Fifth Summer School in
Analysis and Applied Mathematics
Rome, June 1-5, 2009

Giovanni Alberti (University of Pisa)

Classical and non-classical tools for minimal surfaces*

* These notes contain some short parts that were not included in the lectures for lack of time.

Lecture 1

1.0 Outline of the lectures

These lectures are aimed to:

- recalling some basic notions related to the Plateau problem & minimal surfaces, i.e., area, area formula, first variation of the area....
- recalling some basic facts from the theory of holomorphic functions and differential geometry of surfaces in the space which will be used extensively in the lectures by prof. Hildebrandt.

1st and 2nd lecture

- Giving a brief and self-contained introduction to the theory of finite perimeter sets as a tool to prove existence results for variational problems related to area-minimization

3rd lecture

- Outlining a few applications of the theory of finite perimeter sets (to give an idea of where this theory can be fruitfully applied....).

4th lecture

1.1 Hausdorff measure

Purpose: intrinsic (and widely accepted) notion of length and area for subsets of higher dimensional spaces (the point is to avoid parametrizations).

Definition of Hausdorff measure

Fix a real number $d \geq 0$ (the dimension).

Given a set $E \subset \mathbb{R}^n$ define, for every $\delta > 0$

$$\boxed{\begin{gathered}\mathcal{H}^d_\delta(E) := \inf\left\{ \sum_i (\operatorname{diam} E_i)^d \;\middle|\; \begin{array}{l}\{E_i\} \text{ countable} \\ \text{cover of } E \\ \text{s.t. } \operatorname{diam}(E_i) \leq \delta \\ \text{for every } i\end{array}\right\} \\ \text{and} \\ \mathcal{H}^d(E) := \lim_{\delta \to 0} \mathcal{H}^d_\delta(E) = \sup_{\delta > 0} \mathcal{H}^d_\delta(E)\end{gathered}}$$

insert renormalization const. $\frac{\alpha_d}{2^d}$

d-dimensional Hausdorff measure of E

Remarks

- $\mathcal{H}^d$ is a σ-additive measure on Borel sets
- $\mathcal{H}^d$ is invariant under isometries (rigid motions) and scales as one would expect from a d-dimensional measure:
  $\mathcal{H}^d(\lambda E) = \lambda^d \mathcal{H}^d(E)$
- $\mathcal{H}^n = \mathcal{L}^n$ = Lebesgue measure on $\mathbb{R}^n$. In fact $\mathcal{H}^d$ = usual d-dimensional volume of subsets of d-dimensional planes (or smooth d-dim. submanifolds). FOR THIS WE NEED THE RINORM. CONSTANT $\frac{\alpha_d}{2^d}$ WITH $\alpha_d :=$ VOL. UNIT BALL IN $\mathbb{R}^d$.

- Why do we need to take the limit $\delta \to 0$ in the definition? Indeed this is not needed for $d=n$: $\mathcal{H}^n = \mathcal{H}^n_\delta = \mathcal{L}^n$ for every $\delta>0$. But it is needed for $d<n$ (for intance, if E is a bounded curve of infinite length, you have $\mathcal{H}^1(E)=+\infty$ but $\mathcal{H}^1_\delta(E)<+\infty$ for every $\delta>0$).
- Finally, $\mathcal{H}^d$ can be defined of $E\subset X$ with X metric space, and it is intrinsic in the sense that it depends only on the restriction of the distance to the set E. In fact there holds more: if d and d' are distances on X such that $d'(x,y) = d(x,y) + o(d(x,y))$ as $d(x,y)\to 0$, then d and d' induce the same Hausdorff measures; that is, $\mathcal{H}^d$ depends only on the asymptotic behaviour of the distance on close points (only the metric matters).

Hausdorff dimension

Note that given E and $d>d'$ then

$\mathcal{H}^d(E)>0 \Rightarrow \mathcal{H}^{d'}(E)=+\infty$ ("a surface has infinite length")

$\mathcal{H}^{d'}(E)<+\infty \Rightarrow \mathcal{H}^d(E)=0$ ("a curve has zero area")

This motivates the following definition of Hausdorff dimension:

$$\dim_H(E) := \inf\{d \mid \mathcal{H}^d(E)=0\} = \sup\{d \mid \mathcal{H}^d(E)=+\infty\}.$$

1.2 Area formula

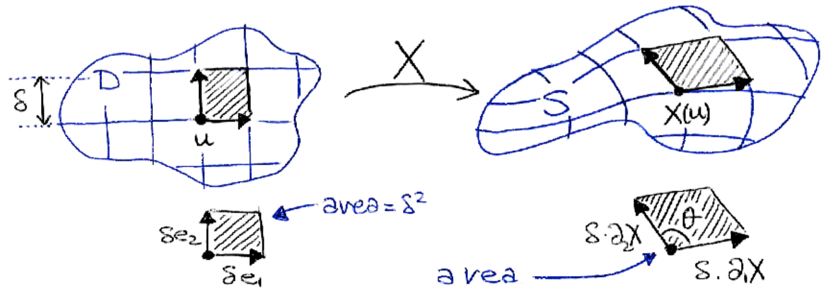
Aim: compute effectively the area (Hausd. meas.) using a parametrization of the set.

Given a map $X: D\to\mathbb{R}^n$, with D open set in $\mathbb{R}^d$, which parametrizes the d-dimensional surface $S:=X(D)$, then $\mathcal{H}^d(S)$ is given by a suitable integral formula.

Standard assumptions: X of class C^1, injective (more or less), maximal rank (in most points).

FIRST CASE: $d=2$ & $n=3$ (surfaces in space)

Let $X=X(u)=X(u_1,u_2)$; $\partial_1 X$, $\partial_2 X$ partial deriv. of X.



area $= \delta^2 \cdot |\partial_1 X|\cdot|\partial_2 X| \cdot \sin\theta$
$= \delta^2 |\partial_1 X \wedge \partial_2 X|$

usual vector product in $\mathbb{R}^3$

Then it is intuitively clear that

$$\boxed{\text{Area}(S) = \mathcal{H}^2(S) = \int_D |\partial_1 X \wedge \partial_2 X|\, du_1\, du_2}$$

In particular $\partial_1 X$ and $\partial_2 X$ span the tangent plane to S at X and therefore

$$N = \frac{\partial_1 X \wedge \partial_2 X}{|\partial_1 X \wedge \partial_2 X|}$$

is a normal (unit) vector to S at X.

Notice that therefore a parametrization X defines implicitely an orientation of the parametrized surface S (either in terms of choice of a basis of $Tan(S,x)$ or, equivalently, in terms of choice of a normal vector).

SECOND CASE:
$d \geq n$ arbitrary.

Let S be a d-dimensional surface in $\mathbb{R}^n$ parametrized by $X : D \subset \mathbb{R}^d \to \mathbb{R}^n$. Then

$$Vol_d(S) = \mathcal{H}^d(S) = \int_D J \, du_1 \dots du_d$$

where $J = J(u)$ is the d-dimensional volume of the parallelogram spanned by the vectors $\partial_1 X, \dots, \partial_d X$ (the columns of the matrix ∇X).

How do we compute the Jacobian determinant J ?

Taking any isometry R from $Tan(S,x) = Span\{\partial_1 X, \dots, \partial_d X\}$ to $\mathbb{R}^d$ we have

$$J = |\det(R \cdot \nabla X)|$$

recall that $\det(A^t) = \det A$ $\longrightarrow$ $= \sqrt{\det((R\cdot\nabla X)^t (R \cdot \nabla X))}$

$R^t R = I$ $\longrightarrow$ $= \sqrt{\det((\nabla X)^t \nabla X)}$

Binet's formula $\longrightarrow$ $= \sqrt{\sum_{\substack{M = d\times d \\ \text{minor} \\ \text{of } \nabla X}} (\det M)^2}$

Hence

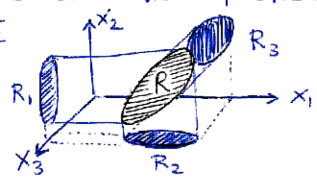
$$\boxed{Vol_d(S) = \mathcal{H}^d(S) = \int_D J = \int_D \sqrt{\det(\nabla X^t \nabla X)} = \int_D \sqrt{\sum_{\substack{M = d\times d \\ \text{minor} \\ \text{of } \nabla X}} (\det M)^2}}$$

Remarks

- Binet's formula state that for a $n \times d$ matrix A there holds $\det(A^t A) = \sum_{\substack{M = d\times d \\ \text{minor of } A}} (\det M)^2$.

For $n=3$, $d=2$ it has the following interpretation. if R is a subset of a plane in $\mathbb{R}^3$ and R_1, R_2, R_3 are the projection on the coordinate planes, then

$$Area(R) = \sqrt{\sum_i (Area(R_i))^2}$$



- Area of the graph of a function.
  Let $X(u) = (u, f(u))$ be the standard parametrization of the graph Γ_f of the function $f : \mathbb{R}^d \to \mathbb{R}^m$.

  Then, if $\boxed{m=1}$ we recover the usual formula

$$\mathrm{Vol}_d(\Gamma_f) = \int_D \sqrt{1+|\nabla f|^2}.$$

  However this formula is no longer correct for $\boxed{m>1}$, indeed in this case

$$\mathrm{Vol}_d(\Gamma_f) = \int_D \sqrt{1+\sum_{\substack{M \text{ square} \\ \text{minor of} \\ \nabla f}} (\det M)^2}$$

- If X is not injective the area formula must be corrected to take into account multiplicity

$$\boxed{\int_{p\in S} \underbrace{\#(X^{-1}(p))}_{\substack{\text{number of } u\in D \text{ s.t.} \\ X(u)=p\text{, i.e.,} \\ \text{multiplicity of } X \text{ at } p}} \, d\mathcal{H}^d(p) = \int_D J = \int_D \sqrt{\det(\nabla X^t \nabla X)}}$$

- Another important variant of the area formula in case X is not injective involves the degree instead of the multiplicity:

$$\boxed{\int_{p\in S} \underbrace{\deg(X,p)}_{\substack{\text{number of } u\in D \text{ s.t.} \\ X(u)=p\text{, counted } 1 \\ \text{if } \nabla X(u) \text{ is orient.} \\ \text{preserving, counted} \\ -1 \text{ if } \nabla X(u) \text{ is orient.} \\ \text{reversing.}}} \, d\mathcal{H}^d(p) = \int_D \underbrace{\pm}_{\substack{\text{the sign } \pm \text{ is } +1 \\ \text{if } \nabla X(u) \text{ is orient.} \\ \text{preserving, and } -1 \\ \text{if it is orientat.} \\ \text{reversing.}}} J(u) \, d\mathcal{L}^d(u)}$$

This is known as "oriented" area formula.

1.3 Plateau's Problem (simplest formulation)

Given a closed (regular) curve Γ in $\mathbb{R}^3$, find the surface S with boundary Γ with minimal area.

(Here "find" means "prove the existence of").

Parametric approach

Among all $X : D \to \mathbb{R}^3$ such that X restricted to ∂D parametrizes Γ, find the one such that

$$F(X) = \mathrm{Area}(X(D)) = \int_D |\partial_1 X \wedge \partial_2 X|$$

is minimal.

(To be discussed: a) which domain D we consider? just a disc? b) what is the regularity of X? c) is X injective?)

Naïve attempt

One can imagine of finding a minimizer of F by a standard semicontinuity-and-compactness strategy, that is, by the following two steps:

1. Proving that F is weakly lower semicontinuous on a suitable class $\mathcal{F}$ of Sobolev maps $X : D \to \mathbb{R}^3$

   Semicontinuity

2. Proving that the set $\mathcal{F}_m := \{X \mid F(X) \leq m\}$ is bounded (in the Sobolev norm under consideration) at least for some $m > \inf F$.

   Compactness

Indeed $\mathcal{F}_m$ is weakly closed because F is weakly lower semicontinuous, and therefore if it is bounded it is weakly compact (at least if $\mathcal{F}$ is a reflexive space). Thus a standard argument shows that F attains a minimum on $\mathcal{F}_m$ and therefore on $\mathcal{F}$.

It is worth to see what happens in the case we are interested in.

It turns out that semicontinuity is (essentially) OK, but the problem is compactness.

1.4 Semicontinuity of $F(X)=\int_D |\partial_1 X \wedge \partial_2 X|$

We recall two basic facts:

FACT 1 If $v_n \xrightarrow{w} v$ in $L^p(D,\mathbb{R}^N)$ and $f : \mathbb{R}^N \to [0,+\infty]$ is a convex lower s.c. function then

$$\int_D f(v) \leq \liminf_{n\to\infty} \int_D f(v_n)$$

this is essentially due to the fact that f is the upper envelope of affine functions, which allows to write the integral $\int f(v)$ as a (localized) upper envelope of weakly continuous functionals.

FACT 2 If $u_n \xrightarrow{w} u$ in $W^{1,p}(D,\mathbb{R}^2)$ with $p \geq 2$, then $\nabla u_n \xrightarrow{w} \nabla u$ in L^p and

$$\det \nabla u_n \xrightarrow{w} \det \nabla u \qquad (*)$$

in $L^{p/2}$ (some extra care should be taken for the case $p=2$).

The key point for $(*)$ is the identity

$$\boxed{\begin{aligned}\det \nabla u &= \partial_1 u_1\, \partial_2 u_2 - \partial_2 u_1\, \partial_1 u_2 \\ &= \partial_1 (u_1\, \partial_2 u_2) - \partial_2 (u_1\, \partial_1 u_2)\end{aligned}}$$

Indeed $u^{(n)} \xrightarrow{w} u$ in $W^{1,p}$ implies $u^{(n)} \to u$ in L^p and $\nabla u^{(n)} \xrightarrow{w} \nabla u$ in L^p, hence $u_1^{(n)}\, \partial_2 u_2^{(n)} \xrightarrow{w} u_1 \partial_2 u_2$ in $L^{p/2}$ and $\partial_1 (u_1^{(n)} \partial_2 u_2^{(n)}) \to \partial_1 (u_1 \partial_2 u_2)$ in the sense of distributions, hence $\det \nabla u^{(n)} \to \det \nabla u$ in the sense of distributions, and the rest follows by the fact that $\det \nabla u^{(n)}$ are unif. bounded in $L^{p/2}$.

Putting together these two facts we obtain that

$$F(X) = \int_D |\partial_1 X \wedge \partial_2 X| = \int \sqrt{\sum_{1\leq i<j\leq 3} (\det \nabla(X_i, X_j))^2}$$

is weakly lower s.c. in $W^{1,p}$ for $p \geq 2$.

The key point is that $|\partial_1 X \wedge \partial_2 X|$ is a convex functions of the determinants of the 2×2 minors of ∇X. At the same way one shows that

$$F(u) = \int_D f(\nabla u)$$

is weakly lower s.c. if f is any convex functions of ∇u and its square minors (what is called a "polyconvex" function).

1.5 Lack of compactness (coercivity) for $F(X)$.

Unfortunately the class $\mathcal{F}_m := \{X \mid F(X) \le m\}$ is not bounded in any reasonable Sobolev norm.

The reason is that F is invariant under re-parametrization, that is,

$$F(X) = F(X \circ \phi)$$

for every diffeomorphism $\phi : D \to D$. But taking suitable ϕ we can make any Sobolev norm of $X \circ \phi$ as large as we want.

On the other hand one can use the invariance of F to restrict the search of a minimizer to a much smaller and better behaved class!

This will indeed be the starting point of prof. Hildebrandt's lectures

Note that indeed a similar approach is routinely used when looking for curves of minimal length

on a surface S (geodesics). Indeed using the fact that every curve $\gamma : [0,1] \to S$ admits a re-parametrization with constant speed one finds out that

$$\boxed{\begin{array}{l} \text{minimization of } \int_0^1 |\dot\gamma| \text{ over all } \gamma : [0,1] \to S \\ \qquad\qquad \parallel \\ \text{minimization of } \int_0^1 |\dot\gamma| \text{ over all} \\ \quad \gamma \text{ such that } |\dot\gamma| \text{ is constant} \\ \qquad\qquad \parallel \\ \text{minimization of } \int_0^1 |\dot\gamma|^2 \text{ over all } \gamma : [0,1] \to S. \end{array}}$$

In the rest of this lecture I'll show that the equivalent of constant-speed parametrization in the case of surfaces are conformal parametrization. I'll then discuss a few questions related to conformal changes of variable in the plane.

1.6 Conformal parametrizations

Note that for every 3×2 matrix $M=(M_1, M_2)$ ← column vectors
there holds

$$|M_1\wedge M_2| = |M_1||M_2| \operatorname{sen}\theta \quad \text{← angle between the vectors } M_1, M_2$$
$$\le |M_1||M_2|$$
$$\le \tfrac{1}{2}(|M_1|^2 + |M_2|^2)$$
$$= \tfrac{1}{2}|M|^2 \leftarrow \text{Euclidean norm of } M = \Big(\sum_{ij} M_{ij}^2\Big)^{\frac{1}{2}}$$

Thus $|M_1\wedge M_2| \le \frac{1}{2}|M|^2$ and moreover
$=$ holds if and only if M satisfies

a) $M_1\cdot M_2 = 0$ (i.e. $\theta = \frac{\pi}{2}$) & $|M_1| = |M_2|$

$\Updownarrow$

b) M is a composition of an isometry and an homothety

$\Updownarrow$

c) M preserves angles between vectors, that is $\dfrac{Mv_1\cdot Mv_2}{|Mv_1|\cdot|Mv_2|} = \dfrac{v_1\cdot v_2}{|v_1|\cdot|v_2|}$

$\Updownarrow$

d) M preserves orthogonality, that is
$v_1\cdot v_2 = 0 \Rightarrow Mv_1\cdot Mv_2 = 0.$ $(*)$

[a) ⇒ b) ⇒ c) ⇒ d) immediate.
d) ⇒ a) : apply $(*)$ with $v_1 = e_1$, $v_2 = e_2$
and then with $v_1 = e_1+e_2$, $v_2 = e_1 - e_2$.]

The matrices M satisfying a) or b) or c) or d) are called conformal.

A consequence of previous computations is that

> for a generic parametrization $X: D \to \mathbb{R}^3$ (we assume X of class C^1...)
> there holds
>
> $$\underbrace{F(X) = \int_D |\partial_1 X \wedge \partial_2 X|}_{\text{area functional}} \le \underbrace{\frac{1}{2}\int_D |\nabla X|^2}_{\text{Dirichlet funct.}}$$
>
> and $=$ holds iff X is conformal, that is, ∇X is a conformal matrix at (almost) every point of D.

Of course the existence of conformal parametrization for all surfaces cannot be taken for granted, but this will not be discussed in this lecture. I will instead spend some time to discuss conformal changes of variables in the plane.

1.7 Conformal maps in the plane

In the following we identify $\mathbb{R}^2$ and the complex field $\mathbb{C}$

A map $f: D \subset \mathbb{R}^2 \to \mathbb{R}^2$ is conformal (assume f of class C^1) if $\partial_1 f \cdot \partial_2 f = 0$ and $|\partial_1 f| = |\partial_2 f|$ in every point, that is

$$\boxed{\nabla f \in \underset{\left\{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| a,b \in \mathbb{R}\right\}}{\mathcal{M}} \cup \underset{\left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix} \middle| a,b \in \mathbb{R}\right\}}{\mathcal{M}'}}$$

Now, $\nabla f \in \mathcal{M}$ is equivalent to say that f satisfies the Cauchy-Riemann equations, that is, f is holomorphic

(and the identification $a+bi \simeq \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ gives the identification of the complex derivative f' with the Jacobian matrix ∇f).

On the other hand, $\nabla f \in \mathcal{M}'$ means that f is anti-holomorphic.

Let us review a few useful facts.

FACT 1

A conformal change of variable preserve Dirichlet energy and conformality.

If $\phi: D' \subset \mathbb{R}^2 \to D \subset \mathbb{R}^2$ is conformal and bijective, for every $u: D \to \mathbb{R}$ (or $\mathbb{R}^m$) there holds

$$\int_D |\nabla u|^2 = \int_{D'} |\nabla (u \circ \phi)|^2 .$$

Moreover for every $X: D \to \mathbb{R}^3$ which is conformal, $X \circ \phi$ is conformal too.

FACT 2

If D is connected and $f: D \to \mathbb{R}^2$ is conformal, then f is either holomorphic or anti-holomorphic.

For f of class C^2, define $A := \{z \in D : \nabla f(z) \in \mathcal{M} \setminus \{0\}\}$, $A' := \{z \in D : \nabla f(z) \in \mathcal{M}' \setminus \{0\}\}$ and $A_0 := \{z \in D : \nabla f(z) = 0\}$. Set

$$g(z) := \begin{cases} \frac{\partial f}{\partial z}(z) & \text{if } z \in A \\ 0 & \text{if } z \in A_0 \\ \frac{\partial f}{\partial \bar z}(z) & \text{if } z \in A' \end{cases}$$

$\left(\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right), \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\right)$. Then g is holomorphic, and therefore is either constantly 0 or $g^{-1}(0)$ is a discrete set, that is A_0 is discrete, which implies that A or A' is empty (because they cannot be separated by a discrete set).

FACT 3

holo $\circ$ holo = holo $(\Leftarrow \mathcal{M} \times \mathcal{M} \subset \mathcal{M})$,

holo $\circ$ antiholo = antiholo $(\Leftarrow \mathcal{M} \times \mathcal{M}' \subset \mathcal{M}')$,

antiholo $\circ$ holo = anti holo $(\Leftarrow \mathcal{M} \times \mathcal{M}' \subset \mathcal{M}')$,

antiholo $\circ$ anti holo = holo $(\Leftarrow \mathcal{M}' \times \mathcal{M}' \subset \mathcal{M})$.

In particular every antiholomorphic function can be written as $f(\bar z)$ or $\overline{f(z)}$ with f holomorphic.

As a corollary of facts 2 and 3, conformal maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ reduce "essentially" to holomorphic maps.

FACT 4

If f is holomorphic & injective then $f' \neq 0$ (i.e. $\nabla f \neq 0$) at every point. Hence f' is holomorphic too.

Assume $f'(z_0) = 0$. Then f can be written as $f(z) = f(z_0) + (z - z_0)^n g(z)$ for some $n \geq 2$, g holomorphic with $g(z_0) \neq 0$ (just write the Taylor series of f at z_0 ...). Since $g(z_0) \neq 0$, it can be locally written as $h^n(z)$ and therefore $f(z) = f(z_0) + ((z - z_0) h(z))^n$, which is clearly NOT injective in any neighbourhood of z_0 ....

FACT 5

If D is the disc ($D = \{z : |z| \leq 1\}$) an holomorphic function $f : D \to \mathbb{C}$ is determined (up to a purely imaginary constant) by the restriction of its Real Part $\mathrm{Re} f$ to the boundary of D.

Indeed $f(z) = \sum_{n=0}^{\infty} a_n z^n \Rightarrow \mathrm{Re} f(e^{i\theta}) =$ $= \sum_n \frac{1}{2} a_n e^{in\theta} + \frac{1}{2} \bar{a}_n e^{-in\theta} \Rightarrow$ the Fourier coefficients of $\mathrm{Re} f(e^{i\theta})$ give $\mathrm{Re}\, a_0$ and a_n for every $m \geq 1$.

QUESTION

How many are the holomorphic bijections from a domain D into itself? and from D into another domain D'?

The answer is: very few!

This is already hinted by fact 5, and indeed we have:

FACT 6

If D is the disc, the holomorphic omeomorfisms of D into itself are of the form

$$\boxed{f(z) = b \, \frac{z + a}{1 + \bar{a} z}} \qquad (*)$$

with $a, b \in \mathbb{C}$, $|b| = 1$ and $|a| < 1$.

We first check that the maps given by $(*)$ are homeomorphisms of D into itself (they are obviously holomorphic). Consider indeed all projective transformations of the projective line $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$, namely the maps $g(z) = \frac{Az + B}{Cz + D}$ with $AD - BC \neq 0$.

Imposing that FOUR points of the circle S^1 (e.g., $\pm 1, \pm i$) are mapped into the circle it is enough to guarantee that all S^1 is mapped onto S^1 because projective transformations maps conics in conics, and one conic is determined by four points. These conditions give all the maps of the form (*) with $|b|=1$. The condition $|a|<1$ is obtained by requiring that 0 is mapped inside D and not outside.

To prove that there are no other holomorphic homeomorphisms of D besides these in (*), it is enough to show that every such homeomorphism f which satisfies the additional constraint $f(0)=0$ is a rotation, that is $f(z)=\lambda z$ for some λ with $|\lambda|=1$ (if f does not satisfies $f(0)=0$, by composing with a suitable map in (*) we get $\tilde{f}$ s.t. $\tilde{f}(0)=0$; if $\tilde{f}$ is a rotation than f is of the form (*)).

Indeed we have $\left|\frac{f(z)}{z}\right|=1$ for every $z\in\partial D=S^1$ (because f maps ∂D in ∂D) and since $\frac{f(z)}{z}$ is holomorphic also in 0, the maximum principle yields $\left|\frac{f(z)}{z}\right|=1$ in D and therefore $\frac{f(z)}{z}$ = constant of modulus 1 in D (an holom. function with constant modulus is constant).

Remark. By fact 6, an holomorphic homeomorphism f of the disc D is uniquely determined by assigning the values $f(z_1)$, $f(z_2)$, $f(z_3)\in S^1$ for given $z_1, z_2, z_3 \in S^1$.

FACT 7

Given the rigidity of holomorphic automorphisms of the disk (fact 6) it is rather surprising that the following holds:

> **RIEMANN MAPPING THEOREM**
> If A is a simply connected open set in $\mathbb{C}$ with $A\neq\mathbb{C}$, then there exists an holom. homeomorphism $f: D\to A$, where D is the disc. Moreover if the boundary of A is sufficiently regular, then f extends to an homeomorphism of $\bar{D}$ in $\bar{A}$.

FACT 8

From the Riemann mapping theorem one might guess that if two open set in $\mathbb{C}$ are homeomorphic they are also isomorphic (holomorphically homeomorphic). It is not so: let $D_{R,r}$ denote the annulus $\{r\le|z|\le R\}$; then $D_{R,r}$ is isomorphic to $D_{R',r'}$ if and only if $R/r = R'/r'$.

Assume for simplicity $R=R'=1$, and let $f: D_{1,r}\to D_{1,r'}$ be an isomorphism. Up to an inversion we can also assume that f maps S^1 in S^1 and rS^1 in $r'S^1$. Then f can be extended by reflection to an isomorphism $\tilde{f}: D_{1,r^2}\to D_{1,r'^2}$.

Take indeed

$$\tilde{f}(z) := \begin{cases} f(z) & \text{if } r \le |z| \le 1, \\ \dfrac{r'^2}{\overline{f(r^2/\bar{z})}} & \text{if } r^2 \le |z| < r. \end{cases}$$

By iterating this reflection procedure one gets an holomorphic homeomorphism $\tilde{f} : D \to D$ such that

$$|f(z^n)| = r'^n = |z^n|^{\frac{\log r'}{\log r}}$$

for every z s.t. $|z| = r$. But then $\frac{\log r'}{\log r}$ must be equal to 1, that is $r = r'$.

Final Remark

We have seen that holomorphic homeomorp. in the plane are quite rigid, and of course the same holds for conformal homeomorphisms (cfr. fact 2 above).

In higher dimension the situation is even worse: let $f : A \subset \mathbb{R}^n \to \mathbb{R}^n$ be a conformal map, that is,

$$\partial_i f \cdot \partial_j f = 0 \quad \forall i \ne j; \quad |\partial_i f| = |\partial_j f| \quad \forall i, j.$$

Liouville's theorem

> Then f is (locally) a composition of omotheties, isometries (rotations and reflections) and inversions $\left(x \mapsto \frac{x}{|x|^2}\right)$.

This class is unsuitable for any variational application.....