having periodic geodesics, closed geodesics, non-closed geodesics accumulating a closed one, and geodesic converging to a pole, thus displaying the behavior described in Theorem 4.6(i), (ii) and (iii).

Example 6.1. Take

$$Q = i\gamma\left(w^1\right)^2 \frac{\partial}{\partial w^1} + (1+i\gamma) w^1 w^2 \frac{\partial}{\partial w^2},$$

where $\gamma \in \mathbb{R}$. In this case $\nu = 1$; hence χ_ν is a biholomorphism, and thus Q is biholomorphically conjugated to the geodesic field G.

The field Q is non-dicritical. It has two characteristic directions: $0 = [1:0]$, which is non-degenerate for $\gamma \neq 0$, and $\infty = [0:1]$, which is always degenerate. In the chart centered at 0 we have

$$X_0 = \zeta_0, \qquad \eta_0 = -\frac{i\gamma}{\zeta_0}\, d\zeta_0, \qquad G = \zeta_0 v_0 \frac{\partial}{\partial \zeta_0} + i\gamma (v_0)^2 \frac{\partial}{\partial v_0};$$

in particular, 0 is a Fuchsian singularity and, denoting by ∇^o the connection induced by ∇ on $\mathbb{P}^1(\mathbb{C})$ via X_Q, we have

$$\mu_0 = \mathrm{ord}_0(X_0) = 1, \qquad \mathrm{Res}_0(\nabla) = -i\gamma, \qquad \mathrm{Res}_0\left(\nabla^o\right) = -1 - i\gamma.$$

If $\gamma \neq 0$ the integral curves of G in the standard chart centered at 0 are of the form $\sigma(t) = (\zeta(t), v(t))$, with

$$\zeta(t) = \zeta_0 \exp\left[\frac{i}{\gamma}\log(1 - i\gamma v_0 t)\right], \qquad v(t) = \frac{v_0}{1 - i\gamma v_0 t},$$

where we have chosen the determination of the logarithm so that $\zeta(0) = \zeta_0$. The curve ζ is a geodesic for the meromorphic connection ∇^o, and we can write

$$\zeta(t) = \zeta_0 \exp\bigl[i\gamma^{-1}\log|1 - i\gamma v_0 t| - \gamma^{-1}\arg(1 - i\gamma v_0 t)\bigr].$$

If $i\gamma v_0 \in \mathbb{R}^*$ then $\arg(1 - i\gamma v_0 t)$ is equal either to 0 or to π depending on the sign of $1 - i\gamma v_0 t$. In both cases $\arg(1 - i\gamma v_0 t)$ is constant, and so ζ is a closed geodesic. Notice that if $i\gamma v_0 \in \mathbb{R}^+$ then ζ is defined only on the half-line $(-\infty, (i\gamma v_0)^{-1})$.

If instead $i\gamma v_0 \notin \mathbb{R}$ then $\arg(1 - i\gamma v_0 t) \to \arg(-i\gamma v_0)$ as $t \to +\infty$, and thus the geodesic ζ accumulates a circumference, which is easily seen to be the support of a closed geodesic.

In all these cases $v(t) \to O$ as $t \to +\infty$ (except when $i\gamma v_0 \in \mathbb{R}^+$; in this case $|v(t)| \to +\infty$ as $t \to (i\gamma v_0)^{-1}$), which means that the integral curves of Q are converging to the origin (or escaping to infinity in the exceptional case) without being tangent to any direction (because $\zeta(t)$ has no limit); in particular, they are not periodic.

However, if $\gamma = 0$ we have $\operatorname{Res}_0(\nabla^o) = -1$, and so we expect to find periodic integral curves of Q. Indeed, if $\gamma = 0$ the integral curves of G in the standard chart centered at 0 are given by

$$\zeta(t) = \zeta_0 \exp(v_0 t), \qquad v(t) \equiv v_0.$$

Therefore if $v_0 \in i\mathbb{R}$ we get a periodic integral curve, as desired. If instead $\operatorname{Re} v_0 < 0$ we have $\zeta(t) \to O$, which means that the geodesic tends to $[1:0]$ and that the integral curve of Q issuing from $(v_0, \zeta_0 v_0)$ tends to a non-zero element of the characteristic leaf $L_{(1,0)} \subset \mathbb{C}^2$. Finally, if $\operatorname{Re} v_0 > 0$ we have $|\zeta(t)| \to +\infty$, which means that the geodesic tends to $[0:1]$ and the integral curve of Q issuing from $(v_0, \zeta_0 v_0)$ escapes to infinity.

7. Local study of the singularities

The next step consists in the study of the geodesic flow nearby singular points. We shall work in the following setting: we have a line bundle E on a Riemann surface S (e.g., $S = \mathbb{P}^1(\mathbb{C})$ and $E = N_S^{\otimes \nu}$); a morphism $X : E \to TS$ which is an isomorphism on $S^o = S \setminus \operatorname{Sing}(X)$, where $\operatorname{Sing}(X)$ contains only isolated points (e.g., $X = X_Q$); and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E.

Fix a local chart $(U_\alpha, z_\alpha, e_\alpha)$ trivializing E and centered at a singular point p_0 of X. In local coordinates we can write

$$X(e_\alpha) = X_\alpha \frac{\partial}{\partial z_\alpha}$$

where $X_\alpha : U_\alpha \to \mathbb{C}$ is holomorphic with $X(p_0) = 0$. The holomorphic 1-form η_α representing ∇ in these coordinates is of the form $\eta_\alpha = k_\alpha\, dz_\alpha$, where k_α is meromorphic with possibly a pole in p_0. The geodesic field G in these coordinates is given by

$$G = X_\alpha v_\alpha \partial_\alpha - (X_\alpha k_\alpha) v_\alpha^2 \frac{\partial}{\partial v_\alpha};$$

saying that G extends holomorphically over the singular point means that $Y_\alpha = X_\alpha k_\alpha$ is holomorphic in U_α. Therefore we shall write

$$\eta_\alpha = \frac{Y_\alpha}{X_\alpha}\,dz_\alpha \quad \text{and} \quad G = X_\alpha v_\alpha \partial_\alpha - Y_\alpha v_\alpha^2 \frac{\partial}{\partial v_\alpha}, \tag{7.1}$$

with $X_\alpha, Y_\alpha \in \mathcal{O}(U_\alpha)$ and $X_\alpha(p_0) = 0$.

We shall denote by ∇^o the meromorphic connection on TS induced by ∇ via X. We have already remarked that the meromorphic form η_α^o representing ∇^o in the chart $(U_\alpha, z_\alpha, \partial/\partial z_\alpha)$ is given by

$$\eta_\alpha^o = \eta_\alpha - \frac{1}{X_\alpha}\,dX_\alpha.$$

In particular,

$$\mathrm{Res}_{p_0}\big(\nabla^o\big) = \mathrm{Res}_{p_0}(\nabla) - \mathrm{ord}_{p_0}(X_\alpha). \tag{7.2}$$

We shall write

$$X_\alpha = z_\alpha^{\mu_X} h_\alpha^X \quad \text{and} \quad Y_\alpha = z_\alpha^{\mu_{\alpha,Y}} h_\alpha^Y,$$

with $\mu_X = \mathrm{ord}_{p_0}(X_\alpha)$, $\mu_{\alpha,Y} = \mathrm{ord}_{p_0}(Y_\alpha)$ and $h_\alpha^X(p_0), h_\alpha^Y(p_0) \neq 0$. Notice that μ_X does not depend on the coordinates, by (7.2), whereas $\mu_{\alpha,Y}$ in general does. Let us recast Definition 6.1 in this context:

Definition 7.1. The *order* $\mu_X \geqslant 1$ of $p_0 \in \mathrm{Sing}(X)$ is the order of vanishing of X at p_0, that is $\mu_X = \mathrm{ord}_{p_0}(X)$. Choosing a local chart $(U_\alpha, z_\alpha, e_\alpha)$ and writing G as in (7.1), we say that p_0 is

- an *apparent singularity* if $\mu_X \leqslant \mu_{\alpha,Y}$;
- a *Fuchsian singularity* if $\mu_X = \mu_{\alpha,Y} + 1$;
- an *irregular singularity* if $\mu_X > \mu_{\alpha,Y} + 1$; in this case $m = \mu_X - \mu_{\alpha,Y} > 1$ is the *irregularity* of p_0;
- a *degenerate singularity* if $\mu_{\alpha,Y} \geqslant 1$.

In Remark 7.3 we shall see that these notions do not depend on the chosen chart. Finally, the *residue* of p_0 is $\mathrm{Res}_{p_0}(\nabla)$, and the *induced residue* of p_0 is $\mathrm{Res}_{p_0}^o(X) = \mathrm{Res}_{p_0}(\nabla) - \mu_X$.

Remark 7.1. With these notations, we have

$$\eta_\alpha^o = \left(\frac{h_\alpha^Y}{z_\alpha^{\mu_X - \mu_{\alpha,Y}} h_\alpha^X} - \frac{\mu_X}{z_\alpha} - \frac{(h_\alpha^X)'}{h_\alpha^X}\right) dz_\alpha.$$

In particular, an apparent singularity is a pole of ∇^o without being a pole of ∇. Furthermore, a Fuchsian singularity of X is a Fuchsian singularity of ∇^o in the classical sense (that is a pole of order 1) unless the induced residue of p_0 vanishes. This is the only possibility for p_0 being a pole of ∇ but not of ∇^o; indeed, is easy to see that this happens if and only if

$$\mathrm{ord}_{p_0}(z_\alpha Y_\alpha - \mu_X X_\alpha) > \mu_X = \mathrm{ord}_{p_0}(X_\alpha),$$

which is equivalent to

$$\mathrm{ord}_{p_0}(Y_\alpha) = \mathrm{ord}_{p_0}(X_\alpha) - 1 \quad \text{and} \quad \mathrm{Res}_{p_0}(\nabla) = \mu_X \geqslant 1,$$

that is to p_0 being Fuchsian with vanishing induced residue.

Remark 7.2. If p_0 is a degenerate singularity, that is $Y_\alpha(p_0) = 0$, then the geodesic field G vanishes identically when restricted to the fiber E_{p_0} over the singularity.

When instead $\mathrm{ord}_{p_0}(Y_\alpha) = 0$, that is $a_0 = Y_\alpha(p_0) \neq 0$, the geodesic field restricted to E_{p_0} is given by $-a_0 v_\alpha^2 \frac{\partial}{\partial v_\alpha}$. In particular, E_{p_0} is G-invariant, and the integral curve of G issuing from $v_0 e_\alpha(p_0) \in E_{p_0} \setminus \{O\}$ is given by

$$v(t) = \frac{v_0}{1 + a_0 v_0 t} e_\alpha(p_0).$$

Notice that $\lim_{t\to\pm\infty} v(t) = O$, and that $v(t)$ is defined for all $t \in \mathbb{R}$ unless $av_0 \in \mathbb{R}^-$.

Our aim is to simplify as much as possible the expression of G by changing the local chart and the local generator of E. Thus we shall use changes of the form

$$(z_\beta, v_\beta) = \varphi(z_\alpha, v_\alpha) = \bigl(\psi(z_\alpha), \xi(z_\alpha) v_\alpha\bigr), \tag{7.3}$$

with $\psi(0) = 0$ and $\psi'(0), \xi(0) \neq 0$ (in the notations of Section 1, we have $\psi' = \psi_{\beta\alpha}$ and $\xi = \xi_{\beta\alpha}$). A quick computation gives

$$X_\beta \circ \psi = \frac{\psi' X_\alpha}{\xi} \quad \text{and} \quad Y_\beta \circ \psi = \frac{1}{\xi} Y_\alpha - \frac{\xi'}{\xi^2} X_\alpha. \tag{7.4}$$

Remark 7.3. In particular, if $\mu_X > \mu_{\alpha,Y}$ then $\mu_{\beta,Y} = \mu_{\alpha,Y}$; if $\mu_X \leqslant \mu_{\alpha,Y}$ then $\mu_X \leqslant \mu_{\beta,Y}$; and $\mu_{\alpha,Y} \geqslant 1$ implies $\mu_{\beta,Y} \geqslant 1$. So Definition 7.1 is well posed.

Remark 7.4. Since we only allow changes of coordinates of the form (7.3), the normal forms we are going to obtain for G are (related to but) different from the usual normal forms for singular holomorphic vector fields in $\mathbb{C}^2$ with respect to unrestricted (formal or) biholomorphic changes of coordinates. Furthermore, as it will become apparent with Theorem 8.1, our classification is also different from the classical classification of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$.

Let us choose $\psi = \mathrm{id}$ and $\xi = h_\alpha^X$. We find

$$X_\beta = z_\beta^{\mu_X} \quad \text{and} \quad Y_\beta = \frac{1}{h_\alpha^X}\bigl(z_\beta^{\mu_{\alpha,Y}} h_\alpha^Y - z_\beta^{\mu_X}\bigl(h_\alpha^X\bigr)'\bigr) = z_\beta^{\mu_{\beta,Y}} h_\beta^Y,$$

where

- if $\mu_{\alpha,Y} < \mu_X$ then $\mu_{\beta,Y} = \mu_{\alpha,Y}$ and $h_\beta^Y = (h_\alpha^Y - z_\beta^{\mu_X - \mu_{\alpha,Y}} (h_\alpha^X)')/h_\alpha^X$;
- if $\mu_{\alpha,Y} \geqslant \mu_X$ then $\mu_{\beta,Y} = \mu_X + \mathrm{ord}_{p_0}((h_\alpha^X)' - z_\beta^{\mu_{\alpha,Y} - \mu_X} h_\alpha^Y) \geqslant \mu_X$.

So we can assume $h_\alpha^X \equiv 1$.

The first result of this section is a complete holomorphic classification of apparent singularities:

Proposition 7.1. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X: E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be an* *apparent singularity of order μ. Then there exists a chart (U, z, e) centered at p_0 such that G in this chart is given by*

$$G = \begin{cases} zv\frac{\partial}{\partial z} & \text{if } \mu = 1, \\ z^\mu(1+az^{\mu-1})v\frac{\partial}{\partial z} & \text{for some } a \in \mathbb{C} \text{ if } \mu > 1. \end{cases}$$

Furthermore, if $\mu > 1$ then $a \in \mathbb{C}$ is a (holomorphic and formal) invariant.

Proof. We have already remarked that we can find a chart $(U_\alpha, z_\alpha, e_\alpha)$ such that $h_\alpha^X \equiv 1$. Furthermore, $\mu_{\alpha,Y} \geqslant \mu$ because p_0 is an apparent singularity.

As a first step, take $\psi = \mathrm{id}$ and ξ solving the Cauchy problem

$$\begin{cases} \frac{1}{\xi} z_\alpha^{\mu_{\alpha,Y}} h_\alpha^Y - \frac{\xi'}{\xi^2} z_\alpha^\mu = 0, \\ \xi(p_0) = 1; \end{cases}$$

this problem has a solution holomorphic in a neighborhood of p_0 because we can rewrite the differential equation in the form

$$\xi' = z^{\mu_{\alpha,Y}-\mu} h_\alpha^Y \xi.$$

Then (7.4) says that in the new coordinates $(U_\beta, z_\beta, e_\beta)$ we have

$$X_\beta = z_\beta^\mu h_\beta^X \quad \text{and} \quad Y_\beta \equiv 0,$$

with $h_\beta^X(p_0) = 1$.

Now, it is known (see, e.g., [18, Theorem 5.25]) that we can find a local change of variable ψ fixing p_0 such that

$$(\psi' X_\beta)(\psi^{-1}(z)) = \begin{cases} z & \text{if } \mu = 1, \\ z^\mu(1+az^{\mu-1}) & \text{for some } a \in \mathbb{C} \text{ if } \mu > 1; \end{cases}$$

hence taking $\xi \equiv 1$ we bring G in the required form. Finally, the last assertion follows from the corresponding fact for one variable vector fields (see, e.g., [18, Section 6$\mathbf{B}_2$]). □

Definition 7.2. The invariant a introduced in the latter proposition is the *apparent index* of the apparent singularity.

As a consequence, we can describe the behavior of the geodesic flow nearby an apparent singularity:

Corollary 7.2. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X: E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be an apparent singularity of order μ and apparent index $a \in \mathbb{C}$ if $\mu > 1$. Then if $\sigma: [0, \varepsilon) \to S^o$ is a geodesic with $\sigma(t) \to p_0$ as $t \to \varepsilon \in (0, +\infty]$, then $\sigma'(t) \to O_{p_0}$ and $X^{-1}(\sigma'(t))$ tends to a non-zero element of E_{p_0}.*

Furthermore, there is a neighborhood $U \subseteq S$ of p_0 such that if $z_0 \in U \setminus \mathrm{Sing}(X)$, and $\sigma_v: [0, \varepsilon_v) \to S^o$ denotes the maximal geodesic issuing from z_0 in the direction $X(v) \in T_{z_0}S$, then

(i) *if $\mu = 1$ then there is a non-zero direction $v_0 \in E_{z_0}$ such that*
- *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}\,\zeta < 0$ then $\sigma_v(t) \to p_0$;*
- *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}\,\zeta > 0$ then σ_v escapes from U;*
- *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}\,\zeta = 0$ then σ_v is a periodic geodesic surrounding p_0;*

(ii) *if $\mu > 1$ and $a = 0$ then there are $\mu - 1$ non-zero directions $v_1, \dots, v_{\mu-1} \in E_{z_0}$ such that*
 – *if $v \notin \mathbb{R}v_1 \cup \dots \cup \mathbb{R}v_{\mu-1}$ then $\sigma_v(t) \to p_0$;*
 – *if $v \in \mathbb{R}v_1 \cup \dots \cup \mathbb{R}v_{\mu-1}$ then $\sigma_v(t)$ escapes from U;*

(iii) *if $\mu > 1$ and $a \neq 0$ then there is a non-zero direction $v_0 \in E_{z_0}$ such that*
 – *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}(\zeta/a) > 0$ then $\sigma_v(t) \to p_0$;*
 – *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}(\zeta/a) = 0$ then either $\sigma_v(t) \to p_0$, or σ_v is a periodic geodesic surrounding p_0, or σ_v escapes from U;*
 – *if $v = \zeta v_0 \in E_{z_0}$ with $\mathrm{Re}(\zeta/a) < 0$ then either $\sigma_v(t) \to p_0$ or σ_v escapes from U.*

Proof. By Proposition 7.1, we can find a chart (U, z, e) centered at p_0 such that a curve $\sigma : [0, \varepsilon) \to U$ is a geodesic if and only if $X^{-1}(\sigma'(t)) = (z(t), v(t))$ satisfies

$$\begin{cases} z' = z^\mu\bigl(1 + az^{\mu-1}\bigr)v, \\ v' = 0, \end{cases} \tag{7.5}$$

with $a = 0$ if $\mu = 1$. In particular, $v(t) \equiv v(0)$, and thus clearly has a finite non-zero limit as $t \to \varepsilon$. Moreover, $\sigma(t) \to p_0$ if and only if $z(t) \to 0$; hence $z(t)^\mu(1 + az(t)^{\mu-1})v(t) \to 0$ as $t \to \varepsilon$. This means exactly that $\sigma'(t) \to O_{p_0}$, and the first assertion is proved.

Assume now $\mu = 1$. Then solving (7.5) we find

$$z(t) = z_0 \exp(vt),$$

where $z_0 = z(0)$ and $v(t) \equiv v$. In particular, $\sigma(t) \to p_0$ as $t \to \varepsilon$ if and only if $\mathrm{Re}\, v < 0$. If $\mathrm{Re}\, v = 0$ we get a periodic geodesic around p_0 (and indeed the induced residue of p_0 is -1). Finally, if $\mathrm{Re}\, v > 0$ then $\sigma(t)$ escapes from U. So we have proved (i), with $v_0 = e(z_0)$.

If instead $\mu > 1$ and $a = 0$ solving (7.5) we get

$$z(t) = z_0\left(1 - \frac{vz_0^{\mu-1}}{\mu - 1}t\right)^{-1/(\mu-1)},$$

where the determination of the root is chosen so that $z(0) = z_0$, and $v(t) \equiv v$. In particular $\sigma(t) \to p_0$ as $t \to +\infty$ unless $vz_0^{\mu-1} \in \mathbb{R}^+$, and we get (ii).

If $\mu > 1$ and $a \neq 0$ the solution of (7.5) satisfies $v(t) \equiv v$ and

$$\left(1 + \frac{1}{az(t)^{\mu-1}}\right)\exp\left[-\left(1 + \frac{1}{az(t)^{\mu-1}}\right)\right] = c_0 \exp\left[\frac{(\mu-1)v}{a}t\right], \tag{7.6}$$

with

$$c_0 = \left(1 + \frac{1}{az_0^{\mu-1}}\right)\exp\left(-\left(1 + \frac{1}{az_0^{\mu-1}}\right)\right).$$

In particular, if z_0 is one of the $\mu - 1$ roots of $az_0^{\mu-1} = -1$, then $c_0 = 0$ and so $z(t) \equiv z_0$. We assume that U is small enough to exclude these points; in particular, we have $c_0 \neq 0$.

Assume $\mathrm{Re}(v/a) > 0$. Then the modulus of the right-hand side of (7.6) goes to $+\infty$ as $t \to +\infty$. Hence the modulus of the left-hand side of (7.6), given by $|w(t)| \exp(-\mathrm{Re}\, w(t))$ where

$$w(t) = 1 + \frac{1}{az(t)^{\mu-1}},$$

goes to $+\infty$ too. This forces $|w(t)| \to +\infty$, and hence $z(t) \to 0$, as required.

If $\mathrm{Re}(\nu/a) = 0$, then the left-hand side of (7.6) has constant modulus $|c_0|$, and argument going to $\pm\infty$. Looking at the level sets of the function $|we^{-w}|$ one sees that there is a critical value $\hat{c}$ so that

- if $|c_0| \geqslant \hat{c}$ then necessarily $\arg w(t)$ is bounded and $|\mathrm{Im}\, w(t)| \to +\infty$;
- if $|c_0| < \hat{c}$ then, depending on which connected component of the level set contains $w(0)$, either $\arg w(t)$ is bounded and $|\mathrm{Im}\, w(t)| \to +\infty$, or $\arg w(t)$ is unbounded and $|w(t)|$ is bounded and bounded away from zero.

If $|\mathrm{Im}\, w(t)| \to +\infty$, then $|w(t)| \to +\infty$, and hence $z(t) \to 0$ as before. If instead $|w(t)|$ is bounded, then $w(t)$ must periodically trace the bounded connected component of the level set of $|we^{-w}|$, and hence $z(t)$ is a periodic geodesic (or escapes U if U is too small).

Finally, if $\mathrm{Re}(\nu/a) < 0$ then the modulus of the right-hand side of (7.6) goes to 0 as $t \to +\infty$, and so either $w(t) \to 0$ or $\mathrm{Re}\, w(t) \to +\infty$. In the former case, $z(t)$ must tend to one of the excluded points, and so σ_v escapes. In the latter case $|w(t)| \to +\infty$, and hence $z(t) \to 0$ as usual. $\square$

The translation of this corollary for homogeneous vector fields is the following:

Corollary 7.3. *Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geqslant 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be an apparent singularity of X_Q of order $\mu \geqslant 1$ (and apparent index $a \in \mathbb{C}$ if $\mu > 1$). Then*:

(i) *if the direction $[\gamma(t)] \in \mathbb{P}^1(\mathbb{C})$ of an integral curve $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{O\}$ of Q tends to $[v_0]$ as $t \to \varepsilon$ then $\gamma(t)$ tends to a non-zero point of the characteristic leaf $L_{v_0} \subset \mathbb{C}^2$;*
(ii) *no integral curve of Q tends to the origin tangent to $[v_0]$;*
(iii) *there is an open set of initial conditions whose integral curves tend to a non-zero point of L_{v_0};*
(iv) *if $\mu = 1$ or $\mu > 1$ and $a \neq 0$ then Q admits periodic orbits of arbitrarily long periods accumulating at the origin.*

Proof. Notice that Q is identically zero on L_{v_0}; therefore either an integral curve is a constant point of L_{v_0} (and then all the assertions are trivial) or does not intersect L_{v_0}. Furthermore, since characteristic leaves are Q-invariant, we are interested only in integral curves γ contained in $\hat{S}_Q$.

Assume that $[\gamma(t)]$ converges to $[v_0]$. Set $\sigma = [\gamma] = p \circ \chi_\nu \circ \gamma$; then σ is a geodesic converging to $[v_0]$, and $X^{-1}(\sigma') = \chi_\nu \circ \gamma$. Then Corollary 7.2 says that $\chi_\nu \circ \gamma$ tends to a non-zero element of E_{v_0}, where $E = N_{\mathbb{P}^1(\mathbb{C})}^{\otimes \nu}$. Now, $E_{v_0} = \chi_\nu(L_{v_0})$, and χ_ν is a ν-to-1 map; since the set of accumulation points of γ is connected and contained in L_{v_0}, it follows that $\gamma(t)$ tends to a non-zero element in L_{v_0}, and (i) is proved. (ii) follows from (i), and (iii) follows from Corollary 7.2(i), (ii) and (iii).

Finally, the periodic geodesics given by Corollary 7.2(i) and (iii) yield periodic integral curves accumulating the origin of arbitrarily long period (because the period is inversely proportional to the modulus of the vector v giving the initial condition of the geodesic), and we get (iv). $\square$

The next section is devoted to the classification (formal and, when possible, holomorphic) of Fuchsian and irregular singularities, and its dynamical consequences. We end this section with a preliminary result on the dynamics of Fuchsian singularities that shall be useful later to deal with resonances.

Proposition 7.4. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X : E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian singularity, and assume that $\mathrm{Res}_{p_0}(\nabla) \in \mathbb{R}^*$. Then*:

(i) *if $\mathrm{Res}_{p_0}(\nabla) < 0$ then all leaves of the metric foliation over $p \in S^o$ tend to the zero section as $p \to p_0$;*
(ii) *if $\mathrm{Res}_{p_0}(\nabla) > 0$ then all leaves of the metric foliation over $p \in S^o$ tend to infinity as $p \to p_0$.*

In particular, if $\sigma : [0, \varepsilon) \to S^o$ is a geodesic with $\sigma(t) \to p_0$ as $t \to \varepsilon$, then

(a) *if* $\mathrm{Res}_{p_0}(\nabla) < 0$ *then* $X^{-1}(\sigma'(t))$ *tends to* O_{p_0} *as* $t \to \varepsilon$;
(b) *if* $\mathrm{Res}_{p_0}(\nabla) > 0$ *then* $|X^{-1}(\sigma'(t))| \to +\infty$ *as* $t \to \varepsilon$.

Proof. Saying that p_0 is Fuchsian means that p_0 is a pole of order 1 of ∇. Therefore in local coordinates $(U_\alpha, z_\alpha, e_\alpha)$ centered at p_0 we can write $\eta_\alpha = k_\alpha\, dz_\alpha$ with

$$k_\alpha = \frac{\rho}{z_\alpha} + k_\alpha^*,$$

where $\rho = \mathrm{Res}_{p_0}(\nabla)$ and k_α^* is holomorphic in a neighborhood of p_0.

In Remark 1.1 we noticed that, since the residue ρ is real, we can find a metric g_α adapted to ∇ in $U_\alpha \setminus \{p_0\}$ by setting

$$g_\alpha(z_\alpha; v_\alpha) = \exp\bigl(2\,\mathrm{Re}\, K_\alpha^*(z_\alpha)\bigr)|z_\alpha|^{2\rho}|v_\alpha|^2, \tag{7.7}$$

where $K_\alpha^* \in \mathcal{O}(U_\alpha)$ is the holomorphic primitive of k_α^* with $K_\alpha^*(p_0) = 0$. The leaves of the metric foliation over $U_\alpha \setminus \{p_0\}$ are the level sets of g_α, and then (i) and (ii) clearly follow. Assertions (a) and (b) are then consequences of the fact that the geodesic foliation is contained in the metric foliation. □

Corollary 7.5. *Let Q be a homogeneous holomorphic vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geqslant 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian characteristic direction of Q with real residue $\rho \in \mathbb{R}^*$. Assume that the direction $[\gamma(t)] \in \mathbb{P}^1(\mathbb{C})$ of an integral curve $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{O\}$ of Q tends to $[v_0]$ as $t \to \varepsilon \in (0, +\infty]$. Then $\gamma(t)$ tends to the origin if $\rho < 0$, and to infinity if $\rho > 0$. In particular, if $\rho > 0$ no integral curve outside the characteristic leaf L_{v_0} can tend to the origin tangent to $[v_0]$.*

Proof. It follows from Proposition 7.4, as usual. □

Remark 7.5. The residue in a Fuchsian singularity p_0 is necessarily different from zero (otherwise p_0 would not be a pole of the connection).

Remark 7.6. In Corollary 8.5 we shall see that the same conclusions can be inferred considering the real part of the residue when the residue is not real; but the proof in the resonant case shall depend on Corollary 7.5.

Remark 7.7. In the irregular case, (7.7) still holds, but now K_α^* is meromorphic in U_α, with a pole at p_0. Therefore the behavior of the leaves of the metric foliation might depend on the way p approaches p_0.

8. Classification of singularities

The next result provides the formal classification of Fuchsian and irregular singularities. Contrarily to the classical case of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, in the Fuchsian case we might have resonances.

Theorem 8.1. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X : E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian or irregular singularity, and in a chart $(U_\alpha, z_\alpha, e_\alpha)$ centered at z_0 write*

$$G = z_\alpha^{\mu_X}(a_0 + a_1 z_\alpha + \cdots)v_\alpha \partial_\alpha - z_\alpha^{\mu_Y}(b_0 + b_1 z_\alpha + \cdots)v_\alpha^2 \frac{\partial}{\partial v_\alpha},$$

with $\mu_X > \mu_Y$ and $a_0, b_0 \neq 0$. Put $\rho = b_0/a_0 \neq 0$; if p_0 is Fuchsian then $\rho = \mathrm{Res}_{p_0}(\nabla)$. Then:

(i) *if p_0 is Fuchsian then*
   (a) *if $\mu_Y - \rho \notin \mathbb{N}^*$ then G is formally conjugated to*

$$z^{\mu_X - 1}\left(zv\partial - \rho v^2 \frac{\partial}{\partial v}\right);$$

   (b) *if $\mu_Y - \rho = n \in \mathbb{N}^*$ then G is formally conjugated to*

$$z^{\mu_X - 1}\left(zv\partial - \rho v^2\left(1 + az^n\right)\frac{\partial}{\partial v}\right)$$

   for a suitable $a \in \mathbb{C}$ which is a formal invariant;

(ii) *if p_0 is irregular then G is formally conjugated to*

$$z^{\mu_X - m}\left(z^m v\partial - \rho v^2\left(1 + az^{m-1}\right)\frac{\partial}{\partial v}\right),$$

where $m = \mu_X - \mu_Y > 1$ is the irregularity and $a = \mathrm{Res}_{p_0}(\nabla)/\rho$ is a formal invariant.

Proof. The proof follows the usual formal Poincaré–Dulac paradigm. Write

$$X_\alpha = z_\alpha^{\mu_X}\sum_{j=0}^{+\infty} a_j z_\alpha^j \quad \text{and} \quad Y_\alpha = z_\alpha^{\mu_Y}\sum_{j=0}^{+\infty} b_j z_\alpha^j;$$

given $n \in \mathbb{N}^*$, we start computing the action on X_α and Y_α of a change of coordinates of the form

$$(z_\beta, v_\beta) = \varphi(z_\alpha, v_\alpha) = \left(z_\alpha + c_1 z_\alpha^{n+1}, v_\alpha + c_2 z_\alpha^n v_\alpha\right),$$

with $(c_1, c_2) \in \mathbb{C}^2 \setminus \{O\}$. Using (7.4) it is easy to see that

$$X_\beta = z_\beta^{\mu_X}\left[\sum_{j=0}^{n-1} a_j z_\beta^j + \left[a_n + a_0\left((n+1-\mu_X)c_1 - c_2\right)\right]z_\beta^n + o\left(z_\beta^n\right)\right]$$

and

$$Y_\beta = \begin{cases} z_\beta^{\mu_Y}\left[\sum_{j=0}^{n-1} b_j z_\beta^j + [b_n - (\mu_Y c_1 + c_2)b_0 - nc_2 a_0]z_\beta^n + o(z_\beta^n)\right] & \text{if } m = \mu_X - \mu_Y = 1, \\ z_\beta^{\mu_Y}\left[\sum_{j=0}^{n-1} b_j z_\beta^j + [b_n - (\mu_Y c_1 + c_2)b_0]z_\beta^n + o(z_\beta^n)\right] & \text{if } m = \mu_X - \mu_Y > 1. \end{cases}$$

So such a change of coordinates does not modify the terms of degree less than n, and acts in the specified way on the terms of degree n. In particular, to get X_β and Y_β without terms of degree n we must choose c_1 and c_2 so that

$$\begin{cases} a_0(\mu_X - n - 1)c_1 + a_0 c_2 = a_n, \\ \mu_Y b_0 c_1 + (na_0 + b_0)c_2 = b_n, \end{cases} \tag{8.1}$$

if $m = 1$, or so that

$$\begin{cases} a_0(\mu_X - n - 1)c_1 + a_0 c_2 = a_n, \\ \mu_Y b_0 c_1 + b_0 c_2 = b_n, \end{cases} \tag{8.2}$$

if $m > 1$.

In case (i), the determinant of the system (8.1) vanishes if and only if

$$n = \mu_Y - \rho.$$

If this happens, the first equation in (8.1) becomes $a_0(\rho c_1 + c_2) = a_n$, and so it can anyway be solved.

In case (ii), the determinant of the system (8.2) vanishes if and only if

$$n + 1 = m.$$

If this happens, the first equation in (8.2) can anyway be solved.

Summing up, in case (i) we can always kill the term of degree n in X_β, whereas we can kill the term of degree n in Y_β if $n \neq \mu_Y - \rho$. In case (ii), we can always kill the term of degree n in X_β, whereas we can kill the term of degree n in Y_β if $n \neq m - 1$. Furthermore, in both cases a quick inspection of (8.1) and (8.2) shows that (as soon as we have killed the terms below the resonance level) we cannot anymore modify the coefficient of the resonant term. Therefore proceeding by induction on n we get the assertion. $\square$

Definition 8.1. The formal invariant $a \in \mathbb{C}$ is called *resonant index.*

Our next aim is to prove that, for Fuchsian singularities, the formal normal forms given by Theorem 8.1(i) are actually holomorphic normal forms, that is that we can find a holomorphic change of coordinates of the form (7.3) bringing G in the given normal form. To do so, we shall adapt the holomorphic Poincaré–Dulac paradigm as described in [18, Section I.5].

Definition 8.2. Given $r > 0$, the *majorant r-norm* of a formal power series in $\mathbb{C}[[z]]$ is defined by

$$\left\| \sum_{j=0}^{\infty} a_j z^j \right\|_r = \sum_{j=0}^{\infty} |a_j| r^j.$$

When $h = (h_1, h_2) \in \mathbb{C}[[z]]^2$ we set

$$\|h\|_r = \|h_1\|_r + \|h_2\|_r.$$

We shall denote by $\mathcal{B}_r$ (respectively, $\mathcal{B}_r^2$) the space of $h \in \mathbb{C}[[z]]$ (respectively, of $h \in \mathbb{C}[[z]]^2$) with finite majorant r-norm, and by $\mathcal{B}_{r,\ell}$ (respectively, $\mathcal{B}_{r,\ell}^2$) the subspace of elements of order at least ℓ. It is easy to check (see [18, Proposition 5.8]) that $\mathcal{B}_r$, $\mathcal{B}_{r,\ell}$, $\mathcal{B}_r^2$ and $\mathcal{B}_{r,\ell}^2$ are Banach spaces.

Majorant norms are multiplicative, that is

$$\|fg\|_r \leqslant \|f\|_r \cdot \|g\|_r \tag{8.3}$$

for all $f, g \in \mathcal{B}_r$; see [18, Lemma 5.10].

Clearly if $r' \leqslant r$ we have $\|h\|_{r'} \leqslant \|h\|_r$, and so the natural inclusion $\mathcal{B}_{r,\ell}^2 \hookrightarrow \mathcal{B}_{r',\ell}^2$ is continuous.

Definition 8.3. Let $S : \mathcal{B}_{r,\ell}^2 \to \mathcal{B}_{r,\ell}^2$ be an operator defined on $\mathcal{B}_{r,\ell}^2$ for fixed $\ell \in \mathbb{N}$ and all r small enough (and commuting with the inclusions $\mathcal{B}_{r,\ell}^2 \hookrightarrow \mathcal{B}_{r',\ell}^2$). We shall say that S is *strongly contracting* if

(i) $\|S(0)\|_r = O(r^2)$, and
(ii) S is Lipschitz on the ball $B_{r,\ell}^2 = \{h \in \mathcal{B}_{r,\ell}^2 \mid \|h\|_r \leqslant r\}$, with Lipschitz constant no greater than $O(r)$ as $r \to 0$.

The point of strongly contracting operators is that, for r small enough, they are a contraction of $B^2_{r,\ell}$ into itself, and hence admit a unique fixed point there.

The next lemma contains examples of strongly contracting operators we shall need later on.

Lemma 8.2. *Given $\ell \in \mathbb{N}$, the following operators are strongly contracting*:

(i) $P_j : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *given by* $P_j(h_1, h_2) = (h_1^j, 0)$, *if* $j \geqslant 2$;
(ii) $Q_j : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *given by* $Q_j(h_1, h_2) = (h_2 h_1^j, 0)$, *if* $j \geqslant 1$;
(iii) $R_{A,t} : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *given by* $R_{A,t}(h) = z^t A(h)$, *where* $t \geqslant 1$ *and* $A : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *satisfies* $\|A(O)\|_r = O(r)$ *and* $\|A(h) - A(k)\|_r \leqslant O(1)\|h - k\|_r$;
(iv) $S_g : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *given by* $S_g(h_1, h_2) = (g_1 \circ (\mathrm{id} + h_1), g_2 \circ (\mathrm{id} + h_1))$, *where* $\ell \geqslant 2$ *and* $g = (g_1, g_2) \in \mathcal{B}^2_{s,\ell}$ *for* s *small enough*;
(v) $T_{p_1,p_2,S} : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *given by* $T_{p_1,p_2,S}(h_1, h_2) = (p_1(h_1, h_2) S_1(h_1, h_2), p_2(h_1, h_2) S_2(h_1, h_2))$, *where* p_1, p_2 *are polynomials, and* $S = (S_1, S_2) : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *is strongly contracting*;
(vi) *linear combinations of strongly contracting operators*;
(vii) *operators of the form* $A \circ S : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$, *with* $S : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *strongly contracting and* $A : \mathcal{B}^2_{r,\ell} \to \mathcal{B}^2_{r,\ell}$ *linear* (*commuting with the inclusions*) *so that* $\|Ah\|_r \leqslant O(1)\|h\|_r$.

Proof. (i) It suffices to check the Lipschitz constant of P_j on $B^2_{r,\ell}$. Using (8.3) we have

$$\left\| P_j(h_1, h_2) - P_j(k_1, k_2) \right\|_r = \left\| h_1^j - k_1^j \right\|_r \leqslant \|h_1 - k_1\|_r \sum_{s=0}^{j-1} \|h_1\|_r^s \|k_1\|_r^{j-s-1} \leqslant j r^{j-1} \|h_1 - k_1\|_r,$$

and so the Lipschitz constant of P_j is $O(r^{j-1})$ on $B^2_{r,\ell}$.

(ii) Analogously, if $(h_1, h_2), (k_1, k_2) \in B^2_{r,\ell}$ we get

$$\begin{aligned} \left\| Q_j(h_1, h_2) - Q_j(k_1, k_2) \right\|_r &= \left\| h_2 h_1^j - k_2 k_1^j \right\|_r \leqslant \|h_2\|_r \left\| h_1^j - k_1^j \right\|_r + \|h_2 - k_2\|_r \|k_1\|_r^j \\ &\leqslant j r^j \|h_1 - k_1\|_r + r^j \|h_2 - k_2\|_r \leqslant j r^j \left\| (h_1, h_2) - (k_1, k_2) \right\|_r, \end{aligned}$$

and so the Lipschitz constant of Q_j is $O(r^j)$ on $B^2_{r,\ell}$.

(iii) It follows immediately from $\|z^t h\|_r = r^t \|h\|_r$.

(iv) See [18, Lemma 5.14].

(v) First of all,

$$\left\| T_{p_1,p_2,S}(O) \right\|_r = \left\| p_1(O) S_1(O) \right\|_r + \left\| p_2(O) S_2(O) \right\|_r \leqslant \max\left\{ \left| p_1(O) \right|, \left| p_2(O) \right| \right\} \left\| S(O) \right\|_r = O\left(r^2\right).$$

Furthermore for $j = 1, 2$ we have

$$\begin{aligned} \left\| p_j(h) S_j(h) - p_j(k) S_j(k) \right\|_r &\leqslant \left\| p_j(h) \right\|_r \left\| S_j(h) - S_j(k) \right\|_r + \left\| p_j(h) - p_j(k) \right\|_r \left\| S_j(k) \right\|_r \\ &\leqslant O(1) \left\| S_j(h) - S_j(k) \right\|_r + \left\| p_j(h) - p_j(k) \right\|_r \left\| S_j(k) - S_j(O) \right\|_r \\ &\quad + \left\| p_j(h) - p_j(k) \right\|_r \left\| S_j(O) \right\|_r \\ &\leqslant O(r) \|h - k\|_r + O(1) \|h - k\|_r O(r) \|k\|_r + O(1) \|h - k\|_r O\left(r^2\right) \\ &\leqslant O(r) \|h - k\|_r, \end{aligned}$$

where we used the fact that $\|p_j(h)\|_r$ is bounded on $B^2_{r,\ell}$, and that

$$p_j(h) - p_j(k) = (h_1 - k_1)q_1(h,k) + (h_2 - k_2)q_2(h,k)$$

for suitable polynomials q_1 and q_2, and the assertion follows.

(vi) and (vii) are obvious. $\square$

We are now ready to prove the main theorem of this section:

Theorem 8.3. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X : E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian singularity, and in a chart $(U_\alpha, z_\alpha, e_\alpha)$ centered at z_0 write*

$$G = z_\alpha^{\mu_X}(a_0 + a_1 z_\alpha + \cdots)v_\alpha \partial_\alpha - z_\alpha^{\mu_Y}(b_0 + b_1 z_\alpha + \cdots)v_\alpha^2 \frac{\partial}{\partial v_\alpha},$$

with $\mu_X = \mu_Y + 1$ and $a_0, b_0 \neq 0$. Put $\rho = b_0/a_0 = \mathrm{Res}_{p_0}(\nabla)$. Then we can find a chart (U, z, e) centered in p_0 in which G is given by

$$z^{\mu_X - 1}\left(z v \partial - \rho v^2 \frac{\partial}{\partial v}\right)$$

if $\mu_Y - \rho \notin \mathbb{N}^*$, or by

$$z^{\mu_X - 1}\left(z v \partial - \rho v^2 \left(1 + a z^n\right) \frac{\partial}{\partial v}\right)$$

for a suitable $a \in \mathbb{C}$ if $n = \mu_Y - \rho \in \mathbb{N}^*$.

Proof. By (the proof of) Theorem 8.1, we can find a chart $(U_\beta, z_\beta, e_\beta)$ centered in p_0 where G is given by $G = X_\beta v_\beta \partial_\beta - Y_\beta v_\beta^2 \frac{\partial}{\partial v_\beta}$ with

$$X_\beta = z_\beta^{\mu_X}\left(1 + g_1(z_\beta)\right) \quad \text{and} \quad Y_\beta = \rho z_\beta^{\mu_X - 1}\left(1 + a z_\beta^n + g_2(z_\beta)\right),$$

where $\mathrm{ord}_{p_0}(g_1)$, $\mathrm{ord}_{p_0}(g_2) > n$ and $a \neq 0$ only if $n = \mu_X - 1 - \rho \in \mathbb{N}^*$. Recalling (7.4), we need to find ψ and ξ with $\psi(0) = 0$, $\psi'(0) = \xi(0) = 1$ such that

$$X_\beta\left(\psi(z)\right) = \frac{\psi'(z)}{\xi(z)} z^{\mu_X} \quad \text{and} \quad Y_\beta\left(\psi(z)\right) = \frac{1}{\xi(z)} \rho z^{\mu_X - 1}\left(1 + a z^n\right) - \frac{\xi'(z)}{\xi(z)^2} z^{\mu_X}. \tag{8.4}$$

Writing $\psi(z) = z + z h_1(z)$ and $\xi(z) = 1 + h_2(z)$ with $(h_1, h_2) \in \mathcal{B}^2_{r,\ell}$ for r small enough and $\ell \geqslant 1$ to be determined later, we can reformulate (8.4) as follows:

$$\begin{cases} (1 + h_2) z^{\mu_X}(1 + h_1)^{\mu_X}\left[1 + g_1(z + z h_1)\right] = \left(1 + z h_1' + h_1\right) z^{\mu_X}, \\ (1 + h_2)^2 \rho z^{\mu_X - 1}(1 + h_1)^{\mu_X - 1}\left[1 + a z^n (1 + h_1)^n + g_2(z + z h_1)\right] \\ \quad = (1 + h_2) \rho z^{\mu_X - 1}\left(1 + a z^n\right) - h_2' z^{\mu_X}. \end{cases}$$

Simplifying the powers of z, expanding the powers of $1+h_1$ and $1+h_2$ and rearranging the terms we get

$$\begin{cases} zh_1' + (1-\mu_X)h_1 - h_2 = \sum_{j=2}^{\mu_X}\binom{\mu_X}{j}h_1^j + h_2\sum_{j=1}^{\mu_X}\binom{\mu_X}{j}h_1^j + (1+h_2)(1+h_1)^{\mu_X}g_1(z+zh_1), \\ -zh_2' + \rho(1-\mu_X)h_1 - \rho h_2 \\ \quad = \rho\sum_{j=2}^{\mu_X-1}\binom{\mu_X-1}{j}h_1^j + 2\rho h_2\sum_{j=1}^{\mu_X-1}\binom{\mu_X-1}{j}h_1^j \\ \qquad + \rho a z^n\left[(\mu_X+n-1)h_1 + h_2 + \sum_{j=2}^{\mu_X+n-1}\binom{\mu_X+n-1}{j}h_1^j + 2h_2\sum_{j=1}^{\mu_X+n-1}\binom{\mu_X+n-1}{j}h_1^j\right] \\ \qquad + \rho(1+h_1)^{\mu_X-1}\bigl(1+az^n(1+h_1)^n\bigr)h_2^2 + \rho(1+h_2)^2(1+h_1)^{\mu_X-1}g_2(z+zh_1). \end{cases}$$

We can write this in a more compact form as

$$Ah = B_1h + B_2h + Ch + Dh + Eh,$$

where A, B_1, B_2, C, D, and E are operators on $\mathcal{B}^2_{r,\ell}$ respectively given by

$$A(h_1,h_2) = \bigl(zh_1' + (1-\mu_X)h_1 - h_2, -zh_2' + \rho(1-\mu_X)h_1 - \rho h_2\bigr),$$

$$B_1(h_1,h_2) = \left(\sum_{j=2}^{\mu_X}\binom{\mu_X}{j}h_1^j, \rho\sum_{j=2}^{\mu_X-1}\binom{\mu_X-1}{j}h_1^j\right),$$

$$B_2(h_1,h_2) = \left(h_2\sum_{j=1}^{\mu_X}\binom{\mu_X}{j}h_1^j, 2\rho h_2\sum_{j=1}^{\mu_X-1}\binom{\mu_X-1}{j}h_1^j\right),$$

$$C(h_1,h_2) = \left(0, \rho a z^n\left[(\mu_X+n-1)h_1 + h_2 + \sum_{j=2}^{\mu_X+n-1}\binom{\mu_X+n-1}{j}h_1^j + h_2\sum_{j=1}^{\mu_X+n-1}\binom{\mu_X+n-1}{j}h_1^j\right]\right),$$

$$D(h_1,h_2) = \bigl(0, \rho(1+h_1)^{\mu_X-1}\bigl(1+az^n(1+h_1)^n\bigr)h_2^2\bigr),$$

$$E(h_1,h_2) = \bigl((1+h_2)(1+h_1)^{\mu_X}g_1(z+zh_1), \rho(1+h_2)^2(1+h_1)^{\mu_X-1}g_2(z+zh_1)\bigr).$$

The operators B_1, B_2, C, D and E are linear combinations of strongly contracting operators, and hence are strongly contracting (by Lemma 8.2). The operator A is linear, and it preserves the degrees: we have

$$A\bigl(c_1z^d, c_2z^d\bigr) = \bigl(\bigl[(d+1-\mu_X)c_1 - c_2\bigr]z^d, \bigl[\rho(1-\mu_X)c_1 - (d+\rho)c_2\bigr]z^d\bigr).$$

So if $d \neq \mu_X - 1 - \rho$ we get

$$A^{-1}\bigl(c_1z^d, c_2z^d\bigr) = \frac{1}{d(d+\rho+1-\mu_X)}\bigl(\bigl[(d+\rho)c_1 - c_2\bigr]z^d, \bigl[\rho(1-\mu_X)c_1 - (d+1-\mu_X)c_2\bigr]z^d\bigr).$$

In particular,

$$\left\|A^{-1}\left(c_1 z^d, c_2 z^d\right)\right\|_r \leqslant O(1/d)\left\|\left(c_1 z^d, c_2 z^d\right)\right\|_r, \tag{8.5}$$

uniformly on r and d. So A is invertible on $\mathcal{B}^2_{r,\ell}$ as soon as $\ell > \mu_X - 1 - \rho = n$ (or $\ell > 1$ if $\mu_X - 1 - \rho \notin \mathbb{N}$), and $\|A^{-1}\|_r = O(1)$ uniformly on r.

Summing up, to solve (8.4) is enough to solve the fixed point problem

$$h = A^{-1}B_1 h + A^{-1}B_2 h + A^{-1}Ch + A^{-1}Dh + A^{-1}Eh$$

on $B^2_{r,\ell}$ for r small enough and $\ell > n$ (or $\ell > 1$ when $a = 0$). Since $A^{-1}B_1$, $A^{-1}B_2$, $A^{-1}C$, $A^{-1}D$ and $A^{-1}E$ are strongly contracting by Lemma 8.2(vii), this is possible, and we are done. □

Remark 8.1. This approach does not work in the irregular case. Working as in the previous proof we reduce the problem to solving an equation of the form $Ah = B_1 h + B_2 h + Ch + Dh + Eh + Fh$, where B_1, B_2, C, D and E are as before (just replace n by $m-1$ and $\mu_X - 1$ by $\mu_X - m$ everywhere), whereas A and F are given by

$$A(h_1, h_2) = \left(zh_1' + (1-\mu_X)h_1 - h_2, \rho(m-\mu_X)h_1 - \rho h_2\right),$$
$$F(h_1, h_2) = \left(0, z^m h_2'\right).$$

In particular

$$A^{-1}\left(c_1 z^d, c_2 z^d\right) = \frac{1}{\rho(d+1-m)}\left([\rho c_1 - c_2]z^d, \left[\rho(m-\mu_X)c_1 - (d+1-\mu_X)c_2\right]z^d\right)$$

and

$$A^{-1}F\left(c_1 z^d, c_2 z^d\right) = \frac{-c_2}{\rho}\left(z^{d+m-1}, (d-\mu_Y)z^{d+m-1}\right),$$

and hence $A^{-1}F$ is not strongly contracting.

As a corollary, we can describe the behavior of the geodesics nearby non-resonant Fuchsian singularities:

Proposition 8.4. *Let E be a line bundle on a Riemann surface S, and assume we have a morphism $X: E \to TS$ which is an isomorphism on $S^o = X \setminus \mathrm{Sing}(X)$, and a meromorphic connection ∇ on E, holomorphic on $E|_{S^o}$, such that the geodesic field G extends holomorphically from $E|_{S^o}$ to the whole of E. Let $p_0 \in \mathrm{Sing}(X)$ be a Fuchsian singularity of order $\mu_X \geqslant 1$, and with vanishing resonant index if $\mu_X - 1 - \rho \in \mathbb{N}^*$, where $\rho = \mathrm{Res}_{p_0}(\nabla)$. Put $\mu_Y = \mu_X - 1$. Then there is a neighborhood $U \subseteq S$ of p_0 such that:*

- (i) *if $\mathrm{Re}\,\rho < \mu_Y$ then all geodesics but one issuing from any point $p \in U \cap S^o$ tend to p_0 staying inside U (the only exception escapes U); furthermore for every geodesic σ going to p_0 inside U we have*
  - (a) *if $\mu_Y\,\mathrm{Re}\,\rho < |\rho|^2$ then $X^{-1}(\sigma'(t)) \to O$ as $\sigma(t) \to p_0$;*
  - (b) *if $\mu_Y\,\mathrm{Re}\,\rho > |\rho|^2$ then $|X^{-1}(\sigma'(t))| \to +\infty$ as $\sigma(t) \to p_0$;*
  - (c) *if $\mu_Y\,\mathrm{Re}\,\rho = |\rho|^2$ then $X^{-1}(\sigma'(t))$ accumulates a circumference in E_{p_0};*
- (ii) *if $\mathrm{Re}\,\rho > \mu_Y$ then all geodesics but one issuing from any point $p \in U \cap S^o$ escapes U; furthermore, the exceptional geodesic σ_0 tends to p_0 in finite time with $|X^{-1}(\sigma_0'(t))| \to +\infty$ as $\sigma_0(t) \to p_0$;*
- (iii) *if $\mathrm{Re}\,\rho = \mu_Y$ but $\rho \neq \mu_Y$ then the geodesics not escaping U are either closed (with $X^{-1}(\sigma')$ either tending to O or diverging to infinity) or accumulate the support of a closed geodesic in U (with $X^{-1}(\sigma')$ tending to O);*

(iv) *if* $\rho = \mu_Y$ *then for all* $p \in S^o \cap U$ *there is a non-zero* $v_p \in E_p$ *such that*
  (a) *if* $v = \zeta v_p \in E_p$ *with* $\operatorname{Re}\zeta < 0$ *then the geodesic* σ_v *issuing from* p *tangent to* $X(v)$ *converges to* p_0 *staying in* U *but with* $|X^{-1}(\sigma_v'(t))| \to +\infty$;
  (b) *if* $v = \zeta v_p \in E_p$ *with* $\operatorname{Re}\zeta > 0$ *then the geodesic* σ_v *issuing from* p *tangent to* $X(v)$ *escapes from* U;
  (c) *if* $v = \zeta v_p \in E_p$ *with* $\operatorname{Re}\zeta = 0$ *then the geodesic* σ_v *issuing from* p *tangent to* $X(v)$ *is periodic and surrounds* p_0.

Proof. By Theorem 8.3 we know that, in a suitable local chart (U, z, e) centered in p_0, a curve σ is a geodesic if and only if $X^{-1}(\sigma'(t)) = (z(t), v(t))$ satisfies

$$\begin{cases} z' = z^{\mu_X} v, \\ v' = -\rho z^{\mu_Y} v^2, \end{cases} \tag{8.6}$$

where $\rho = \operatorname{Res}_{p_0}(\nabla)$ and $\mu_Y = \mu_X - 1$.

Assume $\rho \neq \mu_Y$. Then the solution of (8.6) is

$$\begin{cases} z(t) = z_0(1+ct)^{1/(\rho-\mu_Y)} = z_0 \exp\left(\dfrac{1}{\rho-\mu_Y}\log(1+ct)\right), \\ v(t) = v_0(1+ct)^{-\rho/(\rho-\mu_Y)} = v_0 \exp\left(\dfrac{-\rho}{\rho-\mu_Y}\log(1+ct)\right), \end{cases}$$

with $c = (\rho-\mu_Y)z_0^{\mu_Y} v_0$, where we have chosen the principal determination of the logarithm (that is $\log 1 = 0$). In particular,

$$\left|z(t)\right| = |z_0| \exp\left[\operatorname{Re}\left(\frac{1}{\rho-\mu_Y}\right)\log|1+ct| - \operatorname{Im}\left(\frac{1}{\rho-\mu_Y}\right)\arg(1+ct)\right]$$

and

$$\left|v(t)\right| = |v_0| \exp\left[\operatorname{Re}\left(\frac{-\rho}{\rho-\mu_Y}\right)\log|1+ct| - \operatorname{Im}\left(\frac{-\rho}{\rho-\mu_Y}\right)\arg(1+ct)\right].$$

Notice that $\arg(1+ct)$ is always bounded.

Suppose $\operatorname{Re}\rho < \mu_Y$, so that $\operatorname{Re}(\rho-\mu_Y)^{-1} < 0$. Given z_0, if v_0 is such that $(\rho-\mu_Y)z_0^{\mu_Y} v_0 \notin \mathbb{R}^-$ then $\sigma(t)$ is defined for all $t > 0$ and $\sigma(t) \to p_0$ as $t \to +\infty$. Moreover, $\rho \neq 0$ because p_0 is Fuchsian, and

$$\operatorname{Re}\left(\frac{-\rho}{\rho-\mu_Y}\right) = \frac{-|\rho|^2 + \mu_Y \operatorname{Re}\rho}{|\rho-\mu_Y|^2};$$

hence (a)–(c) follow. If instead v_0 is such that $c = (\rho-\mu_Y)z_0^{\mu_Y} v_0 \in \mathbb{R}^-$, then $|z(t)|$ explodes as $t \to -c^{-1}$, which means that $\sigma(t)$ escapes U; so (i) is proved.

If instead $\operatorname{Re}\rho > \mu_Y$ the situation is reversed: if $(\rho-\mu_Y)z_0^{\mu_Y} v_0 \notin \mathbb{R}^-$ then the geodesic escapes, while if $c = (\rho-\mu_Y)z_0^{\mu_Y} v_0 \in \mathbb{R}^-$ then $\sigma(t) \to p_0$ as $t \to -c^{-1}$. Moreover, $\operatorname{Re}\rho > \mu_Y$ implies $|\rho|^2 \geqslant (\operatorname{Re}\rho)^2 > \mu_Y \operatorname{Re}\rho$, and so $|v(t)| \to +\infty$ as $t \to -c^{-1}$. This completes the proof of (ii).

In case (iii) we have $1/(\rho-\mu_Y) = i\gamma \in i\mathbb{R}^*$, and thus

$$\begin{cases} z(t) = z_0 \exp\bigl[-\gamma \arg(1+ct) + i\gamma \log|1+ct|\bigr], \\ v(t) = v_0 \exp\bigl[-\log|1+ct| + \mu_Y\gamma \arg(1+ct)\bigr] \exp\bigl[-i\bigl(\arg(1+ct) + \mu_Y\gamma \log|1+ct|\bigr)\bigr]. \end{cases}$$

If $c = -i\gamma^{-1} z_0^{\mu_Y} v_0 \in \mathbb{R}^-$ (respectively, $c \in \mathbb{R}^+$) then $\arg(1+ct)$ is constant, and we get a closed geodesic, with $v(t)$ exploding as $t \to -c^{-1}$ (respectively, with $v(t) \to O$ as $t \to +\infty$). If instead $c \notin \mathbb{R}^*$ we get a geodesic accumulating a circumference of positive radius, which is the support of the geodesic issuing from $z_1 = z_0 \exp(-\gamma \arg c)$ in the direction $v_1 = -iz_0^{-\mu_Y}$. We also have $v(t) \to O$. Notice that $\operatorname{Re}\rho = \mu_Y$ implies that the induced residue has real part -1, in accord with the results of Section 4.

Finally, when $\rho = \mu_Y > 0$ (when $\mu_Y = 0$ we have $\rho \neq 0$ by definition), the solution of (8.6) is

$$\begin{cases} z(t) = z_0 \exp\bigl(z_0^{\mu_Y} v_0 t\bigr), \\ v(t) = v_0 \exp\bigl(-\mu_Y z_0^{\mu_Y} v_0 t\bigr). \end{cases}$$

In this case, $\sigma(t) \to p_0$ as $t \to +\infty$ if and only if $\operatorname{Re}(z_0^{\mu_Y} v_0) < 0$, and then $|v(t)| \to +\infty$. If $\operatorname{Re}(z_0^{\mu_Y} v_0) > 0$ then the geodesic escapes, and if $\operatorname{Re}(z_0^{\mu_Y} v_0) = 0$ then the geodesic is periodic, completing the proof of (iv). □

Remark 8.2. In the resonant case, a curve σ is a geodesic if and only if, in suitable local coordinates, $X^{-1}(\sigma'(t)) = (z(t), v(t))$ satisfies

$$\begin{cases} z' = z^{n+\rho+1} v, \\ v' = -\rho z^{n+\rho}\bigl(1 + az^n\bigr) v^2, \end{cases}$$

where $a \in \mathbb{C}^*$, $n \in \mathbb{N}^*$ and $\rho \in \mathbb{Z}$ with $\rho \geqslant -n$. Then

$$\frac{v'}{v} = -\rho\left(\frac{1}{z} + az^{n-1}\right) z' \quad \Longrightarrow \quad v = c_0 z^{-\rho} \exp\left(-\frac{\rho a}{n} z^n\right)$$

with

$$c_0 = v_0 z_0^{\rho} \exp\left(\frac{\rho a}{n} z_0^n\right) \neq 0,$$

where $(z_0, v_0) = (z(0), v(0))$. So $z(t)$ satisfies

$$z' = c_0 z^{n+1} \exp\left(-\frac{\rho a}{n} z^n\right).$$

Setting $w = \frac{\rho a}{n} z^n$ we find

$$w' = \frac{n^2}{\rho a} c_0 w^2 e^{-w} \quad \Longrightarrow \quad -\frac{e^{w(t)}}{w(t)} + \operatorname{Ei}\bigl(w(t)\bigr) = \frac{n^2 c_0}{\rho a} t + c_1,$$

where $\operatorname{Ei}(w)$ is the holomorphic primitive of $w^{-1}e^w$ vanishing at $w_0 = w(0)$, and $c_1 = -w_0^{-1} e^{w_0}$. A numerical study of this equation suggests that for every z_0 one has $w(t) \to O$ (and hence $z(t) \to O$) as $t \to +\infty$ for an open (and possibly dense) set of $v_0 \in \mathbb{C}^*$. So we conjecture that Proposition 8.4(i) holds in this case too (in the resonant case $\operatorname{Re}\rho = \rho < \mu_Y$ necessarily).

Remark 8.3. Notice that a Fuchsian singularity with $\operatorname{Re}\rho < \mu_Y$ cannot appear as a vertex in a cycle of saddle connections which is accumulated by a geodesic. Indeed, such a behavior requires the existence of geodesics arbitrarily close to the singularity and escaping in both forward and backward time. So case (iv) of Theorem 4.6 cannot involve Fuchsian singularities – or apparent singularities of order 1, for the same reason.

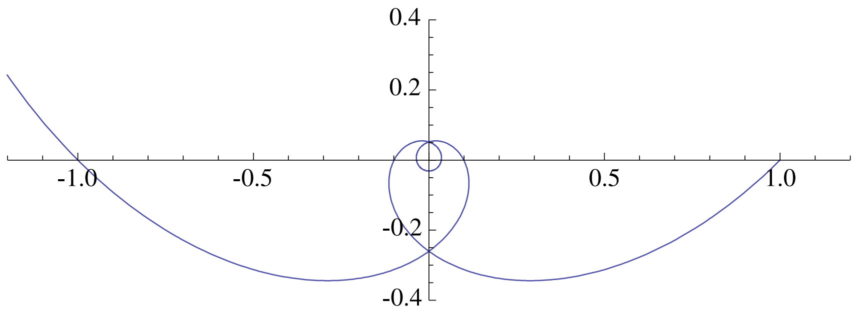

Fig. 1.

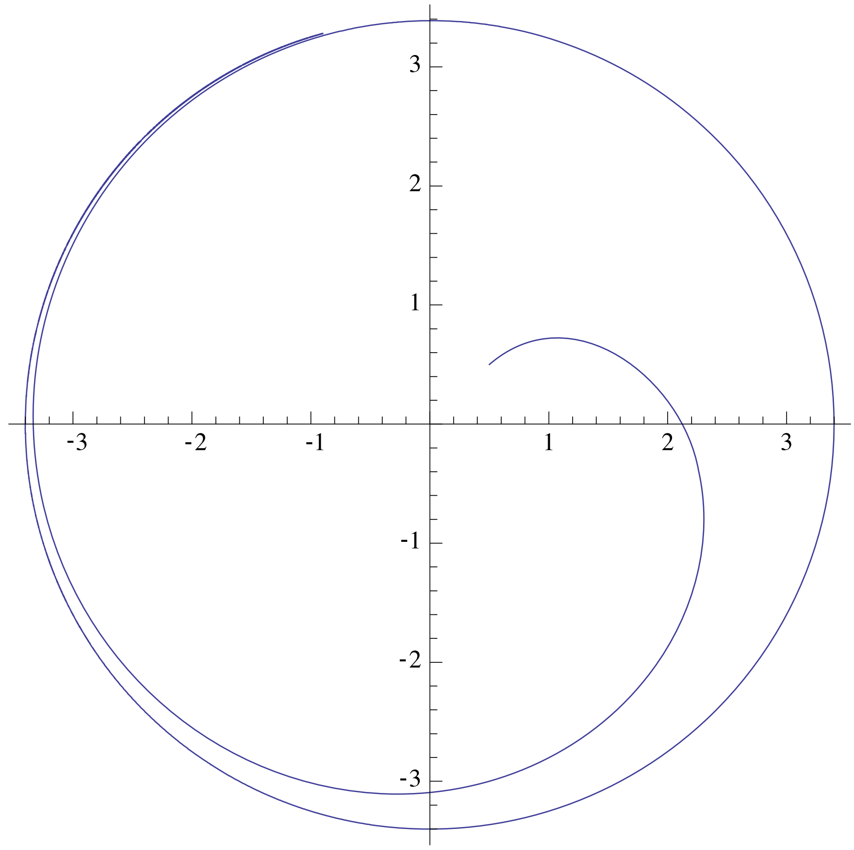

Fig. 2.

Example 8.1. A computation with Mathematica shows that, if we take $\mu_X = 1$ and $\rho = 0.1$, then the geodesic issuing from $z_0 = 1$ in the direction $v_0 = 1 + i$ intersect itself twice before escaping to infinity; see Fig. 1. On the other hand, if we take $\mu_X = 1$ and $\rho = i$, the geodesic issuing from $z_0 = (1 + i)/2$ in the direction $v_0 = 1$ accumulates a closed geodesic; see Fig. 2.

Translating Proposition 8.4 to the case of homogeneous vector fields we get:

Corollary 8.5. *Let Q be a homogeneous vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geqslant 2$. Let $[v_0] \in \mathbb{P}^1(\mathbb{C})$ be a Fuchsian singularity of X_Q of order $\mu_X \geqslant 1$, residue $\rho \in \mathbb{C}^*$ (and resonant index $a \in \mathbb{C}$ if $\mu_X - 1 - \rho \in \mathbb{N}^*$). Put $\mu_Y = \mu_X - 1$. Then:*

(i) *if the direction $[\gamma(t)] \in \mathbb{P}^1(\mathbb{C})$ of an integral curve $\gamma : [0, \varepsilon) \to \mathbb{C}^2 \setminus \{O\}$ of Q tends to $[v_0]$ as $t \to \varepsilon$ and γ is not contained in the characteristic leaf L_{v_0} then*
   (a) *if $\operatorname{Re}\rho < \mu_Y$ and $\mu_Y \operatorname{Re}\rho < |\rho|^2$ then $\gamma(t)$ tends to the origin;*
   (b) *if $\rho = \mu_Y > 0$, or $\operatorname{Re}\rho > \mu_Y$, or $\operatorname{Re}\rho < \mu_Y$ and $\mu_Y \operatorname{Re}\rho > |\rho|^2$, then $\|\gamma(t)\|$ tends to $+\infty$;*
   (c) *if $\operatorname{Re}\rho < \mu_Y$ and $\mu_Y \operatorname{Re}\rho = |\rho|^2$ then $\gamma(t)$ accumulates a circumference in L_{v_0}.*

Furthermore there is a neighborhood $U \subset \mathbb{P}^1(\mathbb{C})$ of $[v_0]$ such that an integral curve γ issuing from a point $z_0 \in \mathbb{C}^2 \setminus L_{v_0}$ with $[z_0] \in U \setminus \{[v_0]\}$ can have one of the following behaviors, where $\hat{U} = \{z \in \mathbb{C}^2 \setminus \{O\} \mid [z] \in U\}$:

(ii) *if* $\operatorname{Re}\rho > \mu_Y$ *then*
 (a) *either $\gamma(t)$ escapes $\hat{U}$, and this happens for a Zariski open dense set of initial conditions in $\hat{U}$*; *or*
 (b) $[\gamma(t)] \to [v_0]$ *but* $\|\gamma(t)\| \to +\infty$;
 in particular, no integral curve outside L_{v_0} converge to the origin tangent to $[v_0]$;
(iii) *if* $\operatorname{Re}\rho = \mu_Y$ *but* $\rho \neq \mu_Y$ *then*
 (a) *either $\gamma(t)$ escapes $\hat{U}$*; *or*
 (b) *$\gamma(t) \to O$ without being tangent to any direction, and $[\gamma(t)]$ is a closed curve or accumulates a closed curve in $\mathbb{P}^1(\mathbb{C})$ surrounding $[v_0]$*; *or*
 (c) *$\|\gamma(t)\| \to +\infty$ without being tangent to any direction, and $[\gamma(t)]$ is a closed curve in $\mathbb{P}^1(\mathbb{C})$ surrounding $[v_0]$*;
 in particular, no integral curve outside L_{v_0} converge to the origin tangent to $[v_0]$;
(iv) *if* $\rho = \mu_Y > 0$ *then*
 (a) *either $\gamma(t)$ escapes $\hat{U}$, and this happens for an open set $\hat{U}_1 \subset \hat{U}$ of initial conditions*; *or*
 (b) *$[\gamma(t)] \to [v_0]$ with $\|\gamma(t)\| \to +\infty$, and this happens for an open set $\hat{U}_2 \subset \hat{U}$ of initial conditions such that $\hat{U}_1 \cup \hat{U}_2$ is dense in $\hat{U}$*; *or*
 (c) *γ is a periodic integral curve with $[\gamma]$ surrounding* $[v_0]$;
 in particular, no integral curve outside L_{v_0} converge to the origin tangent to $[v_0]$, but we have periodic integral curves of arbitrarily long period accumulating the origin;
(v) *if* $\operatorname{Re}\rho < \mu_Y$ *and $a = 0$ then $[\gamma(t)] \to [v_0]$ for an open dense set $\hat{U}_0$ of initial conditions in $\hat{U}$, and γ escapes $\hat{U}$ for $z \in \hat{U} \setminus \hat{U}_0$*; *more precisely,*
 (a) *if* $\mu_Y \operatorname{Re}\rho < |\rho|^2$ *then $\gamma(t) \to O$ tangent to $[v_0]$ for all* $z \in \hat{U}_0$;
 (b) *if* $\mu_Y \operatorname{Re}\rho > |\rho|^2$ *then $|\gamma(t)| \to +\infty$ tangent to $[v_0]$ for all* $z \in \hat{U}_0$;
 (c) *if* $\mu_Y \operatorname{Re}\rho = |\rho|^2$ *then $\gamma(t)$ accumulates a circumference in* L_{v_0}.

Proof. Notice that $\mu_Y - \rho \in \mathbb{N}^*$ implies $\mu_Y > \rho \in \mathbb{Z}$, and so in cases (ii), (iii) and (iv) the resonant index vanishes by definition. Then the only part that does not follow immediately from Proposition 8.4 is part (i) when $a \neq 0$. But in that case $\mu_Y \operatorname{Re}\rho < |\rho|^2$ if and only if $\rho < 0$, and (c) can never happen. The assertion then follows from Proposition 7.4. □

Remark 8.4. This result must be compared with a theorem due to Hakim [16], saying that if $[v_0]$ is a non-degenerate characteristic direction whose director has positive real part then there is an open set of points whose orbits converge to the origin tangentially to $[v_0]$. A non-degenerate characteristic direction $[v_0]$ with non-zero director δ is a Fuchsian singularity of order 1. Then Corollary 8.5 says that if $\operatorname{Re}\delta < 0$ (that is $\operatorname{Re}\rho > 0 = \mu_Y$) then no orbit of the time-1 map of Q outside of L_{v_0} (that is, outside of the parabolic curve whose existence is ensured by Écalle and Hakim's results [11–15]) converges to the origin tangent to $[v_0]$, whereas if $\operatorname{Re}\delta > 0$ (that is $\operatorname{Re}\rho < 0 = \mu_Y$) and $a = 0$ then the orbits under the time-1 map of Q converge to the origin tangent to $[v_0]$ for an open (and dense in a conical neighborhood of $[v_0]$) set of initial conditions.

Remark 8.5. If the conjecture mentioned in Remark 8.2 is true then Corollary 8.5(v) holds in the resonant case too.

Remark 8.6. Since (as already observed in the proof of Corollary 7.3) the periods of the periodic integral curves in Corollary 8.5(iv) tend continuously to infinity as the curves approach the origin, this yields for the time-1 map of Q both periodic orbits accumulating at the origin (when the period of the periodic integral curve is rational) and orbits whose closure is a circle accumulating the origin (when the period of the periodic integral curve is irrational).

Putting together the Poincaré–Bendixson theorems discussed in Section 4 together with the local results in this and the previous sections, one can now say a lot about the dynamics of homogeneous vector fields. In the next section we shall discuss in detail quadratic vector fields; we end this section with an example of application of our results giving a complete description of the dynamics of a substantial class of homogeneous vector fields:

Corollary 8.6. *Let Q be a non-dicritical homogeneous vector field on $\mathbb{C}^2$ of degree $\nu + 1 \geqslant 2$. Assume that all characteristic directions of Q are non-degenerate with non-zero director (or, equivalently, that they are Fuchsian singularities of order 1). Assume moreover that for no set of $g \geqslant 1$ characteristic directions the real part of the sum of the residues is equal to $g - 1$.*

Let $\gamma : [0, \varepsilon_0) \to \mathbb{C}^2$ be a maximal integral curve of Q. Then:

(a) *If $\gamma(0)$ belongs to a characteristic leaf L_{v_0}, then the image of γ is contained in L_{v_0}. Moreover, either $\gamma(t) \to O$ (and this happens for a Zariski open dense set of initial conditions in L_{v_0}), or $\|\gamma(t)\| \to +\infty$.*
(b) *If $\gamma(0)$ does not belong to a characteristic leaf then either*
   (i) *γ converges to the origin tangentially to a characteristic direction $[v_0]$ whose residue has negative real part (and hence whose director has positive real part); or*
   (ii) *$\|\gamma(t)\| \to +\infty$ tangentially to a characteristic direction $[v_0]$ whose residue has positive real part (and hence whose director has negative real part); or*
   (iii) *$[\gamma] : [0, \varepsilon_0) \to \mathbb{P}^1(\mathbb{C})$ intersects itself infinitely many times.*

   Furthermore, if (iii) *never occurs then* (i) *holds for a Zariski open dense set of initial conditions.*

Proof. Statement (a) follows immediately from Lemma 5.4.

For (b), first of all notice that by assumption all characteristic directions of Q are Fuchsian singularities of order 1 (see Remark 6.5). In particular, the induced residues are one less than the residues of the connection induced by Q for all characteristic directions. So the assumption on the residues implies that for no set of characteristic directions the sum of the induced residues has real part equal to -1; therefore Theorem 4.6 implies that either (iii) holds or $[\gamma(t)]$ tends to a characteristic direction $[v_0]$, whose residue cannot be purely imaginary. If the real part of the residue of $[v_0]$ is positive, then we apply Corollary 8.5(ii), showing in particular that this can happen only for a nowhere dense set of initial conditions. If instead the real part of the residue of $[v_0]$ is negative, we apply Corollary 8.5(i)(a) to finish the proof. □

Example 8.2. By Theorem 4.6, if for no set of characteristic directions the real part of the sum or the induced residues belongs to $(-3/2, -1/2)$ then the case (b)(iii) cannot occur, and thus we get a complete description of the dynamics of Q. For instance, assume that Q is a quadratic field (that is $\nu = 1$) with three characteristic directions, necessarily of order 1 (see Remark 5.4), with residues ρ_1, ρ_2 and ρ_3 respectively. By the classical residue theorem for meromorphic connections (see, e.g., [18, Theorem III.17.33]) we know that

$$\rho_1 + \rho_2 + \rho_3 = 1.$$

Then an easy computation shows that case (b)(iii) cannot occur if

$$\operatorname{Re}\rho_1, \operatorname{Re}\rho_2 \notin \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \text{and} \quad \operatorname{Re}(\rho_1 + \rho_2) \notin \left(\frac{1}{2}, \frac{3}{2}\right).$$

As a consequence, at least one (and at most two) of the three residues must have negative real part, and thus case (b)(i) can actually occur. In particular, if only one residue has negative real part then almost all integral curves converge to the origin tangentially to that characteristic direction.

Example 8.3. The vector field

$$Q = \left(-\frac{1}{3}(w^1)^2 + \frac{2}{3}w^1w^2\right)\frac{\partial}{\partial w^1} + \left(\frac{2}{3}w^1w^2 - \frac{1}{3}(w^2)^2\right)\frac{\partial}{\partial w^2}$$

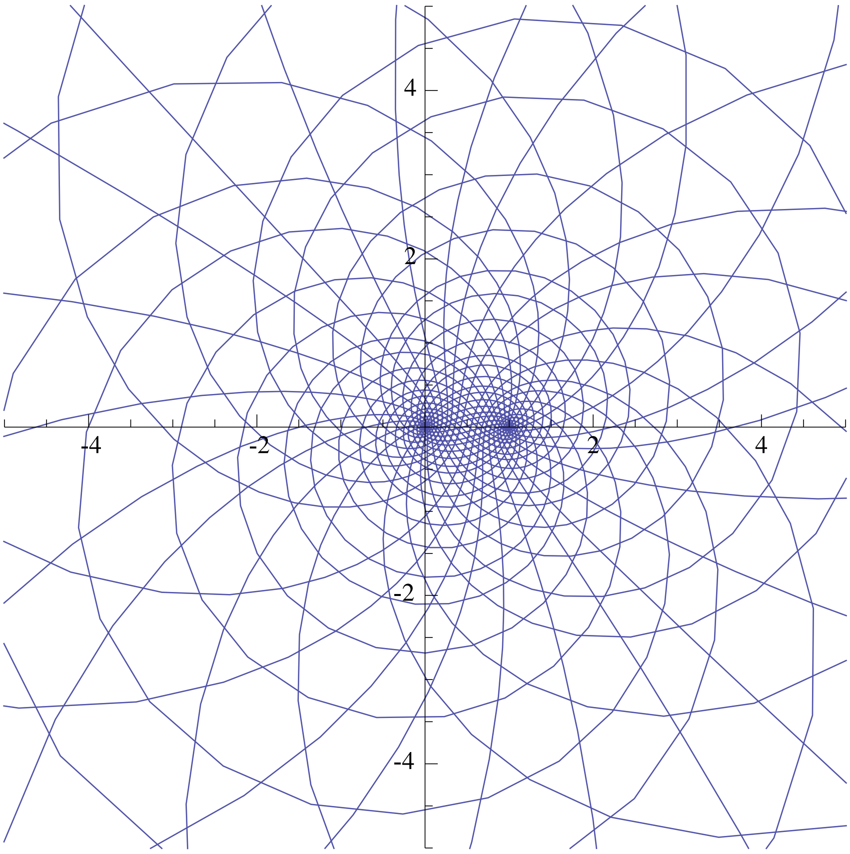

Fig. 3.

has three Fuchsian characteristic directions ($[1:0]$, $[0:1]$ and $[1:1]$) of order 1 and residue $1/3$. In particular, Corollary 8.5 says that for most integral curves γ the geodesic $[\gamma]$ does not converge to a characteristic direction. Since no sum of induced residues has real part equal to -1, Theorem 4.6 implies that for most integral curves the induced geodesic must intersect itself infinitely many times. For instance, Fig. 3 shows $[\gamma]$ for the integral curve γ issuing from $(i, i-1)$.

9. Quadratic vector fields

In this section we shall present, as an example of application of our methods, what we can infer on the dynamics of homogeneous quadratic vector fields.

Arguing as in [3], it is not difficult to see that any not identically zero homogeneous quadratic vector field is linearly conjugated to one of the following:

(∞) $Q(z,w) = z^2\frac{\partial}{\partial z} + zw\frac{\partial}{\partial w}$;
(1_{00}) $Q(z,w) = -z^2\frac{\partial}{\partial w}$;
(1_{10}) $Q(z,w) = -z^2\frac{\partial}{\partial z} - (z^2+zw)\frac{\partial}{\partial w}$;
(1_{11}) $Q(z,w) = -zw\frac{\partial}{\partial z} - (z^2+w^2)\frac{\partial}{\partial w}$;
(2_{001}) $Q(z,w) = zw\frac{\partial}{\partial w}$;
(2_{011}) $Q(z,w) = zw\frac{\partial}{\partial z} + (zw+w^2)\frac{\partial}{\partial w}$;
($2_{10\rho}$) $Q(z,w) = -\rho z^2\frac{\partial}{\partial z} + (1-\rho)zw\frac{\partial}{\partial w}$, with $\rho \neq 0$;
($2_{11\rho}$) $Q(z,w) = (\rho z^2+zw)\frac{\partial}{\partial z} + ((1+\rho)zw+w^2)\frac{\partial}{\partial w}$, with $\rho \neq 0$;
(3_{100}) $Q(z,w) = (z^2-zw)\frac{\partial}{\partial z}$;
($3_{\rho 10}$) $Q(z,w) = \rho(-z^2+zw)\frac{\partial}{\partial z} + (1-\rho)(zw-w^2)\frac{\partial}{\partial w}$, with $\rho \neq 0, 1$;
($3_{\rho\tau 1}$) $Q(z,w) = (-\rho z^2+(1-\tau)zw)\frac{\partial}{\partial z} + ((1-\rho)zw-\tau w^2)\frac{\partial}{\partial w}$, with $\rho, \tau \neq 0$ and $\rho+\tau \neq 1$.

In this list, the vector field (∞) is dicritical; the vector fields $(1_{\bullet\bullet})$ have exactly one characteristic direction $[0:1]$; the vector fields $(2_{\bullet\bullet\bullet})$ have exactly two characteristic directions $[1:0]$ and $[0:1]$; and the vector fields $(3_{\bullet\bullet\bullet})$ have exactly three characteristic directions $[1:0]$, $[1:1]$ and $[0:1]$. Let us see what we can say in the various cases. In our computations, we shall often use Remark 1.7.

Case (∞)**.** The field

$$Q(z,w) = z^2\frac{\partial}{\partial z} + zw\frac{\partial}{\partial w}$$

is dicritical; in particular all complex lines L_v with $v \in \mathbb{C}^2 \setminus O$ are invariant, and the dynamics on each L_v is described by Lemma 5.4. So the line $L_{(0,1)}$ is pointwise fixed, whereas on any other line the integral curves of Q goes to O in both forward and backward time, with the exception of one integral curve going to O in forward time and diverging to infinity in backward time, and of one integral curve diverging to infinity in forward time and going to O in backward time.

Case $(\mathbf{1_{00}})$**.** The field

$$Q(z,w) = -z^2\frac{\partial}{\partial w}$$

has only one (degenerate) characteristic direction $[0:1]$, necessarily of order 3, index -1, residue $\rho = 1$ and induced residue -2. Then Theorems 4.6 and 5.3 imply that the direction $[\gamma]$ of an integral curve γ of Q outside the characteristic leaf $L_{(0,1)}$ goes to $[0:1]$ in both forward and backward time, whereas the characteristic leaf is pointwise fixed.

In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = \zeta_\infty^3 v_\infty\frac{\partial}{\partial\zeta_\infty} - \zeta_\infty^2 v_\infty^2\frac{\partial}{\partial v_\infty};$$

therefore G is already in normal form with a Fuchsian singularity of order 3 (and vanishing resonant index). Furthermore $\mu_Y = 2 > 1 = \rho$; hence Corollary 8.5(i)(b) applies, and it follows that the norm $\|\gamma\|$ of any integral curve outside the characteristic leaf goes to $+\infty$ in forward and backward time.

In this particular case it is easy to explicitly write down the integral curves. Indeed, the integral curve issuing from (z_0, w_0) is given by

$$\gamma(t) = \bigl(z_0, w_0 - z_0^2 t\bigr),$$

whose behavior is exactly as predicted.

Case $(\mathbf{1_{10}})$**.** The field

$$Q(z,w) = -z^2\frac{\partial}{\partial z} - \bigl(z^2 + zw\bigr)\frac{\partial}{\partial w}$$

has only one (degenerate) characteristic direction $[0:1]$, necessarily of order 3, index -1, residue $\rho = 1$ and induced residue -2. Then Theorems 4.6 and 5.3 again imply that the direction $[\gamma]$ of an integral curve γ of Q outside the characteristic leaf $L_{(0,1)}$ goes to $[0:1]$ in both forward and backward time, whereas the characteristic leaf is again pointwise fixed.

In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = \zeta_\infty^3 v_\infty\frac{\partial}{\partial\zeta_\infty} - \zeta_\infty(1+\zeta_\infty)v_\infty^2\frac{\partial}{\partial v_\infty};$$

therefore G is in (formal) normal form with an irregular singularity of order 3, irregularity 2 and resonant index 1. We do not have (yet) general results about the dynamics nearby irregular singularities; however, in this case we can work directly. Let $\sigma(t) = (\zeta_\infty(t), v_\infty(t))$ be an integral curve of G. We know that $\sigma(t)$ is contained in the horizontal foliation, and a quick computation shows that in this case the horizontal foliation is given by

$$\exp\left(-\frac{1}{\zeta_\infty}\right)\zeta_\infty v_\infty \equiv c_0.$$

It follows that σ must satisfy the equation

$$\zeta_\infty' = c_0\zeta_\infty^2 \exp\left(\frac{1}{\zeta_\infty}\right).$$

Separating the variables we get $\exp(-1/\zeta_\infty(t)) = c_0 t + c_1$, that is

$$\zeta_\infty(t) = -\frac{1}{\log(c_0 t + c_1)}, \qquad v_\infty(t) = -\frac{c_0 \log(c_0 t + c_1)}{c_0 t + c_1},$$

and c_0 and c_1 are determined by the initial conditions

$$\log c_1 = -\frac{1}{\zeta_\infty(0)}, \qquad c_0 = c_1 \zeta_\infty(0) v_\infty(0).$$

It follows that if $\zeta_\infty(0)v_\infty(0) \notin \mathbb{R}$ then $v_\infty(t)$ tends to 0 for $t \to \pm\infty$; if instead $\zeta_\infty(0)v_\infty(0) \in \mathbb{R}^*$ then $|v_\infty(t)|$ diverges to $+\infty$ in finite time on one side and converges to 0 on the other side. Recalling that $\chi_1^{-1}(\zeta_\infty, v_\infty) = (\zeta_\infty v_\infty, v_\infty)$, it follows that if $(z_0, w_0) \in \mathbb{C}^2 \setminus \{O\}$ is such that $z_0 \notin \mathbb{R}$ then the integral curve γ of Q issuing from (z_0, w_0) goes to the origin tangent to $[0:1]$ in forward and backward time; if instead $z_0 \in \mathbb{R}^*$ then $\gamma(t)$ goes to the origin in forward time and diverges in backward time, or conversely.

In particular, the origin has an open basin of attraction even though the index of the characteristic direction has negative real part; this cannot happen in the Fuchsian case (see Remark 8.3).

Case ($\mathbf{1_{11}}$). The field

$$Q(z, w) = -zw\frac{\partial}{\partial z} - \left(z^2 + w^2\right)\frac{\partial}{\partial w}$$

has only one (non-degenerate) characteristic direction $[0:1]$, necessarily of order 3, index -1, residue $\rho = 1$ and induced residue -2. Then Theorems 4.6 and 5.3 still imply that the direction $[\gamma]$ of an integral curve γ of Q outside the characteristic leaf $L_{(0,1)}$ goes to $[0:1]$ in both forward and backward time; however this time the characteristic leaf is not pointwise fixed, and the dynamics there is described by Lemma 5.4.

In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = \zeta_\infty^3 v_\infty \frac{\partial}{\partial \zeta_\infty} - \left(1 + \zeta_\infty^2\right)v_\infty^2 \frac{\partial}{\partial v_\infty};$$

therefore G is in (formal) normal form with an irregular singularity of order 3, irregularity 3 and resonant index 1. Let $\sigma(t) = (\zeta_\infty(t), v_\infty(t))$ be an integral curve of G. We know that $\sigma(t)$ is contained in the horizontal foliation, and a quick computation shows that in this case the horizontal foliation is given by

$$\exp\left(-\frac{1}{2\zeta_\infty^2}\right)\zeta_\infty v_\infty \equiv c_0.$$

It follows that σ must satisfy the equation

$$\zeta_\infty' = c_0\zeta_\infty^2 \exp\left(\frac{1}{2\zeta_\infty^2}\right).$$

If F denotes the primitive of the function $\exp(-w^2/2)$ with $F(0)=0$, separating the variables we get

$$F\left(-\frac{1}{\zeta_\infty(t)}\right) = c_0 t + c_1.$$

Since F' is never vanishing, we can find a well-defined branch of F^{-1} along the line $t \mapsto c_0 t + c_1$, and thus

$$\zeta_\infty(t) = -\frac{1}{F^{-1}(c_0 t + c_1)}, \qquad v_\infty(t) = -c_0 F^{-1}(c_0 t + c_1)\exp\left(\frac{1}{2}F^{-1}(c_0 t + c_1)^2\right),$$

where c_0 and c_1 are determined by the initial conditions

$$F^{-1}(c_1) = -\frac{1}{\zeta_\infty(0)}, \qquad c_0 = \exp\left(-\frac{1}{2\zeta_\infty(0)^2}\right)\zeta_\infty(0)v_\infty(0).$$

Writing $F^{-1}(c_0 t + c_1) = R(t) + iI(t)$, with $R(t), I(t) \in \mathbb{R}$, we have

$$\left|v(t)\right|^2 = |c_0|^2\left[R(t)^2 + I(t)^2\right]\exp\left[R(t)^2 - I(t)^2\right].$$

We know that $\zeta_\infty(t) \to 0$ as $t \to \pm\infty$ (if $c_1/c_0 \in \mathbb{R}$ there is one value of t in which $\zeta_\infty(t)$ is not defined, because the geodesic has left the canonical chart, but beyond that point the geodesic re-enters the canonical chart); it follows that $|F^{-1}(c_0 t + c_1)| \to +\infty$. Furthermore, if $\limsup_{t\to\pm\infty} |R(t)|/|I(t)| < 1$ then $|v(t)| \to 0$. Numerical experiments seem to suggest that this can happen for an open set of initial conditions; if this is correct, then we would have an open set of integral curves of Q converging to the origin tangent to $[0:1]$.

Case ($\mathbf{2_{001}}$). The field

$$Q(z, w) = zw\frac{\partial}{\partial w}$$

has two (both degenerate) characteristic directions $[1:0]$ and $[0:1]$. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0 v_0 \frac{\partial}{\partial \zeta_0};$$

therefore $[1:0]$ is an apparent singularity of order 1, residue $\rho = 0$, induced residue -1, and G is in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = -\zeta_\infty^2 v_\infty \frac{\partial}{\partial \zeta_\infty} + \zeta_\infty v_\infty^2 \frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is a Fuchsian singularity of order 2, residue $\rho = 1$, induced residue -1, and G is in normal form (up to a change of sign in v_∞). Both characteristic leaves are pointwise fixed.

This time Theorems 4.6 and 5.3 say that the direction $[\gamma]$ of an integral curve γ of Q outside the characteristic leaves $L_{(1,0)}$ and $L_{(0,1)}$ either goes to $[1:0]$ or $[0:1]$ in both forward and backward time, or is a closed geodesic.

In the coordinate chart centered at $[1:0]$ we can apply Corollary 7.2; it follows that the geodesics are either a saddle connection between $[1:0]$ and $[0:1]$ or periodic; furthermore, $v_0(t)$ is constant. Recalling that $\chi_1^{-1}(\zeta_0, v_0) = (v_0, \zeta_0 v_0)$, it follows that the first coordinate of an integral curve γ is always constant, and the second coordinate is either (constant or) periodic or goes to 0 on one side and diverges to infinity on the other side. So we have periodic integral curves; and the non-periodic integral curves go from a point in the characteristic leaf $L_{(1,0)}$ off to infinity toward the other characteristic leaf $L_{(0,1)}$; in particular, no integral curve converges to the origin, and we have periodic integral curves (and hence periodic points for the time-1 map) accumulating at the origin. These results are confirmed by the explicit expression of the integral curve γ issuing from (z_0, w_0):

$$\gamma(t) = \bigl(z_0, w_0 e^{z_0 t}\bigr).$$

Case ($\mathbf{2_{011}}$). The field

$$Q(z, w) = zw \frac{\partial}{\partial z} + \bigl(zw + w^2\bigr) \frac{\partial}{\partial w}$$

has two characteristic directions: $[1:0]$ is degenerate whereas $[0:1]$ is non-degenerate. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0 v_0 \frac{\partial}{\partial \zeta_0} + \zeta_0 v_0^2 \frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is an apparent singularity of order 1, residue $\rho = 0$, induced residue -1, but G is not in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = -\zeta_\infty^2 v_\infty \frac{\partial}{\partial \zeta_\infty} + (1 + \zeta_\infty) v_\infty^2 \frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is an irregular singularity of order 2, irregularity 2, residue $\rho = 1$, induced residue -1, resonant index 1, and G is in normal form (up to a change of sign in v_∞). The characteristic leaf $L_{(1,0)}$ is pointwise fixed, while the dynamics on the characteristic leaf $L_{(0,1)}$ is described by Lemma 5.4.

Theorems 4.6 and 5.3 say again that the direction $[\gamma]$ of an integral curve γ of Q off the characteristic leaves $L_{(1,0)}$ and $L_{(0,1)}$ either goes to $[1:0]$ or $[0:1]$ in both forward and backward time, or is a closed geodesic. In particular, self-intersecting geodesics are necessarily closed.

The description of the behavior of the integral curves of G nearby $[1:0]$ is provided by Corollary 7.2: we have periodic integral curves, integral curves converging to a non-zero element of the fiber over the singularity, and integral curves escaping toward $[0:1]$. In the chart centered at $[0:1]$, an integral curve $\sigma(t) = (\zeta_\infty(t), v_\infty(t))$ of G satisfies

$$\exp\left(-\frac{1}{\zeta_\infty(t)}\right)\zeta_\infty(t)v_\infty(t) \equiv c_0 \quad \text{and} \quad \zeta_\infty' = c_0\zeta_\infty e^{1/\zeta_\infty};$$

we leave to the reader an analysis similar to the one we did in case (1_{11}).

Case ($\mathbf{2_{10\rho}}$). The field

$$Q(z,w) = -\rho z^2\frac{\partial}{\partial z} + (1-\rho)zw\frac{\partial}{\partial w}$$

with $\rho \neq 0$ has two characteristic directions: $[1:0]$ is non-degenerate whereas $[0:1]$ is degenerate. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0 v_0\frac{\partial}{\partial \zeta_0} - \rho v_0^2\frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is a Fuchsian singularity of order 1, residue $\rho \neq 0$, induced residue $\rho - 1$, and G is in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = -\zeta_\infty^2 v_\infty\frac{\partial}{\partial \zeta_\infty} + (1-\rho)\zeta_\infty v_\infty^2\frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is a Fuchsian singularity of order 2, residue $1-\rho$, induced residue $-1-\rho$, and G is in normal form (up to a change of sign in v_∞). The characteristic leaf $L_{(0,1)}$ is pointwise fixed, while the dynamics on the characteristic leaf $L_{(1,0)}$ is described by Lemma 5.4.

Since both singularities are already in normal form with vanishing resonant index, instead of using Theorem 4.6 we can rely directly on Proposition 8.4 and Corollary 8.5 to describe the behavior of the integral curves of Q. We have:

- if $\operatorname{Re}\rho < 0$ then almost all integral curves of Q converge to the origin tangent to $[1:0]$ both in forward and backward time; each complex line L_v that is not a characteristic leaf contains exactly one real line of initial values of exceptional integral curves, which are converging to the origin tangent to $[1:0]$ on one side and diverging to infinity toward $L_{(0,1)}$ on the other side;
- if $\operatorname{Re}\rho > 0$ then the roles of $[1:0]$ and $[0:1]$ are exchanged;
- if $\operatorname{Re}\rho = 0$ then almost all integral curves of Q converge to the origin both in forward and backward time without being tangent to any direction; each complex line L_v that is not a characteristic leaf contains exactly one real line of initial values of exceptional integral curves, which cover closed geodesics and are converging to the origin on one side and diverging to infinity on the other side.

Case ($\mathbf{2_{11\rho}}$). The field

$$Q(z,w) = \left(-\rho z^2 + zw\right)\frac{\partial}{\partial z} + \left((1-\rho)zw + w^2\right)\frac{\partial}{\partial w}$$

with $\rho \neq 0$ has two characteristic directions, both non-degenerate: $[1:0]$ and $[0:1]$. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0 v_0\frac{\partial}{\partial \zeta_0} - (\rho - \zeta_0)v_0^2\frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is a Fuchsian singularity of order 1, residue $\rho \neq 0$, induced residue $\rho - 1$, and G is not in normal form (unless $\rho = -1$, when G is in normal form with resonant index 1). In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = -\zeta_\infty^2 v_\infty \frac{\partial}{\partial \zeta_\infty} + \bigl(1 + (1-\rho)\zeta_\infty\bigr) v_\infty^2 \frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is an irregular singularity of order 2, irregularity 2, residue (and resonant index) $1-\rho$, induced residue $-1-\rho$, and G is in normal form (up to a change of sign in v_∞). The dynamics on the characteristic leaves $L_{(1,0)}$ and $L_{(0,1)}$ is described by Lemma 5.4.

Theorem 4.6 says that we can have closed geodesics only if $\operatorname{Re}\rho = 0$, and that if $\operatorname{Re}\rho \notin (-1/2, 1/2)$ then no geodesic is self-intersecting; so if $\operatorname{Re}\rho \notin (-1/2, 1/2)$ then necessarily all geodesics are saddle connections.

If $\operatorname{Re}\rho > 0$, Corollary 8.5(ii) applies, and we see that for almost all integral curves γ of Q the direction $[\gamma]$ is escaping from $[1:0]$ (the exceptional curves are escaping to infinity toward $[1:0]$). In particular, if $\operatorname{Re}\rho > 1/2$ it follows that for almost all integral curves γ the direction $[\gamma]$ is going to $[0:1]$ both in forward and backward time.

If $\operatorname{Re}\rho < 0$ and $\rho \neq -1$ then Corollary 8.5(v)(a) applies, and almost all integral curves whose direction starts close enough to $[1:0]$ converge to the origin tangent to $[1:0]$ (and if the conjecture mentioned in Remark 8.2 is true than this holds for $\rho = -1$ too).

If $\operatorname{Re}\rho = 0$ then Corollary 8.5(iii) applies, and we have integral curves going to the origin or escaping to infinity without being tangent to any direction.

To complete the picture of this case, one needs to understand what happens nearby the irregular singularity. We sketch the approach suggested in Remark 1.7. The horizontal foliation is given by

$$e^{-1/\zeta_\infty} \zeta_\infty^{1-\rho} v_\infty \equiv c_0;$$

and a geodesic $\zeta_\infty(t)$ must satisfy

$$\zeta_\infty' = -c_0 \zeta_\infty^{\rho+1} \exp(1/\zeta_\infty).$$

Separating the variables one gets

$$\Gamma\left(\rho, \frac{1}{\zeta_\infty(t)}\right) = -(c_0 t + c_1),$$

where $\Gamma(\rho, w)$ is the incomplete Gamma function. So one is left with studying the behavior of the inverse of the incomplete Gamma function.

Case ($\mathbf{3_{100}}$). The field

$$Q(z,w) = \bigl(z^2 - zw\bigr)\frac{\partial}{\partial z}$$

has three characteristic directions: $[1:0]$ is non-degenerate, whereas $[1:1]$ and $[0:1]$ are degenerate. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0(\zeta_0 - 1) v_0 \frac{\partial}{\partial \zeta_0} - (\zeta_0 - 1) v_0^2 \frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is a Fuchsian singularity of order 1, residue 1, induced residue 0, and G is not in normal form; on the other hand, $[1:1]$ is an apparent singularity of order 1, residue 0, induced residue -1, and G is not in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = \zeta_\infty(\zeta_\infty - 1)v_\infty \frac{\partial}{\partial \zeta_\infty};$$

thus $[0:1]$ is an apparent singularity of order 1, residue 0, induced residue -1, and G is not in normal form. The characteristic leaves $L_{(1,1)}$ and $L_{(0,1)}$ are pointwise fixed, while the dynamics on the characteristic leaf $L_{(1,0)}$ is described by Lemma 5.4.

Putting together Theorem 4.3, Corollary 7.2 and Corollary 8.5(ii) we see that almost all integral curves go from a non-zero element of $L_{(1,1)}$ to a non-zero element of $L_{(0,1)}$; the exceptions are periodic integral curves surrounding $L_{(1,1)}$, and integral curves diverging to infinity toward $[1:0]$. This description is confirmed by the explicit expression of the integral curve of Q issuing from $(z_0, w_0) \notin L_{(1,0)} \cup L_{(1,1)} \cup L_{(0,1)}$ given by

$$\gamma(t) = \left(\frac{z_0 w_0}{z_0 - (z_0 - w_0)e^{w_0 t}}, w_0 \right).$$

Case ($3_{\rho 10}$). The field

$$Q(z,w) = \rho\left(-z^2 + zw\right)\frac{\partial}{\partial z} + (1-\rho)\left(zw - w^2\right)\frac{\partial}{\partial w}$$

with $\rho \neq 0, 1$ has three characteristic directions: $[1:0]$ and $[0:1]$ are non-degenerate, whereas $[1:1]$ is degenerate. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G = \zeta_0(1-\zeta_0)v_0\frac{\partial}{\partial \zeta_0} - \rho(1-\zeta_0)v_0^2\frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is a Fuchsian singularity of order 1, residue ρ, induced residue $\rho - 1$, and G is not in normal form; on the other hand, $[1:1]$ is an apparent singularity of order 1, residue 0, induced residue -1, and G is not in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G = \zeta_\infty(1-\zeta_\infty)v_\infty\frac{\partial}{\partial \zeta_\infty} - (1-\rho)(1-\zeta_\infty)v_\infty^2\frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is a Fuchsian singularity of order 1, residue $1-\rho$, induced residue $-\rho$, and G is not in normal form. The characteristic leaf $L_{(1,1)}$ is pointwise fixed, while the dynamics on the characteristic leaves $L_{(1,0)}$ and $L_{(0,1)}$ is described by Lemma 5.4.

If $\operatorname{Re}\rho \in (0,1)$ then Corollary 7.2 and Corollary 8.5(ii) say that almost all integral curves connect non-zero elements of $L_{(1,1)}$; the exceptions diverge to infinity toward $[1:0]$ or $[0:1]$, or are periodic integral curves around $L_{(1,1)}$. The local dynamics around the singularities allows to exclude geodesics accumulating closed geodesics or cycles of saddle connections; there might exist geodesics self-intersecting infinitely many times, however.

If $\operatorname{Re}\rho < 0$ (respectively, $\operatorname{Re}\rho > 1$) then $[1:0]$ (respectively, $[0:1]$) becomes attracting. Again, the local dynamics around the singularities allows to exclude geodesics accumulating closed geodesics or cycles of saddle connections. If $\operatorname{Re}\rho \leqslant -1/2$ (respectively, $\operatorname{Re}\rho \geqslant 3/2$) then we cannot have geodesics self-intersecting infinitely many times. Therefore almost all integral curves either converge to O tangentially to $[1:0]$ (respectively, to $[0:1]$) or converge to a non-zero element of $L_{(1,1)}$; the exceptions either diverge to infinity toward $[0:1]$ (respectively, $[1:0]$) or are periodic integral curves around $L_{(1,1)}$. If instead $\operatorname{Re}\rho \in (-1/2, 0)$ (respectively, $\operatorname{Re}\rho \in (1, 3/2)$) then we have the same description, but there might exist geodesics self-intersecting infinitely many times. But every simple loop in such a geodesic must surround $[1:0]$ (respectively, $[0:1]$), must have the same external angle, and cannot get too close to $[1:0]$ or $[1:1]$, and so the existence of such a geodesic seems unlikely.

Case $(\mathbf{3}_{\rho\tau\mathbf{1}})$. The field

$$Q(z,w)=\left(-\rho z^2+(1-\tau)zw\right)\frac{\partial}{\partial z}+\left((1-\rho)zw-\tau w^2\right)\frac{\partial}{\partial w}$$

with $\rho,\ \tau\neq 0$ and $\rho+\tau\neq 1$ has three characteristic directions: $[1:0]$ and $[0:1]$ and $[1:1]$, all non-degenerate. In the canonical coordinates (ζ_0, v_0) centered at $[1:0]$ the geodesic field G is given by

$$G=\zeta_0(1-\zeta_0)v_0\frac{\partial}{\partial\zeta_0}-\left(\rho-(1-\tau)\zeta_0\right)v_0^2\frac{\partial}{\partial v_0};$$

therefore $[1:0]$ is a Fuchsian singularity of order 1, residue ρ, induced residue $\rho-1$, and G is not in normal form. Also $[1:1]$ is a Fuchsian singularity of order 1, residue $1-\rho-\tau$, induced residue $-\rho-\tau$, and G is not in normal form. In the canonical coordinates (ζ_∞, v_∞) centered at $[0:1]$ the geodesic field G is given by

$$G=\zeta_\infty(1-\zeta_\infty)v_\infty\frac{\partial}{\partial\zeta_\infty}-\left(\tau-(1-\rho)\zeta_\infty\right)v_\infty^2\frac{\partial}{\partial v_\infty};$$

thus $[0:1]$ is a Fuchsian singularity of order 1, residue τ, induced residue $\tau-1$, and G is not in normal form. The dynamics on the characteristic leaves $L_{(1,0)}$, $L_{(1,1)}$ and $L_{(0,1)}$ is described by Lemma 5.4.

Thanks to the symmetry of the situation, and recalling that the sum of the residues is 1, it suffices to consider the following cases:

(a) $\operatorname{Re}\rho, \operatorname{Re}\tau>0$ and $\operatorname{Re}\rho+\operatorname{Re}\tau<1$ (three residues with positive real part). In this case Theorem 4.6 and Corollary 8.5(ii) implies that almost all geodesics intersect themselves infinitely many times (as in Example 8.3); the exceptions are saddle connections corresponding to integral curves escaping to infinity in both forward and backward time.

(b) $\operatorname{Re}\rho<0$, $\operatorname{Re}\tau>0$ and $\operatorname{Re}\rho+\operatorname{Re}\tau<1$ (two residues with positive real part, one residue with negative real part). Then $[1:0]$ is attracting; this means that we have at least an open set of initial conditions whose integral curves converge to the origin tangent to $[1:0]$. If no induced residue belongs to $(-3/2,-1/2)$, that is $\operatorname{Re}\rho\leqslant-1/2$, $\operatorname{Re}\tau\geqslant 1/2$ and $\operatorname{Re}\rho+\operatorname{Re}\tau\leqslant 1/2$, then Theorem 4.6 implies that almost all integral curves converge to the origin tangent to $[1:0]$; the exceptions diverge to infinity toward $[1:1]$ or $[1:0]$, and thus in this case Corollary 8.6 yields a complete description of the dynamics. If instead there is at least one induced residue belonging to $(-3/2,-1/2)$ there might also be geodesics intersecting themselves infinitely many times.

(c) $\operatorname{Re}\rho, \operatorname{Re}\tau<0$ (two residues with negative real part). A similar description applies to this case. The only differences are: we have two attracting characteristic directions, $[1:0]$ and $[0:1]$; the condition excluding induced residues belonging to $(-3/2,-1/2)$ is $\operatorname{Re}\rho$, $\operatorname{Re}\tau\leqslant-1/2$; and the exceptional integral curves diverge to infinity toward $[1:1]$.

(d) $\operatorname{Re}\rho=0$, $\operatorname{Re}\tau\in(0,1)$ (one purely imaginary residue, two residues with positive real part). In this case we always have exactly one induced residue whose real part belongs to $(-3/2,-1)\cup(-1,-1/2)$ except when $\operatorname{Re}\tau=1/2$. So Theorem 4.6 and Corollary 8.5 say that we have integral curves converging to the origin without being tangent to any direction, and some exceptional integral curve diverging to infinity toward $[1:1]$ or $[0:1]$ or without being tangent to any direction; and we might have integral curves corresponding to geodesics intersecting themselves infinitely many times (this case is excluded if $\operatorname{Re}\tau=1/2$).

(e) $\operatorname{Re}\rho=0$, $\operatorname{Re}\tau>1$ (one purely imaginary residue, one residue with positive real part, one residue with negative real part). In this case we have integral curves converging to the origin tangent to $[1:1]$ or without being tangent to any direction, some exceptional integral curve diverging to infinity toward $[0:1]$ or without being tangent to any direction; and, if $1 < \operatorname{Re}\tau < 3/2$, we might have integral curves corresponding to geodesics intersecting themselves infinitely many times.

(f) $\operatorname{Re}\rho = 0$, $\operatorname{Re}\tau = 1$ (two purely imaginary residues). Finally, in this case almost all integral curves converge to the origin without being tangent to any direction; the exceptional integral curves diverge to infinity either toward $[0:1]$ or without being tangent to any direction.

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