8.5.2012

Es 1 $\int_0^1 \frac{1}{2x+1}\, dx$

$y = \varphi(x) = 2x+1$, $dy = \varphi'(x)\, dx = 2\, dx \Rightarrow dx = \frac{1}{2} dy$

$x = 0 \Rightarrow y = 1$, $x = 1 \Rightarrow y = 3$

$$\int_0^1 \frac{1}{2x+1}\, dx = \int_1^3 \frac{1}{y} \cdot \frac{1}{2}\, dy = \frac{1}{2}\int_1^3 \frac{1}{y}\, dy = \frac{1}{2}\left[\log|y|\right]_1^3$$

$$= \frac{1}{2}(\log 3 - \log 1) = \frac{1}{2}\log 3$$

Es 2 $\int_0^1 \sqrt{1-x^2}\, dx$

$x = \cos t \Rightarrow dx = -\sin t\, dt$

$x = 0$ per $t = \frac{\pi}{2}$, $x = 1$ per $t = 0$

$$\int_0^1 \sqrt{1-x^2}\, dx = \int_{\frac{\pi}{2}}^0 \sqrt{1-\cos^2 t}\,(-\sin t)\, dt =$$

$$= \int_0^{\pi/2} \sqrt{\sin^2 t}\, \sin t\, dt = \int_0^{\pi/2} \sin^2 t\, dt$$

Calcoliamo l'ultimo integrale:

$$\int_0^{\pi/2} \sin^2 t\, dt = \int_0^{\pi/2} \sin t \cdot \sin t\, dt \overset{\text{per parti}}{=} \left[-\cos t \cdot \sin t\right]_0^{\pi/2} - \int_0^{\pi/2} (-\cos t)\cos t\, dt$$

$$= -\cos\frac{\pi}{2}\sin\frac{\pi}{2} + \cos 0 \sin 0 + \int_0^{\pi/2} \cos^2 t\, dt = \int_0^{\pi/2} \cos^2 t\, dt$$

$$= \int_0^{\pi/2} (1 - \sin^2 t)\, dt = \frac{\pi}{2} - \int_0^{\pi/2} \sin^2 t\, dt$$

$$\Longrightarrow 2\int_0^{\pi/2} \sin^2 t\, dt = \frac{\pi}{2} \Rightarrow \int_0^{\pi/2} \sin^2 t\, dt = \frac{\pi}{4}$$

$$\Rightarrow \int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$$

Es 3 Determinare $\int x e^x \, dx$

Per parti: $\int x e^x = x e^x - \int 1 \cdot e^x \, dx = x e^x - e^x + c$

Es 4 Determinare $\int x^\alpha \log x \, dx$ con $\alpha \neq -1$, $x > 0$

Per parti: $\int x^\alpha \log x \, dx = \frac{1}{\alpha+1} x^{\alpha+1} \log x - \int \frac{1}{\alpha+1} x^{\alpha+1} \cdot \frac{1}{x} \, dx$

$$= \frac{1}{\alpha+1} x^{\alpha+1} \log x - \frac{1}{\alpha+1} \int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} \log x - \frac{x^{\alpha+1}}{(\alpha+1)^2} + c$$

$$= \frac{x^{\alpha+1}}{\alpha+1} \left(\log x - \frac{1}{\alpha+1} \right) + c$$

Es 4 bis Stesso integrale con $\alpha = -1$, $x > 0$

$$\int \frac{1}{x} \log x \, dx = \log x \cdot \log x - \int \log x \cdot \frac{1}{x} \, dx$$

$$\Rightarrow \quad 2 \int \frac{1}{x} \log x \, dx = \log^2 x$$

$$\Rightarrow \quad \int \frac{1}{x} \log x \, dx = \frac{1}{2} \log^2 x + c$$

Es 5 $\int \arcsin x \, dx$

Per parti: $\int \arcsin x \, dx = \int 1 \cdot \arcsin x \, dx =$

$$= x \arcsin x - \int x \cdot \frac{1}{\sqrt{1-x^2}} \, dx =$$

$$= x \arcsin x + \int \frac{-2x}{2\sqrt{1-x^2}} \, dx =$$

$$= x \arcsin x + \int \frac{d}{dx}\left(\sqrt{1-x^2}\right) dx$$

$$= x \arcsin x + \sqrt{1-x^2} + c$$

Qui si è usata la regola

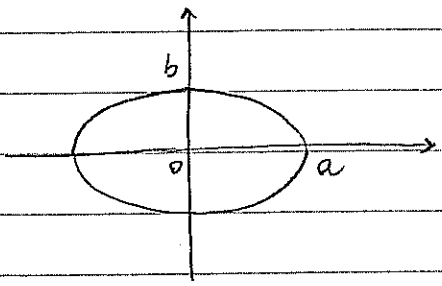
$$\boxed{\int f(x)^{\alpha} f'(x)\, dx = \frac{1}{\alpha+1} f(x)^{\alpha+1} + c \, , \quad \forall \alpha \neq -1}$$

che si verifica derivando il secondo membro.
Per $\alpha = -1$ si ha invece

$$\boxed{\int \frac{f'(x)}{f(x)}\, dx = \log |f(x)| + c}$$

Es 6 Determinare l'area dell'ellisse

$$E = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \right\} \, , \quad a, b > 0$$



Ricaviamo y da $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} \; ; \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \pm b \sqrt{1 - \frac{x^2}{a^2}} \; ; \quad |x| \leq a$$

Perciò $E = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; -a \leq x \leq a \, , \; -b\sqrt{1-\frac{x^2}{a^2}} \leq y \leq b\sqrt{1-\frac{x^2}{a^2}} \right\}$

da cui Area $E = 2\int_{-a}^{a} b \sqrt{1 - \frac{x^2}{a^2}}\, dx = 4b \int_0^a \sqrt{1 - \frac{x^2}{a^2}}\, dx$

Calcoliamo $\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx$ con la sostituzione

$x = a\sin t$: $0 = a\sin 0$, $a = a\sin\frac{\pi}{2}$

$dx = a\cos t\,dt$

$$\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = \int_{a\cdot\sin 0}^{a\cdot\sin\frac{\pi}{2}} \sqrt{1-\frac{x^2}{a^2}}\,dx = \int_0^{\pi/2} \sqrt{1-\sin^2 t}\; a\cos t\,dt$$

$$= \int_0^{\pi/2} a\cos^2 t\,dt$$

Determiniamo l'integrale indefinito $\int \cos^2 t\,dt$

$$\int \cos^2 t\,dt = \cos t\cdot\sin t - \int(-\sin t)\sin t\,dt =$$

$$= \sin t\cos t + \int \sin^2 t\,dt = \sin t\cos t + \int(1-\cos^2 t)\,dt$$

$$= \sin t\cos t + t - \int\cos^2 t\,dt$$

$$\Rightarrow \quad 2\int\cos^2 t\,dt = \sin t\cos t + t$$

$$\Rightarrow \int\cos^2 t\,dt = \frac{1}{2}\sin t\cos t + \frac{t}{2} \quad (+\,c)$$

Concludiamo che

$$\text{Area } E = 4b\int_0^a \sqrt{1-\frac{x^2}{a^2}}\,dx = 4ab\int_0^{\pi/2}\cos^2 t\,dt$$

$$= 4ab\left[\frac{1}{2}\sin t\cos t + \frac{t}{2}\right]_0^{\pi/2} = 4ab\cdot\frac{\pi}{4} = \pi ab\,.$$

7) Determinare $\int e^{\lambda x} \sin(\mu x)\, dx$

Integriamo due volte per parti:

$$\int e^{\lambda x} \sin(\mu x)\, dx = e^{\lambda x}\left(-\frac{1}{\mu}\cos(\mu x)\right) - \int \lambda e^{\lambda x}\left(-\frac{1}{\mu}\cos(\mu x)\right) dx$$

$$= -\frac{1}{\mu} e^{\lambda x}\cos(\mu x) + \frac{\lambda}{\mu}\int e^{\lambda x}\cos(\mu x)\, dx$$

$$= -\frac{1}{\mu} e^{\lambda x}\cos(\mu x) + \frac{\lambda}{\mu}\left(e^{\lambda x}\frac{1}{\mu}\sin(\mu x) - \int \lambda e^{\lambda x}\frac{1}{\mu}\sin(\mu x)\, dx\right.$$

$$= -\frac{1}{\mu} e^{\lambda x}\cos(\mu x) + \frac{\lambda}{\mu^2} e^{\lambda x}\sin(\mu x) - \frac{\lambda^2}{\mu^2}\int e^{\lambda x}\sin(\mu x)\, dx$$

$$\Longrightarrow \left(1 + \frac{\lambda^2}{\mu^2}\right)\int e^{\lambda x}\sin(\mu x)\, dx = -\frac{1}{\mu} e^{\lambda x}\cos(\mu x) + \frac{\lambda}{\mu^2} e^{\lambda x}\sin(\mu x)$$

e moltiplicando per μ^2:

$$(\mu^2 + \lambda^2)\int e^{\lambda x}\sin(\mu x)\, dx = -\mu e^{\lambda x}\cos(\mu x) + \lambda e^{\lambda x}\sin(\mu x)$$

$$\Longrightarrow \int e^{\lambda x}\sin(\mu x)\, dx = \frac{e^{\lambda x}}{\lambda^2+\mu^2}\left(\lambda\sin(\mu x) - \mu\cos(\mu x)\right) + c$$

8) Determinare $\int \frac{1}{x^2 - 1}\, dx$

$\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$ e dunque

$$\int \frac{1}{x^2-1}\, dx = \frac{1}{2}\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx = \frac{1}{2}\left(\log|x-1| - \log|x+1|\right) + c$$

$$= \frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + c$$