Ricerche di Matematica
Vol. LIV, fasc. 2°, (2005), 375-394 [version: March 26, 2006]

Distributional Jacobian and singularities of Sobolev maps

Giovanni Alberti (*)

Abstract. – We review the definition and main properties of the distributional Jacobians, in particular for Sobolev maps $u : \mathbb{R}^n \to \mathbb{R}^k$ with values in the $(k-1)$-dimensional sphere S^{k-1}.

Keywords: *Distributional determinant, Distributional Jacobian, Topological singularities, Sobolev maps, Rectifiability, Integral currents, Finite perimeter sets.*

A.M.S. Classification: 49Q15 (53C65, 58A25)

1. Introduction

This note contains an expanded version of the lecture I gave at the "Renato Caccioppoli Conference" (Naples, September 23-25, 2004).

Purpose of that lecture was to advertise the notion of distributional Jacobian for Sobolev maps $u : \mathbb{R}^n \to \mathbb{R}^k$ with $n \geq k \geq 2$, and in particular for maps with values in the $(k-1)$-dimensional sphere S^{k-1}. For these maps, the distributional Jacobian Ju is supported on the singular set (points of discontinuity) of u, and reflects its geometric structure. This result is reminiscent of a well-known fact about the derivatives of characteristic functions: the distributional gradient $D1_E$ of the characteristic function of a set E is supported on the topological boundary of E (the singular set of 1_E), and if

(*) Dipartimento di Matematica, Università di Pisa, L.go Pontecorvo 5, 56127 Pisa (Italy).
E-mail: `alberti@dm.unipi.it`.

the latter is sufficiently regular, then $D1_E$ is the product of the area measure on ∂E times the inner normal to ∂E. In a way, S^{k-1}-valued maps with bounded distributional Jacobian can be regarded as the vector-valued analogue of finite perimeter sets; this analogy will be the guideline for this presentation.

The notion of distributional Jacobian appears naturally in the study of variational problems for vector valued maps (and indeed, it has been discovered under different names by several authors). For maps $u : \mathbb{R}^k \to \mathbb{R}^k$, the distributional Jacobian is known as distributional determinant (see [5]), and has been widely studied in recent years in the context of semicontinuity problems for polyconvex integrands and their applications to nonlinear elasticity (see, e.g., [32], [20] and references therein). A similar notion was introduced in [10] to study the singularities of Sobolev and harmonic maps from the ball B^3 into the sphere S^2.

More recently, it has been observed that the distrbutional Jacobian is related to the topological structure of the classes of Sobolev maps between manifolds. For instance, the triviality of Ju is a necessary condition – and sometimes also a sufficient one – for the approximation of u by smooth maps (see [15], [35], [34], and references therein). For maps u with values in the circle S^1, the triviality of Ju is a necessary and sufficient condition for the existence of a lifting of u, that is, a real function θ in the same Sobolev class as u which satisfies $u = \exp(i\theta)$ (see [7], [8], [9]). Last but not least, the distributional Jacobian has been successfully used as a tool for tracking energy concentration of minimizers of functionals of Ginzburg-Landau type (cf. [29]), in particular in the variational approach proposed in [28], [3] (see [1] for an informal presentation of some of these results).

2. BV functions and finite perimeter sets

Finite perimeter sets provide a class of generalized boundaries which is large enough to have good compactness properties, hence fitting the framework of the direct method of the Calculus of Variations. The theory was first developed in the 50's by Caccioppoli [11] and De Giorgi [13], [14]. Few years later, the theory of integral currents developed by Federer and Fleming [19] provided a class of generalized (oriented) surfaces of arbitrary dimension and codimension with similar compactness properties.

In this section I just recall the basic definitions and results concerning the theory of BV functions and finite perimeter sets. For the sake of simplicity, I consider only functions defined on the euclidean space $\mathbb{R}^n$ (for more details, see [4], [16], [22]).

2.1 – Functions of bounded variation

According to the current definition (cf. [4], Sect. 3.1, [16], Sect. 5.1), the space $BV(\mathbb{R}^n)$ of functions of bounded variations on $\mathbb{R}^n$ consists of all $u \in L^1(\mathbb{R}^n)$ whose *distributional gradient* Du is (represented by) a bounded measure on $\mathbb{R}^n$ with values in $\mathbb{R}^n$.[1] This means that there exists a vector measure, denoted by Du, which satisfies

$$\int_{\mathbb{R}^n} \phi \cdot Du = -\int_{\mathbb{R}^n} u \,\mathrm{div}\phi \tag{2.1}$$

for every smooth vector-field $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with compact support.[2] The mass of the vector measure Du is then given by

$$\|Du\| = \int_{\mathbb{R}^n} |Du| = \sup_{|\phi|\le 1} \int_{\mathbb{R}^n} u \,\mathrm{div}\phi \;. \tag{2.2}$$

The space $BV(\mathbb{R}^n)$, endowed with the norm $\|u\|_{BV} := \|u\|_1 + \|Du\|$, is a non-separable Banach space.

2.2 – Remarks

(i) The relevance of the space $BV(\mathbb{R}^n)$ lies in the following compactness property, which allows to apply the direct method of the Calculus of Variations and prove the existence of minimizers in BV for a large class of variational problems: let (u_h) be a sequence of functions in $BV(\mathbb{R}^n)$ with uniformly bounded BV-norms and uniformly bounded supports; then, upon extraction of a subsequence, u_h converge in the L^1-norm to some $u \in BV(\mathbb{R}^n)$, while the derivatives Du_n converge to Du in the sense of measures.[3] The latter property implies in addition that $\|Du\| \le \liminf \|Du_h\|$.

[1] By 'measure' we mean a σ-additive measure on Borel sets. Every measure λ with values in $\mathbb{R}^n$ can be written as $\lambda = f\mu$ where μ is a positive measure and f is a vector valued function, that is, $\lambda(E) := \int_E f\,d\mu$ for every Borel set E. The function f is the Radon-Nikodym derivative of λ w.r.t. μ. There exists a unique positive measure μ such that $|f| = 1$ μ-a.e.; this measure is called total variation of λ, and denoted by $|\lambda|$.

[2] The integral at the left-hand side of (2.1) should be understood as $\int \phi \cdot f\,d\mu$, where $Du = f\mu$ is the decomposition described in footnote 1, and the dot $(\cdot)$ stands for the scalar product in $\mathbb{R}^n$. The measure in the integral at the right-hand side of (2.1) is the Lebesgue measure $\mathscr{L}^n$.

[3] The existence of a convergent subsequence follows by the fact that $BV(\Omega)$ embeds compactly in $L^1(\Omega)$ for every bounded open set Ω. The rest of the statement follows by the compactness of the unit closed ball of measures with respect to the weak* topology induced by the duality with continuous functions (Banach-Alaoglu theorem).

(ii) The space $BV(\mathbb{R}^n)$ can be equivalently defined as the space of all functions u in $L^1(\mathbb{R}^n)$ such that the supremum in (2.2) is finite (cf. [22], Definition 1.1; see also the historical note in [4], Sect. 3.12). Note that the space $BV(\mathbb{R})$ *does not agree* with the classical space of functions of bounded variation of one variable.

2.3 – Finite perimeter sets

A Borel set E contained in $\mathbb{R}^n$ is a set with *finite perimeter*, or a *Caccioppoli set*, if the characteristic function 1_E belongs to $BV(\mathbb{R}^n)$ (cf. [16], Sect. 5.2). We call *perimeter* of E the number

$$\mathrm{Per}(E) := \|D1_E\| = \int_{\mathbb{R}^n} |D1_E| \ .$$

2.4 – Why 'perimeter'?

The reason for calling perimeter the mass of the measure $D1_E$ can be understood by computing the distributional derivative $D1_E$ when E is a bounded set with smooth boundary. In this case, the divergence theorem states that for every vector-field ϕ on $\mathbb{R}^n$ there holds

$$\int_{\partial E} \phi \cdot \nu_E = \int_E \mathrm{div}\phi \tag{2.3}$$

where ν_E is the *inner normal* to the boundary of E, and the measure underlying the first integral is the area measure on ∂E, or, more precisely, the $(n-1)$-dimensional Hausdorff measure $\mathscr{H}^{n-1}$. In view of (2.1), formula (2.3) means that $D1_E$ is the restriction of the positive measure $\mathscr{H}^{n-1}$ to the set ∂E, multiplied by the vector density ν_E, that is,

$$D1_E = \nu_E \cdot \mathscr{H}^{n-1} \llcorner \partial E \ . \tag{2.4}$$

In particular $|D1_E|$ is the restriction of $\mathscr{H}^{n-1}$ to the boundary of E, and

$$\mathrm{Per}(E) = \|D1_E\| = \mathscr{H}^{n-1}(\partial E) \ . \tag{2.5}$$

The use of the term 'perimeter' is thus justified.

2.5 – Remarks

(i) The compactness result for BV functions described in Remark 2.2(i) applies in particular to finite perimeter sets: let (E_h) be a sequence of sets with uniformly bounded perimeters, contained in a fixed bounded subset of $\mathbb{R}^n$; then, upon extraction of a subsequence, the sets E_h converge in the L^1-distance to a finite perimeter set E (that is, $\mathscr{L}^n(E \triangle E_h) \to 0$) and $\mathrm{Per}(E) \le \liminf \mathrm{Per}(E_h)$. This result can be used to prove the existence of sets with minimal perimeter, thus providing a possible (weak) solution to the Plateau problem for surfaces with codimension one.

(ii) Many variants of the definition of finite perimeter sets are currently in use, all differing by minor details (cf. [4], Sect. 3.3, and [22], Chap. 1).

(iii) According to [11], a set E is of finite perimeter if it can be approximated in the L^1-distance by a sequence of smooth sets with bounded perimeters. This definition is equivalent to the one in §2.3.

2.6 – Essential boundary

Formula (2.5) hold for sets with smooth and even Lipschitz boundary, but not for all sets of finite perimeter. Take indeed a countable dense set E in $\mathbb{R}^n$: the characteristic function 1_E agrees almost everywhere with the constant function 0, and therefore its distributional derivative is 0, while ∂E is the entire $\mathbb{R}^n$.

However, formula (2.5) hold for any finite perimeter set E provided that the topological boundary ∂E is replaced by the *essential boundary* $\partial_* E$, namely the complement of the set of all points x where the density[4] of E exists and is either 1 or 0. In particular, $\partial_* E$, unlike ∂E, is always $\mathscr{H}^{n-1}$-finite, and therefore its Hausdorff dimension is at most $n-1$.

In fact there holds more, $\partial_* E$ is $(n-1)$-*rectifiable*, which means that it can be covered by countably many hypersurfaces of class C^1, except for an $\mathscr{H}^{n-1}$-negligible subset.[5] This result is important because it shows that the essential boundary shares some of the properties of smooth boundaries; in particular it admits an inner normal and a tangent space at $\mathscr{H}^{n-1}$-almost every point, in a suitable approximate sense.

[4] Namely, the limit as $r \to 0$ of the ratio $\mathscr{L}^n(E \cap B(x,r))/\mathscr{L}^n(B(x,r))$.

[5] This result is due to Federer [17] and De Giorgi [14]. More precisely, the latter proved the rectifiability of the *reduced boundary* of E, which the former had shown to agree with the essential boundary up to an $\mathscr{H}^{n-1}$-negligible subset. Note that the usual definition of rectifiability in Geometric Measure Theory (see [36], [18]), although equivalent to this one, looks quite different.

3. Distributional Jacobian

In this section I briefly recall the notion of distributional Jacobian for Sobolev maps. The general definition can only be written using the language of k-forms. However, I also include alternative definitions for the special cases $n = k$ and $n = 3$, $k = 2$, where no multilinear algebra is needed.

In the following, n, k are fixed integers with $n \geq k \geq 2$.

3.1 – Jacobian of smooth maps

The differential of a function $v : \mathbb{R}^n \to \mathbb{R}$ of class C^1 is the 1-form

$$dv := \sum_{j=1}^{n} \frac{\partial v}{\partial x_j}\, dx_j \ .$$

The Jacobian of a map $u : \mathbb{R}^n \to \mathbb{R}^k$ of class C^1, is the wedge-product of the differentials of its components $u_1, \dots, u_k$, that is, the k-form

$$Ju := du_1 \wedge \ldots \wedge du_k \ . \tag{3.1}$$

3.2 – Special cases

For maps $u : \mathbb{R}^k \to \mathbb{R}^k$, the Jacobian is a k-form on $\mathbb{R}^k$, and identifying k-forms with scalar functions we get[6]

$$Ju := \det(\nabla u) \ . \tag{3.2}$$

For maps $u : \mathbb{R}^3 \to \mathbb{R}^2$, the Jacobian Ju can be identified with the vector product

$$Ju := \nabla u_1 \times \nabla u_2 \ ; \tag{3.3}$$

this means that Ju is a 2-form on $\mathbb{R}^3$ whose coefficients agree (up to a certain change of sign) with those of the vector $\nabla u_1 \times \nabla u_2$. This identification will be made clear in Remark 4.7(i).

[6] Note that, contrary to the common definition of Jacobian determinant, Ju is not $|\det(\nabla u)|$.

3.3 – Jacobian of Sobolev maps, I

Definition (3.1) makes sense even for maps of class $W^{1,k}$ because the product of k-functions in L^k (the differentials du_i) is a well defined function in L^1. More precisely, formula (3.1) defines a *continuous* nonlinear operator J from $W^{1,k}(\mathbb{R}^n;\mathbb{R}^k)$ into the space $L^1(\mathbb{R}^n; \wedge^k\mathbb{R}^n)$ of k-forms with L^1-coefficients. By the density of smooth maps in $W^{1,k}$, J turns out to be the (unique!) continuous extension of the Jacobian operator for maps of class C^1.

However, definition (3.1) does not make sense for maps in $W^{1,p}$ with $p < k$ because the product of k functions in L^p with $p < k$ is not well-defined in any reasonable function space, not even distributions. In the next paragraph we write the Ju in a different form, which paves the way to a well-posed definition of Jacobians for bounded maps in $W^{1,k-1}$.

3.4 – A fundamental identity

Let $u : \mathbb{R}^n \to \mathbb{R}^k$ be a map of class C^2. A simple computation gives

$$Ju = \frac{1}{k} d\Big[\sum_{i=1}^{k} (-1)^{i-1} u_i\, \widehat{du_i} \Big] \ , \tag{3.4}$$

where $\widehat{du_i}$ stands for the wedge-product of all du_j with $j \neq i$.

For $n = 3$ and $k = 2$, formula (3.4) can be written as

$$\nabla u_1 \times \nabla u_2 = \nabla \times \frac{1}{2}(u_1\, \nabla u_2 - u_2 \nabla u_1) \ , \tag{3.5}$$

while for $n = k = 2$ it becomes

$$\det(\nabla u) = \frac{1}{2}\frac{\partial}{\partial x_1}\Big(u_1 \frac{\partial u_2}{\partial x_2} - u_2 \frac{\partial u_1}{\partial x_2}\Big) + \frac{1}{2}\frac{\partial}{\partial x_2}\Big(u_2 \frac{\partial u_1}{\partial x_1} - u_1 \frac{\partial u_2}{\partial x_1}\Big) \ . \tag{3.6}$$

The analogue of (3.6) for $n = k > 2$ is slightly more involved.

3.5 – Jacobian of Sobolev maps, II

Consider the right-hand side of (3.4) when u is a map of class $L^\infty \cap W^{1,k-1}$: $\widehat{du_i}$ is the product of $k-1$ functions (1-forms) in L^{k-1} and therefore belongs to L^1, and since u_i belongs to L^∞, then $u_i\, \widehat{du_i}$ belongs to L^1, too. Hence the sum within brackets in (3.4) belongs to L^1, and its differential is a well-defined distribution (recall that the differential is a *linear* operator).

Following [27], for every $u \in L^\infty \cap W^{1,k-1}(\mathbb{R}^n;\mathbb{R}^k)$, we call *distributional Jacobian* of u the distribution at the right-hand side of (3.4), and we denote it by Ju. Clearly, this definition extends to maps which are locally of class $L^\infty \cap W^{1,k-1}$. For bounded maps in $W^{1,1}(\mathbb{R}^3;\mathbb{R}^2)$ and in $W^{1,1}(\mathbb{R}^2;\mathbb{R}^2)$, the distributional Jacobian can be alternatively defined as the distribution at the right-hand side of (3.5) and (3.6), respectively.

It is easy to show that J is a continuous nonlinear operator from $L^\infty \cap W^{1,k-1}$ into the space of k-forms with coefficients in the space of distributions. More precisely, given a sequence of uniformly bounded maps u_h which converge to u in $W^{1,k-1}$, then Ju_h converge to Ju in the sense of distributions.

3.6 – Remarks

(i) The 'pointwise' and the distributional Jacobian, defined respectively in §3.1 and §3.5, agree for maps of class C^2, and equality carries over by continuity to all maps of class $L^\infty \cap W^{1,k}$. However, equality does not hold in general: for $u(x) := x/|x|$ the distributional Jacobian is a Dirac mass at the origin, while $\det(\nabla u(x)) = 0$ for every $x \neq 0$ (see §4.2).

(ii) If the distributional Jacobian Ju is a measure, then the Radon-Nikodym derivative w.r.t. the Lebesgue measure agrees a.e. with the pointwise Jacobian. This was proved in [31] for $n = k$; the case $n > k$ follows by a suitable dimension-reduction argument.

(iii) The distributional Jacobian is continuous also with respect to the weak convergence: given a sequence of uniformly bounded maps u_h which converge *weakly* to u in $W^{1,p}$ for some $p > k-1$, then Ju_h converge to Ju in the sense of distributions.

(iv) For every $m < p$, the operator $u \mapsto M(\nabla u)$, where M is the determinant of a fixed $m \times m$ minor of the matrix ∇u, is sequentially continuous from $W^{1,p}$ to $L^{p/m}$, where both spaces are endowed with the corresponding weak topologies. This fact has an important consequence in the Calculus of Variations: a functional of the form $F(u) := \int f(\nabla u)$ is sequentially weakly lower-semicontinuous whenever f is polyconvex, that is, when f can be written as a convex functions of the minors $M(\nabla u)$ (cf. [12], Chap. 4).

(v) It can be shown that the Jacobian operator does not admit any continuous extension to $L^\infty \cap W^{1,p}$ for $p < k-1$. To prove this claim in the case $n = k$, it suffices to exhibit a sequence of smooth maps $u_h : \mathbb{R}^n \to S^{k-1}$ which converge in $W^{1,p}_{\mathrm{loc}}$ to the map $u(x) := x/|x|$: the Jacobians of these maps are all null (see §4.1), and therefore cannot converge in any sense to the distributional Jacobian of u, which is a Dirac mass (see §4.2).

(vi) When $u : \mathbb{R}^n \to \mathbb{R}^k$ is a smooth map, $Ju := du_1 \wedge \ldots \wedge du_k$ is the

pull-back according to u of the standard volume form on $\mathbb{R}^k$, $dy_1 \wedge \ldots \wedge dy_k$. Since this form is the differential of the $(k-1)$-form

$$\omega(y) := \frac{1}{k} \sum_{i=1}^{k} (-1)^{i-1} y_i \, \widehat{dy_i} \ ,$$

and the differential commutes with the pull-back, then Ju agrees with the differential of the pull-back of ω according to u: this is precisely identity (3.4). Now, $dy_1 \wedge \ldots \wedge dy_k$ can be written as the differential of many other $(k-1)$-forms ω, each one yielding a different variant of identity (3.4).[7] Our choice of ω happens to be particularly convenient when dealing with S^{k-1}-valued maps.

(vii) The definition of distributional Jacobian for general n and k was given, in a slightly different form, by R. Jerrard and H.M. Soner in [26], [27]; the presentation given here follows closely that in [2]. For $n = k$, the distributional Jacobian was introduced by J. Ball [5] as *distributional determinant* (see also [24]). This definition hinges on a variant of identity (3.6) which was known before (C.B. Morrey used it to prove the weak semicontinuity of polyconvex functionals).

(viii) F.-B. Hang and F.-H. Lin proposed in [23] an alternative definition of distributional Jacobian for maps in the fractional Sobolev spaces $W^{1-1/k,k}(\mathbb{R}^n;\mathbb{R}^k)$ (see also [3], Sect. 5). Note that for $k > 2$ this space includes $W^{1,k-1}(\mathbb{R}^n;\mathbb{R}^k)$. Special cases of this definition appeared before in [7], [35]; in these papers the distributional Jacobian is known as *topological singularity.*

4. Jacobian of maps valued in spheres

For the rest of the paper we restrict our attention to maps u from $\mathbb{R}^n$ into the sphere S^{k-1}. We will see that for maps with 'nice' singularities, the distributional Jacobian is related to the singularities by explicit formulas.

4.1 – Jacobian of smooth maps valued in S^{k-1}

Let us begin with a simple remark: if $u : \mathbb{R}^n \to S^{k-1}$ is a map of class C^1 then $Ju = 0$. Indeed, the matrix $\nabla u(x)$ has rank at most $k-1$ because its columns belong to the tangent space to S^{k-1} at the point $u(x)$; this means

[7] Accordingly, there are many variants of formula (3.6); the one which is most frequently used is $\det(\nabla u) = \frac{\partial}{\partial x_1}\Big(u_1 \frac{\partial u_2}{\partial x_2}\Big) - \frac{\partial}{\partial x_2}\Big(u_1 \frac{\partial u_2}{\partial x_1}\Big)$.

that the linear functionals $du_1(x), \dots, du_k(x)$ are linearly dependent, and therefore their wedge product is null.

The same argument applies to the Jacobian of maps of class $W^{1,k}$, which is defined according to the pointwise formula (3.1). However, the conclusion may not hold for the distributional Jacobian of maps of class $W^{1,k-1}$. This is clarified by the following fundamental example.

4.2 – Example: the map $x/|x|$

The map $u : \mathbb{R}^k \to S^{k-1}$ defined by $u(x) := x/|x|$ belongs to $W^{1,p}_{\mathrm{loc}}$ for every $p < k$. Its pointwise Jacobian is 0, but the distributional Jacobian is

$$Ju = \alpha_k \delta_0 \tag{4.1}$$

where α_k is the volume of the unit ball in $\mathbb{R}^k$, and δ_0 is the Dirac mass at the origin. Formula (4.1) can be proved by approximation: consider the maps

$$u_\varepsilon(x) := \begin{cases} x/|x| & \text{if } |x| \geq \varepsilon \\ x/\varepsilon & \text{if } |x| < \varepsilon \end{cases} ;$$

these maps are Lipschitz and converge to u in $W^{1,p}_{\mathrm{loc}}$ for every $p < k$; moreover the Jacobians $Ju_\varepsilon = \varepsilon^{-k} 1_{B(0,\varepsilon)}$ converge in the sense of measure to $\alpha_k \delta_0$, which therefore must agree with Ju by the continuity of the Jacobian.

4.3 – Maps with 'nice' singularities, case $n = k$

Formula (4.1) is a particular case of a more general formula proved in [10]: if $u : \mathbb{R}^k \to S^{k-1}$ is a map in $W^{1,k-1}_{\mathrm{loc}}$, smooth outside a finite singular set $S := \{x_j\}$, then

$$Ju = \alpha_k \sum_j d_j \delta_{x_j} \quad \text{where } d_j := \deg(u, \partial B_j, S^{k-1}). \tag{4.2}$$

In this formula, B_j is a ball such that $S \cap B_j = \{x_j\}$, and $\deg(u, \partial B_j, S^{k-1})$ stands for the *Brower degree* of the restriction of u to the sphere ∂B_j (and is often called degree of the singularity of u at x_j). The invariance of degree under homotopy shows that the value of d_j does not depend on the choice of B_j.

4.4 – Proof of identity (4.2)

First of all, we remark that u can be approximated in $W^{1,k-1}$ by maps $\tilde{u}$ which are 'radial' close to the singularities, that is, $\tilde{u}(x)$ depends only on the direction of $x - x_j$ when $|x - x_j|$ is sufficiently small.

By the continuity of the Jacobian, it suffices to prove (4.2) for such maps $\tilde{u}$. To this end, we proceed as in the proof of (4.1): for every positive ε sufficiently small we set

$$\tilde{u}_\varepsilon(x) := \begin{cases} \tilde{u}(x) & \text{if } |x - x_j| \geq \varepsilon \text{ for every } j, \\ \varepsilon^{-1}|x - x_j|\, \tilde{u}(x) & \text{if } |x - x_j| < \varepsilon \text{ for some } j. \end{cases}$$

It is not difficult to show that the maps $\tilde{u}_\varepsilon$ are Lipschitz and converge to $\tilde{u}$ in $W^{1,k-1}$. Moreover the Jacobians $J\tilde{u}_\varepsilon$ are uniformly bounded in L^1 and supported on the union of the closures of the balls $B(x_j, \varepsilon)$, and finally the integral of $J\tilde{u}_\varepsilon$ on $B(x_j, \varepsilon)$ must be equal to $\alpha_k d_j$ for every j by the area formula.[8] This information is enough to conclude that the Jacobians $J\tilde{u}_\varepsilon$ converge to the measure $\alpha_k \sum_j d_j \delta_{x_j}$, which therefore must agree with $J\tilde{u}$.

4.5 – Maps with 'nice' singularities, case $n = 3$, $k = 2$

Let be given a map $u : \mathbb{R}^3 \to S^1$ in $W^{1,1}_{\mathrm{loc}}$ which is smooth outside a smooth, closed, oriented curve M. Then the Jacobian Ju, viewed as a distribution with values in $\mathbb{R}^3$, is a measure supported on the curve M of the form

$$Ju = \pi\, d\, \tau_M\, \mathscr{H}^1 \llcorner M \ , \tag{4.3}$$

where τ_M is the tangent unit vector that orients M, $\mathscr{H}^1 \llcorner M$ is the length measure on M, and finally $d := \deg(u, \partial E, S^1)$ with E any 2-dimensional disk that intersects M transversally at one point only.[9] In particular, the mass of the measure Ju is

$$\|Ju\| = \pi |d| \mathscr{H}^1(M) \ .$$

Identity (4.3) can be derived from (4.2) by a suitable slicing argument (see [27]).

4.6 – A Hodge-type operator

In order to write formulas (4.2) and (4.3) for general n and k, we must first introduce a variant of the Hodge operator: for every k-covector λ, $\star\lambda$

[8] The integral of Ju_ε on $B(x_j, \varepsilon)$ is equal to the integral on $\mathbb{R}^k$ of $\deg_j(y) := \deg(\tilde{u}_\varepsilon, B(x_j, \varepsilon), y)$ – the degree of the restriction of $\tilde{u}_\varepsilon$ to the ball $B(x_j, \varepsilon)$ computed at the value $y \in \mathbb{R}^k$. One easily checks that $\deg_j(y) = 0$ for $|y| > 1$, and $\deg_j(y) = d_j$ for $|y| < 1$.

[9] Clearly, formula (4.3) holds provided that E and its boundary are properly oriented (see [2], Sect. 3). The invariance of degree under homotopy shows that the value of d does not depend on the particular choice of E.

is the k-vector defined by duality as[10]

$$\langle\omega;\star\lambda\rangle := \langle\omega\wedge\lambda; e_1\wedge\ldots\wedge e_n\rangle \quad \text{for every } (n-k)\text{-covector } \omega.$$

Despite a rather cryptic definition, the operator $\star$ has a clear geometric meaning: when λ is *simple* – that is, $\lambda = \lambda_1\wedge\ldots\wedge\lambda_k$ where the λ_i are 1-covectors (linear functionals on $\mathbb{R}^n$) – then $\star\lambda$ can be written as a product of vectors $v_1\wedge\ldots\wedge v_{n-k}$ which span the kernel of $\Lambda := (\lambda_1,\ldots,\lambda_k) : \mathbb{R}^n \to \mathbb{R}^k$.[11] In particular, given a smooth map $u:\mathbb{R}^n\to\mathbb{R}^k$, $\star Ju(x)$ is a simple k-vector that spans the tangent plane to the level surface of u passing through x.

4.7 – Remarks

(i) Using the operator $\star$, we can write identities (3.2) and (3.3) in a more precise way: for a map $u:\mathbb{R}^k\to\mathbb{R}^k$ of class C^1 we have $\star Ju = \det(\nabla u)$, and for a map $u:\mathbb{R}^3\to\mathbb{R}^3$ we have $\star Ju = \nabla u_1\times\nabla u_2$. Identities (4.1), (4.2), and (4.3) should be corrected accordingly.

(ii) Given a map $u:\mathbb{R}^n\to\mathbb{R}^k$ of class $L^\infty\cap W^{1,k-1}$, $\star Ju$ is a distribution with values in $(n-k)$-covectors. In other words, $\star Ju$ is an $(n-k)$-*current*, that is, a generalized (oriented) surface of dimension $n-k$. Moreover, since Ju is a differential, then $\star Ju$ is a boundary, and therefore it has no boundary. Clearly, these statements should be intended in the proper distributional sense. Elementary presentations of the theory of currents can be found in [30], [36].

4.8 – Maps with 'nice' singularities, general case

Let be given a map $u:\mathbb{R}^n\to S^{k-1}$ in $W^{1,k-1}_{\mathrm{loc}}$, smooth outside a regular $(n-k)$-dimensional surface (submanifold) M which is oriented, connected, and without boundary. Then

$$\star Ju := \alpha_k\, d\,\tau_M\,\mathscr{H}^{n-k}\llcorner M\ , \tag{4.4}$$

where τ_M is a simple $(n-k)$-vector with norm 1 that spans the tangent space M with the right orientation (cf. footnote 11), and $d := \deg(u,\partial E, S^{k-1})$ is the degree of the restriction of u to the boundary of a k-dimensional disk E that intersects M transversally at one point only.[12] In particular, the mass of the measure Ju is

$$\|Ju\| = \alpha_k |d| \mathscr{H}^{n-k}(M)\ . \tag{4.5}$$

The number d is called degree of the singularity of u at M. Identity (4.4) can be derived from (4.2) by a slicing argument (see [27]).

4.9 – Remarks

(i) If M is not connected, the number d that appears in (4.4) may be different for each connected component.

(ii) Identity (4.4) implies that the support of Ju is contained in the singularity of M, but since d can be 0, the inclusion may be strict. For this reason, we say that Ju represents the singularity of u which is topologically necessary. This explains the term 'topological singularity' used by some authors to denote Ju.

(iii) Identities (4.4) and (4.5) hold also if M is a finite union (not necessarily disjoint) of Lipschitz surface of dimension $n-k$, and u is continuous (or even locally $W^{1,k}$) in the complement of M.

(iv) What can be said about the structure of Ju when u is smooth (continuous) in the complement of a generic closed set C? In this case, $\star Ju$ is an $(n-k)$-current without boundary supported on C. It follows that $\star Ju$ must be 0 if the Hausdorff dimension of C is strictly smaller than $n-k$,[13] or if C is a connected $(n-k)$ dimensional surface with non-empty boundary.[14]

(v) If we view characteristic functions of a set E in $\mathbb{R}^n$ with regular boundary as a map valued in the 0-dimensional sphere S^0 which is smooth outside a smooth hypersurface M without boundary, then identity (2.4) can be interpreted as a special case of (4.4), and (2.5) becomes a special case of (4.5). Thus the class of S^{k-1}-valued maps whose Jacobian is a measure can be regarded as a generalization of the class of finite perimeter sets.

[10] In this formula, $\langle\ ;\ \rangle$ stands for the duality product of covectors and vectors, and $\{e_j\}$ is the standard basis of $\mathbb{R}^n$.

[11] A product $v := v_1\wedge\ldots\wedge v_{n-k}$ is uniquely identified by the (oriented) linear space spanned by the vectors v_i, and by the volume of the parallelogram spanned by these vectors (which is the norm of v). In particular, $v_1\wedge\ldots\wedge v_{n-k} = 0$ if the vectors $\{v_j\}$ are linearly dependent.

[12] Clearly, E and its boundary must be properly oriented for (4.4) to hold, cf. [3], Sect. 3.

[13] A set C which is $\mathscr{H}^{n-k}$-negligible cannot support any $(n-k)$-current with boundary of finite mass. In fact, this is true even if C is $(n-k)$-purely unrectifiable, that is, $\mathscr{H}^{n-k}(C\cap M) = 0$ for every $(n-k)$-dimensional surface M of class C^1.

[14] A variant of the constancy lemma (cf. [36], Theorem 26.27) shows that an $(n-k)$-current T without boundary supported on a smooth, connected surface M with dimension $n-k$ can be written as $T = d\,[M]$ where d is a constant and $[M]$ denotes the current associated to M; if M has non-empty boundary, then the only possibility is $d = 0$.

5. Geometric structure of Jacobians

As pointed out in §3.1, identity (2.5) holds for any set of finite perimeter, provided the set ∂E is replaced by the essential boundary $\partial_* E$ and the inner normal ν_E is suitably defined. A similar situation occurs with identity (4.4):

5.1 – Rectifiability of Jacobians

If $u : \mathbb{R}^n \to S^{k-1}$ belongs to $W^{1,k-1}$ and the Jacobian Ju is a bounded measure, then it can be written as

$$\star Ju := \alpha_k \, d \, \tau_M \, \mathscr{H}^{n-k} \llcorner M \ , \tag{5.1}$$

where M is an $(n-k)$-*rectifiable set*,[15] τ_M is an *orientation* of M,[16] and finally d is an *integer* multiplicity function.

In the language of currents, this statement is summarized by saying that if $\star Ju$ has finite mass, then it agrees up to a factor α_k with a rectifiable $(n-k)$-current with integer multiplicity (and since $\star Ju$ has no boundary, it is also an integral current).

The proof of this result presented in [2] is based on a simple geometric intuition. Looking at the map $u(x) := x/|x|$, one immediately sees that the singular set – which supports the Jacobian – is the boundary of *every* level curve of the map u. In fact, the same is true for every map $u : \mathbb{R}^n \to S^{k-1}$ with a singularity of degree $d \neq 0$ at a smooth surface M with dimension $n-k$.

This observation can be turned into a rigorous statement which is valid for every map $u : \mathbb{R}^n \to S^{k-1}$ of class $W^{1,k-1}$ (see [2], Theorem 3.8): the $(n-k)$-current $\star Ju$ agrees, up to the usual factor α_k, with the boundary of a generic level surface of u, which is a rectifiable current of dimension $n-k+1$. Then the rectifiability of $\star Ju$ follows immediately by the boundary rectifiability theorem of Federer and Fleming (see [36], Theorem 30.3).

5.2 – Remarks

(i) For $n = k$, the rectifiability result in §5.1 simply states that $\star Ju$ agrees, up to a factor α_k, with a finite sum of Dirac masses with integer multiplicity (cf. [10]).

[15] Namely, it can be covered by countably many $(n-k)$-dimensional surfaces of class C^1, except at most an $\mathscr{H}^{n-k}$-negligible subset.

[16] A Borel map such that $\tau_M(x)$ is a simple $(n-k)$-vector with norm 1 that spans the approximate tangent space to M at x for $\mathscr{H}^{n-k}$-almost every $x \in M$.

(ii) The rectifiability of Jacobians can be viewed as a generalization of De Giorgi's rectifiability theorem for finite perimeter sets. However, there is an important difference: no characterization of M as discontinuity set of u has been found yet (in other words, there is no analogue of the essential boundary of finite perimeter sets).

(iii) The rectifiability of Jacobians was first proved by Jerrard and Soner [26] using a nice dimension-reduction argument, which relied on a rectifiability criterion due to B. White [38] (see also [25]). Hang and Lin [23] gave another proof, based on a different definition of distributional Jacobian and, once again, on the boundary rectifiability theorem. Special cases of this result were proved in [7], [8], [33].

5.3 – Relation with cartesian currents

In recent years, M. Giaquinta, G. Modica, and J. Souček proposed a different and more geometric approach to variational problems for vector-valued maps $u : \mathbb{R}^n \to \mathbb{R}^k$ where the unknown variable is the graph of u, viewed as an n-dimensional surface (more precisely, a rectifiable n-current), rather than the map u itself. This point of view is the basis of the theory of cartesian currents [21].

In particular, for maps $u : \mathbb{R}^n \to S^{k-1}$ of class $W^{1,k-1}$, the distributional Jacobian can be recovered as part of the boundary of the graph of u,[17] and the rectifiability result stated in §5.1 is a corollary of the boundary rectifiability theorem of Federer and Fleming.

5.4 – Which surfaces can support a Jacobian?

We have seen in §4.5 that the Jacobian of a map $u : \mathbb{R}^k \to S^{k-1}$ with finitely many singularities $\{x_j\}$ is (up to a factor α_k) a sum of Dirac masses at x_j with integer multiplicities. Conversely, given finitely many points x_j and integers d_j, there exists a map $u : \mathbb{R}^k \to S^{k-1}$ with singular set $\{x_j\}$ such that the degree of the singularity of u at x_j is d_j for every j. The construction of such u is not difficult. In particular, for $k = 2$, it suffices to take

$$u(x) := \prod_j \left(\frac{x - x_j}{|x - x_j|} \right)^{d_j} ,$$

[17] The *regular part* of the graph of u is the set of all points $(x, u(x))$ such that u is approximately differentiable at x; to this set is canonically associated a rectifiable n-current with multiplicity 1, and its boundary is an $(n-1)$-current of the form $T \times S^{k-1}$, where T is a $(n-k)$-current in $\mathbb{R}^n$. T agrees, up to some constant, with $\star Ju$.

where the product is defined by identifying $\mathbb{R}^2$ and the complex field.

The question becomes more interesting for maps $u : \mathbb{R}^n \to \mathbb{R}^k$ with $n > k$: we have seen in §4.8 that the Jacobian of a map with a smooth singular set M of codimension k is supported on this set, and more precisely is an integer multiple of (the current associated to) M itself. Then it is natural to ask if the converse is true:

QUESTION. – Given an integer d and an $(n-k)$-dimensional surface M in $\mathbb{R}^n$, connected, oriented, and without boundary, is it possible to find a map $u : \mathbb{R}^n \to S^{k-1}$, of class $W^{1,k-1}$ and smooth outside M, with a singularity of degree d at M (so that in particular (4.4) holds)?

This and related questions have been studied extensively in [2]; I refer the reader to the original paper for precise statements and detailed proofs, and just recall here one of the main results:

The answer to the question above is positive for $k = 2$ (see [2], §4.1).

A proof of this result is sketched in the next paragraph. But first, let me underline the essential point: given a smooth surface M of codimension k, it is always possible to construct a map on a tubular neighbourhood of M with singularity of prescribed degree d at M.[18] The real difficulty is to extend u to the rest of $\mathbb{R}^n$ without introducing new singularities! This extension can always be done only if $k = 2$ (see §5.6), and it is not obvious even when M is the usual threefold knot in $\mathbb{R}^3$.

5.5 – A construction for $k = 2$ (see [2], §4.1)

Given M an oriented, compact smooth surface in $\mathbb{R}^n$ with codimension two and no boundary, we look for a map $u : \mathbb{R}^n \to S^1$ with singularity of degree d at M. I claim that it suffices to construct a smooth 1-form ω on $\mathbb{R}^n \setminus M$ such that

$$\int_\gamma \omega = d\,\mathrm{link}(M,\gamma) \quad \text{for every closed curve } \gamma \text{ in } \mathbb{R}^n \setminus M, \tag{5.2}$$

where $\mathrm{link}(M,\gamma)$ is the *linking number* of M and γ. Indeed, the integral of such a form ω on every closed curve in the complement of M would be an integer, and then ω would be the differential of a smooth map θ from $\mathbb{R}^n \setminus M$ into the quotient $\mathbb{R}/\mathbb{Z}$.[19] Moreover, setting $u := \exp(2\pi i\theta)$, (5.2) implies that the degree of the singularity of u around M is d.

It remains to construct ω such that (5.2) holds. Recall that

$$\mathrm{link}(M,\gamma) = \deg(\Phi, M\times\gamma, S^{n-1}) ,$$

where $\Phi(x,y) := (x-y)/|x-y|$ for every $x,y \in \mathbb{R}^n$ with $x \neq y$. Denoting by $\tilde\omega$ the pull-back of the volume form on S^{n-1} according to Φ, the area formula implies that $\deg(\Phi, M\times\gamma, S^{n-1})$ times the volume of S^{n-1} (that is, $n\alpha_n$) agrees with the integral of $\tilde\omega$ on $M\times\gamma$, that is

$$n\alpha_n\,\mathrm{link}(M,\gamma) = \int_{M\times\gamma} \tilde\omega = \int_{y\in\gamma} \Big[\int_{x\in M} \tilde\omega(x,y) \Big] . \tag{5.3}$$

Since the integral of an $(n-1)$-form over an $(n-2)$-dimensional surface M is a 1-covector (for a precise definition, see [2], Sect. 2), the integral within square brackets in (5.3) defines for every $y \notin M$ a smooth 1-form that satisfies (5.2) up to a factor $d/(n\alpha_n)$. To conclude, it suffices to set

$$\omega(y) := \frac{d}{n\alpha_n} \int_{x\in M} \tilde\omega(x,y) \quad \text{for every } y \in \mathbb{R}^n \setminus M.$$

5.6 – Connections to topology

The problem of constructing (or extending) a map u with prescribed singularity M has a clear topological flavour, and indeed it is related to well-known problems in algebraic topology.

Let M be an oriented surface in $\mathbb{R}^n$ of codimension k and without boundary, and assume that there exists a map $u : \mathbb{R}^n \to S^1$ which is smooth in the complement of M, and can be written as $u(t,x) = x/|x|$ in a tubular neighbourhood of M identified with the product $M\times B$ as in footnote 18. Then, given a regular value y of u, the set

$$N := u^{-1}(y)$$

is a smooth surface of dimension $n-k+1$ in $\mathbb{R}^n$ with boundary M.

[18] U can be taken diffeomorphic to the product $M\times B$ where B is the unit disk in $\mathbb{R}^k$. For $d = 1$, we can take $u(t,x) := x/|x|$ for every $(t,x) \in M\times B$. For general d we can take $u(t,x) := \phi_d(x/|x|)$ where $\phi_d : S^{k-1} \to S^{k-1}$ is a smooth map with degree d.

[19] Compare with this well-known statement: a 1-form ω on an open set Ω in $\mathbb{R}^n$ is the differential of a real function on Ω if (and only if) the integral of ω on every closed curve in Ω is null.

In fact, the existence of u yields even more: if $S_1, \dots, S_k$ are transversal $(k-2)$-dimensional spheres of radius 1 on S^{k-1} (for $k=2$, just two couples of antipodal points), then the sets

$$N_i := u^{-1}(S_i) \cap M$$

are transversal, smooth hypersurfaces[20] in $\mathbb{R}^n$ without boundary, whose intersection is exactly M. That is, M is a *complete intersection*.

Hence the construction in §5.5 implies that every oriented smooth surface M of codimension two in $\mathbb{R}^n$ is the boundary of an oriented smooth surface of codimension 1, and is a complete intersection.

Since this result does not hold for all surfaces M of codimension $k > 2$ (cf. [37], [6]), in general the answer to the question posed in §5.4 should be negative. However, it is possible to construct a map u with singularity of degree d at a given surface M of codimension k, provided we allow u to be singular also in some additional set S of codimension $k+1$, see [2], Theorem 5.10.[21] Note that formula (4.4) holds for such maps, because the set S is to small too support the Jacobian of u.

5.7 – Remarks

(i) If M is not connected, the construction described in §5.5 can be modified so that u has singularities of different degree on each connected component of M.

(ii) For $k=1$, the question considered in §5.4 becomes: given a smooth oriented hypersurface M in $\mathbb{R}^n$ without boundary, is it the a boundary of an open set? the answer is positive if we assume that M is connected, and negative otherwise (consider M made of two parallel lines in the plane, with same orientation).

(iii) The following question is related to the converse of the rectifiability of Jacobian stated in §5.1, and can be viewed as a generalization of the problem considered in §5.4: given an integral current T in $\mathbb{R}^n$, of codimension k and without boundary, can we find $u \in W^{1,k-1}(\mathbb{R}^n; S^{k-1})$ such that $\star Ju = \alpha_k T$? The answer is positive for all n and k, see [2], Theorem 5.6.

[20] This may not be true for u, but it is true for a generic perturbation of u.

[21] In this theorem it is actually assumed that M is a polyhedral chain.

REFERENCES

[1] G. Alberti: Un risultato di convergenza variazionale per funzionali di tipo Ginzburg-Landau in dimensione qualunque (A variational convergence result for functionals of Ginzburg-Landau type in any dimension). *Boll. Un. Mat. Ital. B (8)*, **4** (2001), 289–310.

[2] G. Alberti, S. Baldo, G. Orlandi: Functions with prescribed singularities. *J. Eur. Math. Soc.*, **5** (2003), 275–311.

[3] G. Alberti, S. Baldo, G. Orlandi: Variational convergence for functionals of Ginzburg-Landau type. *Indiana Univ. Math. J.*, **54** (2005), 1411–1472.

[4] L. Ambrosio, N. Fusco, D. Pallara: *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. Oxford Science Publications, Oxford, 1999.

[5] J.M. Ball: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, **63**(1977), 337–403.

[6] J. Bochnak, W. Kucharz: Complete intersections in differential topology and analytic geometry. *Boll. Un. Mat. Ital. B (7)*, **10** (1996), 1019–1041.

[7] J. Bourgain, H. Brezis, P. Mironescu: On the structure of the Sobolev space $H^{1/2}$ with values into the circle. *C. R. Acad. Sci. Paris Sér. I Math.*, **310** (2000), 119–124.

[8] J. Bourgain, H. Brezis, P. Mironescu: $H^{1/2}$-maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation. *Publ. Math. Inst. Hautes Études Sci.*, **99** (2004), 1–115.

[9] J. Bourgain, H. Brezis, P. Mironescu: Lifting, degree, and distributional Jacobian revisited. *Comm. Pure Appl. Math.*, **58** (2005), 529–551.

[10] H. Brezis, J.-M. Coron, E. Lieb: Harmonic maps with defects. *Commun. Math. Phys.*, **107** (1986), 649–705.

[11] R. Caccioppoli: Misura e integrazione sugli insiemi dimensionalmente orientati, I e II. *Rend. Acc. Naz. Lincei (8)*, **12** (1952), 3–11, and 137–146.

[12] B. Dacorogna: *Direct Methods in the Calculus of Variations*. Applied Math. Sciences, 78. Springer-Verlag, New York 1989.

[13] E. De Giorgi: Su una teoria generale della misura $(r-1)$-dimensionale in uno spazio a r dimensioni. *Ann. Mat. Pura Appl. (4)*, **36** (1954), 191–213.

[14] E. De Giorgi: Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio a r dimensioni. *Ricerche Mat.*, **4** (1955), 95–113.

[15] F. Demengel: Une caractérisation des applications de $W^{1,1}(B^n; S^1)$ qui peuvent être approchées par des fonctions régulières [A characterization of maps in $W^{1,1}(B^n; S^1)$ that can be approximated by smooth maps]. *C. R. Acad. Sci. Paris Sér. I Math.* **310** (1990), 553–557.

[16] L.C. Evans, R.F. Gariepy: *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[17] H. Federer: A note on the Gauss Green theorem. *Proc. Amer. Math. Soc.*, **93** (1959), 418–491.

[18] H. Federer: *Geometric measure theory*. Grundlehren der mathematischen Wissenschaften, 153. Springer, Berlin-New York, 1969. Reprinted in the series Classics in Mathematics. Springer, Berlin-Heidelberg, 1996.

[19] H. Federer, W.H. Fleming: Normal and integral currents. *Ann. of Math. (2)*, **72** (1960), 458–520.

[20] I. Fonseca, N. Fusco, P. Marcellini: On the total variation of the Jacobian. *J. Funct. Anal.*, **207** (2004), 1–32.

[21] M. Giaquinta, G. Modica, J. Souček: *Cartesian currents in the calculus of variations. I. Cartesian currents*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, 37. Springer, Berlin, 1998.

[22] E. Giusti: *Minimal surfaces and functions of bounded variation*. Monographs in Mathematics, 80. Birkhäuser, Boston, 1984.

[23] F.-B. Hang, F.-H. Lin: A remark on the Jacobians. *Commun. Contemp. Math.*, **2** (2000), 35–46.

[24] T. Iwaniec: On the concept of the weak Jacobian and Hessian. in *Papers on analysis*, 181–205. Rep. Univ. Jyväskylä Dep. Math. Stat., 83. Univ. Jyväskylä, Jyväskylä, 2001.

[25] R.L. Jerrard: A new proof of the rectifiable slices theorem. *Ann. Scuola Norm. Sup. Pisa (5)*, **1** (2002), 905–924.

[26] R.L. Jerrard, H.M. Soner: Rectifiability of the distributional Jacobian for a class of functions. *C. R. Acad. Sci. Paris Sér. I Math.*, **329** (1999), 683–688.

[27] R.L. Jerrard, H.M. Soner: Functions of bounded higher variation. *Indiana Univ. Math. J.*, **51** (2002), 645–677.

[28] R.L. Jerrard, H.M. Soner: The Jacobian and the Ginzburg-Landau energy. *Calc. Var. Partial Differential Equations*, **14** (2002), 151–191.

[29] F.-H. Lin, T. Rivière: Complex Ginzburg-Landau equations in high dimensions and codimension-two area minimizing currents. *J. Eur. Math. Soc.*, **1** (1999), 237–311. Erratum, *Ibid.*, **2** (2000), 87–91.

[30] F. Morgan: *Geometric measure theory. A beginner's guide*. Third edition. Academic Press, San Diego, 2000.

[31] S. Müller: Det = det. A remark on the distributional determinant. *C. R. Acad. Sci. Paris Sér. I Math.*, **311** (1990), 13–17.

[32] S. Müller, S.J. Spector: An existence theory for nonlinear elasticity that allows for cavitation. *Arch. Rational Mech. Anal.*, **131** (1995), 1–66.

[33] M.R. Pakzad: On topological singular set of maps with finite 3-energy into S^3. *Z. Anal. Anwendungen*, **21** (2002), 529–530.

[34] M.R. Pakzad, T. Rivière: Weak density of smooth maps for the Dirichlet energy between manifolds. *Geom. Funct. Anal.*, **13** (2003), 223–257.

[35] T. Rivière: Dense subsets of $H^{1/2}(S^2; S^1)$. *Ann. Global Anal. Geom.*, **18** (2000), 517–528.

[36] L. Simon: *Lectures on geometric measure theory*. Proceedings of the Centre for Mathematical Analysis 3. Australian National University, Canberra, 1983.

[37] R. Thom: Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, **28** (1954), 17–86.

[38] B. White: Rectifiability of flat chains. *Ann. Math. (2)*, **150** (1999), 165–184.