Introduction

Let X be a complex manifold, and f a holomorphic map of X into itself. A natural object associated to f is the sequence $\{f^k\}$ of iterates of f, where f^k is the composition of f with itself k times. In particular, the asymptotic behavior of the sequence as k goes to infinity deserves investigation, both for its own sake and for the information we can infer on f itself.

The investigation of this subject began with the works of Schröder [1870, 1871] and Kœnigs [1883]. They were mainly interested in the local situation for, of course, holomorphic functions of one variable. Let z_0 be a point of the complex plane $\mathbf{C}$, and f a holomorphic function defined in a neighbourhood of z_0 such that $f(z_0) = z_0$. Then the behavior of the sequence of iterates of f near z_0 depends on the value of the derivative of f at z_0. More specifically, if $|f'(z_0)| < 1$ every point z sufficiently close to z_0 is attracted by z_0 (i.e., $f^k(z) \to z_0$ as $k \to +\infty$); if $|f'(z_0)| > 1$, the points are repelled away from z_0 — or, if you prefer, they are attracted by z_0 under the action of f^{-1}, which is defined in a neighbourhood of z_0. Finally, if $|f'(z_0)| = 1$, the behavior of $\{f^k\}$ is cyclic, with a finite period if $f'(z_0)$ is a root of unity.

The first really deep work on the global situation is Julia [1918]. He investigated the iteration of rational functions defined on the extended complex plane $\widehat{\mathbf{C}}$, and discovered that the global behavior of the sequence of iterates was both complicated and fascinating. Near fixed points it is possible to adapt and clarify the local description, but new phenomena arise, linked for instance to the distribution of fixed points of higher order (that is, fixed points of f^k with $k > 1$). A main problem was the description of the *Julia set* of f, that is of the set of points $z_0 \in \widehat{\mathbf{C}}$ such that the sequence of iterates $\{f^k\}$ is equicontinuous in no neighbourhood of z_0. The idea is that if $\{f^k\}$ is equicontinuous in a neighbourhood of z_0, then there is a subsequence $\{f^{k_\nu}\}$ converging uniformly near z_0, and then the behavior of the sequence of iterates is somehow under control. In other words, the Julia set is in some sense the singular set for the asymptotic behavior of $\{f^k\}$.

Fatou [1919, 1920a, b, 1926] extended and deepened Julia's work, also investigating the iteration of entire functions. Again, a main role is played by the Julia set, defined replacing the notion of equicontinuity by the notion of normality.

After Fatou, the iteration theory of rational and entire functions momentarily lost its impetus. Besides the works of Baker [1955, 1958, 1959, 1960], Siegel [1942] and Töpfer [1949], mainly devoted to the study of fixed points of higher order using Nevanlinna's distribution value theory, and Brolin [1965], devoted to a deep investigation of the iteration of polynomials of low degree, nothing really new appeared. Only recently, the work of Sullivan, Hubbard, Douady and others has shed a completely new light on the argument, showing its deep relationship with quasi-conformal mappings and Mandelbrot's theory of fractals. Two recent surveys showing (also pictorially) what is going on are Blanchard [1984] and Peitgen and Richter [1986].

But iteration theory has another, totally different, aspect. As we mentioned before, investigating the iteration of a rational or entire function f a main problem is the structure of the set of points where the sequence $\{f^k\}$ is not normal. But if we study the iteration of a holomorphic function defined in a smaller domain like, for instance, the unit disk Δ of $\mathbf{C}$, this problem disappears: the whole family of holomorphic functions of Δ into itself is normal. This fact (quite non trivial for domains more general than Δ) completely changes the situation. In fact, by normality, the function f can be naturally embedded in a relatively compact set (of a suitable space of functions) depending only on f, namely the sequence of iterates $\{f^k\}$, and, as always, the compactness has strong consequences on the structure of f. For instance, if f is a holomorphic function sending a bounded but not necessarily simply connected domain into itself, and z_0 is a fixed point of f, then $|f'(z_0)| \leq 1$, $|f'(z_0)| = 1$ iff f is an automorphism, and $f'(z_0) = 1$ iff f is the identity. In fact, the idea is that the sequence of iterates should have a converging subsequence, and therefore the coefficients of the Taylor expansion of f^k at z_0 (which essentially are powers of the coefficients of the Taylor expansion of f) cannot tend to infinity as $k \to +\infty$. It should be remarked that the strength of this approach was completely revealed only after its application (due to Carathéodory [1932] and H. Cartan [1930b, 1932]) to the theory of holomorphic maps of several complex variables, probably because in one variable it was somehow concealed by Schwarz's lemma.

However, there is still the problem of describing the asymptotic behavior of the sequence of iterates. As already remarked by Julia [1918], if f is a holomorphic function of Δ into itself with a fixed point, that we can assume to be the origin, then the behavior of $\{f^k\}$ can be easily derived by Schwarz's lemma: if $|f'(0)| < 1$ then 0 is attractive, and if $|f'(0)| = 1$ then f is a rotation about the origin. If f has no fixed points in Δ, the problem was solved by Wolff [1926a, b, c] and Denjoy [1926]. Let $x \in \partial\Delta$; then as $z \in \Delta$ tends to x, the Poincaré disks of center z and fixed euclidean radius tend to a *horocycle* at x, that is to an euclidean disk internally tangent to $\partial\Delta$ at x. Then Wolff proved a sort of Schwarz's lemma for holomorphic functions without fixed points, using the horocycles: if f sends Δ into itself without fixed points, then there exists a unique point $x \in \partial\Delta$ (called the *Wolff point* of f) such that f sends every horocycle at x into itself. It is then an easy matter to prove that the sequence of iterates $\{f^k\}$ converges, uniformly on compact sets, to the constant map sending all Δ in x; this is the Wolff-Denjoy theorem.

For a multiply connected domain D different from $\mathbf{C}^*$ and, more generally, for multiply connected hyperbolic Riemann surfaces*, the asymptotic behavior of the sequence of iterates was described by Heins [1941a, 1988]. If f has a fixed point, the local picture forces the global one, exactly as in Δ; if f has no fixed points then the sequence of iterates tends to the boundary. In particular, if the boundary of D is sufficiently smooth, then either the sequence of iterates converges, uniformly on compact sets, to a constant map $x \in \overline{D}$ or f is an automorphism.

In several variables, the natural setting for this kind of iteration theory is the class of taut manifolds, a concept introduced by Wu [1967]. A complex manifold X is taut if the family of holomorphic maps of Δ into X is normal; for instance, a strongly pseudoconvex domain is taut, as well as any bounded convex domain. If X is a taut manifold, then the family of holomorphic maps of X into itself is normal, and hence we can investigate the

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* Rådstrom [1953] and Keen [1988] studied the iteration of holomorphic functions in $\mathbf{C}^*$, miming Julia and Fatou. I do not know any paper on iteration theory of holomorphic functions defined on a torus.

iteration theory there.

As we already mentioned, if f is a holomorphic map of a taut manifold into itself, we can study f imbedding it into its sequence of iterates. In this way Carathéodory [1932] and H. Cartan [1930b, 1932] developed the classical theory of holomorphic mappings in bounded domains of $\mathbf{C}^n$, later generalized by Wu [1967] to taut manifolds, and by Kobayashi [1967a] to even more general manifolds. Furthermore, using H. Cartan's ideas, (Hervé [1951] for taut domains in $\mathbf{C}^2$ and) Bedford [1983b] characterized the maps of X into X that are limit of a subsequence $\{f^{k_\nu}\}$ of iterates of a holomorphic map f of X into itself. It turns out that the role played by the constant functions in one variable is now played by the *holomorphic retractions*, that is by the holomorphic maps ρ of X into itself such that $\rho^2 = \rho$. Then, as already observed by Hervé [1951] for taut domains in $\mathbf{C}^2$, the sequence of iterates of a holomorphic map f defined on a taut manifold X converges to a holomorphic map ρ of X into itself iff f has a fixed point z_0 such that the spectrum of the differential of f at z_0 is contained in $\Delta \cup \{1\}$; moreover, ρ is a holomorphic retraction.

If X actually is a bounded domain D of $\mathbf{C}^n$, we can ask something more: is it possible that the sequence of iterates converge to a map sending D into its boundary? In other words, is it possible to generalize the Wolff-Denjoy theorem to several variables? Clearly, the first thing we need is a substitute for the horocycles. In the unit ball B^n of $\mathbf{C}^n$ we can use the Bergmann metric to define the horospheres at a point of ∂B^n exactly as we did with the Poincaré metric in Δ. Then we can practically repeat the proof of the Wolff-Denjoy theorem and show that if f is a holomorphic map of B^n into itself without fixed points then the sequence of iterates $\{f^k\}$ converges, uniformly on compact sets, to a point of ∂B^n, as first shown by Hervé [1963a].

For a more general bounded domain D, the situation is not so clear. The Bergmann metric in B^n was the right metric to use because it satisfies a Schwarz lemma, i.e., it is contracted by holomorphic maps; in other domains, the Bergmann metric does not have this property. On the other hand, Carathéodory [1926] before and Kobayashi [1967a, b] after have defined two (pseudo)distances on any complex manifold which are, practically by definition, contracted by holomorphic maps. Using the Kobayashi distance, then, it is possible to define a notion of horosphere in any bounded domain of $\mathbf{C}^n$, and it turns out that this is the generalization we need. In particular, we can prove that if f is a holomorphic map of a bounded strongly convex domain into itself without fixed points, then the sequence of iterates of f converges, uniformly on compact sets, to a point of the boundary. The restriction to strongly convex domains is essential: the theorem is false for non-convex strongly pseudoconvex domains.

The iteration theory on hyperbolic Riemann surfaces and, more generally, on taut manifolds is strongly linked to two other topics. First of all, as we have already remarked, the asymptotic behavior of the sequence of iterates of a map f is strongly affected by the structure of the fixed point set of f. The influence is reciprocal: iteration theory provides several effective tools for the construction of fixed points. For instance, let $\mathcal{F}$ be a family of continuous functions of $\overline{\Delta}$ into itself holomorphic in Δ such that $f \circ g = g \circ f$ for every $f,\, g \in \mathcal{F}$; then Shields [1964], using iteration theory, proved that there always exists a common fixed point of $\mathcal{F}$, that is a point $z_0 \in \overline{\Delta}$ such that $f(z_0) = z_0$ for all $f \in \mathcal{F}$. Suffridge [1974] extended Shields' theorem to B^n, and we shall present a proof valid in strongly convex domains.

The second topic is the study of the so-called angular derivative. Another application of the horocycles in Δ, mainly developed by Julia [1920], Wolff [1926d] and Carathéodory [1929], is the study of the behavior of the ratio $\big(f(z)-y\big)/\big(z-x\big)$ as $z \to x \in \partial\Delta$, where f is a holomorphic function sending Δ into itself, and $y \in \partial\Delta$. The Julia-Wolff-Carathéodory theorem states that, under some mild hypotheses on f and x, there exists a unique point $y \in \partial\Delta$ such that the aforementioned ratio converges non-tangentially to a finite limit $L \in \mathbf{C}$, and, furthermore, the derivative $f'(z)$ tends to L as $z \to x$ within any angular region of vertex x contained in Δ; for this reason L is called the *angular derivative* of f at x. Besides its own interest, this theorem has many important applications to function theory of hyperbolic Riemann surfaces, and some of them concern iteration theory.

The corresponding problem in several variables is quite complicated, and reveals new interesting features concerning the behavior of the differential of f along complex tangential directions. Hervé [1963a] and Rudin [1980] investigated the situation in the unit ball of $\mathbf{C}^n$, and we shall present a generalization to strongly convex domains; in both cases, the main tool in the proofs is provided by the horospheres, exactly as in iteration theory.

Some comments and remarks about the structure of this work are in order. I have written this book keeping in mind two different goals (and audiences). First of all, this is intended as a reference book on iteration theory on taut manifolds and related topics. During my own investigations, I found many beatiful theorems never appeared in book form (Heins' theorem on iteration theory of hyperbolic Riemann surfaces, or the Lempert-Royden-Wong theory on complex geodesics, just to quote two of them); furthermore, the whole theory seemed to me requiring a comprehensive exposition collecting several results scattered around in the literature. Of course, one could not discuss every single topic in only one book; so I decided to concentrate the exposition around three theorems, the Wolff-Denjoy, Shields', and the Julia-Wolff-Carathéodory theorems, following them from the origin to their most recent generalizations. This allowed an unified exposition of the main results, and a clearer discussion of the threads connecting the three principal themes of this book, iteration theory, common fixed points and angular derivatives.

So, the first audience of this book is mainly composed by research workers in geometric function theory; they will find an up-to-date description of the field, open problems to solve, and even references to several topics not discussed here. But, as already anticipated, I also had another goal in mind. This book would like to be an introduction to geometric function theory in several complex variables for, say, first-year graduate students (or, in Italy, for good fourth-year university students too), giving them both a first sample of typical features and techniques of the several variables theory, and strong (and aestethical) motivations provided by the classical one variable theory. Mainly for this reason I devoted the first part of the book to a comprehensive discussion of one variable theory, starting from Schwarz's lemma; besides its own beauty, iteration theory on Δ is the model inspiring the whole development of the theory in several variables.

I tried to keep prerequisites down; besides a good knowledge of function theory of one complex variable and a first acquaintance with function theory of several complex variables, only a good topological background (up to covering spaces and fundamental group) and the basic notions of differential geometry, measure theory and functional analysis are necessarily required. Here and there we shall use some specialized result taken from other fields (like differential equations, or algebraic topology); however, we shall always quote explicitely the needed theorem, and provide a reference to a place where a proof can be found. Furthermore, I scattered in the book advices to the novice reader, warning him of particularly dangerous situations. Here is the first one: the last two chapters are considerably more technical than the others, and probably require a working knowledge of complex analysis in one and several variables to be fully understood (and appreciated).

Let us now briefly describe the actual content of this book; more details can be found in the introductions to each chapter.

The first part is devoted to the one-variable theory. Chapter 1.1 contains the results of complex analysis that are particularly important for the understanding of the theory: Schwarz's lemma, the Poincaré metric and the structure of the automorphism group of Δ; Montel's, Vitali's and Picard's theorems; the classification of Riemann surfaces and the boundary behavior of the universal covering map of multiply connected hyperbolic domains.

In chapter 1.2 we introduce the horocycles in Δ and their main properties, Julia's and Wolff's lemmas. We use them to study the angular derivative of a holomorphic function of Δ into itself, proving the Julia-Wolff-Carathéodory theorem, and then to investigate the structure of the automorphism group of a hyperbolic Riemann surface.

Chapter 1.3 is devoted to iteration theory. We describe the theory of holomorphic functions with a fixed point; we present two proofs of the Wolff-Denjoy theorem; we develop the iteration theory on hyperbolic Riemann surfaces, and its version in finitely connected hyperbolic domains; finally, we prove Shields' theorem and other facts about fixed points.

In chapter 1.4 we investigate the one-parameter semigroups of holomorphic functions on a Riemann surface, a natural generalization of the concept of sequence of iterates. Following Berkson and Porta [1978] and Heins [1981], we present a characterization of one-parameter semigroups in Δ, their interpretation in terms of translations of domains of a particular kind, and the classification on Riemann surfaces other than the disk.

The second part is devoted to the several variables theory. In chapter 2.1 we officially meet taut manifolds, and we describe the basic theory of holomorphic maps on them; in particular, we characterize the maps arising as a limit of a subsequence of iterates, a main tool for iteration theory in several variables.

In chapter 2.2 we study the unit ball B^n of $\mathbf{C}^n$. We introduce the Bergmann metric, the horospheres and the generalizations of Schwarz's, Julia's and Wolff's lemmas; we study the angular derivatives of a holomorphic map of B^n into itself; we prove the Wolff-Denjoy theorem in B^n, and Suffridge's version of Shields' theorem.

Chapter 2.3 is devoted to the invariant distances, metrics and measures introduced by Carathéodory and Kobayashi, which are the main tools needed to develop our three main themes in domains other than B^n. Besides their basic properties, we describe the behavior of the Kobayashi distance and metric at the boundary of a strongly pseudoconvex domain, we prove a weak form of Fefferman's theorem, and a characterization of B^n due to Graham and Wu [1985a].

In chapter 2.4 we deal with iteration theory in several variables. We study the general situation in taut manifolds; we define the horospheres in general domains and we describe their main properties; finally, we investigate iteration theory in bounded convex domains of $\mathbf{C}^n$.

In chapter 2.5 we investigate fixed point sets. We prove a characterization of the ball in terms of its automorphism group due to Wong [1977] and Rosay [1979], and Lempert's theorem about fixed points of compact groups of automorphisms of a convex domain; we describe the structure of the fixed point set of a holomorphic map of a convex domain into itself, generalizing Shields' theorem to strongly convex domains, and we end with one-parameter semigroups on taut manifolds.

Chapter 2.6 is devoted to complex geodesics. After their basic properties, we present the Lempert-Royden-Wong theory of complex geodesics in convex domains, proving the facts we shall need to extend the Julia-Wolff-Carathéodory theorem to strongly convex domains.

Finally, in chapter 2.7 we deal with angular derivatives. After the introduction of suitable boundary approach regions, we shall discuss a version of Lindelöf's theorem in strongly convex domains, and we shall end the book with the last generalization of the Julia-Wolff-Carathéodory theorem.

Each chapter is equipped with notes, containing history, comments, remarks, indications of related topics and references to the bibliography. I tried to sistematically trace who did what when; I shall appreciate any comment regarding missing (or wrong. . .) references.

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